Approximate Aggregation in Dynamic Economies*

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Abstract

We provide a characterization of aggregation in the neoclassical growth model with aggregate shocks and uninsurable employment risk. The extent to which an economy aggregates can be determined by examining the (lack of) linearity with respect to wealth in the household's savings function. We take advantage of this fact and separate the consumption-savings problem of the individual household from the Walrasian auctioneer's problem of clearing the aggregate capital market. This allows us to sidestep the curse of dimensionality associated with models of this type while rigorously analyzing household behavior. Our main theorem isolates the nonlinear elements of the savings function, which contain higher-order moments of the wage distribution. As wage dispersion increases, markets become less complete, precautionary savings motives become stronger and aggregation breaks down. Standard calibrations do not contain sufficient wage dispersion to generate substantial disaggregation. Our insights have broader implications for models with heterogeneous agents and can be used to shed light on existing computational techniques.

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1 Introduction

Dynamic models with non-insurable idiosyncratic shocks have become standard in macroeconomics. Early versions of the model were used to examine the role of incomplete insurance on the permanent income hypothesis, monetary and fiscal policy, precautionary savings, the cost of business cycles, asset pricing, etc. [Bewley (1977), Bewley (1986), İmrohoroğlu (1989), Huggett (1993), Aiyagari (1994), Diaz-Gimiez and Prescott (1997), Marcet and Singleton (1999)]. These papers have spawned a vast literature which has demonstrated the importance of the incomplete markets assumption and heterogeneity in agent behavior.

One particularly popular variant of this setting is the one sector neoclassical growth model with idiosyncratic shocks to labor supply of Krusell and Smith (1998) (KS, hereafter). Despite the popularity, few analytical results have been established with respect to the model's properties. Perhaps the best example of this pertains to aggregation. The key message of KS is that the neoclassical growth model with idiosyncratic risk and aggregate shocks features "approximate aggregation." Using a numerical approach that is now well known, KS show that most agents can self-insure through the accumulation of capital. These agents have nearly affine policy functions in the state variables, which permits the aggregation of Gorman (1953, 1961). In equilibrium, there exists a small fraction of agents who are close to their borrowing constraints, but their overall contribution to the aggregate capital stock is so small that it is nearly negligible; hence *approximate* aggregation attains. While this aggregation result is a robust numerical finding, a formal treatment is missing. This paper serves to fill this void.

Our main theorem delivers conditions under which aggregation can be rigorously studied in the neoclassical growth model with idiosyncratic labor shocks. We derive an expression for the savings function of the household which highlights explicitly the permanent income component and the nonlinear adjustment arising from the incomplete market. We show that the nonlinear components are well-behaved functions which are decreasing and convex in household resources, and which depend on higher-order moments of the exogenous wage process. As the wage shock becomes more dispersed, precautionary savings motives become stronger, markets become less complete and aggregation breaks down. The distribution of wealth plays a key role in our analysis. In a simple two-period setting, we show analytically that the combination of incomplete markets (i.e., households cannot borrow) and substantial wealth heterogeneity leads to a departure of aggregation even when the wage is deterministic. As the number of periods increases, aggregation becomes more sensitive to the degree of contemporaneous wealth inequality as households compare current wealth to a discounted stream of future income. Relatively poorer households will want to borrow heavily against future income in order to smooth consumption but will be unable to do so due to the incomplete markets assumption.

Our encompasses that of KS with a finite number of time periods. The primary challenge in solving this model is that the future price of capital depends upon the asset holdings and employment status of each agent. Therefore, the distribution of wealth is a relevant state variable. Since our focus in this paper is purely the aggregation properties of the economy, we introduce a Walrasian auctioneer who sets the level of aggregate capital and labor in advance for all time periods and all outcomes of the shocks. This sequence is then communicated to the agents. Armed with this knowledge, agents are free to make investment-consumption allocations. We show that this approach is amenable to many other economic settings.

Our results and analysis have broader implications for this literature. By understanding the nature of the approximate aggregation result of KS, we can introduce elements to the model that serve to break aggregation. This would allow us to address important research questions that require a departure from the representative agent framework. Our approach offers an alternative (and potentially promising) method to solving models with heterogeneity. Our decomposition of the savings function allows us to intelligently bin agents according to the extent to which their savings function deviates from the permanent income component. In lieu of a representative agent, households would predict aggregates using a coarse representation of the distribution consisting of a few of these representative bins. This would serve to increase the accuracy of these predictions relative to the classic KS algorithm at the cost of a manageable computational penalty.

2 The Economic Environment

Time is discrete and finite, consisting of T periods and indexed by $t = 1, 2, ..., T$. We will use the convention that a new period commences with the arrival of new information. Any variable known or chosen at date t will be indexed by t.

2.1 Households There is a large, measure one, population of households that live for T periods. Households value consumption according to

$$U(c_1, c_2, ..., c_T) = \mathbb{E}_1 \sum_{t=1}^{T} \beta^{t-1} u(c_t) \tag{1}$$

where $0 < \beta < 1$ is the intertemporal discount factor and period utility takes the constant relative risk aversion (CRRA) form

$$u(c_t) = \begin{cases} \frac{c_t^{1-\sigma}}{1-\sigma} & \sigma > 0,\ \sigma \neq 1 \\ \log(c_t) & \sigma = 1 \end{cases}$$

Household income in each period is composed of proceeds from a single savings asset and an endowment which is driven by an individual and exogenous stochastic process. Our primary focus will be on production economies in which savings come in the form of capital and endowments in the form of time or efficiency units to devote to labor. As such, for the general discussion we denote the level of savings brought into period $t+1$ by k_t and the endowment in period t by $\ell_t \geq 0$. We allow for the possibility of savings to depreciate at the rate $\delta \in [0, 1]$.

Letting R_t denote the return on savings net of depreciation and letting W_t denote the price of a unit of endowment in terms of the consumption good, the period resource constraint can now be written as

$$c_t + k_t \leq (1 - \delta + R_t)k_{t-1} + W_t \ell_t$$

where the left hand side is expenditures at time t and the terms on the right hand side are savings and endowment income, respectively.

Households maximize their preferences subject to the above budget constraint and a borrowing limit which will be discussed below, given initial savings k_0 and outcomes ℓ_1, R_1, and W_1. To do so, they must have access to a predictive distribution for the stochastic endowment as well as for the prices R_t and W_t, through which to compute expectations. Since the endowment is exogenous, we will assume that the households have direct knowledge about this process. Prices, however, will be determined in equilibrium. We treat households as price takers in the general sense in that they take the predictive distributions of future prices as given. That is, they take random variables representing current and all possible future returns on savings $\{R_t\}_{t=2}^T$ and endowments $\{W_t\}_{t=2}^T$ as given.

In particular, the value of both savings and endowments may be subject to some aggregate risk. We will maintain this approach throughout our theoretical content in order to preserve generality, while in the applications households will compute current and expected future prices using sufficient aggregate statistics and aggregate laws of motion. To visualize this predictive distribution approach, it is helpful to think of the tree diagram shown in Figure 1 for the case $T = 3$. We assume households know the probabilities with which all possible realizations for prices (R and W) occur.

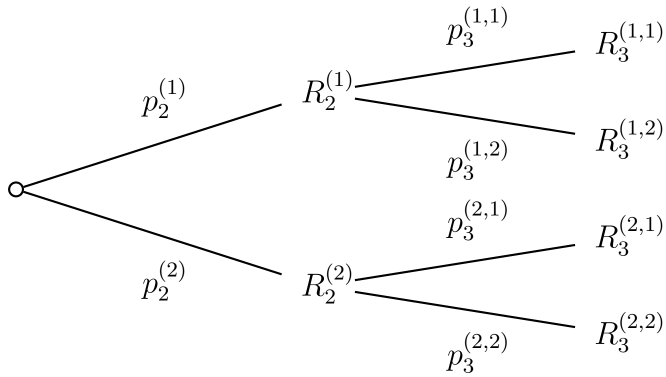

Figure 1: Example of a predictive distribution for $T = 3$. We assume households know all possible future realizations for the return on savings (R), and the corresponding probabilities (p) with which they occur. Subscripts denote time periods and superscripts denote states of the world.

Shocks can enter the model in various forms (e.g., idiosyncratic, aggregate) and can take various correlation structures (e.g., idiosyncratic shocks can be correlated with aggregate shocks). Our primary restriction on the shocks is that there be a finite number of possible income and price outcomes in each future period.

Assumption 1: For each period $t = 2, ..., T$ there is some finite set $\mathcal{S}_t \subset \mathbb{R}^3$ such that the household predictive distribution assigns probability one to the event $(\ell_t, R_t, W_t) \in \mathcal{S}_t$.

This restriction on household predictions is general and allows for rational exectations equilibria in those aggregate environments most commonly found in the literature. Exogenous idiosyncratic and aggregate shocks are not restricted to be Markov and can take general forms of correlation. For example, employment shocks can be serially correlated as well as correlated with exogenous aggregate shocks. The theorems below do not rely on independence unless explicitly stated. While this restriction does rule out commonly used continuous support processes, for example autoregressive series with normal innovations, it admits finite Markov approximations to such series commonly used in numerical solutions.

Our formal analysis will rely heavily on the intertemporal Euler equation being a necessary condition for optimality in the household's problem. It will therefore facilitate the analysis to subject the households to a natural borrowing limit, in which case the asymptote in the period utility function for zero consumption prevents this constraint from binding.

Assumption 2: Natural Borrowing Limit. Household asset holdings must satisfy $k_t \geq \underline{k}_t$ for $t = 1, ..., T$, where $\underline{k}_t$ denotes the natural borrowing limit in period t.

The natural borrowing limit equals the infimum of resource holdings for which the household can guarantee repayment according to its income stream. This is simply the minimum possible outcome of discounted future endowment income. In particular, if the labor endowments are bounded below by zero and there is a positive probability that all future endowments will be null, then the natural borrowing limit is a no-borrowing constraint. In the general case, the borrowing limit in a given period will depend on predictive distributions for both endowments and aggregates.

We can now write the household's sequential problem formally as follows: Given predictive distributions for the random variables $\{R_t\}_{t=2}^T$, $\{W_t\}_{t=2}^T$, and $\{\ell_t\}_{t=2}^T$, the household solves

$$\begin{aligned} \max_{\{c_t\}_{t=1}^T, \{k_t\}_{t=1}^T} \quad & \mathbb{E}_1 \sum_{t=1}^{T} \beta^{t-1} u(c_t) \\ \text{subject to} \quad & c_t + k_t \leq (1 - \delta + R_t) k_{t-1} + W_t \ell_t, \quad t = 1, ..., T \\ & k_t \geq \underline{k}_t, \quad t = 1, ..., T \\ & k_0, \ \ell_1, \ R_1, \ W_1 \ \text{ given} \end{aligned}$$

The inclusion of idiosyncratic uncertainty in the household's problem suggests that the equilibrium will feature ex-post heterogeneity. When considering savings across the distribution of households, it will be convenient to frame a solution to a typical household's problem in terms of an optimal rule which households use to select their savings in each period taking account of their income in that period and their predictions about future income.

To this end, we reformulate a typical household's optimization as a sequence of dynamic programming problems, one for each period in the model. In each such period, the household state will be comprised of its current resource holdings (the combined value of its assets and current endowment) as well as distributions from which to predict future outcomes. In order for the solution to this sequence to produce the same optimal allocations as the sequential problem outlined above, household forecasts will need to be time consistent. Tomorrow's predictions should be derived from today's while incorporating any new information which is available.

Let $\mathcal{L}_1$ and $\mathcal{F}_1$ denote the household predictive distributions in the initial period of life, as in the above sequential problem. These distributions associate to each possible idiosyncratic history of nature a corresponding sequence of endowments and prices respectively. Upon the arrival of new information at the start of the second period, predictions about the future will evolve depending on the prevailing second period state of nature. Specifically, the new predictions $\mathcal{L}_2$ and $\mathcal{F}_2$ will associate to each possible idiosyncratic history reachable from the period 2 state the same sequence of endowments and prices as $\mathcal{L}_1$ and $\mathcal{F}_1$. From information available in period 1, then, the household may compute probabilities that its predictions will evolve to a particular outcome of $\mathcal{L}_2$ and $\mathcal{F}_2$ tomorrow; denote these transition probabilities

$\pi(\mathcal{L}_2, \mathcal{F}_2 | \mathcal{L}_1, \mathcal{F}_1)$, and similarly as additional periods of uncertainty are resolved.

Extending the above, let $\mathcal{L}_t$ and $\mathcal{F}_t$ denote the predictive distributions of future endowments and prices (respectively) at time t,

$$\mathcal{L}_t = \{\ell_s\}_{s=t+1}^{T}, \quad t = 1, ..., T$$
$$\mathcal{F}_t = \{R_s, W_s\}_{s=t+1}^{T}, \quad t = 1, ..., T$$

with corresponding transition probabilities $\pi(\mathcal{L}_{t+1}, \mathcal{F}_{t+1} | \mathcal{L}_t, \mathcal{F}_t)$ Furthermore, let x_t denote period resources at time t, that is

$$x_t = (1 - \delta + R_t)k_{t-1} + W_t \ell_t$$

Then the dynamic programming formulation consists of the sequence of problems

$$\begin{aligned} V^{(t)}(x_t, \mathcal{L}_t, \mathcal{F}_t) \quad &= \quad \max_{c_t, k_t} \Big(u(c) + \beta \mathbb{E}_t V^{(t+1)} (x_{t+1}, \mathcal{L}_{t+1}, \mathcal{F}_{t+1}) \Big) \\ \text{subject to} \quad & \quad c_t + k_t \leq x_t \\ & \quad x_{t+1} = (1 - \delta + R_{t+1})k_t + W_{t+1}\ell_{t+1} \\ & \quad k_t \geq \underline{k}_t \end{aligned} \tag{2}$$

for $t = 1, ..., T$, along with the terminal condition $V^{(T+1)} \equiv 0$.

A solution to this sequence indicates savings functions $k^{(t)}(x_t, \mathcal{L}_t, \mathcal{F}_t)$, $t = 1, ..., T$ giving a typical household's choice of additional asset holdings as a function of current resources and predictions, and this will be our primary object of focus. It will therefore be important that the households' problems have a unique, well-behaved solution, as our first proposition establishes.

Proposition 1: Household Existence and Uniqueness. There is a unique solution to the household's dynamic programming problem (2). The associated savings functions $k^{(t)}$ are increasing (strictly for $t < T$) with respect to x_t and satisfy

$$\lim_{x_t \to \underline{k}_t} k^{(t)}(x_t, \mathcal{L}_t, \mathcal{F}_t) = \underline{k}_t,$$
$$\lim_{x_t \to \infty} k^{(t)}(x_t, \mathcal{L}_t, \mathcal{F}_t) = \infty, \quad t < T$$

The corresponding value functions are strictly increasing and strictly concave with respect to x_t and satisfy

$$\lim_{x_t \to \underline{k}_t} V^{(t)}(x_t, \mathcal{L}_t, \mathcal{F}_t) = -\infty$$

Proof. See Appendix I. □

Our assumptions ensure that the inter-temporal Euler equations are a necessary condition for optimality. In terms of the asset choice, these conditions read

$$(x_t - k_t)^{-\sigma} = \beta \mathbb{E}_t (1 - \delta + R_t)(x_{t+1} - k_{t+1})^{-\sigma}, \quad t = 1, ..., T-2 \tag{3}$$

$$(x_{T-1} - k_{T-1})^{-\sigma} = \beta \mathbb{E}_t (1 - \delta + R_T) x_T^{-\sigma} \tag{4}$$

Our main theorem will explore the properties of a household's savings function, and thus admits interpretation in a variety of economic environments. Our eventual focus will be on the neoclassical growth model with idiosyncratic labor shocks. Before narrowing the perspective to production economies, however, we pause to note that the framework developed thus far can be adapted to a pure credit environment, which is a continuous asset space version of Huggett (1993).

Example 1: Pure Credit, No Aggregate Uncertainty.

In this environment, households receive a stochastic endowment of consumption goods in each period and trade one-period bonds ($\delta = 1$) with a risk free rate of return r_t, $t = 1, ..., T$ in order to smooth consumption and partially insure against low endowment outcomes. Consider the case in which the endowment is independently, identically distributed across some finite set of values $0 < y_1 < ... < y_L$ in each period with associated probabilities $\pi_1, ..., \pi_L$ satisfying $\pi_1 + ... + \pi_L = 1$.

The predictive distribution of the endowment in each period is then the discrete distribution placing probability π_j on the value y_j. The predictive distribution of R_t simply places mass 1 on the constant $1 + r_t$, $t = 2, ..., T$, and the predictive distribution of W_t places mass 1 at unity, so that there is no aggregate risk. Since these distributions are both degenerate, we may collapse the state space of the household problem to include only current period resources x_t and future returns $r_{t+1}, ..., r_T$.

2.2 Firms In the context of a production economy, the savings vehicle available to the households will be capital and the endowment will come in the form of labor efficiency units. The income from these assets will come from renting them out to firms which operate in perfectly competitive factor and product markets. The aggregate production technology is Cobb-Douglas,

$$Y_t = F(Z_t, K_{t-1}, L_t) = Z_t K_{t-1}^{\alpha} L_t^{1-\alpha}, \quad t = 1, ..., T$$

with $\alpha \in [0, 1]$. Aggregate capital and labor are denoted K and L respectively, and Z is an aggregate productivity shock. Profit maximization delivers the rental rate of capital and the wage rate as

$$R_t = \alpha Z_t \left(\frac{K_{t-1}}{L_t} \right)^{\alpha - 1} \tag{5}$$

$$W_t = (1 - \alpha) Z_t \left(\frac{K_{t-1}}{L_t} \right)^{\alpha} \tag{6}$$

for $t = 1, ..., T$.

Provided firms are optimizing, households can forecast prices by predicting total productivity and firm demand for capital and labor. Like individual efficiency, we assume that the aggregate productivity shock follows some exogenous but unspecified stochastic process with strictly positive support. In order to maintain Assumption 1 we will restrict it to some finite set in each period.

Assumption 3: Technology Shock. *For $t = 2, .., T$, the aggregate technology shock Z_t takes on values in a finite set $\mathcal{Z}_t \subset (0, \infty)$.*

Assuming that, as with individual efficiency, the household predictive distribution for productivity is derived from the underlying exogenous stochastic process, they are left needing a way to formulate expectations regarding firm inputs. This plays a central role in the equilibrium concept for this economy.

2.3 Competitive Equilibrium The general equilibrium concept will involve finding predictive distributions for prices such that when the economic agents (households in a credit model, or households and firms in a production economy) take these as given and solve their respective optimization problems, supply and demand are balanced so that markets clear. Market clearing is complicated by the fact that ex-post heterogeneity on the household side means that computing aggregates requires us to keep track of the entire distribution of households across resources and endowments, beginning from some exogenous initial distribution. Allowing for general aggregate fluctuations means that at time $t \geq 2$ this distribution could depend on the entire history of aggregate shock outcomes. The number of such histories grows exponentially with t, so that we are faced with a substantial curse of dimensionality.

Our interest in building a theoretical description of aggregation allows us to sidestep this issue by using the fact that a competitive equilibrium requires households to behave as price takers, implying that they do not take into account how their consumption / savings behavior impinges on aggregates. We can characterize aggregation by studying the typical household's savings function taking the predictive distributions as given.

2.4 Equilibrium in Pure Credit Economies

Since equilibrium is relatively straightforward to construct in the pure credit environment, we begin by providing several examples in this case before moving on to a general treatment of production economies. We begin by noting that the definition of equilibrium in the context of Example 1 is familiar:

Example 1: Pure Credit, No Aggregate Uncertainty.

Recall that in this example the idiosyncratic endowments are IID, taking value y_l with probability π_l for $l = 1, ..., L$. Let λ_1 denote an exogenous initial distribution of households across incomes. Then equilibrium consists of a sequence of returns $\{r_t\}_{t=1}^T$, consumption, savings, and value functions $\{V^{(t)}, k^{(t)}, c^{(t)}\}_{t=1}^T$ and household distributions $\{\lambda_t\}_{t=2}^T$ across resources such that: [i.] taking current and future prices as given, the functions $\{V^{(t)}, k^{(t)}, c^{(t)}\}$ solve the household's dynamic programming problems; [ii.] market clearing holds in each period:

$$\int k^{(t)}(x_t)\; d\lambda_t(x_t) = 0, \quad t = 1, ..., T-1$$

[iii.] the distributions are consistent with the household savings behavior:

$$\lambda_{t+1}(x_{t+1}) = \sum_{l=1}^{L} \int_{\underline{k}_t}^{\infty} 1\big([1 + r_{t+1}]k^{(t)}(x_t) + y_l = x_{t+1}\big)\pi_l \; d\lambda_t(x_t)$$

□

Allowing for serial correlation in endowments in the above example complicates the situation, owing to the fact that predictive probabilities will vary across the population of households. Since we are treating predictions as a high dimensional state variable in the household problem, we need to include the distribution $\lambda_t(x_t, \mathcal{L}_t)$ of households over both income and idiosyncratic predictions as an equilibrium object. In order to keep the problem manageable, we will assume that there are a finite set of household idiosyncratic futures in the economy.

Assumption 4: Finite Idiosyncratic Predictions. *For each time t there is a finite set $\{\mathcal{L}_{1t}, ..., \mathcal{L}_{Lt}\}$*

such that

$$\sum_{i=1}^{L} \int_{\underline{k}_t}^{\infty} d\lambda_t(x_t, \mathcal{L}_{it}) \equiv 1.$$

To illustrate the formulation of equilibrium with idiosyncratic uncertainty but no aggregate uncertainty, we consider a two state Markov process in place of the IID shocks to endowments of Example 1.

Example 2: Pure Credit, No Aggregate Uncertainty, Markov Endowments.

Consider a pure credit economy in which the endowment follows a two-step Markov process. This process consists of two states y_{low} and y_{high} with probability of transitioning from a low efficiency state to a high efficiency state given by $\pi(y_{\text{high}}|y_{\text{low}})$, and vice versa for $\pi(y_{\text{low}}|y_{\text{high}})$. In each period, household predictive probabilities fall into one of two categories $\mathcal{L}_{\text{low}}$ and $\mathcal{L}_{\text{high}}$, depending on that period's realization.

Let λ_1 denote an exogenous initial distribution of households across income and expectations. Then competitive equilibrium consists of a deterministic sequence of returns $\{r_t\}_{t=1}^T$, consumption, savings, and value functions $\{V^{(t)}, k^{(t)}, c^{(t)}\}_{t=1}^T$ and household distributions $\{\lambda_t\}_{t=2}^T$ across income and predictive probabilities such that: [i.] taking current and future prices as given, the functions $\{V^{(t)}, k^{(t)}, c^{(t)}\}$ solve the household's dynamic programming problems; [ii.] market clearing holds in each period:

$$\sum_{i \in \{\text{low, high}\}} \int_{\underline{k}_t}^{\infty} k^{(t)}(x_t, \mathcal{L}_i)\, d\lambda_t(x_t, \mathcal{L}_i) = 0, \quad t = 1, ..., T-1$$
$$\sum_{i \in \{\text{low, high}\}} \int_{\underline{k}_t}^{\infty} c^{(t)}(x_t, \mathcal{L}_i)\, d\lambda_t(x_t, \mathcal{L}_i) = \sum_{i \in \{\text{low, high}\}} \int_{\underline{k}_t}^{\infty} y_i\, d\lambda_t(x_t, \mathcal{L}_i), \quad t = 1, ..., T$$

[iii.] the distributions are consistent with the household savings behavior:

$$\lambda_{t+1}(x_{t+1}, \mathcal{L}_i) = \sum_{j \in \{\text{low, high}\}} \int_{\underline{k}_t}^{\infty} 1\big([1 + r_{t+1}]k^{(t)}(x_t, \mathcal{L}_j) + y_i = x_{t+1}\big)\pi(y_i|y_j)\, d\lambda_t(x_t, \mathcal{L}_j)$$

Note that, in this case, the current period endowment realization is a sufficient statistic to determine predictive distributions. For practical applications, then, it is much simpler to collapse the household state space to current income and endowment, and reformulate the above definition in terms of these variables.

When aggregate uncertainty is present, households also need to form nondegenerate expectations about future prices. Since prices are an aggregate feature of the economy, these expectations will be constant across the distribution.

Assumption 5: Uniform Aggregate Predictions. *All households take as given the same predictive distributions for* $\{R_t,\ W_t\}$.

Our third example illustrates equilibrium with aggregate uncertainty.

Example 3: Pure Credit, Aggregate and Idiosyncratic Uncertainty.

Consider a pure credit economy in which the aggregate state Z has two outcomes, good and bad, and the endowment also has two values, y_{low} and y_{high}. Assume that the aggregate state and endowment jointly follow a Markov process which is calibrated so that the fraction of households which receive a high endowment is higher in a good aggregate state than in a bad one (see, for example, the process in Krusell and Smith, 1998).

In each period, household idiosyncratic predictive probabilities fall into one of four categories $\mathcal{L}_{\text{low,bad}}$, $\mathcal{L}_{\text{low,good}}$, $\mathcal{L}_{\text{high,bad}}$, and $\mathcal{L}_{\text{high,good}}$. Let the probability of transitioning from endowment state y_j to endowment state y_i conditional on today's aggregate state being Z_t and tomorrow's being Z_{t+1} be given by $\pi(y_i|Z_t, y_j, Z_{t+1})$.

Let λ_1 denote an exogenous initial distribution of households across income and endowment expectations. Then competitive equilibrium consists of predictive distributions for returns $\mathcal{F}_t = \{r_s\}_{s=t}^T$ for $t = 1, ..., T$, consumption, savings, and value functions $\{V^{(t)},\ k^{(t)},\ c^{(t)}\}_{t=1}^T$ and household distributions $\{\lambda_t\}_{t=2}^T$ across income and endowment predictions such that: [i.] taking predictions as given, the functions $\{V^{(t)},\ k^{(t)},\ c^{(t)}\}$ solve the household's dynamic programming problems; [ii.] market clearing holds in each period and for each aggregate history:

$$\sum_i \int_{\underline{k}_t}^{\infty} k^{(t)}(x_t, \mathcal{L}_i, \mathcal{F}_t)\, d\lambda_t(x_t, \mathcal{L}_i) = 0, \quad t = 1, ..., T-1$$
$$\sum_i \int_{\underline{k}_t}^{\infty} c^{(t)}(x_t, \mathcal{L}_i, \mathcal{F}_t)\, d\lambda_t(x_t, \mathcal{L}_i) = \sum_{i \in \{\text{low, high}\}} \int_{\underline{k}_t}^{\infty} y_i\, d\lambda_t(x_t, \mathcal{L}_i), \quad t = 1, ..., T$$

[iii.] the distributions are consistent with the household savings behavior:

$$\lambda_{t+1}(x_{t+1}, \mathcal{L}_i) = \sum_j \int_{\underline{k}_t}^{\infty} 1\big([1 + r_{t+1}]k^{(t)}(x_t, \mathcal{L}_j, \mathcal{F}_t) + y_i = x_{t+1}\big)\pi(y_i|Z_t, y_j, Z_{t+1})\, d\lambda_t(x_t, \mathcal{L}_j)$$

Note that, in this case, the sequence $\{\lambda_t\}_{t=2}^T$ retains a stochastic element which is driven by the aggregate uncertainty, so that it is a sequence of random distributions. Also notice that the Markov structure allows for the collapse of the household state space to current income, current endowment, and current aggregate state. □

2.5 Equilibrium in Production Economies With a representative firm as in Section 2.2, we could formulate an equilibrium in much the same fashion as in the pure credit case: find predictive distributions for prices and household distributions across income and predictions such that in each period (and for each aggregate history) the capital and labor markets clear. Alternatively, we can formulate an equilibrium concept by imposing the market clearing and firm optimization conditions and focusing on the household side in a manner consistent with previous literature.

In equilibrium, firm demand for capital and labor will equal the aggregate factors supplied by households. Since firms are optimizing in equilibrium, then, the factor prices are given by the expressions (5) and (6) with firm demands replaced by aggregates. It follows that in order to form beliefs about future prices in equilibrium it is enough for the households to have access to predictive distributions regarding aggregate capital and labor from which to compute prices using the firm optimization expressions.

Assumption 6: Aggregate Predictive Distributions. *Each household takes as given the same predictive distribution of aggregate variables* $\{K_t, L_t\}_{t=2}^{T}$.

Invoking a cross sectional law of large numbers in this setting, analogous to condition [iii] of equilibrium in Example 2 and Example 3,[1] aggregate labor will derive from the exogenous idiosyncratic processes, so that we may assume that households have rational expectations about future aggregate labor. The predictive distribution for aggregate capital is more complicated, as it derives endogenously from the household decisions. An additional equilibrium condition will therefore ensure that household predictions about next period capital come true.

Definition 1: Competitive Equilibrium. Let λ_1 denote an exogenous initial distribution of households across income and labor efficiency expectations. Then competitive equilibrium consists of predictive distributions for aggregate capital $\mathcal{F}_t = \{K_s\}_{s=t+1}^{T}$ for $t = 1, ..., T$, consumption, savings, and value functions $\{V^{(t)},\ k^{(t)},\ c^{(t)}\}_{t=1}^{T}$ and household distributions $\{\lambda_t\}_{t=2}^{T}$ across income and efficiency predictions such that: [i.] taking predictions as given, the functions $\{V^{(t)},\ k^{(t)},\ c^{(t)}\}$ solve the household's dynamic programming problems; [ii.] prices are given by (5) and (6); [iii.] the distributions are consistent with the household

[1]The consistency of such a law of large numbers with the behavior of a continuum of households is a measure-theoretic question, the discussion of which would take us too far abroad of our objective. See, for example, Judd (1985) and Uhlig (1996).

savings behavior:

$$\lambda_{t+1}(x_{t+1},\mathcal{L}_i)=\sum_{j=1}^{L}\int_{\underline{k}_t}^{\infty}1\big([1-\delta+R_{t+1}]k^{(t)}(x_t,\mathcal{L}_j,\mathcal{F}_t)+W_{t+1}\ell_{t+1}=x_{t+1}\big)$$
$$\times\ \pi(\mathcal{L}_i|\mathcal{F}_t,\mathcal{L}_j,\mathcal{F}_{t+1})\,d\lambda_t(x_t,\mathcal{L}_j)$$

[iv.] household one-period forward aggregate predictions are realized:

$$K_{t+1}=\sum_{j=1}^{L}\int_{\underline{k}_t}^{\infty}k^{(t)}(x_t,\mathcal{L}_j,\mathcal{F}_t)\,d\lambda_t(x_t,\mathcal{L}_j)$$

The idea of adjusting prices to clear markets in equilibrium may be thought of in terms of a Walrasian auctioneer (henceforth, the auctioneer) who fixes aggregate levels of the capital stock K_t for all times and all possible outcomes of shocks.[2] The households then solve their problems, optimizing expected discounted utility, taking the auctioneer's forecasts as given (condition [i]) and computing the corresponding prices (condition [ii]). This produces savings and consumption rules $\{k^{(t)}, c^{(t)}\}$ in each period t, which in turn imply the manner in which the household distribution evolves over time and shocks (condition [iii]). The auctioneer thereby obtains a listing of desired aggregates by integrating the values of $k^{(t)}$ over households. If these desired aggregates agree with the original forecasts, the auctioneer has solved her problem (condition [iv]). Otherwise, she will adjust her forecasts and try again.

A complete characterization of the equilibrium must contain a careful analysis of the auctioneer's problem.[3] However our goal here is to study how the individual savings decisions of households aggregate in the economy. Insofar as our theoretical claims will be robust to whatever future beliefs about prices the households may possess, we do not need to solve the auctioneer's problem in order to discuss aggregation.

2.6 Examples of Production Economies There are three successively more general production economies that we will examine at various stages in order to illustrate the content of our theory based on the nature of risk that the economic agents face. These economies closely parallel the pure credit examples previously introduced.

First, a degenerate example involves no uncertainty whatsoever. This example will play a central theme in the general theory, and so we take a moment to explore its properties.

[2]This idea is not new and dates back to at least to Prescott and Mehra (1980). Ljungqvist and Sargent (2004) contains a textbook treatment referring to it as the "Big K, little k trick."

[3]We should note that we currently do not have a proof of uniqueness for the auctioneer's problem.

Example 4: No Uncertainty.

In this environment, households receive a constant, nonstochastic labor efficiency endowment ℓ in each period. They rent capital k_t and labor ℓ to the firms at rates R_t and W_t, respectively. Total factor productivity is constant, so that output is $Y_t = ZK_{t-1}^{\alpha}\ell^{1-\alpha}$.

In this instance, the predictive distribution of efficiency places unit mass on ℓ in every period. The predictive distribution of capital place unit mass on each of T values $K_1, ..., K_T$ respectively, one value for each period. As in the pure credit example, determinacy allows us to collapse a household's state space to include only current period resources while treating endowments and prices as embedded.

Within this environment the household Euler equations (3) and (4) simplify to

$$(x_t - k_t)^{-\sigma} = \beta(1 - \delta + R_t)(x_{t+1} - k_{t+1})^{-\sigma}, \quad t = 1, ..., T-2 \tag{7}$$

$$(x_{T-1} - k_{T-1})^{-\sigma} = \beta(1 - \delta + R_T)x_T^{-\sigma} \tag{8}$$

These equations may be algebraically simplified to become linear in endogenous variables, admitting a linear solution for time t capital accumulation in terms of time t resources. In the simplest case, when $\sigma = 1$ (log utility), this solution has the form

$$k^{(t)}(x_t) = \frac{\beta + ... + \beta^{T-t}}{1 + \beta + ... + \beta^{T-t}} x_t - \frac{1}{1 + \beta + ... + \beta^{T-t}} \left(\sum_{s=t+1}^{T} \frac{W_s \ell}{\prod_{r=t+1}^{s}(1 - \delta + R_r)} \right) \tag{9}$$

The savings function is characterized by permanent income behavior. The household saves a constant fraction of wealth (x_t) and a constant fraction of discounted future wages. The difference between the two measures the extent to which the household would like to borrow against future labor income to fund additional consumption today.

In the general CRRA case, a more complicated but analogous formula holds as income and substitution effects no longer offset. In particular, if we define effective discount factors

$$Q_{T-1} = \left[\beta(1 - \delta + R_T)^{1-\sigma}\right]^{1/\sigma} \tag{10}$$

$$Q_t = \left[\beta(1 - \delta + R_{t+1})^{1-\sigma}\right]^{1/\sigma} (1 + Q_{t+1}), \quad t = 1, ..., T-2 \tag{11}$$

then we obtain

$$k^{(t)}(x_t) = \frac{Q_t}{1 + Q_t} x_t - \frac{1}{1 + Q_t} \left(\sum_{s=t+1}^{T} \frac{W_s \ell}{\prod_{r=t+1}^{s}(1 - \delta + R_r)} \right) \tag{12}$$

Observe that this expression reduces to the one for logarithmic utility upon substituting

$\sigma = 1$. For details of these derivations, see Proposition 2 in Appendix I.

The next example adds idiosyncratic uncertainty into the household's labor efficiency process but maintains the assumption of constant TFP, which is the setting of Aiyagari (1994). Considering such a setting will allow us to illustrate the content of our theory while removing complications which result with aggregate uncertainty.

Example 5: No Aggregate Uncertainty.

In this environment, household labor efficiency follows an exogenous Markov chain. This process consists of two states ℓ_{low} and ℓ_{high} with probability of transitioning from a low efficiency state to a high efficiency state given by $\pi(\ell_{\text{high}}|\ell_{\text{low}})$, and vice versa for $\pi(\ell_{\text{low}}|\ell_{\text{high}})$. A cross-sectional law of large numbers ensures that aggregate efficiency follows a deterministic series given an initial distribution of households across efficiency states. Households rent capital k_t and labor ℓ_t to the firms at rates R_t and W_t, respectively. Total factor productivity is constant, so that output is $Y_t = ZK_{t-1}^{\alpha}L_t^{1-\alpha}$.

In this instance, the predictive distribution of efficiency in period t for a household with efficiency $\ell_1 \in \{\ell_{\text{low}}, \ell_{\text{high}}\}$ places a mass equal to the probability of transitioning from ℓ_1 to ℓ_{low} in $t-1$ transitions on ℓ_{low}, and likewise for ℓ_{high}. Since TFP and aggregate efficiency are deterministic, the predictive distribution for capital places unit mass on each of T values $K_1, ..., K_T$ respectively, one value for each period. In this instance, the Markov structure allows us to collapse a household's state space to include only current period resources and current efficiency, insofar as these are sufficient to determine all future transition probabilities. □

Finally, we record our motivating example and overall focus, a finite horizon version of the economy of Krusell and Smith (1998). This economy adds aggregate uncertainty in the form of business cycle fluctuations in total factor productivity which are moreover correlated with the level of aggregate labor efficiency.

Example 6: Idiosyncratic and Aggregate Uncertainty.

In this environment, factor productivity and household labor efficiency follow an exogenous joint Markov chain. This process consists of four states consisting all possible pairs (Z_t, ℓ_t) with $Z_t \in \{Z_{\text{low}}, Z_{\text{high}}\}$ and $\ell_t \in \{\ell_{\text{low}}, \ell_{\text{high}}\}$. Using notation analogous to the previous example, we label transition probabilities by $\pi(Z_{t+1}, \ell_{t+1}|Z_t, \ell_t)$ for $t = 1, ..., T-1$. A cross-sectional law of large numbers ensures that aggregate efficiency follows a deterministic series given an initial distribution of households across efficiency states and a full aggregate history. Households rent capital k_t and labor ℓ_t to the firms at rates R_t and W_t, respectively. Output is $Y_t = Z_tK_{t-1}^{\alpha}L_t^{1-\alpha}$.

In this instance, the predictive distribution of efficiency in period t for a household with efficiency $\ell_1 \in \{\ell_{\text{low}}, \ell_{\text{high}}\}$ when the aggregate state is Z_1 places on ℓ_{low} a mass equal to the probability of transitioning from ℓ_1 and Z_1 to ℓ_{low} and any aggregate state in $t-1$ transitions, and likewise for ℓ_{high}. The predictive distribution of capital places positive mass on each of 2^{t-1} values, one value for each aggregate history up to time t, the mass being equal to the probability of that history being realized. In this instance, the Markov structure allows us to collapse a household's state space to include only current period resources, efficiency, and aggregate state, insofar as these are sufficient to determine all future transition probabilities. □

In the next section, we begin to explore the phenomenon of approximate aggregation with a focus on the solution to the household's problem. Our main result will show that the entire class of models described thus far share the feature that adding risk to the economy results in a well-behaved perturbation to the closed form solution of Example 4. This perturbation will be seen to reflect household risk aversion and prudence.

3 Approximate Aggregation

This section builds sequentially to our main aggregation theorem.

3.1 Aggregation in a Deterministic Environment In the economy of Example 4 with the natural borrowing constraint, we showed above that, given any sequence of prices, the savings function which solves the household's problem is linear. Integrating the expressions in (9) and (12) with respect to the household distribution, we therefore obtain a corresponding law of motion for aggregate capital

$$K_t = \begin{cases} \frac{\beta+\ldots+\beta^{T-t}}{1+\beta+\ldots+\beta^{T-t}} X_t - \frac{1}{1+\beta+\ldots+\beta^{T-t}} \left(\sum_{s=t+1}^{T} \frac{W_s \ell}{\prod_{r=t+1}^{s}(1-\delta+R_r)} \right) & \text{if } \sigma = 1 \\ \frac{Q_t}{1+Q_t} X_t - \frac{1}{1+Q_t} \left(\sum_{s=t+1}^{T} \frac{W_s \ell}{\prod_{r=t+1}^{s}(1-\delta+R_r)} \right) & \text{if } \sigma \neq 1 \end{cases}$$

where $X_t = (1-\delta+R_t)K_{t-1} + W_t\ell$ is aggregate resources at time t, and where Q_t was defined in (10)-(11).

The above law of motion states the familiar fact that the real business cycle model without idiosyncratic uncertainty admits a representative agent: we obtain the same value for all aggregates in every period if we consider a unit mass of households all holding initial resources X_1. This aggregation result relies heavily on the fact that an ex ante resource poor household has the ability to borrow against a high future wage stream, up to its natural borrowing limit. We now relax this assumption and investigate the consequences for aggregation if borrowing is simply disallowed.

Consider the simple case with just a single savings decision ($T = 2$) and log utility ($\sigma = 1$), and ad-hoc borrowing constraint $k_1 \geq 0$. Further, we will consider the case in which capital depreciates completely every period, $\delta = 1$. While this is not in line with standard calibrations of the model, it has the attractive feature that aggregate resources in each period is just aggregate output in that period, which leads to a convenient simplification in the equilibrium prices.

Specifically, a typical household's problem can now be written as

$$\begin{aligned} \max_{c_1,c_2,k_1} \quad & u(c_1) + \beta u(c_2) \\ \text{subject to} \quad & c_1 + k_1 \leq x_1 \\ & c_2 = x_2 = R_2 k_1 + W_2 \ell_2 \\ & k_1 \geq 0 \end{aligned}$$

with x_1 given.

The effect of the tighter borrowing constraint is simply that a household which would borrow under a natural borrowing limit can no longer substitute across periods. It will now behave in a hand-to-mouth fashion, consuming all of its resource endowment in the first period and its labor income in the second. Applying the fact that, with total depreciation, the ratio of wage rate to rental rate in equilibrium is just the ratio of shares going to these factors, $W_t/R_t = (1-\alpha)K_2/\alpha L_2$, we obtain the savings function

$$k_1 = \begin{cases} \frac{1}{1+\beta}\left(\beta x_1 - \frac{(1-\alpha)K_2}{\alpha L_2}\ell_2\right), & \text{if } x_1 \geq \frac{(1-\alpha)K_2}{\alpha\beta L_2}\ell_2 \\ 0, & \text{otherwise} \end{cases} \tag{13}$$

This savings function is nonlinear: it has a kink where the previous linear function intersects the x axis. For the purposes of this exercise, we will refer to the households which choose a positive level of savings as wealthy, and those for which the constraint binds as poor.

Turning now to the equilibrium we take as given an initial distribution λ_1 of households across resources. If this distribution is such that every household chooses a positive level of savings (every household is wealthy), then this level of savings is linear in initial resources for all households and aggregation once again obtains, with aggregate savings being given by

$$K_2 = \frac{\alpha\beta}{1+\alpha\beta} X_1$$

In turn, for this to hold in equilibrium, given this prediction for aggregate capital all households must decide to save a positive level of capital,

$$x_1 \geq \frac{(1-\alpha)K_2}{\alpha\beta L_2}\ell_2 = \left(\frac{1-\alpha}{1+\alpha\beta}\right)\frac{\ell_2}{L_2}X_1$$

for almost every x_1 in the support of λ_1. Rearranging slightly, we obtain a necessary condition for the economy to aggregate in this simple example,

$$\frac{x_1}{X_1} \geq \left(\frac{1-\alpha}{1+\alpha\beta}\right)\frac{\ell_2}{L_2} \tag{14}$$

Namely, (almost) every household must have a share of initial resources which is sufficiently large relative to its share of the aggregate wage bill. Certainly this will not hold for every initial distribution which we might entertain, and in particular not for those which place a positive measure of households in every neighborhood of 0 initial resource share.

We are interested, then, in cases for which there is a positive measure set of households which will choose to save nothing in the first period,

$$\lambda\,(\text{poor}) > 0$$

where the set of poor agents we define as the set

$$\left\{(x_1, \ell_2) \;:\; x_1 < \frac{(1-\alpha)K_2}{\alpha\beta L_2}\ell_2\right\}$$

with K_2 equal to equilibrium aggregate capital. Integrating the defining expression of this set, we obtain an equilibrium bound on the resources held by the poor households,

$$\int_{\text{poor}} x_1 \, d\lambda_1 < \frac{(1-\alpha)K_2}{\alpha\beta}\int_{\text{poor}} \frac{\ell_2}{L_2}\, d\lambda_1 \tag{15}$$

This bound depends on the endogenous outcome for aggregate capital, however this is entirely driven by the savings behavior of the wealthy households. Integrating the savings function over the complement of the set of poor households, we obtain

$$K_2 = \frac{1}{1+\beta}\left[\beta\int_{\text{wealthy}} x_1 d\lambda_1 - \frac{(1-\alpha)K_2}{\alpha}\int_{\text{wealthy}}\frac{\ell_2}{L_2}\, d\lambda_1\right]$$

which may be solved for

$$K_2 = \left(\frac{\alpha\beta}{\alpha(1+\beta) + (1-\alpha)\int_{\text{wealthy}} \frac{\ell_2}{L_2}\, d\lambda_1} \right) \int_{\text{wealthy}} x_1\, d\lambda_1$$

Substituting this expression into (15), we obtain a necessary condition for equilibrium,

$$\frac{\int_{\text{poor}} \frac{x_1}{X_1}\, d\lambda_1}{\int_{\text{wealthy}} \frac{x_1}{X_1}\, d\lambda_1} < \frac{(1-\alpha)\int_{\text{poor}} \frac{\ell_2}{L_2}\, d\lambda_1}{\alpha(1+\beta) + (1-\alpha)\int_{\text{wealthy}} \frac{\ell_2}{L_2}\, d\lambda_1} \tag{16}$$

In other words, for disaggregation to obtain, the shares of initial resources must be sufficiently skewed relative to labor market outcomes.

In the case of non-total depreciation of capital, $\delta < 1$, we cannot explicitly solve for aggregate capital as above. Nonetheless, it is possible to get an analogous inequality, which reads as

$$\frac{\int_{\text{poor}} \frac{x_1}{X_1}\, d\lambda_1}{\int_{\text{wealthy}} \frac{x_1}{X_1}\, d\lambda_1} < \frac{(1-\alpha)\int_{\text{poor}} \frac{\ell_2}{L_2}\, d\lambda_1}{\alpha(1+\beta)\left[1 + \frac{1-\delta}{R_2}\right] + (1-\alpha)\int_{\text{wealthy}} \frac{\ell_2}{L_2}\, d\lambda_1} \tag{17}$$

From this we can see that partial depreciation has a tendency to promote aggregation by making the necessary condition for disaggregation a stronger inequality. For example, if we fix $\ell_2 = L_2$ to be constant across the population and further fix λ_1 to consist of two equal measure point masses holding initial resources x_{poor} and x_{wealthy}, the bound becomes

$$\frac{x_{\text{poor}}}{x_{\text{wealthy}}} < \frac{(1-\alpha)/2}{\alpha(1+\beta)\left[1 + \frac{1-\delta}{R_2}\right] + (1-\alpha)/2}$$

This is a looser bound on disparity when capital depreciates completely than when it depreciates partially or not at all.

We conclude this discussion by summarizing our results for the deterministic economy in a proposition, after which we will proceed to add back the stochastic elements of our setup and explore the consequences for aggregation.

Proposition 2:Aggregation in Deterministic Economies. In the deterministic setting of Example 4 with the natural borrowing limit, the economy aggregates in the sense that it admits a representative household.

With an ad-hoc borrowing limit at 0, aggregation cannot hold in the model with two periods and total depreciation unless (almost) all households are sufficiently resource-wealthy relative to their future wages, as in (14). If aggregation does not hold, then the distribution of resources is sufficiently skewed relative to the distribution of future wages, as in (16).

Finally, in the model with two periods and partial depreciation, if aggregation does not hold, then the distribution of resources is sufficiently skewed relative to the distribution of future wages, as in (17).

3.2 Uncertainty We now introduce idiosyncratic labor shocks to the two-period setup described above, while returning to a natural borrowing limit. Capital must now serve the dual role of the savings vehicle to intertemporally smooth consumption and as insurance against employment shocks. This, combined with the risk-averse nature of our households, will result in savings functions which are nonlinear across the entire feasible set of resources. Hence the savings decisions will not aggregate for non-degenerate distributions of households across resources.

For algebraic convenience, we will assume for the present discussion that we are in a two-period lived economy of the type described in Example 5, with $\sigma = 1$. The typical household's problem takes the form

$$\begin{aligned} \max_{c_1,c_2,k_1} \quad & u(c_1) + \beta\,\mathbb{E}\,u(c_2) \\ \text{subject to} \quad & c_1 + k_1 = x_1 \\ & c_2 = x_2 = (1-\delta+R_2)k_1 + W_2\ell_2 \\ & \ell_2 = \begin{cases} \ell_{\text{low}} & \text{with probability } p \\ \ell_{\text{high}} & \text{with probability } 1-p \end{cases} \\ & k_1 \geq \underline{k}_1 \end{aligned} \tag{18}$$

with x_1 given. The value of p in the above formulation may vary across the distribution of households if, for example, we are considering x_1 to be generated by initial rental and employment income with a Markov labor transition, as in Example 5.

Recall that the Euler equation (4) is necessary for optimality with the natural borrowing constraint. We write the expected value out explicitly to obtain

$$\frac{1}{x_1 - k_1} = \beta\left(\frac{p(1-\delta+R_2)}{(1-\delta+R_2)k_1 + W_2\ell_{\text{low}}} + \frac{(1-p)(1-\delta+R_2)}{(1-\delta+R_2)k_1 + W_2\ell_{\text{high}}}\right)$$

Taking reciprocals and rearranging the aggregate quantities under the expected value rewrites this as

$$\beta(x_1 - k_1) = \left(\frac{p}{k_1 + \frac{W_2 \ell_{\text{low}}}{1-\delta+R_2}} + \frac{(1-p)}{k_1 + \frac{W_2 \ell_{\text{high}}}{1-\delta+R_2}} \right)^{-1}$$

The left hand side is an expression which is linear in x_1 and k_1, while the right hand side is necessarily nonlinear in k_1 due to the idiosyncratic uncertainty. Nonetheless, we can add the fractions under the brackets to obtain

$$\beta(x_1 - k_1) = \frac{\left(k_1 + \frac{W_2 \ell_{\text{low}}}{1-\delta+R_2}\right)\left(k_1 + \frac{W_2 \ell_{\text{high}}}{1-\delta+R_2}\right)}{k_1 + \frac{W_2}{1-\delta+R_2}\left[p\ell_{\text{high}} + (1-p)\ell_{\text{low}}\right]} \tag{19}$$

This form clarifies the structure of the right hand side: it consists of a rational function in k_1 which is formed from the ratio of a quadratic polynomial to a linear one. The asymptotic behavior of such a function is such that it approaches a linear asymptote as $k_1 \to \infty$. With some additional algebra, we can extract this asymptote, rewriting the above as

$$\begin{aligned} k_1 = &\frac{\beta}{1+\beta} x_1 - \frac{1}{1+\beta} \frac{W_2}{1-\delta+R_2}\left[p\ell_{\text{low}} + (1-p)\ell_{\text{high}}\right] \\ &+ \frac{1}{1+\beta}\left(\frac{W_2}{1-\delta+R_2}\right)^2 \left(\frac{p(1-p)(\ell_{\text{high}} - \ell_{\text{low}})^2}{k_1 + \frac{W_2}{1-\delta+R_2}\left[p\ell_{\text{high}} + (1-p)\ell_{\text{low}}\right]} \right) \end{aligned} \tag{20}$$

To facilitate interpretation of this expression, let $n_2 := \frac{W_2}{1-\delta+R_2}(\ell - \ell_{\text{low}})$ denote the household's discounted excess earnings, so that

$$n_2 = \begin{cases} 0 & \text{with probability } p \\ \frac{W_2}{1-\delta+R_2}(\ell_{\text{high}} - \ell_{\text{low}}) & \text{with probability } 1-p \end{cases}$$

Then a straightforward calculation gives

$$\text{Var}(n_2) = p(1-p)\left[\frac{W_2}{1-\delta+R_2}(\ell_{\text{high}} - \ell_{\text{low}})\right]^2$$

$$\frac{W_2}{1-\delta+R_2}\ell_{\text{low}} + \frac{\text{Var}(n_2)}{\mathbb{E}(n_2)} = \frac{W_2}{1-\delta+R_2}\left[p\ell_{\text{high}} + (1-p)\ell_{\text{low}}\right]$$

so that our rearranged Euler equation gives

$$k_1 = \frac{1}{1+\beta}\left(\beta x_1 - \frac{W_2}{1-\delta+R_2}\,\mathbb{E}\,\ell_2 + \left(\frac{\mathrm{Var}(n_2)}{k_1 + \frac{W_2}{1-\delta+R_2}\ell_{\text{low}} + \frac{\mathrm{Var}(n_2)}{\mathbb{E}(n_2)}}\right)\right) \tag{21}$$

The first two terms on the right-hand side of this equation are the same as the deterministic case (9), with the deterministic wage replaced by the expected wage. The final term is strictly positive and captures the precautionary savings motive of the household. It is a nonlinear function of household capital choice, k_1. Thus, the savings function is nonlinear in wealth and aggregation will not hold. The extent to which the economy aggregates depends upon the size of this term. We therefore refer to it as the nonlinear error in the otherwise linear savings rule. If this error is small, the economy will "approximately aggregate." If the error is large, aggregation breaks down.

The nonlinear error term has the disadvantage that it is endogenous. However, from Proposition 1 we know that there is some well-behaved function $k^{(1)}$ of resources[4] x_1 which solves the household's problem, and so we can write the above equation as

$$k^{(1)}(x_1) = \frac{1}{1+\beta}\left(\beta x_1 - \frac{W_2}{1-\delta+R_2}\,\mathbb{E}\,\ell_2 + \epsilon^{(1)}(x_1)\right)$$

where

$$\epsilon^{(1)}(x_1) := \left(\frac{\mathrm{Var}(n_2)}{k^{(1)}(x_1) + \frac{W_2}{1-\delta+R_2}\ell_{\text{low}} + \frac{\mathrm{Var}(n_2)}{\mathbb{E}(n_2)}}\right)$$

is a well-defined function whose closed form is inaccessible to us.[5] Nonetheless, we can derive properties of this error from what we know about the savings function. For example, differentiating $\epsilon^{(1)}(x_1)$ and recalling that $k^{(1)}$ is strictly increasing, we obtain that the nonlinear error is strictly decreasing in resources. A more sophisticated argument shows that it is strictly convex, and we can moreover calculate its limits as resources approach the endpoints of the domain. We summarize this as a theorem.

Theorem 1: Nonlinear Error, Log Utility. The savings function $k^{(1)}(x_1)$ which solves model

[4] In this section and the next, for simplicity of notation we suppress the dependence of the partial equilibrium objects on predictive distributions.

[5] As we will see below, this environment is sufficiently simple that we could, in principle, derive a closed form expression for $k^{(1)}$. However, this will not be the case in general.

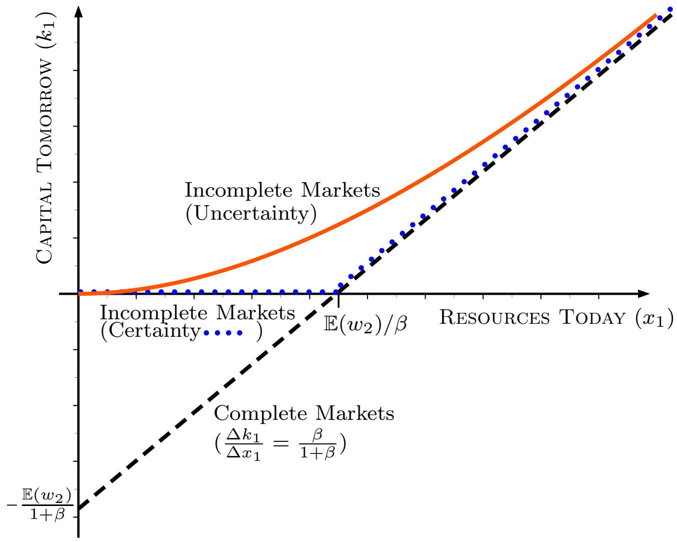

Figure 2: Savings Function ($k^{(1)}$) plotted against resources (x_1) for complete markets (dashed line), incomplete markets with certainty (dotted), and incomplete markets with uncertainty (solid). We define $w_2 = W_2\ell_2/(1-\delta+R_2)$.

(18) can be written in the form

$$k^{(1)}(x_1) = \frac{1}{1+\beta}\left(\beta x_1 - \frac{W_2}{1-\delta+R_2}\,\mathbb{E}\,\ell_2 + \epsilon^{(1)}(x_1)\right) \tag{22}$$

where the nonlinear error term is strictly decreasing, strictly convex, and satisfies

$$\lim_{x_1\to\underline{k}_1}\epsilon^{(1)}(x_1) = \frac{W_2}{1-\delta+R_2}\mathbb{E}_1\ell_2 + \underline{k}_1, \qquad \lim_{x_1\to\infty}\epsilon^{(1)}(x_1) = 0$$

Proof. See Appendix I. □

Adopting the notation $w_2 = W_2\ell_2/(1-\delta+R_2)$, Figure 2 plots the savings function $k^{(1)}$ against initial resources x_1 under complete markets (dashed line), incomplete markets with certainty (dotted line, (13)) and incomplete markets with uncertainty (solid line, (22)), for simplicity taking $\ell_{low} = 0$. This figure is a pseudo-replica of Figure 2 of Krusell and Smith (1998).

With complete markets, households can fully insure across states of nature via state-contingent assets. Savings turn negative when the household's wealth (x_1) falls below tomorrow's discounted expected wage, $\mathbb{E}(w_2)/\beta$. Households can borrow up to the discounted expected payout from next period's asset holdings. Under complete markets and for any point on the distribution of wealth, the slope of the savings function is constant at $\beta/(1+\beta)$. Aggregation holds.

When we do not allow households to borrow but future wages are known with certainty (dotted line), households save the same constant fraction until wealth falls below the discounted wage. At that point, savings equal zero as discussed in the previous subsection. With uncertainty, the savings function limits to zero as wealth falls to this natural borrowing limit, and asymptotes to the complete markets savings function as wealth increases. The discrepancy in the savings functions under complete and incomplete markets when wealth is evaluated at the expected discounted wage may be expressed as a function of the variance and mean of discounted excess wages.[6]

As in the deterministic case, the relationship between the expected wage and the distribution of wealth is critical. As the expected discounted wage decreases relative to the distribution of wealth, the x-axis intercept of the complete markets equilibrium shifts to the left. Households are better able to smooth consumption and the incomplete markets equilibrium approaches the complete markets counterpart. As the variance of the wage diminishes, the vertical distance between the complete and incomplete markets equilibrium also diminishes.

As a final note, we observe that, in this simple setting, were we so inclined we could explicitly solve (21) for k_1 in terms of exogenous expressions rather than leaving it as an equation which defines the savings function implicitly. Indeed, some algebraic manipulation rewrites it as

$$\left(k_1 - \frac{1}{1+\beta}(\beta x_1 - \mathbb{E}\, w_2)\right)\left(k_1 + w_{\text{low}} + \frac{\operatorname{Var}(n_2)}{\mathbb{E}(n_2)}\right) = \operatorname{Var}(n_2)$$

which is just a quadratic equation which we could solve using the quadratic formula. This procedure does not generalize beyond the two period model. However, it is interesting to note that if we extend the resource space to the entire real line, the above equation is that of a hyperbola with asymptotes

$$k_1 = \frac{1}{1+\beta}(\beta x_1 - \mathbb{E}\, w_2) \quad \text{and} \quad k_1 = -\left(w_{\text{low}} + \frac{\operatorname{Var}(n_2)}{\mathbb{E}(n_2)}\right)$$

[6] Precisely, the measurement of the gap with $\ell_2 = 0$ can be calculated via the quadratic formula to be

$$\frac{1}{2}\frac{\operatorname{Var}(n_2)}{\mathbb{E}(n_2)}\left(\sqrt{1 + \frac{4\,\mathbb{E}(n_2)^2}{(1+\beta)\operatorname{Var}(n_2)}} - 1\right)$$

Rationalizing the numerator and taking a derivative reveals this to be increasing in the variance.

3.3 General CRRA Utility With log utility, income and substitution effects offset and the savings function is simplified. Precisely, we can combine budget constraints in the model (18) to obtain the intertemporal constraint

$$c_1 + \frac{c_2}{1-\delta+R_2} = x_1 + \frac{W_2 \ell_2}{(1-\delta+R_2)} := y$$

where the household's expenses appear on the left and its discounted future income stream on the right. In this form, it is evident that the price of consumption in period 2 relative to that in period 1 is $1/(1-\delta+R_2)$, and moreover using this to substitute for c_2 in the household's objective function gives

$$\log(c_1) + \beta\,\mathbb{E}\log\left[(1-\delta+R_2)\,(y-c_1)\right] = \log(c_1) + \beta\log(1-\delta+R_2) + \beta\,\mathbb{E}\log(y-c_1)$$

From the expansion on the right side of the equality, it is evident that with logarithmic utility the price of period 2 consumption relative to period 1 consumption does not enter the household optimality conditions, apart from its impact on total income. A similar calculation goes through when allowing for aggregate uncertainty. With $\sigma \neq 1$, the income and substitution effects do not offset, which complicates our analysis. Nonetheless, the general structure remains the same.

The analogue of the rearranged Euler equation (19) in this setting is

$$\left[\beta(1-\delta+R_2)^{1-\sigma}\right]^{1/\sigma}(x_1-k_1) = \frac{\left(k_1 + \frac{W_2\ell_{\text{low}}}{1-\delta+R_2}\right)\left(k_1 + \frac{W_2\ell_{\text{high}}}{1-\delta+R_2}\right)}{\left[p\left(k_1 + \frac{W_2\ell_{\text{high}}}{1-\delta+R_2}\right)^{\sigma} + (1-p)\left(k_1 + \frac{W_2\ell_{\text{low}}}{1-\delta+R_2}\right)^{\sigma}\right]^{1/\sigma}} \tag{23}$$

There are two main complications relative to the log case: first, the factor of $1-\delta+R_2$ on the left side, which arises due to the non-offsetting income and substitution effects, and second, the denominator on the right hand side now takes the form of an L^σ norm.

The first of these is benign for deriving our linear-plus-error expansion, insofar as it is just enters as an effective discount factor (although aggregate uncertainty will complicate this somewhat). The second is a more serious obstacle, although the denominator still behaves linearly in the limit $k_1 \to \infty$. To extract an expression for the asymptote, we can make use of the following simple observation.

Lemma 1. *Let $A, B \in \mathbb{R}$ and let $f : \mathbb{R} \to \mathbb{R}$ satisfy*

$$\lim_{x\to\infty}(f(x) - Ax) = B$$

Then

$$\lim_{x\to\infty}(f(x)-(Ax+B))=0.$$

Proof. See Appendix I. □

We apply this to the function of k_1 on the right side of the Euler equation with $A=1$. We can then calculate the limit B through some algebra and an application of l'Hospital's rule. The result is a linear-plus-error expansion of the savings function in the two period CRRA case without aggregate uncertainty, where the nonlinear error term satisfies the properties previously established for the log case.

Theorem 2: Nonlinear Error, CRRA Utility. The savings function $k^{(1)}(x_1)$ which solves model (18) with log replaced by a general CRRA utility function can be written in the form

$$k^{(1)}(x_1)=\frac{1}{1+Q_1}\left(Q_1x_1-\frac{W_2}{1-\delta+R_2}\,\mathbb{E}\,\ell_2+\epsilon^{(1)}(x_1)\right), \qquad \text{with}\quad Q_1=\left[\beta(1-\delta+R_2)^{1-\sigma}\right]^{1/\sigma} \tag{24}$$

where the nonlinear error term is strictly decreasing, strictly convex, and satisfies

$$\lim_{x_1\to\underline{k}_1}\epsilon^{(1)}(x_1)=\frac{W_2}{1-\delta+R_2}\,\mathbb{E}[\ell_2]+\underline{k}_1, \qquad \lim_{x_1\to\infty}\epsilon^{(1)}(x_1)=0$$

As an immediate corollary, Figure 2 and the related discussion in the previous section apply to household savings across all values of the intertemporal substitution/risk aversion parameter $\sigma>0$. Our full main theorem will extend this result in several dimensions, allowing for many periods, many shock outcomes, and aggregate uncertainty.

The above theorem gives some basic qualitative properties of the nonlinear error, however a closer examination gives some insight about the impact of household risk aversion on savings decisions. It will be helpful to once again write w_2 for discounted wages (with w_{low} and w_{high} defined appropriately), n_2 for discounted excess wages (with $n_{\text{low}}=0$ and n_{high} defined appropriately), and to introduce the notation

$$h_\sigma=\left[p\left(1+\frac{n_{\text{high}}}{k_1+w_{\text{low}}}\right)^\sigma+(1-p)\left(1+\frac{n_{\text{low}}}{k_1+w_{\text{low}}}\right)^\sigma\right]^{1/\sigma}$$

Using this, the right side of (23) can be rewritten in the linear-plus-error form as

$$\begin{aligned}\left[\beta(1-\delta+R_2)^{1-\sigma}\right]^{1/\sigma}(x_1-k_1) = \;& k_1 + \frac{W_2}{1-\delta+R_2}\,\mathbb{E}\,\ell_2 \\ & + \left(\frac{h_1-h_\sigma}{h_\sigma}\right)(k_1+w_{\text{low}}) + \left(\frac{1-h_\sigma}{h_\sigma}\right)\mathbb{E}\,n_2 \end{aligned} \tag{25}$$

The first line here is just the linear contribution. The remaining two terms on the right are the components of the nonlinear portion; as such, their sum is negative, and moreover both vanish as $k_1 \to \infty$. Understanding the contribution of each term involves understanding the norms h_σ. These norms have the following properties which can be found in graduate texts in mathematical analysis (for example, Rudin (1986)).

Lemma 2. *For fixed W_2, R_2, and k_1, the function h_σ is strictly increasing for $0 < \sigma < \infty$ with*

$$\lim_{\sigma\to 0} h_\sigma = \left(1+\frac{n_{\text{high}}}{k_1+w_{\text{low}}}\right)^p \left(1+\frac{n_{\text{low}}}{k_1+w_{\text{low}}}\right)^{1-p}$$

and

$$\lim_{\sigma\to\infty} h_\sigma = \max\left(1+\frac{n_{\text{high}}}{k_1+w_{\text{low}}}, 1+\frac{n_{\text{low}}}{k_1+w_{\text{low}}}\right)$$

We therefore consider what happens as σ increases from 0 (risk neutral) to 1 (offsetting income and substitution effects), and from 1 to ∞ (extreme risk aversion). First, note that (25) can be rewritten as

$$c_1 = \frac{1}{\left[\beta(1-\delta+R_2)^{1-\sigma}\right]^{1/\sigma} h_\sigma}\left[h_1(k_1+w_{\text{low}}) + \mathbb{E}\,n_2\right] \tag{26}$$

with c_1 just denoting consumption in the first period. This form is informative about the interaction between intertemporal substitution and risk aversion in determining the household consumption allocation.

In particular, the single parameter governing both of these features enters only the first factor on the right side of the equation. Consider holding savings k_1 fixed while adjusting this parameter. For the optimality condition to hold, then, initial consumption must vary to balance the equation. Since k_1 is fixed, this amounts to adjusting initial resources to justify the fixed level of savings as the coefficient of risk aversion/intertemporal substitution increases.

Differentiating the first factor on the right side of (26), we see that the direction in which initial resources adjust to a change in household substitution motive or risk aversion is given by the sign of

$$\frac{\beta(\sigma-1)}{\sigma}\left(\beta(1-\delta+R_2)^{1-\sigma}\right)^{\frac{1}{\sigma}-1}(1-\delta+R_2)^{-\sigma}h_\sigma-\left(\beta(1-\delta+R_2)^{1-\sigma}\right)^{1/\sigma}\frac{\partial h_\sigma}{\partial\sigma}$$

The first term here dictates how the former impacts required resources for holding assets equal to k_1 as σ increases, while the second term dictates how the latter does.

In the range $0<\sigma<1$, throughout which the income effect dominates in the household's intertemporal substitution, both terms in this expression are negative. In this case, then, required resources are unambiguously decreasing in σ: as σ increases, the dominant income effect weakens and risk aversion strengthens. The result is that a smaller initial allocation of resources is required for the household to choose a given level of substitutive/precautionary savings.

When $\sigma=1$, substitution and income effects offset, and only the risk aversion component in the above derivative remains, and this is once again negative. Increasing risk aversion continues to require lower resource holdings for a given level of savings.

As σ increases past one, the term reflecting changing intertemporal motives becomes positive as a dominant substitution effect takes hold and strengthens. At first this term continues to be dominated by increasing household risk aversion. However, for a fixed endowment process and savings level, h_σ increases to the finite limit given in the above lemma. Meanwhile, the factor $\sigma-1$ in the first term increases without bound. It follows, then, that eventually increasing σ reduces the initial resources required to justify a given savings level, as a rapidly increasing substitution effect overwhelms increasing risk aversion.

3.4 Many Periods and Aggregate Uncertainty

We now present the extension of Theorem 1 and Theorem 2 to the full generality of the household dynamic programming problems (2). Simply put, a typical household's savings functions can be deconstructed into two components. The first corresponds to the savings of a household in the deterministic environment of Example 4, (9) and (12), with discounted future wages and discount factors replaced by time t expected values. The second is a nonlinear error term with the same features as that in the two period setting.

The statement is clearest with logarithmic utility.

Theorem 3: Main Theorem, $\sigma=1$. The savings functions $k^{(t)}(x_t,\mathcal{L}_t,\mathcal{F}_t)$, $t=1,...,T$ which solve the dynamic programming problems (2) with $\sigma = 1$ can be written in the form

$$k_t(x_t, \mathcal{L}_t, \mathcal{F}_t) = \frac{\beta + ... + \beta^{T-t}}{1 + \beta + ... + \beta^{T-t}} x_t - \frac{1}{1 + \beta + ... + \beta^{T-t}} \mathbb{E}_t \left(\sum_{s=t+1}^{T} \frac{W_s \ell_s}{\prod_{r=t+1}^{s} (1 - \delta + R_r)} \right) + \epsilon^{(t)}(x_t, \mathcal{L}_t, \mathcal{F}_t)$$

where the nonlinear error term $\epsilon^{(t)}$ is identically zero without uncertainty, and is strictly decreasing, strictly convex, and satisfies

$$\lim_{x_1 \to \underline{k}_t} \epsilon^{(t)}(x_t) = \frac{1}{1 + \beta + ... + \beta^{T-t}} \left[\mathbb{E}_t \left(\sum_{s=t+1}^{T} \frac{W_s \ell_s}{\prod_{r=t+1}^{s} (1 - \delta + R_r)} \right) + \underline{k}_t \right]$$

and

$$\lim_{x_1 \to \infty} \epsilon^{(t)}(x_t) = 0$$

with uncertainty.

As in the two period cases, the relative simplicity of the log utility case stems from offsetting income and substitution effects. In particular, the prices of future consumption (and hence future aggregate fluctuations) do not impact current period consumption, and hence do not impact current period savings, apart from the need to discount expected future income.

With $\sigma \neq 1$, future prices directly impact current savings/consumption decisions, so that household expectations about aggregate fluctuations enter the savings function.

Theorem 4: Main Theorem, $\sigma \neq 1$. Make the sequence of recursive definitions

$$\begin{aligned} M_T &= (1 - \delta + R_T)^{1-\sigma} \\ Q_{T-1} &= (\beta \mathbb{E}_{T-1} M_T)^{1/\sigma} \\ M_t &= (1 - \delta + R_t)^{1-\sigma} (1 + Q_{t+1})^{\sigma}, \quad & t = 2, ..., T \\ Q_{t-1} &= [\beta \mathbb{E}_{t-1} M_t]^{1/\sigma}, & t = 2, ..., T \end{aligned}$$

The savings functions $k^{(t)}(x_t, \mathcal{L}_t, \mathcal{F}_t)$, $t = 1, ..., T$ which solve the dynamic programming problems (2) with can be written in the form

$$k^{(t)}(x_t, \mathcal{L}_t, \mathcal{F}_t) = \frac{Q_t}{1+Q_t} x_t - \frac{1}{1+Q_t} \mathbb{E}_t \left(\sum_{s=t+1}^{T} \left(\prod_{r=t+1}^{s} \frac{M_r}{\mathbb{E}_{r-1} M_r} \right) \frac{W_s \ell_s}{\prod_{r=t+1}^{s} (1-\delta+R_r)} \right) + \epsilon^{(t)}(x_t, \mathcal{L}_t, \mathcal{F}_t)$$

where the nonlinear error term $\epsilon^{(t)}$ is identically zero without uncertainty, and is strictly decreasing, strictly convex, and satisfies

$$\lim_{x_t \to \underline{k}_t} \epsilon^{(t)}(x_t) = \frac{1}{1+Q_t} \left[\mathbb{E}_t \left(\sum_{s=t+1}^{T} \left(\prod_{r=t+1}^{s} \frac{M_r}{\mathbb{E}_{r-1} M_r} \right) \frac{W_s \ell_s}{\prod_{r=t+1}^{s} (1-\delta+R_r)} \right) + \underline{k}_t \right]$$

and

$$\lim_{x_t \to \infty} \epsilon^{(t)}(x_t) = 0$$

with uncertainty.

Observe that the expressions Q_t, $t = 1, ..., T-1$ in the above theorem are stochastic analogues of (10) and (11) which capture non-offsetting income and substitution effects, in which the factors $(1-\delta+R_s)^{1-\sigma}$, $s = t+1, ..., T$ are replaced by their time $s-1$ expected values, which are in turn projected back to time t in a nonlinear way (so that the law of iterated expectation is powerless to simplify the expressions). An additional feature present is a re-weighting of wages which magnifies the importance of wages which are paid in states of nature for which current and future rental rates are high compared to the average of such.

The above theorems are stated and proven for the finite horizon model constructed earlier. This leaves the natural question about whether a similar statement holds in the infinite horizon analogue.[7] The form that such a theorem would take is evident upon the limit as $T \to \infty$. For log utility, as before, things are somewhat simpler.

Conjecture 1: Infinite Horizon, Log Utility. The savings functions $k^{(t)}(x, \mathcal{L}_t, \mathcal{F}_t)$ which solve the infinite horizon analogue of the model (18) with log utility can be written in the form

$$k^{(t)}(x_t, \mathcal{L}_t, \mathcal{F}_t) = \beta x_t - (1-\beta) \mathbb{E}_t \left(\sum_{s=2}^{\infty} \frac{W_s \ell_s}{\prod_{r=t+1}^{s} (1-\delta+R_r)} \right) + \epsilon^{(t)}(x_t, \mathcal{L}_t, \mathcal{F}_t)$$

[7] The only substantial difference here is that the household now must be endowed with predictive probabilities into the infinite future.

where the nonlinear error term $\epsilon^{(t)}$ is identically zero without uncertainty, and is strictly decreasing, strictly convex, and satisfies

$$\lim_{x_t \to \underline{k}_t} \epsilon^{(t)}(x_t) = (1-\beta)\left[\mathbb{E}_t\left(\sum_{s=t+1}^{\infty} \frac{W_s \ell_s}{\prod_{r=t+1}^{s}(1-\delta+R_r)}\right) + \underline{k}_t\right]$$

and

$$\lim_{x_t \to \infty} \epsilon^{(t)}(x_t) = 0$$

with uncertainty.

For general CRRA utility, we suggest the following.

Conjecture 2: Infinite Horizon, $\sigma \neq 1$. Let

$$M_t = (1-\delta+R_t)^{1-\sigma}(1+(\beta\mathbb{E}_{t+1}(1-\delta+R_{t+2})^{1-\sigma}(1+...)^{1/\sigma})^{\sigma})^{1/\sigma}, \quad t = 2, 3, ...$$
$$Q_{t-1} = [\beta \mathbb{E}_{t-1} M_t]^{1/\sigma}, \quad t = 2, 3, ...$$

The savings functions $k^{(t)}(x, \mathcal{L}_t, \mathcal{F}_t)$ which solve the infinite horizon analogue of the model (18) can be written in the form

$$k^{(t)}(x_t, \mathcal{L}_t, \mathcal{F}_t) = \frac{Q_t}{1+Q_t}x_t - \frac{1}{1+Q_t}\mathbb{E}_t\left(\sum_{s=t+1}^{\infty}\left(\prod_{r=t+1}^{s}\frac{M_r}{\mathbb{E}_{r-1}M_r}\right)\frac{W_s\ell_s}{\prod_{r=t+1}^{s}(1-\delta+R_r)}\right) + \epsilon^{(t)}(x_t, \mathcal{L}_t, \mathcal{F}_t)$$

where the nonlinear error term $\epsilon^{(t)}$ is identically zero without uncertainty, and is strictly decreasing, strictly convex, and satisfies

$$\lim_{x_t \to \underline{k}_t} \epsilon^{(t)}(x_t) = \frac{1}{1+Q_t}\left[\mathbb{E}_t\left(\sum_{s=t+1}^{\infty}\left(\prod_{r=t+1}^{s}\frac{M_r}{\mathbb{E}_{r-1}M_r}\right)\frac{W_s\ell_s}{\prod_{r=t+1}^{s}(1-\delta+R_r)}\right) + \underline{k}_t\right]$$

and

$$\lim_{x_t \to \infty} \epsilon^{(t)}(x_t) = 0$$

with uncertainty.

Naturally, for the above conjectures to make sense, the limits which define the sums and products out into the infinite future must be well defined, which in turn requires that household predictive distributions must be restricted to be well-behaved in an appropriate sense. Putting aside this difficulty, the strategy for proving this seems evident. If we consider Euler equation iteration to be an operator on an appropriate function space, our main theorems essentially state that this operator leaves invariant a particular subspace. If this subspace is closed in some topology in which repeated application of the operator leads to convergence, then the above conjecture will hold. While we have not yet established the details of the proof, the numerical exercises in the next section will provide substantial numerical evidence for this statement.

3.5 Outline of the Proofs The proofs of Theorem 3 and Theorem 4 follow the same general approach alluded to when considering their two period variants. The main complications introduced in the general environment are: i) Aggregate uncertainty means we cannot pull aggregates (prices) out from expected values; ii) Many shock outcomes requires more complicated indexing of future labor and aggregate outcomes; and iii) Having several periods means that a household's time $t < T - 1$ decision must factor in its future savings behavior so that there is an additional endogenous component in the Euler equation.

The first of these issues is benign in the log case due to the offsetting effects, and is dealt with in the $\sigma \neq 1$ case by means of the definitions of M_t and Q_t in Theorem 4. The second issue is simply an algebraic nuisance. The third is potentially more serious; however, by the results of the previous sections we already know that the 2 period savings functions are well behaved, and consequently so will be the penultimate period savings functions in the general cases. There is therefore hope that our understanding of the second-to-last period savings function will allow us to control the additional endogenous component in the Euler equation for the third-to-last period savings function, and so on. The proofs of the main theorems therefore adopt an inductive approach.

Specifically, we directly establish the theorem for $k^{(T-1)}$ as a base case using the Euler equation linking consumption in periods $T - 1$ and T. This Euler equation does not include any additional endogenous component, since the household saves nothing in the final period of life. We then proceed to assume that the theorem has been proved for the savings function in period $t + 1$ for some value of $t > 1$. Writing out the expected value in the Euler equation (3) linking period t and period $t + 1$ consumption, we have

$$\beta(x_t - k_t) = \left(\sum_j \frac{p_j}{k_t + \frac{W_j \ell_j}{1-\delta+R_j} - \frac{k_j}{1-\delta+R_j}} \right)^{-1}, \quad \sigma = 1$$

$$\beta(x_t - k_t) = \left(\sum_j \frac{p_j (1-\delta+R_j)^{1-\sigma}}{\left(k_t + \frac{W_j \ell_j}{1-\delta+R_j} - \frac{k_j}{1-\delta+R_j} \right)^{\sigma}} \right)^{-1/\sigma}, \quad \sigma \neq 1$$

where $j = 1, ..., J$ indexes the possible outcomes of the random variables $(\ell_{t+1}, R_{t+1}, W_{t+1})$ in period $t+1$, and where p_j is the probability assigned to the jth outcome by the household's time t predictive probabilities.[8] Here, we are using the notation

$$k_j = k^{(t+1)}(x_j, \mathcal{L}_j, \mathcal{F}_j) \quad \text{with} \quad x_j = (1 - \delta + R_j)k_t + W_j \ell_j$$

for household savings at time $t+1$ when the state indexed j prevails, with $\mathcal{L}_j$ and $\mathcal{F}_j$ the predictive distributions which associated with the jth state.

Applying the inductive hypothesis, we can write

$$k_j = \frac{\beta + ... + \beta^{T-t-1}}{1 + \beta + ... + \beta^{T-t-1}} x_j - \frac{1}{1 + \beta + ... + \beta^{T-t-1}} \mathbb{E}_{t+1} \left(\sum_{s=t+2}^{T} \frac{W_s \ell_s}{\prod_{r=t+2}^{s} (1 - \delta + R_r)} \right) + \epsilon_j$$

when $\sigma = 1$, with ϵ_j defined in analogy with k_j, and similarly for $\sigma \neq 1$. We can then combine linear expressions in denominators on the right hand sides of Euler equations to get

$$\begin{aligned} k_t + \frac{W_j \ell_j}{1 - \delta + R_j} - \frac{k_j}{1 - \delta + R_j} = & \frac{k_t}{1 + \beta + ... + \beta^{T-t-1}} \\ & + \frac{1}{1 - \delta + R_j} \left(W_j \ell_j + \frac{1}{1 + \beta + ... + \beta^{T-t-1}} \mathbb{E}_{t+1} \left(\sum_{s=t+2}^{T} \frac{W_s \ell_s}{\prod_{r=t+2}^{s} (1 - \delta + R_r)} \right) \right) \\ & - \frac{\epsilon_j}{1 - \delta + R_j} \end{aligned}$$

for $\sigma = 1$ and again, similarly for $\sigma \neq 1$. To simplify the notation for the arguments to follow, we simply write this as

$$k_t + \frac{W_j \ell_j}{1 - \delta + R_j} - \frac{k_j}{1 - \delta + R_j} = \begin{cases} A k_t + B_j + \delta_j & \sigma = 1 \\ A_j k_t + B_j + \delta_j & \sigma \neq 1 \end{cases}$$

where we note that the coefficient on k_t will depend on the prevailing time $t+1$ state for general coefficients of relative risk aversion. We can then rewrite the Euler equations as

$$\beta(x_t - k_t) = \begin{cases} \frac{\prod_j (A k_t + B_j + \delta_j)}{\sum_j p_j \prod_{i \neq j} (A k_t + B_i + \delta_i)} & \sigma = 1 \\ \frac{\prod_j (A_j k_t + B_j + \delta_j)}{\left(\sum_j p_j \prod_{i \neq j} (A_j k_t + B_i + \delta_i)^{\sigma} \right)^{1/\sigma}} & \sigma \neq 1 \end{cases}$$

Intuitively, in each of these the numerator has highest degree one larger than the denominator

[8] In a more conventional Markov setting, p_j is just the probability of the j outcome conditional on time t information.

so that they should approach a linear asymptote as $k_t \to \infty$. This intuition is somewhat obfuscated by the additional nonlinear factors δ_j, but nonetheless we may apply Lemma 1, the asymptotic properties of the time $t+1$ savings function errors, and some delicate calculus to extract the asymptote.

What remains is to determine the properties of the nonlinear error term. After some calculus and algebra, one sees that in the general case the monotonicity argument amounts to deriving a bound

$$\mathbb{E}_t\left[\frac{1-\delta+R_{t+1}}{c_{t+1}^{\sigma}}\right] \leq \left[\mathbb{E}_t M_{t+1}^{1-\sigma}\right]^{1/(1+\sigma)} \left[\mathbb{E}_t \frac{(1-\delta+R_{t+1})\partial c_{t+1}/\partial k_t}{c_{t+1}^{1+\sigma}}\right]^{\sigma/(1+\sigma)}$$

and the convexity argument likewise amounts to deriving another bound

$$\mathbb{E}_t\left[\frac{(1-\delta+R_{t+1})\partial c_{t+1}/\partial k_t}{c_{t+1}^{1+\sigma}}\right]^2 \leq \mathbb{E}_t\left[\frac{1-\delta+R_{t+1}}{c_{t+1}^{\sigma}}\right]\mathbb{E}_t\left[\frac{(1-\delta+R_{t+1})(\partial c_{t+1}/\partial k_t)^2}{c_{t+1}^{2+\sigma}}\right]$$

The veracity of the first of these follows from Hölder's inequality, and that of the second from the Cauchy-Schwartz inequality, which closes the induction and hence the proof.

Notice that the above bounds can be rewritten in terms of the period utility function as

$$\frac{\sigma}{\left[\mathbb{E}_t M_{t+1}^{1-\sigma}\right]^{\frac{1}{1+\sigma}}} \leq \frac{-\left[\mathbb{E}_t(1-\delta+R_{t+1})\frac{\partial c_{t+1}}{\partial k_t}u''(c_{t+1})\right]^{\frac{\sigma}{1+\sigma}}}{\mathbb{E}_t\left[(1-\delta+R_{t+1})u'(c_{t+1})\right]}$$

and

$$\left(\frac{1+\sigma}{\sigma}\right)\frac{\mathbb{E}_t\left[(1-\delta+R_{t+1})\frac{\partial c_{t+1}}{\partial k_t}u''(c_{t+1})\right]}{\mathbb{E}_t\left[(1-\delta+R_{t+1})u'(c_{t+1})\right]} \leq \frac{\mathbb{E}_t\left[(1-\delta+R_{t+1})\left(\frac{\partial c_{t+1}}{\partial k_t}\right)^2 u'''(c_{t+1})\right]}{\mathbb{E}_t\left[(1-\delta+R_{t+1})\frac{\partial c_{t+1}}{\partial k_t}u''(c_{t+1})\right]}$$

respectively. Drawing the evident parallel with coefficients of risk aversion and prudence, we therefore put forth the intuition that the monotonicity of the nonlinear correction to household savings with idiosyncratic uncertainty follows from a dynamic form of risk aversion while its convexity follows from a dynamic form of household prudence. In the two period case, the consumption derivatives in the numerators under expected values are constants, sharpening the analogy.

4 Numerical Results for Equilibrium Distributions

The Theorems in Sections 3.2-3.4 are statements about household savings behavior across the entire, unspecified distribution of wealth. Since this savings behavior is nonlinear across the entire distribution, the economy will not perfectly aggregate for any nondegenerate distribution of wealth. On the other hand, we have seen that the nonlinear component of savings strictly decreases in household resources. Consequently, if the distribution places most of the resources in the possession of wealthy households, for whom the nonlinear error is small, the savings of these households will approximately aggregate and economic aggregates should behave closely to those predicted by a deterministic analogue of the economy. On the other hand, if the resource share held by poor households is substantial, the nonlinear error will significantly impact aggregates and this approximation will fail.

In the initial period of our model, the distribution of wealth is exogenous, so that we can aggregate or disaggregate the economy according to our wishes. From period 2 onward, however, the distribution evolves endogenously. We are therefore concerned with whether or not approximate aggregation prevails throughout part or all of the equilibrium history of the economy.[9] The main challenges to answering this question lie in finding equilibrium predictive distributions and distributions of wealth, namely those which satisfy Definition 1. At present, we do not have a sufficient theoretical description of these objects, and so we resort to numeric methods.[10]

While the finite horizon model specification in Section 2 is attractive from a theoretical perspective, in that it captures a general setting in which our main theorem applies, it is somewhat impervious to computing the equilibrium numerically. We have enlarged the state space in order to convert a potentially non-Markov environment to a Markov one, thereby subjecting ourselves to the curse of dimensionality. In order to have workable equilibrium objects with which to operationalize our theory, we will therefore narrow our focus to a single example in which we can collapse the state space to something manageable. Since it is both our motivating example and a sufficiently rich environment to demonstrate all of the main features of the theory, this will be the economy of Krusell and Smith (1998), previously cited as our Example 6.

In the next part, we outline in more detail the numerical models that we solve and the algorithms used. We then conduct a number of exercises to evaluate the claim that these economies approximately aggregate. These results suggest that, for a standard calibration

[9] Note that, in the final period, savings are zero across the entire distribution, so that the economy aggregates in a trivial way.

[10] Combining our main theorem with a discrete time version of the results of Gabaix, Lasry, Lions, and Moll (Forthcoming) may fill the theory gap. This will be one objective of future work.

of the model, approximate aggregation provides an excellent (if partial) description.

4.1 Numerical Models The baseline neoclassical growth model with idiosyncratic and aggregate uncertainty of Krusell and Smith (1998) differs from our Example 6 in three major features. It is an infinite horizon model, the household state space consists of contemporaneous statistics rather than predictive probabilities, and prices are computed from these statistics via an aggregate law of motion.

To make the connection with our setup, we may interpret the first feature in our context as taking T sufficiently large that, to numerical precision, the household savings function does not change from period to period.[11] As we have previously discussed, the second feature is just a collapsing of the state space of our setup, and the last we can interpret in our context as a particular Walrasian auctioneer.

To illustrate the last point, we briefly pause to consider examples common in the literature.

Example 7: Steady State Auctioneer.

The steady state auctioneer specifies a single, constant value for all prices in all periods. In the pure credit setting, this amounts to an auctioneer which fixes a constant interest rate. In the production setting, this can take the form of the Aiyagari Auctioneer which specifies a single aggregate capital labor ratio K/N for every future period and computes prices following (5) and (6).

This is an appropriate auctioneer for environments with no uncertainty, or those with idiosyncratic uncertainty but no aggregate uncertainty. □

With aggregate uncertainty, aggregate shocks prevent a steady state from prevailing, leading to the following two examples.

Example 8: Krusell Smith Auctioneer.

This auctioneer computes uses a distributional law of motion which forecasts tomorrow's distribution of households across capital and labor states from today's distribution given shock outcomes. Future prices are then computed by integrating the sequence of household distributions in each variable to determine aggregate capital and labor, and then using (5) and (6).

This is the auctioneer which is used in Krusell and Smith (1998) to formulate their general equilibrium concept. Unfortunately, as of this writing, existence of an equilibrium distributional law of motion remains an open problem. □

[11] We will not concern ourselves here with late-life behavior of the households, however we do note that an alternative way to explore the equilibrium implications of our theory would involve T overlapping generations of households, in which case we could collapse the state space while incorporating this feature.

Example 9: Approximate Aggregation Auctioneer.

This auctioneer forecasts aggregate capital in future periods by using a law of motion which depends only on aggregate capital in the current period and the current aggregate state. Aggregate labor is forecast using its exogenous stochastic properties. Future prices are, once again, calculated using (5) and (6).

This auctioneer was initially proposed in Krusell and Smith (1998), and is now commonly used in numerics to compute approximate equilibria. It allows for collapsing the household state to a four dimensional space. This formulation generalizes to the Higher Order Moments Auctioneer which forecasts capital using further distributional statistics, such as the variance of the distribution, in addition to the mean. □

Note that our theory applies to any of the above auctioneer's rules, and indeed to any price setting mechanic that the household takes as given, regardless of whether or not it's an equilibrium rule, insofar as our main theorem gives a linear-plus-error structure in which expected values are computed using the forecast prices. Having solved the numerical model for general equilibrium with any of the above specifications, then, and given a way to compute the linear expression in the main theorems, we can perform several experiments to evaluate the claim that the economy approximately aggregates.

Computing the linear part of the linear-plus-error expansion of the savings function will come down to the following proposition. We will focus on the log utility case to simplify this calculation.[12]

Proposition 3: Let $k_{\text{lin}}^{(t)}$ denote the linear component of the savings function in Theorem 3. Then $k_{\text{lin}}^{(t)}$ solves the series of intertemporal equations

$$\frac{1}{x_t - k_{\text{lin}}^{(t)}} = \frac{\beta}{k_{\text{lin}}^{(t)} + \mathbb{E}_t\left[\left(\frac{1}{(1-\delta+R_{t+1})}\right)\left(W_{t+1}\ell_{t+1} - k_{\text{lin}}^{(t+1)}\right)\right]}$$

for $t = 1, ..., T-1$ with $k_{\text{lin}}^{(T)} \equiv 0$.

Proof. See Appendix II. □

[12]For the $\sigma \neq 1$ case, we can replace the equation in this proposition with

$$\frac{1}{\left(x_t - k_{\text{lin}}^{(t)}\right)^{\sigma}} = \frac{Q_t^{\sigma}}{\left(k_{\text{lin}}^{(t)} + \mathbb{E}_t\left[\left(\frac{M_{t+1}}{(1-\delta+R_{t+1})\mathbb{E}_t M_{t+1}}\right)\left(W_{t+1}\ell_{t+1} - \left[k_{\text{lin}}^{(t+1)} - \frac{Q_{t+1}}{1+Q_{t+1}} x_{\text{lin}}^{(t+1)}\right]\right)\right]\right)^{\sigma}}$$

$$x_{\text{lin}}^{(t+1)} = (1-\delta+R_{t+1})k_{\text{lin}}^{(t)} + W_{t+1}\ell_{t+1}$$

The key here is that the expression under the expected value will compute expected future wages when the equation is iterated, yet only requires us to calculate expectations about variables one period forward. In each iteration we can therefore compute $k_{\text{lin}}^{(t)}$ and Q_t on our idiosyncratic and aggregate state grids, respectively and interpolate them off the grid if necessary in the following iteration.

The main feature of the above proposition is that it removes the most difficult challenge to computing the linear portion of the household savings function, namely the need to calculate the entire tree of possible future incomes, rental rates, and wages. Instead, we only need to compute one-period-foward expected values. This motivates the following numeric procedure.

1. Solve the model for (approximate) general equilibrium using any of the algorithms from the literature which provides a law of motion for computing future prices. For the purposes of our exercises we use a typical simulation method with a law of motion based on a finite vector of moments.

   The standard approach provides us with an equilibrium savings function defined on a four dimensional grid of individual capital, individual labor, total factor productivity, and aggregate capital (the collapsed state space) along with an equilibrium law of motion for aggregate capital (the auctioneer's rule).

2. Using the equilibrium aggregate law of motion from step 1, iterate backwards on Proposition 3 until convergence to obtain the infinite horizon (or large T) asymptote. Specifically, given an initial guess or the result of the previous iteration, k', compute a new function k'' at each grid point $[ik, i\ell, iZ, iK]$ via

$$k''[ik, i\ell, iZ, iK] = (1 - \delta + R[iZ, iK])k[ik] + W[iZ, iK]\ell[i\ell] \\ - \frac{1}{\beta}\left(k'[ik, i\ell, iZ, iK] + \mathbb{E}\left[\frac{1}{1 - \delta + R'}(W'\ell' - k'(k'))\right]\right)$$

   Here we can compute R' and W' under the expected value on the right side by using the law of motion for aggregate capital to predict K' for each value of $K[iK]$, and we can compute the iterated savings $k'(k')$ by interpolation off the idiosyncratic and aggregate capital grids.

3. Compute relevant statistics (position of the distribution compared to the intercept, one period ahead forecasts, many period ahead forecasts, etc.)

We next describe a few statistics regarding Step 1 above, before proceeding to report our results for a number of exercises in Step 3.

4.2 Calculation of General Equilibrium An approximate competitive equilibrium was computed via a modified version of the stochastic-simulation algorithm of Maliar, Maliar, and Valli (2010) which combines that article's household solving method with the simulation procedure of Young (2010). In short, this algorithm proceeds as follows:

1. Guess an initial savings function, aggregate law of motion, and cross-sectional distribution of households. Generate a long sequence of total factor productivity shocks once and for all.

2. Solve the household's problem by Euler equation iteration, beginning from the initial savings function and using the aggregate law of motion to forecast one-period-forward prices. Iterate until the savings functions converge up to some tolerance.

3. Use the savings function from Step 3 to simulate the cross-sectional distribution for the sequence of TFP shocks generated in Step 1 via the procedure of Young (2010).

4. Use the time series of distributional statistics generated in Step 3 to update the aggregate law of motion, for example by ordinary least squares regression (in the case of a law which is linear in coefficients).

5. Repeat steps 2-4 until the aggregate law of motion converges within some tolerance.

6. Test for equilibrium. For example, one can compute the R squared fit of the regression in Step 4, or use the procedure of Den Haan (2010).

We follow Maliar, Maliar, and Valli (2010) in our grid choices: with deterministic steady state capital given by

$$K_{ss} = \left(\frac{\frac{1}{\beta} - (1-\delta)}{\alpha} \right)^{-\frac{1}{1-\alpha}}$$

the idiosyncratic grid points are distributed on the interval $[0, 25K_{ss}]$ according to the polynomial rule

$$k_j = \left(\frac{j}{100} \right)^7 25 \cdot K_{ss}, \quad j = 1, ..., 100$$

while four aggregate grids are distributed linearly on the interval $[0.8K_{ss}, 1.3K_{ss}]$. Parameters were chosen in line with the computational literature: $\beta = 0.99$, $\alpha = 0.36$, $\delta = 0.025$, and $\sigma = 1$. Productivity shocks take values in the set $\{0.99,\ 1.01\}$ and efficiency shocks take values in the set $\{0,\ 1.111\}$, and these shocks follow the joint Markov process of Krusell and Smith (1998). We do not include taxes or unemployment insurance for this exercise.

The algorithm was implemented in the programming language Julia and calculations performed on an MSI GT70 2QD Laptop with an Intel Core i7-4710MQ processor and 16 GB of RAM. Interpolation in step 2 uses cubic splines via the Julia package Dierckx, which simply acts as a wrapper for the FORTRAN package of the same name. The model solved in 43.22 seconds, with 34 iterations on the aggregate law of motion and 1909 iterations on the Euler equation for the initial aggregate law of motion loop.

The aggregate law of motion we took to be log linear in the mean, with coefficients dependent on TFP regime. The equilibrium law of motion was computed, for good and bad aggregate states respectively, to be

$$\log(K') = 0.140040 + 0.962644 \log(K), \quad R^2 = 0.9999991$$
$$\log(K') = 0.128928 + 0.964224 \log(K), \quad R^2 = 0.9999989$$

The equilibrium test of Den Haan (2010) gave an average aggregate law of motion forecast error of 0.065% and a maximum of 0.233% for a sequence of 10000 on-equilibrium aggregate shocks. For a sequence of off-equilibrium shocks[13], the average forecast error was 0.125% and the maximum was 0.230%.

4.3 Approximate Aggregation in General Equilibrium In Krusell and Smith (1998), when addressing the question of why an accurate equilibrium is attained with a log linear law of motion for the mean of capital depending only on last period's mean and the aggregate state, the authors offer the intuition that

"...the marginal propensities to consume are almost identical for agents with different employment states and levels of capital...Most of the capital, however, is held by agents with essentially the same savings propensity. Very few agents-the very poorest ones-have a much lower propensity, and the capital that they hold is negligible. For these reasons, aggregation is almost perfect."

We have verified the first part of this claim in our model by precisely identifying the constant (conditional on aggregate predictive probabilities in the $\sigma \neq 1$ case) slope of the linear asymptote of household savings. In this section, we will explore the assertions that little capital is held by agents with a low propensity to save, and that this is a satisfactory explanation for why only the mean matters in the aggregate law of motion.

We begin by reproducing Figure 2 (Figure 2 in Krusell and Smith (1998)) for our numerically computed equilibrium savings function, along with the linear part computed via Step 2 in the strategy outlined in Section 4.1. This plot, shown in our Figure 3, demonstrates clearly the asymptotic trend of the savings function towards the linear part of the linear-plus-error expansion. Observe that the linear part of the savings function of an employed household never crosses the k axis. The reason for this is clear. The linear part consists of household resources net of discounted expected future wages. For an employed household, current resources includes positive current wages, which exceed discounted expected future wages for this economy. A reverse argument makes clear why the linear part of the savings function of an unemployed household is negative when capital holdings are low.

[13]These shocks consisted of 100 low TFP states followed by 100 high TFP states.

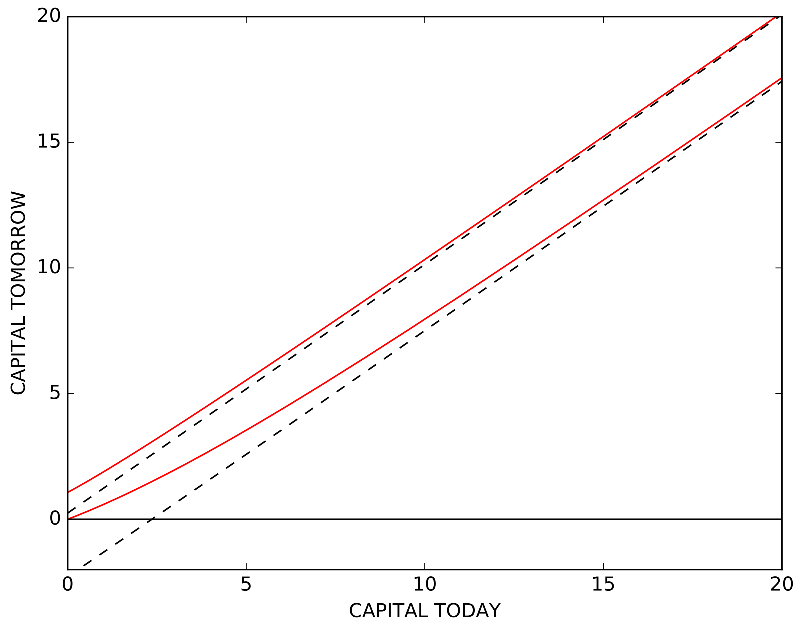

Figure 3: Savings function with incomplete markets (red) and complete markets (black) with aggregate capital $K \approx 43$ in a good aggregate state. The top curves are for an employed household and the bottom curves for an unemployed household.

Recall that in Section 3.1 the deterministic economy exactly aggregated in the absence of a positive measure of borrowing-constrained households. It is therefore of some interest to examine what fraction of our households fall below the point at which a zero borrowing limit would bind for a household using the complete markets savings function. As noted above, this will never be the case for an employed household in this economy, and so we will specifically look at the fraction of households which fall below the point at which the constraint would bind for an unemployed household.

Figure 4 plots the distribution of households across capital holdings in a typical period of a long simulation, along with a vertical line denoting the level of capital at which the unemployed household's asymptote intersects zero. It is immediately evident that nearly the entire distribution of wealth sits to the right of this line. Virtually all households are wealthy relative binding point in the borrowing constrained deterministic analogue of this economy.

To quantify this, we conducted a simulation of household savings decisions for 10,000 aggregate shocks, distributed according to the exogenous stochastic process. For each period and aggregate state, the level of capital holdings for which a zero borrowing limit would bind for an unemployed household in the deterministic analogue of the economy was computed. Integration of the distribution then provided the percentage of households with

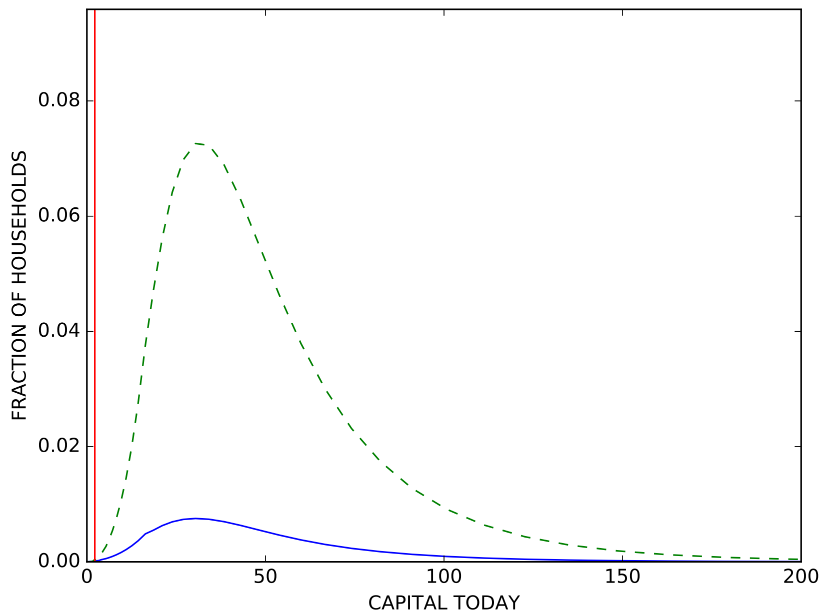

Figure 4: Distribution of wealth in a typical period. The solid curve is the distribution of the unemployed households, and the dashed curve is that of the employed households.

capital holdings below this level, as well as the capital held (as expressed as a percentage of the aggregate).

The mean and maximum values of these series across the simulation are reported in Table 1. We see that, on average, only 0.0607% of households fall below the level for which the constraint would bind in the deterministic analogue. The maximum percentage of households falling below the binding point is significantly higher, at 0.2265% . Both the average and maximum percentage of the aggregate held by these households is negligible, at less than a thousandth of a percent. This, then, would appear to be strong evidence in favor of the third assertion in the quote above.

To gain some additional insight, we repeat the above exercise for a simulation of "off-equilibrium" shocks, consisting of 100 low aggregate states followed by 100 high aggregate states.[14] This, then, is a sequence of shocks which is very unlikely to be encountered in equilibrium. Such an event leads to a substantial additional buildup of households which would be borrowing constrained in the deterministic analogue. However, these housholds still hold very little capital as a whole.

Next, we consider the size of the nonlinear error across the entire distribution. Figure 5

[14] This is similar to an exercise inMaliar, Maliar, and Valli (2010) used to investigate the accuracy of their aggregate law of motion.

plots this error as a function of capital holdings for employed and unemployed households. This plot demonstrates the monotonicity and convexity properties of the nonlinear error which were established in the theory. It is also evident that the nonlinear error for an unemployed household exceeds that for an employed one. Table 1 also reports the average

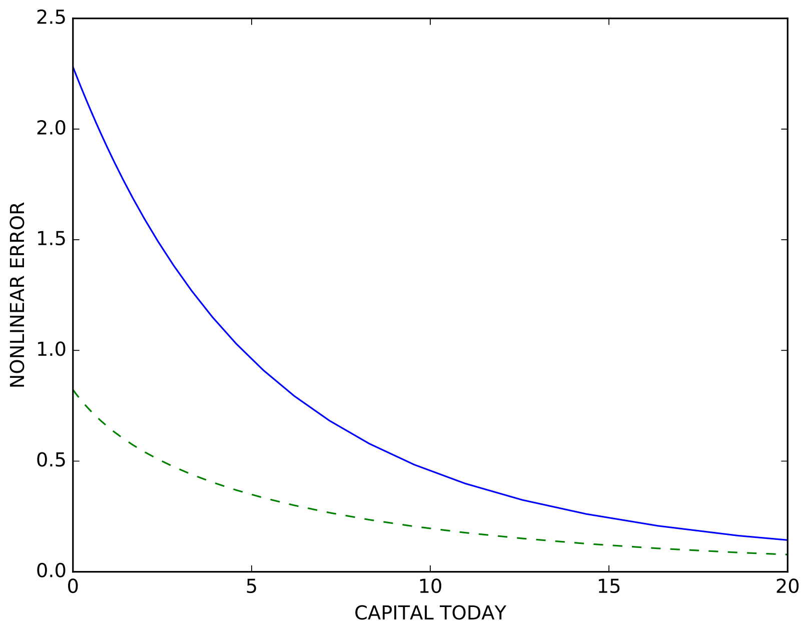

Figure 5: Nonlinear error in the household savings function with aggregate capital $K \approx 43$ in a good aggregate state. The solid curve is for an unemployed household and the dashed curve is for an employed household.

and maximum values of the nonlinear error integrated across the cross-sectional distributions for the on- and off- equilibrium simulation exercises described above. These values are reported as a percentage of aggregate capital savings in each period. At less than .15% in all cases, the nonlinear error would seem not to have a large impact in the aggregate.

Next, Figure 6 shows three aggregate capital series for on-equilibrium aggregate shocks. One of these is simply the series generated in the last 1,000 periods of a 10,000 period simulation of the household decisions in the economy,

$$K_t = \int k \, d\lambda(k, \ell), \quad t = 2, ..., 10,000$$

with

$$\lambda_{t+1}(k', \ell') = \int 1(k'(k, \ell, Z, K) = k')\pi(\ell'|\ell, Z, Z') \, d\lambda_t(k, \ell), \quad t = 1, ..., 9,999$$

λ_1 given.

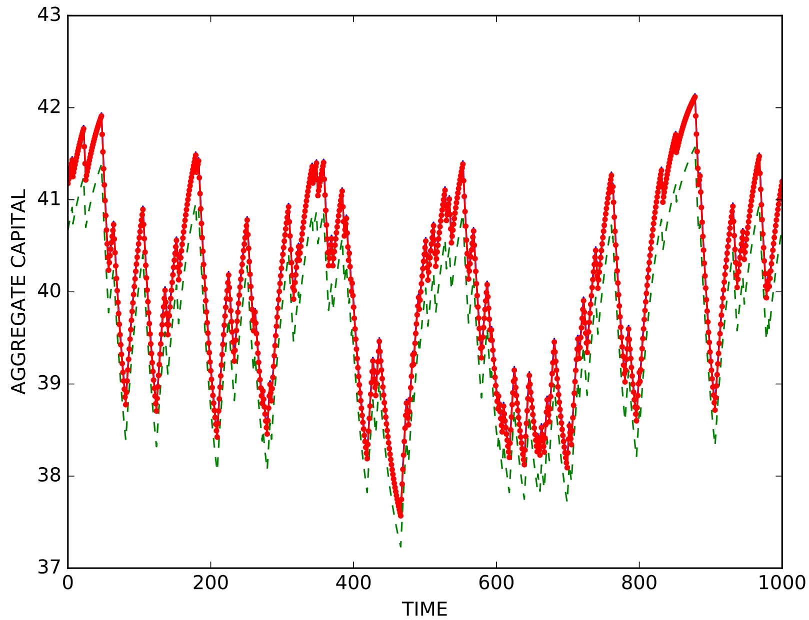

Figure 6: Aggregate capital series for equilibrium shocks generated by household decisions (solid line, obscured), one period ahead approximate aggregation forecasts (solid line with circles), and many period ahead approximate aggregation forecasts (dashed).

The second is a series of one period forecasts, in which we take the distribution generated in each period of the previous simulation, and predict next period's aggregate capital by summing the linear part of the savings function with a zero borrowing limit imposed. Precisely, we define the approximately aggregating savings function

$$k'_{\text{aa}}(k, \ell, Z, K) := \max\left[k'_{\text{lin}}(k, \ell, Z, K), 0\right]$$

and compute aggregates

$$K_{t+1} = \int k'_{\text{aa}}(k, \ell, Z, K)\; d\lambda_t(k, \ell), \quad t = 1, ..., 9, 999$$

with the sequence of distributions simulated above.

This series provides a simple but weak test of approximate aggregation in the intuitive sense of the quote given above. Specifically, the quote asserts that aggregation is almost perfect due to most of the capital being held by households with similar savings propensities, while the very poorest hold negligible capital. Our second simulation interprets this statement in a precise way. Having provided a natural definition of the very poor, we treat them as negligible by asserting that they add nothing to the aggregate. Moreover, we treat households which are not very poor as having identical propensities, while maintaining continuity and convexity of savings behavior. We may then examine how this series differs from the first.

Visually, the first two series are almost identical in the plot, confirming that aggregates one period forward are very well predicted by the approximately aggregating savings behavior. A quantitative exercise verifies this, giving that the average and maximum absolute differences across the simulation are of comparable size to the aggregate nonlinear errors calculated previously (see again Table 1). Apart from the binding borrowing limit, these statistics are the same, so the similarity is just reasserting the intuition that the capital held by the very poor is negligible. The results are similar for off-equilibrium shocks, whose associated capital series are plotted in Figure 7.

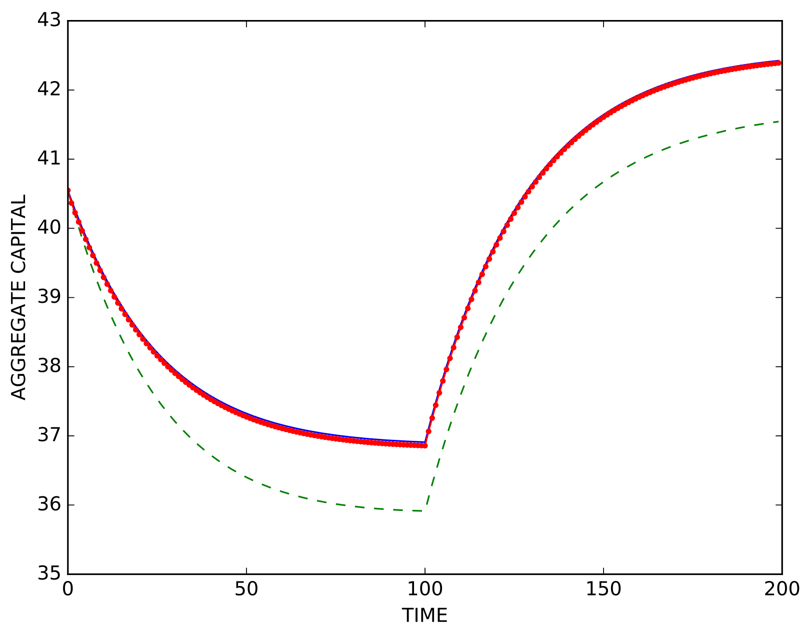

Figure 7: Aggregate capital series for off-equilibrium shocks generated by household decisions (solid line, obscured), one period ahead complete markets forecasts (solid line with circles), and many period ahead approximate aggregation forecasts (dashed).

The weakness of the previous test comes from the fact that it incorporates only a one-period forecast, while households have need of forecasting into the infinite future when making savings decisions. Our third series therefore provides a stronger test of aggregation by simply simulating a unit mass of households using the approximately aggregating savings

Statistic	Equilibrium Average (%)	Equilibrium Max (%)	Off Equilibrium Average (%)	Off Equilibrium Max (%)
Mass of Very Poor	0.0607	0.2265	0.1028	0.2490
Capital Held by Very Poor	0.0021	0.0076	0.0035	0.0084
Aggregate Nonlinear Error	0.1058	0.1390	0.1097	0.1427
One Period Forecast Error	0.1052	0.1360	0.1084	0.1395
Many Period Forecast Error	1.3578	2.4489	2.2301	2.7054

Table 1: Aggregation Statistics for equilibrium and off-equilibrium simulations. The first row lists percentages of households, while the remaining ones are given as a percentage of aggregate capital in the simulated economy.

function to make their decisions. That is, the third aggregate series follows

$$K_t = \int k \; d\lambda_t(k, \ell)$$

with

$$\lambda_{t+1}(k', \ell') = \int 1\Big(k_{\text{aa}} = k'\Big)\pi(\ell'|\ell, Z, Z') \; d\lambda_t(k, \ell), \quad t = 1, ..., 9,999$$

λ_1 given.

The series thus generated differs visually, and typically sits slightly below the other two. This is not too surprising, as the savings behavior which is being simulated is dominated by the incomplete markets savings function for all capital levels. Nonetheless, the aggregate remains close to that determined by actual household savings and shows similar responses to the various sequences of shocks, albeit at a lower level. The difference between this series and the other two is magnified for off-equilibrium shocks, as shown in Figure 7. While the one-period ahead forecasts remain close, the divergence of the many-period ahead forecasts is magnified for long sequences of identical.

Quantitative results for the strong test are shown in the last row of Table 1. Put simply, the results indicate that in a strong sense the intuition regarding approximate aggregation is capable of explaining over 98.5% of equilibrium aggregate time series behavior in this economy on average, and over 97.5% in the worst case scenario. Off equilibrium, this intuitive idea still manages to capture over 97.5% on average and over 97% at worst. In both cases, the approximate aggregation series is biased downwards relative to the actual household decisions, as it fails to completely capture the extent of household risk aversion and prudence.

We conclude with a short investigation of the effect of changing parameter values on approximate aggregation. This exercise consisted of adjusting each of β, α, and δ by ten percent of its baseline value (keeping other parameters at their baseline values), in the

Statistic	Mass (Capital) of Poor Ave (%)	Mass (Capital) of Poor Max (%)	Nonlinear Error Ave (%)	Nonlinear Error Max (%)	Forecast Error Ave (%)	Forecast Error Max (%)
β	1.2284 (0.1703)	2.5673 (0.3433)	1.3816	1.7864	6.3071	7.2508
α	0.0704 (0.0029)	0.2931 (0.0117)	0.1572	0.2087	1.8828	2.3825
δ	0.0743 (0.0030)	0.3022 (0.0121)	0.1236	0.1663	1.4695	1.8898
Z	0.0604 (0.0021)	0.2588 (0.0089)	0.1078	0.1468	1.2163	2.3199
ℓ	0.0595 (0.0022)	0.2212 (0.0081)	0.0906	0.1193	1.6489	2.3001

Table 2: Comparative statics experiments. Mass is given as a percentage of households, while remaining quantities are given as a percentage of aggregate capital in the simulated economy.

direction that intuition would suggest weakens aggregation. Specifically, we've considered $\beta = 0.891$ (impatient households), $\alpha = 0.324$ (lower return to savings), and $\delta = 0.0275$ (higher rate of depreciation). In addition, we consider wider dispersion in the aggregate shock processes, letting productivity take values in the set $\{0.979,\ 1.021\}$. We also consider the effect of increasing the labor efficiency endowment in good states, so that it takes values in the set $\{0,\ 1.22\}$.

Table 2 shows the results for a sequence of equilibrium shocks. The first two columns give the mass of poor households along with the capital they hold, the second two columns give the aggregate nonlinear error, and the last two columns give the many period forecast error.

It is evident that the parameter adjustments result in a larger poor population which holds a more substantial percentage of capital, a larger aggregate nonlinear error, and more drastic forecast errors. The largest change in each measure comes from the decrease in β, while the adjustments to α and δ cause very similar increases to the mass of poor households. This is somewhat misleading: in comparative statics terms, the marginal effect of adjusting δ is much closer in magnitude to that of adjusting β in every case.

Widening the possible values for total factor productivity has less of an impact on all measures, and in fact serves to lowers the forecast error, which is sensible given the offsetting income and substitution effects ($\sigma = 1$), insofar as total factor productivity impacts household decisions only in through the discounting of future income. Results are mixed when we increase efficiency in good labor states: each period's shock is now more disperse, even as overall efficiency of employed households is higher.

Summarizing, the economic features which are most likely to obstruct approximate aggregation are impatient households (low β) or a poor savings vehicle, namely one which depreciates quickly.

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APPENDIX I - PROOFS OF PRELIMINARY RESULTS

In this appendix next we provide rigorous proofs of our theoretical results leading up to the main theorems. We begin with Proposition 1, which establishes existence and uniqueness for the households' dynamic programming problems. This proof is a straightforward exercise in finite horizon dynamic programming. For convenience, we recall the statement here.

Proposition 1: Household Existence and Uniqueness. There is a unique solution to the household's dynamic programming problem (2). The associated savings functions $k^{(t)}$ are increasing (strictly for $t < T$) with respect to x_t and satisfies

$$\lim_{x_t \to \underline{k}_t} k^{(t)}(x_t, \mathcal{L}_t, \mathcal{F}_t) = \underline{k}_t,$$

$$\lim_{x_t \to \infty} k^{(t)}(x_t, \mathcal{L}_t, \mathcal{F}_t) = \infty, \quad t < T$$

The corresponding value functions are strictly increasing and strictly concave with respect to ω and satisfy

$$\lim_{x_t \to \underline{k}_t} V^{(t)}(x_t, \mathcal{L}_t, \mathcal{F}_t) = -\infty$$

Proof. The proof is an induction, beginning with the terminal period T. Since $V^{(T+1)} \equiv 0$, it is immediate that the unique solution to the terminal problem is $c^{(T)}(x_T, \mathcal{L}_t, \mathcal{F}_t) = x_T$ and $k^{(T)} \equiv 0$, with corresponding value function $V^{(T)} = u(x_T)$. The savings function is trivially increasing, while the value function is strictly increasing and strictly concave by our selection of preferences. Moreover, in this case we have $\underline{k}_T = 0$, and so that the stated limits at the borrowing constraint hold, trivially in the case of $k^{(T)}$ and due to the asymptote of the period utility function in the case of $V^{(T)}$.

Having established the base case, we now suppose that, given $t < T$, we have a unique solution $(c^{(t+1)}, k^{(t+1)}, V^{(t+1)})$ satisfying the stated properties. As discussed in the main text, the natural borrowing limit implies that the first order conditions are necessary and sufficient for a solution to the household problem. Writing this condition in period t, we have

$$\frac{1}{(x_t - k_t)^\sigma} = \beta \mathbb{E}_t (1 - \delta + R_{t+1}) \frac{\partial V^{(t+1)}(x_{t+1}, \mathcal{L}_{t+1}, \mathcal{F}_{t+1})}{\partial x_{t+1}}$$

For a fixed value of x_t, the left side of this equation is strictly increasing in $k_t \in (\underline{k}_t, x_t)$, and increases without bound as $k_t \to x_t^-$. On the right side, the limit at $\underline{k}_t$ is ∞ due to the asymptote in the value function at the borrowing constraint. Moreover, since x_{t+1} is strictly increasing in k_t, strict concavity of the value function in resources implies that the right side of the equation is strictly decreasing. It follows from these observations that there is a unique value $k^{(t)}(x_t, \mathcal{L}_t, \mathcal{F}_t)$ such that the equation balances.

To see that the savings function is increasing, we differentiate implicitly the first order condition to get

$$\frac{-\sigma}{(x_t - k_t)^{\sigma+1}} \left(1 - \frac{\partial k^{(t)}}{\partial x_t}\right) = \beta \mathbb{E}_t (1 - \delta + R_{t+1})^2 \left(\frac{\partial^2 V}{\partial x_{t+1}^2}\right) \frac{\partial k^{(t)}}{\partial x_t}$$

Solving for the derivative of the savings function, we have

$$\left(\frac{\sigma}{(x_t - k_t)^{\sigma+1}} - \beta \mathbb{E}_t (1 - \delta + R_{t+1})^2 \left(\frac{\partial^2 V}{\partial x_{t+1}^2}\right)\right) \frac{\partial k^{(t)}}{\partial x_t} = \frac{\sigma}{(x_t - k_t)^{\sigma+1}}$$

Once again applying concavity of the value function, we see that all terms here are positive, which establishes (strict) monotonicity.

The limit of the savings function at resources amounting to the natural borrowing limit follows immediately from the squeeze theorem, since $\underline{k}_t \leq k^{(t)} \leq x_t$. We can translate this inequality to read

$$0 \leq x_t - k^{(t)}(x_t, \mathcal{L}_t, \mathcal{F}_t) \leq x_t - \underline{k}_t$$

which show in turn that $c^{(t)} \to 0$ as resources approach the borrowing limit. Then, writing

$$V^{(t)}(x_t, \mathcal{L}_t, \mathcal{F}_t) = u\left(x_t - k^{(t)}(x_t, \mathcal{L}_t, \mathcal{F}_t)\right) + \beta \mathbb{E}_t V^{(t+1)}(x_{t+1}, \mathcal{L}_{t+1}, \mathcal{F}_{t+1})$$

the asymptote for the period t value function follows from inspection of the first term. The limit of the savings function at ∞ is evident from the Euler equations (3)-(4) along with the definition of x_{t+1}. In particular, if household resources in period t are increased without bound, $x_t \to \infty$, while household savings (and hence period $t+1$ resources) remain bounded, the left side of the period t Euler equation would vanish while the right side remained strictly positive, a contradiction.

Monotonicity and convexity of the value function follow from the envelope conditions. Explicitly, differentiating the above expression

$$\begin{aligned}
\frac{\partial V^{(t)}}{\partial x_t} &= \frac{1}{(x_t - k^{(t)})^{\sigma}}\left(1 - \frac{\partial k^{(t)}}{\partial x_t}\right) + \beta \mathbb{E}_t \frac{\partial V^{(t+1)}}{\partial x_{t+1}}(1 - \delta + R_{t+1})\frac{\partial k^{(t)}}{\partial x_t} \\
\frac{\partial^2 V^{(t)}}{\partial x_t^2} &= \frac{-\sigma}{(x_t - k^{(t)})^{\sigma+1}}\left(1 - \frac{\partial k^{(t)}}{\partial x_t}\right)^2 + \beta \mathbb{E}_t \frac{\partial^2 V^{(t+1)}}{\partial x_{t+1}^2}(1 - \delta + R_{t+1})^2 \left(\frac{\partial k^{(t)}}{\partial x_t}\right)^2 \\
&\quad - \frac{1}{(x_t - k^{(t)})^{\sigma}}\frac{\partial^2 k^{(t)}}{\partial x_t^2} + \beta \mathbb{E}_t \frac{\partial V^{(t+1)}}{\partial x_{t+1}}(1 - \delta + R_{t+1})\frac{\partial^2 k^{(t)}}{\partial x_t^2}
\end{aligned}$$

Using first order conditions to simplify these, we therefore get (respectively)

$$\begin{aligned}
\frac{\partial V^{(t)}}{\partial x_t} &= \frac{1}{(x_t - k^{(t)})^{\sigma}} > 0 \\
\frac{\partial^2 V^{(t)}}{\partial x_t^2} &= \frac{-\sigma}{(x_t - k^{(t)})^{\sigma+1}}\left(1 - \frac{\partial k^{(t)}}{\partial x_t}\right)^2 + \beta \mathbb{E}_t \frac{\partial^2 V^{(t+1)}}{\partial x_{t+1}^2}(1 - \delta + R_{t+1})^2 \left(\frac{\partial k^{(t)}}{\partial x_t}\right)^2 < 0
\end{aligned}$$

as desired. This closes the induction. □

Next, we derive the savings function in the deterministic environment with a natural borrowing limit. We will do so in the case of general risk aversion parameters σ. First we recall the expression in the form of a proposition.

Proposition 2: Savings in a Deterministic Production Economy. Let

$$\begin{aligned}
Q_{T-1} &= \left[\beta(1 - \delta + R_T)^{1-\sigma}\right]^{1/\sigma} \\
Q_t &= \left[\beta(1 - \delta + R_{t+1})^{1-\sigma}\right]^{1/\sigma}(1 + Q_{t+1}), \quad t = 1, ..., T-2
\end{aligned}$$

In the production economy without uncertainty, the household savings function is given by

$$k_t(x_t) = \frac{Q_t}{1+Q_t}x_t - \frac{1}{1+Q_t}\left(\sum_{s=t+1}^{T} \frac{W_s \ell}{\prod_{r=t+1}^{s}(1-\delta+R_r)}\right)$$

Proof. Taking σth roots in the terminal Euler equation (8), we obtain

$$\begin{aligned} x_{T-1} - k_{T-1} &= \frac{x_T}{(\beta(1-\delta+R_T))^{1/\sigma}} \\ &= \frac{(1-\delta+R_T)k_{T-1} + W_T\ell}{(\beta(1-\delta+R_T))^{1/\sigma}} \\ &= \frac{k_{T-1} + \frac{W_T\ell}{1-\delta+R_T}}{(\beta(1-\delta+R_T)^{1-\sigma})^{1/\sigma}} \end{aligned}$$

Solving for k_{T-1} gives

$$k_{T-1} = \frac{(\beta(1-\delta+R_T)^{1-\sigma})^{1/\sigma}}{1+(\beta(1-\delta+R_T)^{1-\sigma})^{1/\sigma}}x_{T-1} - \frac{1}{1+(\beta(1-\delta+R_T)^{1-\sigma})^{1/\sigma}}\frac{W_T\ell}{1-\delta+R_T}$$

In terms of the definition of Q_{T-1} and the savings function, this says

$$k^{(T-1)}(x_t) = \frac{Q_{T-1}}{1+Q_{T-1}}x_{T-1} - \frac{1}{1+Q_{T-1}}\frac{W_T\ell}{1-\delta+R_T}$$

We can now proceed by induction, using the formula for the period $t+1$ savings function to simplify the period t Euler equation (7). We can write this equation as

$$x_t - k_t = \frac{k_t + \frac{W_{t+1}\ell}{1-\delta+R_{t+1}} - \frac{k_{t+1}}{1-\delta+R_{t+1}}}{(\beta(1-\delta+R_{t+1})^{1-\sigma})^{1/\sigma}}$$

This will be satisfied taking k_t and k_{t+1} given by the period t and $t+1$ savings functions, respectively. Using the inductive hypothesis for the latter, with $x_{t+1} = (1-\delta+R_{t+1})k_t + W_{t+1}\ell$, we can rewrite the Euler equation as

$$\begin{aligned} x_t - k_t &= \frac{\frac{1}{1+Q_{t+1}}k_t + \frac{1}{1+Q_{t+1}}\frac{W_{t+1}\ell}{1-\delta+R_{t+1}} + \frac{1}{1-\delta+R_{t+1}}\frac{1}{1+Q_{t+1}}\left(\sum_{s=t+2}^{T}\frac{W_s\ell}{\prod_{r=t+2}^{s}(1-\delta+R_r)}\right)}{(\beta(1-\delta+R_{t+1})^{1-\sigma})^{1/\sigma}} \\ &= \frac{\frac{1}{1+Q_{t+1}}k_t + \frac{1}{1+Q_{t+1}}\sum_{s=t+1}^{T}\frac{W_s\ell}{\prod_{r=t+1}^{s}(1-\delta+R_r)}}{(\beta(1-\delta+R_{t+1})^{1-\sigma})^{1/\sigma}} \end{aligned}$$

Solving for k_t, we obtain the desired expression. ∎

Next, we fill in the details of Theorem 1. This version of the theorem admits direct calculations which will be replaced by more circumspect arguments as we allow for more generality. We begin from the rearranged Euler equation (19).

Theorem 1: Nonlinear error, Log Utility. The savings function $k^{(1)}(x_1)$ which solves model (18) can be written in the form

$$k^{(1)}(x_1) = \frac{1}{1+\beta}\left(\beta x_1 - \frac{W_2}{1-\delta+R_2}\,\mathbb{E}\,\ell_2 + \epsilon^{(1)}(x_1)\right)$$

where the nonlinear error term is strictly decreasing, strictly convex, and satisfies

$$\lim_{x_1 \to \underline{k}_1} \epsilon^{(1)}(x_1) = \frac{W_2}{1-\delta+R_2}\mathbb{E}_1\ell_2 + \underline{k}_1, \qquad \lim_{x_1 \to \infty} \epsilon^{(1)}(x_1) = 0$$

Proof. Having established (19) via the calculations in Section 3.2, we may express the right side as

$$\begin{aligned} &k_1 + \frac{W_2}{1-\delta+R_2}(p\ell_{\text{low}} + (1-p)\ell_{\text{high}}) \\ &\quad + \frac{\left(k_1 + \frac{W_2\ell_{\text{low}}}{1-\delta+R_2}\right)\left(k_1 + \frac{W_2\ell_{\text{high}}}{1-\delta+R_2}\right)}{k_1 + \frac{W_2}{1-\delta+R_2}\Big[p\ell_{\text{high}} + (1-p)\ell_{\text{low}}\Big]} - \left(k_1 + \frac{W_2}{1-\delta+R_2}(p\ell_{\text{low}} + (1-p)\ell_{\text{high}})\right) \end{aligned}$$

Writing the last two terms as a single fraction gives

$$\frac{\left(k_1 + \frac{W_2\ell_{\text{low}}}{1-\delta+R_2}\right)\left(k_1 + \frac{W_2\ell_{\text{high}}}{1-\delta+R_2}\right) - \left(k_1 + \frac{W_2}{1-\delta+R_2}(p\ell_{\text{low}} + (1-p)\ell_{\text{high}})\right)\left(k_1 + \frac{W_2}{1-\delta+R_2}\Big(p\ell_{\text{high}} + (1-p)\ell_{\text{low}}\Big)\right)}{k_1 + \frac{W_2}{1-\delta+R_2}\Big[p\ell_{\text{high}} + (1-p)\ell_{\text{low}}\Big]}$$

which after some cancellation in the numerator yields

$$\left(\frac{W_2}{1-\delta+R_2}\right)^2 \frac{\ell_{\text{low}}\ell_{\text{high}} - (p\ell_{\text{low}} + (1-p)\ell_{\text{high}})\,(p\ell_{\text{high}} + (1-p)\ell_{\text{low}})}{k_1 + \frac{W_2}{1-\delta+R_2}\Big[p\ell_{\text{high}} + (1-p)\ell_{\text{low}}\Big]}$$

so that the right side of (19) is now

$$\begin{aligned} &k_1 + \frac{W_2}{1-\delta+R_2}(p\ell_{\text{low}} + (1-p)\ell_{\text{high}}) \\ &\qquad + \left(\frac{W_2}{1-\delta+R_2}\right)^2 \frac{\ell_{\text{low}}\ell_{\text{high}} - (p\ell_{\text{low}} + (1-p)\ell_{\text{high}})\,(p\ell_{\text{high}} + (1-p)\ell_{\text{low}})}{k_1 + \frac{W_2}{1-\delta+R_2}\Big[p\ell_{\text{high}} + (1-p)\ell_{\text{low}}\Big]} \end{aligned}$$

The numerator in the trailing expression can be written as

$$\begin{aligned}\ell_{\text{low}}\ell_{\text{high}} - (p\ell_{\text{low}} + (1-p)\ell_{\text{high}})\,(p\ell_{\text{high}} + (1-p)\ell_{\text{low}})\\ &= (1 - p^2 - (1-p)^2)\ell_{\text{low}}\ell_{\text{high}} - p(1-p)(\ell_{\text{high}}^2 + \ell_{\text{low}}^2)\\ &= (2p - 2p^2)\ell_{\text{low}}\ell_{\text{high}} - p(1-p)(\ell_{\text{high}}^2 + \ell_{\text{low}}^2)\\ &= -p(1-p)\left(\ell_{\text{high}} - \ell_{\text{low}}\right)^2\end{aligned}$$

We can no write out (19) in its entirety as

$$\begin{aligned}&\beta(x_1 - k_1)\\ &\quad = k_1 + \frac{W_2}{1-\delta+R_2}(p\ell_{\text{low}} + (1-p)\ell_{\text{high}}) - \left(\frac{W_2}{1-\delta+R_2}\right)^2 \frac{p(1-p)\left(\ell_{\text{high}} - \ell_{\text{low}}\right)^2}{k_1 + \frac{W_2}{1-\delta+R_2}\left[p\ell_{\text{high}} + (1-p)\ell_{\text{low}}\right]}\end{aligned}$$

Collecting the terms linear in k_1, this rearranges to give (20). As alluded to in the main text, then, we apply Proposition 1 to conclude that there is a well-defined savings function $k^{(1)}$ which satisfies this equation and the conditions of that proposition. Letting

$$\epsilon^{(1)} = \left(\frac{W_2}{1-\delta+R_2}\right)^2 \frac{p(1-p)\left(\ell_{\text{high}} - \ell_{\text{low}}\right)^2}{k^{(1)} + \frac{W_2}{1-\delta+R_2}\left[p\ell_{\text{high}} + (1-p)\ell_{\text{low}}\right]} \tag{27}$$

we have

$$k^{(1)}(x_1) = \frac{1}{1+\beta}\left(\beta x_1 - \frac{W_2}{1-\delta+R_2}\mathbb{E}_1\ell_2 + \epsilon^{(1)}(x_1)\right) \tag{28}$$

We must now establish properties of the nonlinear error term. The limit of this error term at the borrowing constraint follows from the limit of the savings function at this constraint. Specifically, taking the limit in the above expression we have

$$\underline{k}_1 = \frac{1}{1+\beta}\left(\beta\underline{k}_1 - \frac{W_2}{1-\delta+R_2}\mathbb{E}_1\ell_2 + \lim_{x_1\to\underline{k}_1}\epsilon^{(1)}(x_1)\right)$$

which we solve for

$$\lim_{x_1\to\underline{k}_1}\epsilon^{(1)}(x_1) = \frac{W_2}{1-\delta+R_2}\mathbb{E}_1\ell_2 + \underline{k}_1$$

To calculate the limit as resources increase without bound, we apply our knowledge that savings increase without bound in this limit along with the definition (27) to get

$$\begin{aligned}\lim_{x_1\to\infty}\epsilon^{(1)}(x_1) &= \left(\frac{W_2}{1-\delta+R_2}\right)^2 \frac{p(1-p)\left(\ell_{\text{high}}-\ell_{\text{low}}\right)^2}{\lim_{x_1\to\infty}k^{(1)}+\frac{W_2}{1-\delta+R_2}\Big[p\ell_{\text{high}}+(1-p)\ell_{\text{low}}\Big]} \\ &= 0\end{aligned}$$

To see monotonicity, observe that

$$\frac{\partial\epsilon^{(1)}(x_1)}{\partial x_1} = -\left(\frac{W_2}{1-\delta+R_2}\right)^2 \frac{p(1-p)\left(\ell_{\text{high}}-\ell_{\text{low}}\right)^2}{\Big(k^{(1)}+\frac{W_2}{1-\delta+R_2}[p\ell_{\text{high}}+(1-p)\ell_{\text{low}}]\Big)^2}\frac{\partial k^{(1)}}{\partial x_1} < 0 \tag{29}$$

where the inequality follows from the fact that the savings function is increasing.

To see convexity, first observe that the linear-plus-error structure implies that

$$\frac{\partial^2 k^{(1)}}{\partial x_1^2} = \frac{\partial^2\epsilon^{(1)}}{\partial x_1^2} \tag{30}$$

Then, taking a second derivative in (29) we have

$$\begin{aligned}\frac{\partial^2\epsilon^{(1)}(x_1)}{\partial x_1^2} = 2\left(\frac{W_2}{1-\delta+R_2}\right)^2 \frac{p(1-p)\left(\ell_{\text{high}}-\ell_{\text{low}}\right)^2}{\Big(k^{(1)}+\frac{W_2}{1-\delta+R_2}[p\ell_{\text{high}}+(1-p)\ell_{\text{low}}]\Big)^3}\frac{\partial k^{(1)}}{\partial x_1} \\ -\left(\frac{W_2}{1-\delta+R_2}\right)^2 \frac{p(1-p)\left(\ell_{\text{high}}-\ell_{\text{low}}\right)^2}{\Big(k^{(1)}+\frac{W_2}{1-\delta+R_2}[p\ell_{\text{high}}+(1-p)\ell_{\text{low}}]\Big)^2}\frac{\partial^2 k^{(1)}}{\partial x_1^2}\end{aligned}$$

Using (30) and solving, we get

$$\frac{\partial^2\epsilon^{(1)}(x_1)}{\partial x_1^2} = 2\frac{\left(\frac{W_2}{1-\delta+R_2}\right)^2 \frac{p(1-p)\left(\ell_{\text{high}}-\ell_{\text{low}}\right)^2}{\left(k^{(1)}+\frac{W_2}{1-\delta+R_2}[p\ell_{\text{high}}+(1-p)\ell_{\text{low}}]\right)^3}}{1+\left(\frac{W_2}{1-\delta+R_2}\right)^2 \frac{p(1-p)\left(\ell_{\text{high}}-\ell_{\text{low}}\right)^2}{\left(k^{(1)}+\frac{W_2}{1-\delta+R_2}[p\ell_{\text{high}}+(1-p)\ell_{\text{low}}]\right)^2}}\frac{\partial k^{(1)}}{\partial x_1}$$

Every expression on the right hand side is positive, which gives convexity of the nonlinear error term. This completes the proof. □

Next we fill in the short proof of Lemma 1. While this lemma amounts to a basic calculus exercise, it will be a vital ingredient in general versions of our theorems.

Lemma 3. *Let $A, B \in \mathbb{R}$ and let $f : \mathbb{R} \to \mathbb{R}$ satisfy*

$$\lim_{x\to\infty}(f(x) - Ax) = B$$

Then

$$\lim_{x\to\infty}(f(x) - (Ax + B)) = 0.$$

Proof. We have

$$\lim_{x\to\infty}(f(x) - (Ax + B)) = \lim_{x\to\infty}(f(x) - Ax) - B = B - B = 0$$

which proves it. □

On a practical note, observe that

$$\lim_{x\to\infty}\left(\frac{f(x)}{x} - A\right) = \lim_{x\to\infty}\left(\frac{f(x) - Ax}{x}\right) = \lim_{x\to\infty}\left(\frac{B}{x}\right) = 0$$

Rearranging gives us a simple way to calculate the slope A:

$$A = \lim_{x\to\infty}\frac{f(x)}{x}$$

We are now ready to complete the proof of Theorem 2, in which we have omitted aggregate uncertainty and restricted ourselves to two periods, while allowing for general period utility.

Theorem 2: Nonlinear Error, CRRA Utility. The savings function $k^{(1)}(x_1)$ which solves model (18) with log replaced by a general CRRA utility function can be written in the form

$$k^{(1)}(x_1) = \frac{1}{1+Q_1}\left(Q_1 x_1 - \frac{W_2}{1-\delta+R_2}\,\mathbb{E}\,\ell_2 + \epsilon^{(1)}(x_1)\right),$$
$$\text{with} \quad Q_1 = \left[\beta(1-\delta+R_2)^{1-\sigma}\right]^{1/\sigma}$$

where the nonlinear error term is strictly decreasing, convex, and satisfies

$$\lim_{x_1\to\underline{k}_1}\epsilon^{(1)}(x_1) = \frac{W_2}{1-\delta+R_2}\mathbb{E}_1\ell_2 + \underline{k}_1, \qquad \lim_{x_1\to\infty}\epsilon^{(1)}(x_1) = 0$$

Proof. To simplify our expressions, we will adopt the notation

$$w_{\text{low}} = \frac{W_2}{1-\delta+R_2}\ell_{\text{low}}, \quad w_{\text{high}} = \frac{W_2}{1-\delta+R_2}\ell_{\text{high}}$$

throughout the proof.

We begin from the rearranged Euler equation (23), which we restate here for convenience:

$$\left[\beta(1-\delta+R_2)^{1-\sigma}\right]^{1/\sigma}(x_1-k_1) = \frac{(k_1+w_{\text{low}})(k_1+w_{\text{high}})}{\left[p(k_1+w_{\text{high}})^{\sigma}+(1-p)(k_1+w_{\text{low}})^{\sigma}\right]^{1/\sigma}}$$

We will apply Lemma 1 to the right hand side of this equation. To this end, we calculate the limit

$$\lim_{k_1\to\infty}\left(\frac{(k_1+w_{\text{low}})(k_1+w_{\text{high}})}{\left[p(k_1+w_{\text{high}})^{\sigma}+(1-p)(k_1+w_{\text{low}})^{\sigma}\right]^{1/\sigma}} - k_1\right)$$

Expressing the argument as a single fraction, rearranged, we wish to calculate

$$\lim_{k_1\to\infty}\left(\frac{k_1^2\left(1-\left[p\left(1+\frac{w_{\text{high}}}{k_1}\right)^{\sigma}+(1-p)\left(1+\frac{w_{\text{low}}}{k_1}\right)^{\sigma}\right]^{1/\sigma}\right)+k_1(w_{\text{low}}+w_{\text{high}})+w_{\text{low}}w_{\text{high}}}{k_1\left[p\left(1+\frac{w_{\text{high}}}{k_1}\right)^{\sigma}+(1-p)\left(1+\frac{w_{\text{low}}}{k_1}\right)^{\sigma}\right]^{1/\sigma}}\right)$$

We split the argument into three separate fractions, one for each term in the numerator, and compute the limit of each. After some thought, the third of these limits is zero, while the second is $w_{\text{low}}+w_{\text{high}}$. The first limit we rewrite as

$$\lim_{k_1\to\infty}\left(\frac{k_1\left(1-\left[p\left(1+\frac{w_{\text{high}}}{k_1}\right)^{\sigma}+(1-p)\left(1+\frac{w_{\text{low}}}{k_1}\right)^{\sigma}\right]^{1/\sigma}\right)}{\left[p\left(1+\frac{w_{\text{high}}}{k_1}\right)^{\sigma}+(1-p)\left(1+\frac{w_{\text{low}}}{k_1}\right)^{\sigma}\right]^{1/\sigma}}\right)$$

The denominator here has limit 1, so the above limit will equal

$$\lim_{k_1\to\infty}\left(k_1\left(1-\left[p\left(1+\frac{w_{\text{high}}}{k_1}\right)^{\sigma}+(1-p)\left(1+\frac{w_{\text{low}}}{k_1}\right)^{\sigma}\right]^{1/\sigma}\right)\right)$$

provided this limit exists. We rewrite the argument as

$$\lim_{k_1\to\infty}\left(\frac{\left(1-\left[p\left(1+\frac{w_{\text{high}}}{k_1}\right)^{\sigma}+(1-p)\left(1+\frac{w_{\text{low}}}{k_1}\right)^{\sigma}\right]^{1/\sigma}\right)}{1/k_1}\right)$$

and observe that the numerator and denominator both have limit equal to zero as $k_1 \to \infty$. We may therefore use l'Hospital's rule to conclude that this limit is equal to that of

$$\frac{-\frac{1}{\sigma}\left[p\left(1+\frac{w_{\text{high}}}{k_1}\right)^{\sigma}+(1-p)\left(1+\frac{w_{\text{low}}}{k_1}\right)^{\sigma}\right]^{\frac{1}{\sigma}-1}}{(-1/k_1^2)} \times\left[\sigma p\left(1+\frac{w_{\text{high}}}{k_1}\right)^{\sigma-1}\left(-\frac{w_{\text{high}}}{k_1^2}\right)+\sigma(1-p)\left(1+\frac{w_{\text{low}}}{k_1}\right)^{\sigma-1}\left(-\frac{w_{\text{low}}}{k_1^2}\right)\right]$$

This expression simplifies to give

$$-\left[p\left(1+\frac{w_{\text{high}}}{k_1}\right)^{\sigma}+(1-p)\left(1+\frac{w_{\text{low}}}{k_1}\right)^{\sigma}\right]^{\frac{1}{\sigma}-1} \times\left[p\left(1+\frac{w_{\text{high}}}{k_1}\right)^{\sigma-1}w_{\text{high}}+(1-p)\left(1+\frac{w_{\text{low}}}{k_1}\right)^{\sigma-1}w_{\text{low}}\right]$$

from which we can read off the limit as

$$-pw_{\text{high}}-(1-p)w_{\text{low}}$$

Combining all three limits, we now get

$$\begin{aligned}\lim_{k_1\to\infty}\left(\frac{(k_1+w_{\text{low}})(k_1+w_{\text{high}})}{\left[p(k_1+w_{\text{high}})^{\sigma}+(1-p)(k_1+w_{\text{low}})^{\sigma}\right]^{1/\sigma}}-k_1\right)&=-pw_{\text{high}}-(1-p)w_{\text{low}}+w_{\text{low}}+w_{\text{high}}\\&=\mathbb{E}_1w_2\end{aligned}$$

Applying Lemma 1, we conclude that

$$\lim_{k_1\to\infty}\left(\frac{(k_1+w_{\text{low}})(k_1+w_{\text{high}})}{\left[p(k_1+w_{\text{high}})^{\sigma}+(1-p)(k_1+w_{\text{low}})^{\sigma}\right]^{1/\sigma}}-k_1-\mathbb{E}_1w_2\right)=0$$

Denoting the argument of the limit by $-\delta(k_1)$, we may therefore write the Euler equation

(23) in this case as

$$\left[\beta(1-\delta+R_2)^{1-\sigma}\right]^{1/\sigma}(x_1-k_1) = k_1 + \mathbb{E}_1 w_2 - \delta(k_1)$$

with $\lim_{k_1\to\infty}\delta(k_1) = 0$. Rearranging, this becomes

$$k_1 = \frac{1}{1+Q_1}\left(Q_1 x_1 - \mathbb{E}_1 w_2 + \delta(k_1)\right)$$

with Q_1 as given in the theorem statement.

Once again invoking the existence proposition as in the proof of the previous theorem and letting $\epsilon^{(1)}(x_1) = \delta\left(k^{(1)}(x_1)\right)$, we obtain the desired decomposition

$$k^{(1)}(x_1) = \frac{1}{1+Q_1}\left(Q_1 x_1 - \mathbb{E}_1 w_2 + \epsilon^{(1)}(x_1)\right)$$

Taking limits at the borrowing constraint,

$$\underline{k}_1 = \frac{1}{1+Q_1}\left(Q_1 \underline{k}_1 - \mathbb{E}_1 w_2 + \lim_{x_1\to\underline{k}_1}\epsilon^{(1)}(x_1)\right)$$

which we solve for

$$\lim_{x_1\to\underline{k}_1}\epsilon^{(1)}(x_1) = \mathbb{E}_1 w_2 + \underline{k}_1$$

We also have

$$\lim_{x_1\to\infty}\epsilon^{(1)}(x_1) = \lim_{x_1\to\infty}\delta\left(k^{(1)}(x_1)\right) = 0$$

since $k^{(1)}(x_1)\to\infty$ as $x_1\to\infty$ (by the existence and uniqueness proposition) and $\delta\to 0$ as its argument increases to ∞.

To establish that $\epsilon^{(1)}$ is decreasing in resources, we observe that

$$\frac{\partial\epsilon^{(1)}}{\partial x_1} = \frac{\partial\delta}{\partial k_1}\frac{\partial k^{(1)}}{\partial x_1}$$

has the same sign as $\partial\delta/\partial k_1$. Next we observe that the definition of $\delta(k_1)$ satisfies

$$\left(\frac{p}{(k_1+w_{\text{low}})^{\sigma}} + \frac{1-p}{(k_1+w_{\text{high}})^{\sigma}}\right)^{-1/\sigma} = k_1 + \mathbb{E}_1 w_2 - \delta(k_1) \tag{31}$$

It therefore suffices to show that the derivative of the left side of the above equality with respect to k_1 is larger than unity. Calculating this derivative, it suffices to show that

$$\left(\frac{p}{(k_1+w_{\text{low}})^\sigma}+\frac{1-p}{(k_1+w_{\text{high}})^\sigma}\right)^{-\frac{1}{\sigma}-1}\left(\frac{p}{(k_1+w_{\text{low}})^{\sigma+1}}+\frac{1-p}{(k_1+w_{\text{high}})^{\sigma+1}}\right)>1$$

Rearranging, this is equivalent to the bound

$$\left(\frac{p}{(k_1+w_{\text{low}})^\sigma}+\frac{1-p}{(k_1+w_{\text{high}})^\sigma}\right)^{\frac{1}{\sigma}}<\left(\frac{p}{(k_1+w_{\text{low}})^{\sigma+1}}+\frac{1-p}{(k_1+w_{\text{high}})^{\sigma+1}}\right)^{\frac{1}{\sigma+1}}$$

That this inequality is true is a consequence of the fact that L^σ norms are increasing in σ (which in turn follows from Jensen's inequality).

The proof of convexity is similar. First, we once again observe that the linear-plus-error structure implies that

$$\frac{\partial^2 k^{(1)}}{\partial x_1^2}=\frac{\partial^2 \epsilon^{(1)}}{\partial x_1^2}$$

so that

$$\begin{aligned}\frac{\partial^2 \epsilon^{(1)}}{\partial x_1^2}&=\frac{\partial^2\delta}{\partial k_1^2}\left(\frac{\partial k^{(1)}}{\partial x_1}\right)^2+\frac{\partial\delta}{\partial k_1}\frac{\partial^2 k^{(1)}}{\partial x_1^2}\\&=\frac{\partial^2\delta}{\partial k_1^2}\left(\frac{\partial k^{(1)}}{\partial x_1}\right)^2+\frac{\partial\delta}{\partial k_1}\frac{\partial^2 \epsilon^{(1)}}{\partial x_1^2}\end{aligned}$$

and consequently

$$\left(1-\frac{\partial\delta}{\partial k_1}\right)\frac{\partial^2 \epsilon^{(1)}}{\partial x_1^2}=\frac{\partial^2\delta}{\partial k_1^2}\left(\frac{\partial k^{(1)}}{\partial x_1}\right)^2$$

Since we have seen above that δ is decreasing in k_1 it suffices to show that δ is convex.

Letting

$$\begin{aligned}h(k_1)&:=\frac{p}{(k_1+w_{\text{low}})^\sigma}+\frac{1-p}{(k_1+w_{\text{high}})^\sigma}\\g(k_1)&:=h(k_1)^{-1/\sigma}\end{aligned}$$

we note from (31) that

$$\frac{\partial^2\delta}{\partial k_1^2}=-\frac{\partial^2 g}{\partial k_1^2}$$

so that it suffices to show that $\partial^2 g/\partial k_1^2 < 0$. We have

$$\begin{aligned}
\frac{\partial g}{\partial k_1} &= -\frac{1}{\sigma}h(k_1)^{-\frac{1}{\sigma}-1}\frac{\partial h}{\partial k_1} \\
\frac{\partial^2 g}{\partial k_1^2} &= -\frac{1}{\sigma}\left(-\frac{1}{\sigma}-1\right)h(k_1)^{-\frac{1}{\sigma}-2}\left(\frac{\partial h}{\partial k_1}\right)^2 - \frac{1}{\sigma}h(k_1)^{-\frac{1}{\sigma}-1}\frac{\partial^2 h}{\partial k_1^2} \\
&= \frac{1}{\sigma}h(k_1)^{-\frac{1}{\sigma}-2}\left(\left(\frac{1}{\sigma}+1\right)\left(\frac{\partial h}{\partial k_1}\right)^2 - h(k_1)\frac{\partial^2 h}{\partial k_1^2}\right)
\end{aligned}$$

It therefore suffices that the expression in brackets in the last line is negative. To do so, we first observe that

$$\begin{aligned}
\frac{\partial h}{\partial k_1} &= \frac{-\sigma p}{(k_1+w_{\text{low}})^{\sigma+1}} + \frac{-\sigma(1-p)}{(k_1+w_{\text{high}})^{\sigma+1}} \\
\frac{\partial^2 h}{\partial k_1^2} &= \frac{\sigma(\sigma+1)p}{(k_1+w_{\text{low}})^{\sigma+2}} + \frac{\sigma(\sigma+1)(1-p)}{(k_1+w_{\text{high}})^{\sigma+2}}
\end{aligned}$$

so it suffices to show that

$$\begin{aligned}
&\left(\frac{1}{\sigma}+1\right)\left[\frac{-\sigma p}{(k_1+w_{\text{low}})^{\sigma+1}} + \frac{-\sigma(1-p)}{(k_1+w_{\text{high}})^{\sigma+1}}\right]^2 \\
&\qquad - \left[\frac{p}{(k_1+w_{\text{low}})^{\sigma}} + \frac{1-p}{(k_1+w_{\text{high}})^{\sigma}}\right]\left[\frac{\sigma(\sigma+1)p}{(k_1+w_{\text{low}})^{\sigma+2}} + \frac{\sigma(\sigma+1)(1-p)}{(k_1+w_{\text{high}})^{\sigma+2}}\right] < 0
\end{aligned}$$

Clearing factors of σ and rearranging, this is equivalent to showing

$$\begin{aligned}
&\left[\frac{p}{(k_1+w_{\text{low}})^{\sigma+1}} + \frac{(1-p)}{(k_1+w_{\text{high}})^{\sigma+1}}\right]^2 \\
&\qquad < \left[\frac{p}{(k_1+w_{\text{low}})^{\sigma}} + \frac{1-p}{(k_1+w_{\text{high}})^{\sigma}}\right]\left[\frac{p}{(k_1+w_{\text{low}})^{\sigma+2}} + \frac{(1-p)}{(k_1+w_{\text{high}})^{\sigma+2}}\right]
\end{aligned}$$

That this inequality holds is a consequence of the Cauchy-Schwartz inequality. This completes the proof of the theorem. □

Appendix II - Proofs of the Main Results

In this appendix we give details of the proofs of our main theorems, Theorem 3 and Theorem 4. As mentioned in the main text, these proofs build on the two period versions by including aggregate uncertainty, arbitrarily many shock outcomes, and multiple periods, leading to substantial additional bookkeeping and the need to track additional endogenous components in the optimality conditions.

The proof of the logarithmic case illustrates the majority of the main features, while eliminating the somewhat tedious tracking of recursive effective discount factors which appear in the general case. For this reason, we present this case in detail before giving the argument in full generality.

We will find it convenient to introduce the notation $o(f(x))$ to denote any function $g(x)$ such that $\lim_{x\to\infty} g(x)/f(x) = 0$. In particular, $o(1)$ denotes any function $g(x)$ such that $\lim_{x\to\infty} g(x) = 0$.

Recall the statement of the theorem in the log case.

Theorem 3: Main Theorem, $\sigma = 1$. The savings functions $k^{(t)}(x_t, \mathcal{L}_t, \mathcal{F}_t)$, $t = 1, ..., T$ which solve the dynamic programming problems (2) with $\sigma = 1$ can be written in the form

$$k^{(t)}(x_t, \mathcal{L}_t, \mathcal{F}_t) = \frac{\beta + ... + \beta^{T-t}}{1 + \beta + ... + \beta^{T-t}} x_t - \frac{1}{1 + \beta + ... + \beta^{T-t}} \mathbb{E}_t \left(\sum_{s=t+1}^{T} \frac{W_s \ell_s}{\prod_{r=t+1}^{s} (1 - \delta + R_r)} \right) + \epsilon^{(t)}(x_t, \mathcal{L}_t, \mathcal{F}_t)$$

where the nonlinear error term $\epsilon^{(t)}$ is identically zero without uncertainty, and is strictly decreasing, strictly convex, and satisfies

$$\lim_{x_1 \to \underline{k}_t} \epsilon^{(t)}(x_t, \mathcal{L}_t, \mathcal{F}_t) = \frac{1}{1 + \beta + ... + \beta^{T-t}} \left[\mathbb{E}_t \left(\sum_{s=t+1}^{T} \frac{W_s \ell_s}{\prod_{r=t+1}^{s} (1 - \delta + R_r)} \right) + \underline{k}_t \right]$$

and

$$\lim_{x_1 \to \infty} \epsilon^{(t)}(x_t, \mathcal{L}_t, \mathcal{F}_t) = 0$$

with uncertainty.

Proof. The proof is by induction, beginning with the period $T-1$ savings function. The base case essentially follows the argument given for the two period model detailed in Appendix I, albeit with additional bookkeeping due to the inclusion of arbitrarily many shock outcomes and aggregate uncertainty. Although it is lengthy, we provide the argument, to illustrate the algebra without the need to worry about additional endogeneity due to future savings.

Base Case (T-1): We begin by writing out the expected value of the terminal Euler equation (4). To facilitate this, for a given state $(x_{T-1}, \mathcal{L}_{T-1}, \mathcal{F}_{T-1})$ of resources and predictive probabilities at time $T-1$ we let

$$\{(\ell_j, R_j, W_j) : \ j = 1, ..., J\}$$

denote the possible time T endowment and price outcomes which are assigned positive predictive probability. We denote the associated predictive probabilities by p_j. Moreover, we let $w_j = W_j \ell_j / (1 - \delta + R_j)$.

With this notation in hand the Euler equation can be written out as

$$\begin{aligned} \frac{1}{x_{T-1} - k_{T-1}} &= \beta \left(\sum_{j=1}^{J} \frac{p_j}{k_{T-1} + w_j} \right) \\ &= \beta \left(\frac{\sum_{j=1}^{J} p_j \prod_{i \neq j} (k_{T-1} + w_i)}{\prod_j (k_{T-1} + w_j)} \right) \end{aligned}$$

Taking the reciprocal, this gives

$$\beta(x_{T-1} - k_{T-1}) = \frac{\prod_j (k_{T-1} + w_j)}{\sum_{j=1}^{J} p_j \prod_{i \neq j} (k_{T-1} + w_i)} \tag{32}$$

The right hand side is a rational function whose numerator has degree J and whose denominator has degree $J-1$. We therefore expect this to approach some linear asymptote as $k_{T-1} \to \infty$. To extract the asymptote, we write out the numerator and denominator, getting

$$\beta(x_{T-1} - k_{T-1}) = \frac{k_{T-1}^{J} + k_{T-1}^{J-1} \sum_j w_j + o(k_{T-1}^{J-1})}{k_{T-1}^{J-1} + k_{T-1}^{J-2} \sum_j p_j \sum_{i \neq j} w_i + o(k_{T-1}^{J-2})}$$

We can now apply Lemma 1 on the right hand side in a transparent way. Specifically, we have

$$\begin{aligned} &\lim_{k_{T-1} \to \infty} \left(\frac{k_{T-1}^{J} + k_{T-1}^{J-1} \sum_j w_j + o(k_{T-1}^{J-1})}{k_{T-1}^{J-1} + k_{T-1}^{J-2} \sum_j p_j \sum_{i \neq j} w_i + o(k_{T-1}^{J-2})} - k_{T-1} \right) \\ &\quad = \lim_{k_{T-1} \to \infty} \left(\frac{k_{T-1}^{J} + k_{T-1}^{J-1} \sum_j w_j + o(k_{T-1}^{J-1}) - k_{T-1}^{J} + k_{T-1}^{J-1} \sum_j p_j \sum_{i \neq j} w_i + o(k_{T-1}^{J-1})}{k_{T-1}^{J-1} + k_{T-1}^{J-2} \sum_j p_j \sum_{i \neq j} w_i + o(k_{T-1}^{J-2})} \right) \\ &\quad = \lim_{k_{T-1} \to \infty} \left(\frac{k_{T-1}^{J-1} \sum_j p_j w_j + o(k_{T-1}^{J-1})}{k_{T-1}^{J-1} + k_{T-1}^{J-2} \sum_j p_j \sum_{i \neq j} w_i + o(k_{T-1}^{J-2})} \right) \\ &\quad = \lim_{k_{T-1} \to \infty} \left(\frac{\sum_j p_j w_j + o(1)}{1 + o(1)} \right) \\ &\quad = \sum_j p_j w_j \\ &\quad = \mathbb{E}_{T-1} w_T \end{aligned}$$

so that the lemma says that

$$\lim_{k_{T-1}\to\infty}\left(\frac{\prod_j(k_{T-1}+w_j)}{\sum_{j=1}^J p_j\prod_{i\neq j}(k_{T-1}+w_i)}-k_{T-1}-\mathbb{E}_{T-1}w_T\right)=0$$

Letting $-\delta(k_{T-1})$ denote the argument of the above limit, we combine with (32) to get

$$\beta(x_{T-1}-k_{T-1})=k_{T-1}+\mathbb{E}_{T-1}w_T-\delta(k_{T-1})$$

By the existence and uniqueness proposition, there is a unique period $T-1$ savings function $k^{(T-1)}$ which solves the household problem in this period. Hence we may define

$$\epsilon^{(1)}(x_1,\mathcal{L}_{T-1},\mathcal{F}_{T-1}):=\delta\left(k^{(1)}(x_1,\mathcal{L}_{T-1},\mathcal{F}_{T-1})\right)$$

and rearrange the above equation into the desired form

$$k^{(T-1)}(x_{T-1},\mathcal{L}_{T-1},\mathcal{F}_{T-1})=\frac{1}{1+\beta}\left(\beta x_{T-1}-\mathbb{E}_{T-1}w_T+\epsilon^{(1)}(x_1,\mathcal{L}_{T-1},\mathcal{F}_{T-1})\right)$$

We can calculate the behavior of the nonlinear error at the domain endpoints identically to the two period CRRA case above. Precisely, taking limits at the borrowing constraint,

$$\underline{k}_{T-1}(\mathcal{L}_{T-1},\mathcal{F}_{T-1})=\frac{1}{1+\beta}\Bigg(\beta\underline{k}_{T-1}(\mathcal{L}_{T-1},\mathcal{F}_{T-1})-\mathbb{E}_{T-1}w_T + \lim_{x_{T-1}\to\underline{k}_{T-1}(\mathcal{L}_{T-1},\mathcal{F}_{T-1})}\epsilon^{(T-1)}(x_{T-1},\mathcal{L}_{T-1},\mathcal{F}_{T-1})\Bigg)$$

which we solve for

$$\lim_{x_{T-1}\to\underline{k}_{T-1}(\mathcal{L}_{T-1},\mathcal{F}_{T-1})}\epsilon^{(T-1)}(x_{T-1},\mathcal{L}_{T-1},\mathcal{F}_{T-1})=\mathbb{E}_{T-1}w_T+\underline{k}_{T-1}(\mathcal{L}_{T-1},\mathcal{F}_{T-1})$$

We also have

$$\lim_{x_{T-1}\to\infty}\epsilon^{(T-1)}(x_{T-1},\mathcal{L}_{T-1},\mathcal{F}_{T-1})=\lim_{x_{T-1}\to\infty}\delta\left(k^{(T-1)}(x_{T-1},\mathcal{L}_{T-1},\mathcal{F}_{T-1})\right)=0$$

since $k^{(T-1)}(x_{T-1})\to\infty$ as $x_{T-1}\to\infty$ (by the existence and uniqueness proposition) and $\delta\to 0$ as its argument increases to ∞.

To establish that $\epsilon^{(T-1)}$ is decreasing in resources, we once again observe that

$$\frac{\partial \epsilon^{(T-1)}}{\partial x_{T-1}} = \frac{\partial \delta}{\partial k_{T-1}} \frac{\partial k^{(T-1)}}{\partial x_{T-1}}$$

has the same sign as $\partial\delta/\partial k_{T-1}$. Next we observe that the definition of $\delta(k_{T-1})$ satisfies

$$\left(\sum_{j=1}^{J} \frac{p_j}{k_{T-1} + w_j}\right)^{-1} = k_{T-1} + \mathbb{E}_{T-1} w_T - \delta(k_{T-1})$$

It therefore suffices to show that the derivative of the left side of the above equality with respect to k_{T-1} is larger than unity. Calculating this derivative, it suffices to show that

$$\left(\sum_{j=1}^{J} \frac{p_j}{k_{T-1} + w_j}\right)^{-2} \left(\sum_{j=1}^{J} \frac{p_j}{(k_{T-1} + w_j)^2}\right) > 1$$

Rearranging, this is equivalent to the bound

$$\sum_{j=1}^{J} \frac{p_j}{k_{T-1} + w_j} < \left(\sum_{j=1}^{J} \frac{p_j}{(k_{T-1} + w_j)^2}\right)^{1/2}$$

That this inequality is true is a consequence of the fact that the L^1 norm on the left is dominated by the L^2 norm on the right (which, again, follows from Jensen's inequality).

To prove convexity, we once again observe that the linear-plus-error structure implies that

$$\frac{\partial^2 k^{(T-1)}}{\partial x_{T-1}^2} = \frac{\partial^2 \epsilon^{(T-1)}}{\partial x_{T-1}^2}$$

so that, exactly as in the two period CRRA case, it suffices to show that δ is convex. Letting

$$h(k_{T-1}) := \sum_{j=1}^{J} \frac{p_j}{k_{T-1} + w_j}$$
$$g(k_{T-1}) := 1/h(k_{T-1})$$

and repeating the derivative calculations in that case, we see that it suffices to show that

$$2\left(\frac{\partial h}{\partial k_{T-1}}\right)^2 - h(k_{T-1}) \frac{\partial^2 h}{\partial k_{T-1}^2} < 0$$

To do so, we calculate

$$\frac{\partial h}{\partial k_{T-1}} = -\sum_{j=1}^{J} \frac{p_j}{(k_{T-1}+w_j)^2}$$
$$\frac{\partial^2 h}{\partial k_{T-1}^2} = 2\sum_{j=1}^{J} \frac{p_j}{(k_{T-1}+w_j)^3}$$

so it suffices to show that

$$\left[\sum_{j=1}^{J} \frac{p_j}{(k_{T-1}+w_j)^2}\right]^2 - \left[\sum_{j=1}^{J} \frac{p_j}{k_{T-1}+w_j}\right]\left[\sum_{j=1}^{J} \frac{p_j}{(k_{T-1}+w_j)^3}\right] < 0$$

which is equivalent to

$$\left[\sum_{j=1}^{J} \frac{p_j}{(k_{T-1}+w_j)^2}\right]^2 < \left[\sum_{j=1}^{J} \frac{p_j}{k_{T-1}+w_j}\right]\left[\sum_{j=1}^{J} \frac{p_j}{(k_{T-1}+w_j)^3}\right]$$

This inequality is once again true by the Cauchy-Schwartz inequality, and this establishes our base case for the induction.

Inductive Step. Suppose now that the theorem is proved for the savings function in period $t+1$, where $1 < t \leq T-1$. For a given state $(x_t, \mathcal{L}_t, \mathcal{F}_t)$ of resources and predictive probabilities at time t, we suppose that there are J time $t+1$ states which are assigned positive predictive probability, and we let

$$\{(\ell_j, R_j, W_j) : \; j = 1, ..., J\}$$

denote the possible time $t+1$ endowment and price outcomes. Note that both J and the set in the above line may change depending on which period t we are focused on - we suppress these dependencies for simplicity of notation. We denote the associated predictive probabilities by p_j. Moreover, we let $w_j = W_j \ell_j/(1-\delta+R_j)$.

The inductive step will, naturally, involve what we already know about the time $t+1$ savings function and its nonlinear error term. For simplicity, we will suppress the dependence of these functions on predictive distributions in the argument.

Having established these notational conventions the time t Euler equation can be written as

$$\frac{1}{x_t - k_t} = \beta \left(\sum_{j=1}^{J} \frac{p_j}{k_t + w_j - \frac{1}{1-\delta+R_j} k_{t+1}} \right)$$

For optimality, time $t+1$ savings must be given by the time $t+1$ savings function, and by the inductive hypothesis we have

$$\begin{aligned}
&\frac{1}{1-\delta+R_j} k^{(t+1)} \Big((1-\delta+R_j)k_t + W_j \ell_j \Big) \\
&\qquad = \frac{\beta + ... + \beta^{T-t-1}}{1+\beta+...+\beta^{T-t-1}}(k_t + w_j) - \frac{1}{1+\beta+...+\beta^{T-t-1}} \mathbb{E}_{t+1} \left(\sum_{s=t+2}^{T} \frac{W_s \ell_s}{\prod_{r=t+2}^{s}(1-\delta+R_r)} \right) \\
&\qquad\quad + \epsilon^{(t+1)} \Big((1-\delta+R_j)k_t + W_j \ell_j \Big)
\end{aligned}$$

so that the denominators on the right side of the Euler equation take the form

$$\begin{aligned}
&\frac{1}{1+\beta+...+\beta^{T-t-1}} k_t + \frac{1}{1+\beta+...+\beta^{T-t-1}} \left(w_j + \mathbb{E}_{t+1} \left(\sum_{s=t+2}^{T} \frac{W_s \ell_s}{\prod_{r=t+2}^{s}(1-\delta+R_r)} \right) \right) \\
&\qquad - \epsilon^{(t+1)} \Big((1-\delta+R_j)k_t + W_j \ell_j \Big)
\end{aligned}$$

In order to simplify the algebraic steps to follow, we let

$$\begin{aligned}
A_j \equiv A &:= \frac{1}{1+\beta+...+\beta^{T-t-1}} \\
B_j &:= \frac{1}{1+\beta+...+\beta^{T-t-1}} \left(w_j + \mathbb{E}_{t+1} \left(\sum_{s=t+2}^{T} \frac{W_s \ell_s}{\prod_{r=t+2}^{s}(1-\delta+R_r)} \right) \right) \\
C_j(k_t) &:= B_j - \epsilon^{(t+1)} \Big((1-\delta+R_j)k_t + W_j \ell_j \Big)
\end{aligned}$$

In this notation, the Euler equation becomes

$$\frac{1}{x_t - k_t} = \beta \left(\sum_{j=1}^{J} \frac{p_j}{A k_t + C_j(k_t)} \right)$$

We now proceed much as in the terminal case, by writing this Euler equation as

$$\beta(x_t - k_t) = \frac{\prod_{j=1}^{J} (A k_t + C_j(k_t))}{\sum_{j=1}^{J} p_j \prod_{i \neq j} (A k_t + C_i(k_t))}$$

In order to apply Lemma 1, we now compute

$$\lim_{k_t \to \infty} \left(\frac{\prod_{j=1}^{J} (Ak_t + C_j(k_t))}{\sum_{j=1}^{J} p_j \prod_{i \neq j} (Ak_t + C_i(k_t))} - Ak_t \right)$$

We may rewrite the argument here as a single fraction. To do so, we first note that

$$\lim_{k_t \to \infty} \frac{C_j(k_t)}{k_t} = \lim_{k_t \to \infty} \left(\frac{B_j}{k_t} - \frac{\epsilon^{(t+1)} \Big((1 - \delta + R_j) k_t + W_j \ell_j \Big)}{k_t} \right) = 0$$

so that $C_j(k_t) = o(k_t)$. After some cancellation, then this single fraction can be written as

$$\begin{aligned} &\frac{A^{J-1} k_t^{J-1} \sum_j C_j(k_t) + o(k^{J-1}) - A^{J-1} k_t^{J-1} \sum_{j=1}^{J} p_j \sum_{i \neq j} C_i(k_t) + o(k^{J-1})}{\sum_{j=1}^{J} p_j \prod_{i \neq j} (Ak_t + C_i(k_t))} \\ &\qquad = \frac{A^{J-1} k_t^{J-1} \sum_j p_j C_j(k_t) + o(k^{J-1})}{A^{J-1} k_t^{J-1} + o(k^{J-1})} \\ &\qquad = \frac{\sum_j p_j C_j(k_t) + o(1)}{1 + o(1)} \\ &\qquad \longrightarrow \sum_j p_j C_j(k_t) = \mathbb{E}_t C_{t+1}(k_t) \text{ as } k_t \to \infty \end{aligned}$$

Noting that

$$\begin{aligned} \mathbb{E}_t C_{t+1}(k_t) = \mathbb{E}_t \frac{1}{1 + \beta + ... + \beta^{T-t-1}} \left(w_{t+1} + \mathbb{E}_{t+1} \left(\sum_{s=t+2}^{T} \frac{W_s \ell_s}{\prod_{r=t+2}^{s} (1 - \delta + R_r)} \right) \right) \\ - \mathbb{E}_t \epsilon^{(t+1)} \Big((1 - \delta + R_{t+1}) k_t + W_{t+1} \ell_{t+1} \Big) \\ = \frac{1}{1 + \beta + ... + \beta^{T-t-1}} \mathbb{E}_t \left(\sum_{s=t+1}^{T} \frac{W_s \ell_s}{\prod_{r=t+1}^{s} (1 - \delta + R_r)} \right) + o(1) \end{aligned}$$

Lemma 1 tells us that

$$\lim_{k_t \to \infty} \left(\beta \left(\sum_{j=1}^{J} \frac{p_j}{Ak_t + C_j(k_t)} \right) - Ak_t - \frac{1}{1 + \beta + ... + \beta^{T-t-1}} \mathbb{E}_t \left(\sum_{s=t+1}^{T} \frac{W_s \ell_s}{\prod_{r=t+1}^{s} (1 - \delta + R_r)} \right) \right) = 0$$

Letting $-\delta(k_t)$ denote the argument of this limit, it follows that we can rewrite the time t

Euler equation as

$$\beta(x_t - k_t) = Ak_t + \frac{1}{1+\beta+...+\beta^{T-t-1}}\mathbb{E}_t\left(\sum_{s=t+1}^{T}\frac{W_s\ell_s}{\prod_{r=t+1}^{s}(1-\delta+R_r)}\right) - \delta(k_t)$$

with $\lim_{k_t\to\infty}\delta(k_t) = 0$. Recalling the definition of A and rearranging, this gives

$$k_t = \frac{\beta+...+\beta^{T-t}}{1+\beta+...+\beta^{T-t}}x_t - \frac{1}{1+\beta+...+\beta^{T-t}}\mathbb{E}_t\left(\sum_{s=t+1}^{T}\frac{W_s\ell_s}{\prod_{r=t+1}^{s}(1-\delta+R_r)}\right) + \delta(k_t)$$

As in previous arguments, we define

$$\epsilon^{(t)}(x_t, \mathcal{L}_t, \mathcal{F}_t) := \delta\left(k^{(t)}(x_t, \mathcal{L}_t, \mathcal{F}_t)\right)$$

which gives us the desired form of the savings function.

The limits of the nonlinear error follow as usual: the limit at the borrowing constraint from rearranging the identity

$$\begin{aligned}\underline{k}_t(\mathcal{L}_t, \mathcal{F}_t) = {} & \frac{\beta+...+\beta^{T-t}}{1+\beta+...+\beta^{T-t}}\underline{k}_t(\mathcal{L}_t, \mathcal{F}_t) - \frac{1}{1+\beta+...+\beta^{T-t}}\mathbb{E}_t\left(\sum_{s=t+1}^{T}\frac{W_s\ell_s}{\prod_{r=t+1}^{s}(1-\delta+R_r)}\right) \\ & + \lim_{x_t\to\underline{k}_t(\mathcal{L}_t,\mathcal{F}_t)}\epsilon^{(t)}(x_t, \mathcal{L}_t, \mathcal{F}_t)\end{aligned}$$

and the limit at ∞ from the definition of $\epsilon^{(t)}$ above, the behavior of $k^{(t)}$ at ∞, and the definition of δ.

By a similar argument to previous cases, monotonicity of the error term will follow from showing that the derivative of

$$\left(\sum_{j=1}^{J}\frac{p_j}{Ak_t + C_j(k_t)}\right)^{-1}$$

with respect to k_t is larger than A.

$$\left(\sum_{j=1}^{J}\frac{p_j}{Ak_t + C_j(k_t)}\right)^{-2}\left(\sum_{j=1}^{J}\frac{p_j(A+\frac{\partial C_j}{\partial k_t})}{(Ak_t + C_j(k_t))^2}\right) > A$$

Noting that

$$\frac{\partial C_j}{\partial k_t} = -(1-\delta+R_j)\frac{\partial \epsilon^{(t+1)}}{\partial x_{t+1}} > 0$$

we have

$$\left(\sum_{j=1}^{J} \frac{p_j}{Ak_t + C_j(k_t)}\right)^{-2} \left(\sum_{j=1}^{J} \frac{p_j(A + \frac{\partial C_j}{\partial k_t})}{(Ak_t + C_j(k_t))^2}\right) > \left(\sum_{j=1}^{J} \frac{p_j}{Ak_t + C_j(k_t)}\right)^{-2} \left(\sum_{j=1}^{J} \frac{p_j A}{(Ak_t + C_j(k_t))^2}\right)$$

so it is sufficient to show that

$$\left(\sum_{j=1}^{J} \frac{p_j}{Ak_t + C_j(k_t)}\right)^{-2} \left(\sum_{j=1}^{J} \frac{p_j A}{(Ak_t + C_j(k_t))^2}\right) \geq A$$

which is implied by showing

$$\left(\sum_{j=1}^{J} \frac{p_j}{Ak_t + C_j(k_t)}\right)^{-2} \left(\sum_{j=1}^{J} \frac{p_j}{(Ak_t + C_j(k_t))^2}\right) \geq 1$$

Rearranging, we get

$$\sum_{j=1}^{J} \frac{p_j}{Ak_t + C_j(k_t)} \leq \left(\sum_{j=1}^{J} \frac{p_j}{(Ak_t + C_j(k_t))^2}\right)^{1/2}$$

which is true by the norm argument given in the base case.

To prove convexity, we once again observe that the linear-plus-error structure implies that

$$\frac{\partial^2 k^{(t)}}{\partial x_t^2} = \frac{\partial^2 \epsilon^{(t)}}{\partial x_t^2}$$

so that, exactly as in previous instances, it suffices to show that δ is convex. Letting

$$h(k_t) := \sum_{j=1}^{J} \frac{p_j}{Ak_t + C_j(k_t)}$$
$$g(k_t) := 1/h(k_t)$$

we can do the familiar calculation from previous cases to conclude that it suffices to demonstrate

$$2\left(\frac{\partial h}{\partial k_{T-1}}\right)^2 < h(k_{T-1})\frac{\partial^2 h}{\partial k_{T-1}^2} \tag{33}$$

To show this we calculate

$$\begin{aligned}
\frac{\partial h}{\partial k_t} &= -\sum_{j=1}^{J} \frac{p_j(A + \frac{\partial C_j}{\partial k_t})}{(Ak_t + C_j(k_t))^2} \\
\frac{\partial^2 h}{\partial k_t^2} &= 2\sum_{j=1}^{J} \frac{p_j(A + \frac{\partial C_j}{\partial k_t})^2}{(Ak_t + C_j(k_t))^3} - \sum_{j=1}^{J} \frac{p_j \frac{\partial^2 C_j}{\partial k_t^2}}{(Ak_t + C_j(k_t))^2}
\end{aligned}$$

Since $\epsilon^{(t+1)}$ is strictly convex in resources, it follows that the second sum here is positive. Consequently, it's contribution is to make the right side of (33) larger, and hence to complete the proof we must only show that

$$\left[\sum_{j=1}^{J} \frac{p_j(A + \frac{\partial C_j}{\partial k_t})}{(Ak_t + C_j(k_t))^2}\right]^2 \leq \left[\sum_{j=1}^{J} \frac{p_j}{Ak_t + C_j(k_t)}\right]\left[\sum_{j=1}^{J} \frac{p_j(A + \frac{\partial C_j}{\partial k_t})^2}{(Ak_t + C_j(k_t))^3}\right]$$

This inequality is yet again a consequence of the Cauchy-Schwartz inequality, proving convexity.

This closes the induction and completes the proof. □

The main theorem in its full generality adds the additional technical complication that future aggregates appear nonlinearly in the effective discount factor, leading to additional bookkeeping.

Theorem 4: Main Theorem, $\sigma \neq 1$. Make the sequence of recursive definitions

$$\begin{aligned}
M_T &= (1 - \delta + R_T)^{1-\sigma} \\
Q_{T-1} &= (\beta \mathbb{E}_{T-1} M_T)^{1/\sigma} \\
M_t &= (1 - \delta + R_t)^{1-\sigma}(1 + Q_{t+1})^{\sigma}, \quad t = 2, ..., T \\
Q_{t-1} &= [\beta \mathbb{E}_{t-1} M_t]^{1/\sigma}, \quad t = 2, ..., T
\end{aligned}$$

The savings functions $k^{(t)}(x_t, \mathcal{L}_t, \mathcal{F}_t)$, $t = 1, ..., T$ which solve the dynamic programming problems (2) with can be written in the form

$$k^{(t)}(x_t, \mathcal{L}_t, \mathcal{F}_t) = \frac{Q_t}{1+Q_t} x_t - \frac{1}{1+Q_t} \mathbb{E}_t \left(\sum_{s=t+1}^{T} \left(\prod_{r=t+1}^{s} \frac{M_r}{\mathbb{E}_{r-1} M_r} \right) \frac{W_s \ell_s}{\prod_{r=t+1}^{s} (1-\delta+R_r)} \right) + \epsilon^{(t)}(x_t, \mathcal{L}_t, \mathcal{F}_t)$$

where the nonlinear error term $\epsilon^{(t)}$ is strictly decreasing, convex, and satisfies

$$\lim_{x_t \to \underline{k}_t} \epsilon^{(t)}(x_t, \mathcal{L}_t, \mathcal{F}_t) = \frac{1}{1+Q_t} \left[\mathbb{E}_t \left(\sum_{s=t+1}^{T} \left(\prod_{r=t+1}^{s} \frac{M_r}{\mathbb{E}_{r-1} M_r} \right) \frac{W_s \ell_s}{\prod_{r=t+1}^{s} (1-\delta+R_r)} \right) + \underline{k}_t \right]$$

and

$$\lim_{x_t \to \infty} \epsilon^{(t)}(x_t, \mathcal{L}_t, \mathcal{F}_t) = 0$$

Proof. The proof is again by induction, beginning with the period $T-1$ savings function.

Base Case (T-1): We begin by writing out the expected value of the terminal Euler equation (4). We continue to use the notation established in the log case: for a given state $(x_{T-1}, \mathcal{L}_{T-1}, \mathcal{F}_{T-1})$ of resources and predictive probabilities at time $T-1$ we let

$$\{(\ell_j, R_j, W_j) : \ j = 1, ..., J\}$$

denote the possible time T endowment and price outcomes which are assigned positive predictive probability. We denote the associated predictive probabilities by p_j. Moreover, we let $w_j = W_j \ell_j / (1-\delta+R_j)$. We also introduce the notation

$$D_j = \frac{1}{(1-\delta+R_j)^{1-\sigma}}$$

The terminal Euler equation (4) can be written as

$$\frac{1}{(x_{T-1} - k_{T-1})^\sigma} = \beta \left(\sum_{j=1}^{J} \frac{p_j}{D_j (k_{T-1} + w_j)^\sigma} \right) = \beta \left(\frac{\sum_{j=1}^{J} p_j \prod_{i \neq j} D_i (k_{T-1} + w_i)^\sigma}{\prod_j D_j (k_{T-1} + w_j)^\sigma} \right)$$

Taking the reciprocal and then taking σth roots, this gives

$$\left(\frac{\beta}{\prod_j D_j}\right)^{1/\sigma}(x_{T-1}-k_{T-1}) = \frac{\prod_j (k_{T-1}+w_j)}{\left(\sum_{j=1}^J p_j \prod_{i\neq j} D_i (k_{T-1}+w_i)^\sigma\right)^{1/\sigma}}$$

Some algebra in the numerator and denominator gives

$$\beta(x_{T-1}-k_{T-1}) = \frac{k_{T-1}^J + k_{T-1}^{J-1}\sum_j w_j + o(k_{T-1}^{J-1})}{k_{T-1}^{J-1}\left(\sum_j p_j \prod_{i\neq j} D_i (1+\frac{w_i}{k_{T-1}})^\sigma\right)^{1/\sigma}}$$

To apply Lemma 1 on the right hand side, we compute

$$\lim_{k_{T-1}\to\infty}\left(\frac{k_{T-1}^J + k_{T-1}^{J-1}\sum_j w_j + o(k_{T-1}^{J-1})}{k_{T-1}^{J-1}\left(\sum_j p_j \prod_{i\neq j} D_i (1+\frac{w_i}{k_{T-1}})^\sigma\right)^{1/\sigma}} - \frac{k_{T-1}}{\left(\sum_j p_j \prod_{i\neq j} D_i\right)^{1/\sigma}}\right)$$
$$= \lim_{k_{T-1}\to\infty}\left(\frac{k_{T-1}^J\left[1-\frac{1}{\left(\sum_j p_j \prod_{i\neq j} D_i\right)^{1/\sigma}}\left(\sum_j p_j \prod_{i\neq j} D_i (1+\frac{w_i}{k_{T-1}})^\sigma\right)^{1/\sigma}\right] + k_{T-1}^{J-1}\sum_j w_j + o(k_{T-1}^{J-1})}{k_{T-1}^{J-1}\left(\sum_j p_j \prod_{i\neq j} D_i (1+\frac{w_i}{k_{T-1}})^\sigma\right)^{1/\sigma}}\right)$$

Considering this as the limit of three separate fractions (one for each term in the numerator), we see that the third converges to 0 and the second converges to

$$\frac{\sum_j w_j}{\left(\sum_j p_j \prod_{i\neq j} D_i\right)^{1/\sigma}}$$

We must therefore calculate the limit of the fraction corresponding to the first term; this simplifies to

$$\lim_{k_{T-1}\to\infty}\left(\frac{k_{T-1}\left[1-\frac{1}{\left(\sum_j p_j \prod_{i\neq j} D_i\right)^{1/\sigma}}\left(\sum_j p_j \prod_{i\neq j} D_i (1+\frac{w_i}{k_{T-1}})^\sigma\right)^{1/\sigma}\right]}{\left(\sum_j p_j \prod_{i\neq j} D_i (1+\frac{w_i}{k_{T-1}})^\sigma\right)^{1/\sigma}}\right)$$

The denominator has limit

$$\left(\sum_j p_j \prod_{i\neq j} D_i\right)^{1/\sigma} \tag{34}$$

so we need only compute the limit of numerator, which we rewrite as

$$\left(\frac{\left[1-\frac{1}{\left(\sum_j p_j \prod_{i\neq j} D_i\right)^{1/\sigma}}\left(\sum_j p_j \left[\prod_{i\neq j} D_i\right]\left(1+\frac{\sum_{i\neq j} w_i}{k_{T-1}}+o(1/k_{T-1})\right)^{\sigma}\right)^{1/\sigma}\right]}{1/k_{T-1}}\right)$$

Both numerator and denominator tend to zero here as k_{T-1} increases without bound, so that we may apply l'Hospital's rule to compute the limit. Taking derivatives, we therefore wish to compute the limit of

$$\begin{aligned}
& k_{T-1}^2 \frac{1}{\left(\sum_j p_j \prod_{i\neq j} D_i\right)^{1/\sigma}} \left(\sum_j p_j \left[\prod_{i\neq j} D_i\right](1+o(1))^{\sigma}\right)^{\frac{1}{\sigma}-1} \\
& \qquad \times \left[\sum_j p_j \left[\prod_{i\neq j} D_i\right]\left(1+\frac{\sum_{i\neq j} w_i}{k_{T-1}}+o(1/k_{T-1})\right)^{\sigma-1}\left(-\frac{\sum_{i\neq j} w_i}{k_{T-1}^2}+o(1/k_{T-1}^2)\right)\right] \\
& = \frac{1}{\left(\sum_j p_j \prod_{i\neq j} D_i\right)^{1/\sigma}} \left(\sum_j p_j \left[\prod_{i\neq j} D_i\right](1+o(1))^{\sigma}\right)^{\frac{1}{\sigma}-1} \\
& \qquad \times \left[\sum_j p_j \left[\prod_{i\neq j} D_i\right](1+o(1))^{\sigma-1}\left(-\sum_{i\neq j} w_i+o(1)\right)\right]
\end{aligned}$$

After some thought, one sees that the limit of the last expression as $k_{T-1}\to\infty$ is

$$\frac{-\sum_j p_j \left[\prod_{i\neq j} D_i\right]\sum_{i\neq j} w_i}{\left(\sum_j p_j \prod_{i\neq j} D_i\right)}$$

Combining this with (34), we now get

$$\lim_{k_{T-1}\to\infty}\left(\frac{k_{T-1}^J + k_{T-1}^{J-1}\sum_j w_j + o(k_{T-1}^{J-1})}{k_{T-1}^{J-1}\left(\sum_j p_j \prod_{i\neq j} D_i(1+\frac{w_i}{k_{T-1}})^\sigma\right)^{1/\sigma}} - \frac{k_{T-1}}{\left(\sum_j p_j \prod_{i\neq j} D_i\right)^{1/\sigma}}\right)$$

$$= \frac{\sum_j w_j}{\left(\sum_j p_j \prod_{i\neq j} D_i\right)^{1/\sigma}} - \frac{\sum_j p_j \left[\prod_{i\neq j} D_i\right]\sum_{i\neq j} w_i}{\left(\sum_j p_j \prod_{i\neq j} D_i\right)^{1+1/\sigma}}$$

$$= \frac{\sum_j p_j \left[\prod_{i\neq j} D_i\right] w_j}{\left(\sum_j p_j \prod_{i\neq j} D_i\right)^{1+1/\sigma}}$$

$$= \frac{1}{\left(\sum_j p_j \prod_{i\neq j} D_i\right)^{1/\sigma}} \sum_j p_j \left(\frac{\prod_{i\neq j} D_i}{\sum_k p_k \prod_{l\neq k} D_l}\right) w_j$$

Now Lemma 1 implies that the limit as $k_{T-1}\to\infty$ of

$$-\delta(k_{T-1}) := \frac{\prod_j (k_{T-1}+w_j)}{\left(\sum_{j=1}^J p_j \prod_{i\neq j} D_i (k_{T-1}+w_i)^\sigma\right)^{1/\sigma}} - \frac{k_{T-1}}{\left(\sum_j p_j \prod_{i\neq j} D_i\right)^{1/\sigma}} - \frac{1}{\left(\sum_j p_j \prod_{i\neq j} D_i\right)^{1/\sigma}} \sum_j p_j \left(\frac{\prod_{i\neq j} D_i}{\sum_k p_k \prod_{l\neq k} D_l}\right) w_j$$

vanishes. It now follows from the terminal Euler equation that we can write

$$\left(\frac{\beta}{\prod_j D_j}\right)^{1/\sigma} (x_{T-1}-k_{T-1}) = \frac{k_{T-1}}{\left(\sum_j p_j \prod_{i\neq j} D_i\right)^{1/\sigma}} + \frac{1}{\left(\sum_j p_j \prod_{i\neq j} D_i\right)^{1/\sigma}} \sum_j p_j \left(\frac{\prod_{i\neq j} D_i}{\sum_k p_k \prod_{l\neq k} D_l}\right) w_j - \delta(k_{T-1})$$

with $\lim_{k_{T-1}\to\infty}\delta(k_{T-1}) = 0$. Rearranging slightly, we get

$$\left(\beta \sum_j p_j/D_j\right)^{1/\sigma} (x_{T-1}-k_{T-1}) = k_{T-1} + \sum_j p_j \left(\frac{\prod_{i\neq j} D_i}{\sum_k p_k \prod_{l\neq k} D_l}\right) w_j - \delta(k_{T-1})$$

Noting that

$$\begin{aligned}\left(\beta\sum_j p_j/D_j\right)^{1/\sigma} &= Q_{T-1}\\ \frac{\prod_{i\neq j} D_i}{\sum_k p_k \prod_{l\neq k} D_l} = \frac{1/D_j}{\sum_k p_k/D_k} &= \frac{M_j}{\mathbb{E}_{T-1}M_T}\end{aligned}$$

and solving for k_{T-1} gives

$$k_{T-1} = \frac{1}{1+Q_{T-1}}\left(Q_{T-1}x_{T-1} - \sum_j p_j\left(\frac{\prod_{i\neq j} D_i}{\sum_k p_k \prod_{l\neq k} D_l}\right) w_j + \delta(k_{T-1})\right)$$

By the existence and uniqueness proposition, there is a unique period $T-1$ savings function $k^{(T-1)}$ which solves the household problem in this period. Hence we may define

$$\epsilon^{(T-1)}(x_{T-1}, \mathcal{L}_{T-1}, \mathcal{F}_{T-1}) := \delta\left(k^{(T-1)}(x_{T-1}, \mathcal{L}_{T-1}, \mathcal{F}_{T-1})\right)$$

so that we obtain the desired form

$$\begin{aligned}k^{(T-1)}(x_{T-1}, \mathcal{L}_{T-1}, \mathcal{F}_{T-1}) = \frac{1}{1+Q_{T-1}}\Bigg(&Q_{T-1}x_{T-1} - \sum_j p_j\left(\frac{\prod_{i\neq j} D_i}{\sum_k p_k \prod_{l\neq k} D_l}\right) w_j\\ &+ \epsilon^{(T-1)}(x_1, \mathcal{L}_{T-1}, \mathcal{F}_{T-1})\Bigg)\end{aligned}$$

As in other cases, the limit of the savings function at the borrowing constraint is determined from solving

$$\begin{aligned}\underline{k}_{T-1}(\mathcal{L}_{T-1}, \mathcal{F}_{T-1}) = \frac{1}{1+Q_{T-1}}\Bigg(&Q_{T-1}\underline{k}_{T-1}(\mathcal{L}_{T-1}, \mathcal{F}_{T-1}) - \sum_j p_j\left(\frac{\prod_{i\neq j} D_i}{\sum_k p_k \prod_{l\neq k} D_l}\right) w_j\\ &+ \lim_{x_{T-1}\to\underline{k}_{T-1}(\mathcal{L}_{T-1},\mathcal{F}_{T-1})} \epsilon^{(T-1)}(x_{T-1}, \mathcal{L}_{T-1}, \mathcal{F}_{T-1})\Bigg)\end{aligned}$$

We also have

$$\lim_{x_{T-1}\to\infty} \epsilon^{(T-1)}(x_{T-1}, \mathcal{L}_{T-1}, \mathcal{F}_{T-1}) = \lim_{x_{T-1}\to\infty} \delta\left(k^{(T-1)}(x_{T-1}, \mathcal{L}_{T-1}, \mathcal{F}_{T-1})\right) = 0$$

since $k^{(T-1)}(x_{T-1}) \to \infty$ as $x_{T-1} \to \infty$ (by the existence and uniqueness proposition) and $\delta \to 0$ as its argument increases to ∞.

To establish that $\epsilon^{(T-1)}$ is decreasing in resources, we once again observe that

$$\frac{\partial \epsilon^{(T-1)}}{\partial x_{T-1}} = \frac{\partial \delta}{\partial k_{T-1}} \frac{\partial k^{(T-1)}}{\partial x_{T-1}}$$

has the same sign as $\partial \delta / \partial k_{T-1}$. Next we observe that the definition of $\delta(k_{T-1})$ satisfies

$$\left(\sum_{j=1}^{J} \frac{p_j}{D_j (k_{T-1} + w_j)^{\sigma}} \right)^{-1/\sigma} = k_{T-1} + \mathbb{E}_{T-1} w_T - \delta(k_{T-1})$$

It therefore suffices to show that the derivative of the left side of the above equality with respect to k_{T-1} is larger than unity. Calculating this derivative, it suffices to show that

$$\left(\sum_{j=1}^{J} \frac{p_j}{D_j (k_{T-1} + w_j)^{\sigma}} \right)^{-\frac{1}{\sigma}-1} \left(\sum_{j=1}^{J} \frac{p_j}{D_j (k_{T-1} + w_j)^{\sigma+1}} \right) > 1$$

Rearranging, this is equivalent to the bound

$$\left(\sum_{j=1}^{J} \frac{p_j}{D_j (k_{T-1} + w_j)^{\sigma}} \right)^{1/\sigma} < \left(\sum_{j=1}^{J} \frac{p_j}{D_j (k_{T-1} + w_j)^{\sigma+1}} \right)^{\frac{1}{\sigma+1}}$$

That this inequality is true is a consequence of the fact that L^{σ} norms are increasing in σ, once again following from Jensen's inequality.

To prove convexity, we once again observe that the linear-plus-error structure implies that

$$\frac{\partial^2 k^{(T-1)}}{\partial x_{T-1}^2} = \frac{\partial^2 \epsilon^{(T-1)}}{\partial x_{T-1}^2}$$

so that, exactly as in the two period CRRA case, it suffices to show that δ is convex. Letting

$$h(k_{T-1}) := \sum_{j=1}^{J} \frac{p_j}{D_j (k_{T-1} + w_j)^{\sigma}}$$
$$g(k_{T-1}) := 1/h(k_{T-1})$$

and repeating the derivative calculations in that case, we see that it suffices to show that

$$\left(\frac{1}{\sigma} + 1 \right) \left(\frac{\partial h}{\partial k_{T-1}} \right)^2 - h(k_{T-1}) \frac{\partial^2 h}{\partial k_{T-1}^2} < 0$$

To do we calculate

$$\frac{\partial h}{\partial k_{T-1}} = -\sigma \sum_{j=1}^{J} \frac{p_j}{D_j(k_{T-1}+w_j)^{\sigma+1}}$$
$$\frac{\partial^2 h}{\partial k_{T-1}^2} = \sigma(\sigma+1) \sum_{j=1}^{J} \frac{p_j}{D_j(k_{T-1}+w_j)^{\sigma+2}}$$

so it suffices to show that

$$\left[\sum_{j=1}^{J} \frac{p_j}{D_j(k_{T-1}+w_j)^{\sigma+1}}\right]^2 - \left[\sum_{j=1}^{J} \frac{p_j}{D_j(k_{T-1}+w_j)^{\sigma}}\right]\left[\sum_{j=1}^{J} \frac{p_j}{D_j(k_{T-1}+w_j)^{\sigma+2}}\right] < 0$$

which is equivalent to

$$\left[\sum_{j=1}^{J} \frac{p_j}{D_j(k_{T-1}+w_j)^{\sigma+1}}\right]^2 < \left[\sum_{j=1}^{J} \frac{p_j}{D_j(k_{T-1}+w_j)^{\sigma}}\right]\left[\sum_{j=1}^{J} \frac{p_j}{D_j(k_{T-1}+w_j)^{\sigma+2}}\right]$$

This inequality is once again true by the Cauchy-Schwartz inequality, and this establishes our base case for the induction.

Inductive Step. Suppose now that the theorem is proved for the savings function in period $t+1$, where $1 < t \leq T-1$. For a given state $(x_t, \mathcal{L}_t, \mathcal{F}_t)$ of resources and predictive probabilities at time t, we suppose that there are J time $t+1$ states which are assigned positive predictive probability, and we let

$$\{(\ell_j, R_j, W_j) : \ j = 1, ..., J\}$$

denote the possible time $t+1$ endowment and price outcomes. Note that both J and the set in the above line may change depending on which period t we are focused on - we suppress these dependencies for simplicity of notation. We denote the associated predictive probabilities by p_j. Moreover, we let $w_j = W_j \ell_j/(1-\delta+R_j)$ and $D_j = \frac{1}{(1-\delta+R_j)^{1-\sigma}} = 1/M_j$.

The inductive step will involve what we already know about the time $t+1$ savings function and its nonlinear error term. For simplicity, we once again suppress the dependence of these functions on predictive distributions in the argument.

Having established these notational conventions the time t Euler equation can be written as

$$\frac{1}{(x_t - k_t)^\sigma} = \beta \left(\sum_{j=1}^{J} \frac{p_j}{D_j \left(k_t + w_j - \frac{1}{1-\delta+R_j} k_{t+1} \right)^\sigma} \right)$$

For optimality, time $t+1$ savings must be given by the time $t+1$ savings function, and by the inductive hypothesis we have

$$\begin{aligned}
&\frac{1}{1-\delta+R_j} k^{(t+1)} \Big((1-\delta+R_j)k_t + W_j \ell_j \Big) \\
&\quad = \frac{Q_j}{1+Q_j}(k_t + w_j) - \frac{1}{1+Q_j} \mathbb{E}_{t+1} \left(\sum_{s=t+2}^{T} \left(\prod_{r=t+2}^{s} \frac{M_r}{\mathbb{E}_{r-1} M_r} \right) \frac{W_s \ell_s}{\prod_{r=t+2}^{s} (1-\delta+R_r)} \right) \\
&\quad + \frac{1}{1-\delta+R_j} \epsilon^{(t+1)} \Big((1-\delta+R_j)k_t + W_j \ell_j \Big)
\end{aligned}$$

so that the bracketed expressions in the denominators on the right side of the Euler equation take the form

$$\begin{aligned}
&\frac{1}{1+Q_j} k_t + \frac{1}{1+Q_j} \left(w_j + \mathbb{E}_{t+1} \left(\sum_{s=t+2}^{T} \left(\prod_{r=t+2}^{s} \frac{M_r}{\mathbb{E}_{r-1} M_r} \right) \frac{W_s \ell_s}{\prod_{r=t+2}^{s} (1-\delta+R_r)} \right) \right) \\
&\quad - \epsilon^{(t+1)} \Big((1-\delta+R_j)k_t + W_j \ell_j \Big)
\end{aligned}$$

In order to simplify the algebraic steps to follow, we let

$$\begin{aligned}
A_j &:= \frac{1}{1+Q_j} \\
B_j &:= \frac{1}{1+Q_j} \left(w_j + \mathbb{E}_{t+1} \left(\sum_{s=t+2}^{T} \left(\prod_{r=t+2}^{s} \frac{M_r}{\mathbb{E}_{r-1} M_r} \right) \frac{W_s \ell_s}{\prod_{r=t+2}^{s} (1-\delta+R_r)} \right) \right) \\
C_j(k_t) &:= B_j - \frac{\epsilon^{(t+1)} \Big((1-\delta+R_j)k_t + W_j \ell_j \Big)}{1-\delta+R_j}
\end{aligned}$$

In this notation, the Euler equation becomes

$$\frac{1}{(x_t - k_t)^\sigma} = \beta \left(\sum_{j=1}^{J} \frac{p_j}{D_j (A_j k_t + C_j(k_t))^\sigma} \right)$$

We now proceed much as in the terminal case, by writing this Euler equation as

$$\left(\frac{\beta}{\prod_j D_j}\right)^{1/\sigma}(x_t - k_t) = \frac{\prod_j (A_j k_t + C_j(k_t))}{\left(\sum_{j=1}^J p_j \prod_{i\neq j} D_i (A_i k_t + C_i(k_t))^\sigma\right)^{1/\sigma}}$$

In order to apply Lemma 1, we now compute

$$\lim_{k_t\to\infty}\left(\frac{\prod_{j=1}^J (A_j k_t + C_j(k_t))}{\left(\sum_{j=1}^J p_j \prod_{i\neq j} D_i (A_i k_t + C_i(k_t))^\sigma\right)^{1/\sigma}} - \frac{\prod_j A_j}{\left(\sum_j p_j \prod_{i\neq j} D_i A_i^\sigma\right)^{1/\sigma}} k_t\right)$$

We may rewrite the argument here as a single fraction. To do so, we first note that

$$\lim_{k_t\to\infty}\frac{C_j(k_t)}{k_t} = \lim_{k_t\to\infty}\left(\frac{B_j}{k_t} - \frac{\epsilon^{(t+1)}\left((1-\delta+R_j)k_t + W_j\ell_j\right)}{k_t}\right) = 0$$

so that $C_j(k_t) = o(k_t)$. We therefore want to compute the limit as $k_t \to \infty$ of

$$\frac{k_t^J\left(\prod_j A_j - \frac{\prod_j A_j}{\left(\sum_j p_j \prod_{i\neq j} D_i A_i^\sigma\right)^{1/\sigma}}\left(\sum_{j=1}^J p_j \prod_{i\neq j} D_i (A_i + \frac{C_i(k_t)}{k_t})^\sigma\right)^{1/\sigma}\right) + k_t^{J-1}\sum_j C_j(k_t)\prod_{i\neq j} A_j + o(k_t^{J-1})}{\left(\sum_{j=1}^J p_j \prod_{i\neq j} D_i (A_i k_t + C_i(k_t))^\sigma\right)^{1/\sigma}}$$

$$= \frac{k_t^J\left(\prod_j A_j - \frac{\prod_j A_j}{\left(\sum_j p_j \prod_{i\neq j} D_i A_i^\sigma\right)^{1/\sigma}}\left(\sum_{j=1}^J p_j \prod_{i\neq j} D_i (A_i + \frac{C_i(k_t)}{k_t})^\sigma\right)^{1/\sigma}\right) + k_t^{J-1}\sum_j C_j(k_t)\prod_{i\neq j} A_j + o(k_t^{J-1})}{k_t^{J-1}\left(\sum_{j=1}^J p_j \prod_{i\neq j} D_i (A_i + \frac{C_i(k_t)}{k_t})^\sigma\right)^{1/\sigma}}$$

$$= \frac{k_t\left(\prod_j A_j - \frac{\prod_j A_j}{\left(\sum_j p_j \prod_{i\neq j} D_i A_i^\sigma\right)^{1/\sigma}}\left(\sum_{j=1}^J p_j \prod_{i\neq j} D_i (A_i + \frac{C_i(k_t)}{k_t})^\sigma\right)^{1/\sigma}\right) + \sum_j C_j(k_t)\prod_{i\neq j} A_j + o(1)}{\left(\sum_{j=1}^J p_j \prod_{i\neq j} D_i (A_i + o(1))^\sigma\right)^{1/\sigma}}$$

Splitting this into three separate fractions and taking limits termwise, the limit of the third term vanishes and that of the second term is

$$\lim_{k_t\to\infty}\frac{\sum_j C_j(k_t)\prod_{i\neq j} A_j}{\left(\sum_{j=1}^J p_j \prod_{i\neq j} D_i (A_i + o(1))^\sigma\right)^{1/\sigma}} = \frac{\sum_j B_j \prod_{i\neq j} A_j}{\left(\sum_j p_j \prod_{i\neq j} D_i A_i^\sigma\right)^{1/\sigma}}$$

The limit of the first term requires more attention once again. The denominator approaches

$$\left(\sum_j p_j \prod_{i\neq j} D_i A_i^\sigma\right)^{1/\sigma}$$

while we can use l'Hospital's rule to calculate that of the numerator upon writing it as

$$\frac{\prod_j A_j - \frac{\prod_j A_j}{\left(\sum_j p_j \prod_{i\neq j} D_i A_i^\sigma\right)^{1/\sigma}} \left(\sum_{j=1}^J p_j \prod_{i\neq j} D_i (A_i + \frac{C_i(k_t)}{k_t})^\sigma\right)^{1/\sigma}}{1/k_t}$$

and observing that both numerator and denominator tend to 0 as k_t tends to ∞. Taking the necessary derivatives, we must compute the limit of

$$\begin{aligned}
&k_t^2 \frac{\prod_j A_j}{\left(\sum_j p_j \prod_{i\neq j} D_i A_i^\sigma\right)^{1/\sigma}} \frac{1}{\sigma} \left(\sum_{j=1}^J p_j \prod_{i\neq j} D_i \left(A_i + \frac{C_i(k_t)}{k_t}\right)^\sigma\right)^{\frac{1}{\sigma}-1} \\
&\quad \times \Big[\sum_{j=1}^J \sigma p_j \left(\prod_{i\neq j} D_i A_i^\sigma\right) \left(1 + \sum_{i\neq j} \frac{C_i(k_t)}{A_i k_t} + o(1/k_t)\right)^{\sigma-1} \left(\sum_{i\neq j} \left[\frac{\partial C_i/\partial k_t}{A_i k_t} - \frac{C_i(k_t)}{A_i k_t^2}\right] + o(1/k_t^2)\right)\Big] \\
&\quad = \frac{\prod_j A_j}{\left(\sum_j p_j \prod_{i\neq j} D_i A_i^\sigma\right)^{1/\sigma}} \left(\sum_{j=1}^J p_j \prod_{i\neq j} D_i \left(A_i + o(1)\right)^\sigma\right)^{\frac{1}{\sigma}-1} \\
&\qquad \times \Big[\sum_{j=1}^J p_j \left(\prod_{i\neq j} D_i A_i^\sigma\right) (1 + o(1))^{\sigma-1} \left(\sum_{i\neq j} \left[\frac{k_t}{A_i}\frac{\partial C_i}{\partial k_t} - \frac{C_i(k_t)}{A_i}\right] + o(1)\right)\Big]
\end{aligned}$$

Taking the limit $k_t \to \infty$ of the last line, we end up with

$$\frac{\prod_j A_j}{\sum_j p_j \prod_{i\neq j} D_i A_i^\sigma} \Big[\sum_{j=1}^J p_j \left(\prod_{i\neq j} D_i A_i^\sigma\right) \lim_{k_t\to\infty} \left(\sum_{i\neq j} \left[\frac{k_t}{A_i}\frac{\partial C_i}{\partial k_t} - \frac{C_i(k_t)}{A_i}\right]\right)\Big] \tag{35}$$

To calculate the remaining limit, we observe from the following short, technical argument that

$$\lim_{k_t\to\infty} k_t \frac{\partial C_i}{\partial k_t} = 0$$

To see this, first observe that we have

$$\begin{aligned}
\frac{\partial C_i}{\partial k_t} k_t &= \frac{\partial \epsilon^{(t+1)}\big((1-\delta+R_i)k_t + w_i\big)}{\partial x_{t+1}} (1-\delta+R_i)k_t \\
&= \Big((1-\delta+R_i)k_t + w_i\Big) \frac{\partial \epsilon^{(t+1)}\big((1-\delta+R_i)k_t + w_i\big)}{\partial x_{t+1}} \\
&\quad - w_i \frac{\partial \epsilon^{(t+1)}\big((1-\delta+R_i)k_t + w_i\big)}{\partial x_{t+1}}
\end{aligned}$$

Making the change of variables $x_{t+1} = (1 - \delta + R_i)k_t + w_i$, this can be written as

$$= x_{t+1}\frac{\partial \epsilon^{(t+1)}}{\partial x_{t+1}} - w_i \frac{\partial \epsilon^{(t+1)}}{\partial x_{t+1}}$$

Then, since $\epsilon^{(t+1)} \to 0$ as $x_{t+1} \to \infty$, the same is true for its derivative, and the second term in the last line above vanishes in this limit. Since $x_{t+1} \to \infty$ and $k_t \to \infty$ it therefore suffices to show that the first term also vanishes in this limit.

Recalling that ϵ^{t+1} is convex in resources, it must be the case that

$$|\epsilon^{(t+1)}(x) - \epsilon^{(t+1)}(x_0)| \geq \left| \frac{\partial \epsilon^{(t+1)}(x_0)}{\partial x_{t+1}}(x - x_0) \right|$$

for any $x_0, x \in (\underline{k}_{t+1}, \infty)$. In particular, we can take $x = 2x_0$ giving

$$|\epsilon^{(t+1)}(2x_0) - \epsilon^{(t+1)}(x_0)| \geq \left| \frac{\partial \epsilon^{(t+1)}(x_0)}{\partial x_{t+1}}(x_0) \right|$$

Taking the limit as $x_0 \to \infty$, the left side vanishes, while the right side remains greater than or equal to zero. It follows from the squeeze theorem that

$$\left| \frac{\partial \epsilon^{(t+1)}(x_0)}{\partial x_{t+1}} x_0 \right| \to 0$$

as needed.

Recalling now the definition of $C_i(k_t)$ and using the above calculation, we arrive at the closed form expression for (35)

$$-\frac{\prod_j A_j}{\sum_j p_j \prod_{i \neq j} D_i A_i^\sigma} \Big[\sum_{j=1}^{J} p_j \left(\prod_{i \neq j} D_i A_i^\sigma \right) \sum_{i \neq j} \frac{B_i}{A_i} \Big]$$

Combining this with the previously calculated limits, we obtain

$$\lim_{k_t \to \infty} \left(\frac{\prod_{j=1}^{J} (A_j k_t + C_j(k_t))}{\left(\sum_{j=1}^{J} p_j \prod_{i \neq j} D_i (A_i k_t + C_i(k_t))^\sigma \right)^{1/\sigma}} - \frac{\prod_j A_j}{\left(\sum_j p_j \prod_{i \neq j} D_i A_i^\sigma \right)^{1/\sigma}} k_t \right)$$

$$= \frac{\sum_j B_j \prod_{i \neq j} A_j}{\left(\sum_j p_j \prod_{i \neq j} D_i A_i^\sigma \right)^{1/\sigma}} - \frac{\prod_j A_j \Big[\sum_{j=1}^{J} p_j \left(\prod_{i \neq j} D_i A_i^\sigma \right) \frac{B_i}{A_i} \Big]}{\left(\sum_j p_j \prod_{i \neq j} D_i A_i^\sigma \right) \left(\sum_j p_j \prod_{i \neq j} D_i A_i^\sigma \right)^{1/\sigma}}$$

Combining the fractions and simplifying, we get

$$\frac{\left(\sum_j B_j \prod_{i\neq j} A_j\right)\left(\sum_j p_j \prod_{i\neq j} D_i A_i^\sigma\right) - \prod_j A_j \left[\sum_{j=1}^J p_j \left(\prod_{i\neq j} D_i A_i^\sigma\right) \sum_{i\neq j} \frac{B_i}{A_i}\right]}{\left(\sum_j p_j \prod_{i\neq j} D_i A_i^\sigma\right)^{1+\frac{1}{\sigma}}}$$

Some thought allows us to rearrange this as

$$\left(\prod_j A_j\right) \frac{\sum_j p_j \left(\prod_{i\neq j} D_i A_i^\sigma\right) \frac{B_j}{A_j}}{\left(\sum_j p_j \prod_{i\neq j} D_i A_i^\sigma\right)^{1+\frac{1}{\sigma}}}$$

Lemma 1 now tells us that

$$\begin{aligned} -\delta(k_t) := & \frac{\prod_{j=1}^J (A_j k_t + C_j(k_t))}{\left(\sum_{j=1}^J p_j \prod_{i\neq j} D_i (A_i k_t + C_i(k_t))^\sigma\right)^{1/\sigma}} - \frac{\prod_j A_j}{\left(\sum_j p_j \prod_{i\neq j} D_i A_i^\sigma\right)^{1/\sigma}} k_t \\ & - \left(\prod_j A_j\right) \frac{\sum_j p_j \left(\prod_{i\neq j} D_i A_i^\sigma\right) \frac{B_j}{A_j}}{\left(\sum_j p_j \prod_{i\neq j} D_i A_i^\sigma\right)^{1+\frac{1}{\sigma}}} \end{aligned}$$

vanishes in the limit $k_t \to \infty$. It follows that we can rewrite the time t Euler equation as

$$\left(\frac{\beta}{\prod_j D_j}\right)^{1/\sigma} (x_t - k_t) = \frac{\prod_j A_j}{\left(\sum_j p_j \prod_{i\neq j} D_i A_i^\sigma\right)^{1/\sigma}} k_t + \frac{\left(\prod_j A_j\right) \sum_j p_j \left(\prod_{i\neq j} D_i A_i^\sigma\right) \frac{B_j}{A_j}}{\left(\sum_j p_j \prod_{i\neq j} D_i A_i^\sigma\right)^{1+\frac{1}{\sigma}}} - \delta(k_t)$$

with $\lim_{k_t\to\infty} \delta(k_t) = 0$. Rearranging somewhat, we get

$$\begin{aligned} x_t - k_t &= \frac{1}{\left(\beta \sum_j \frac{p_j}{D_j A_j^\sigma}\right)^{1/\sigma}} k_t + \frac{1}{\left(\beta \sum_j \frac{p_j}{D_j A_j^\sigma}\right)^{1/\sigma}} \sum_j p_j \left(\frac{\prod_{i\neq j} D_i A_i^\sigma}{\sum_l p_l \prod_{m\neq l} D_m A_m^\sigma}\right) \frac{B_j}{A_j} - \delta(k_t) \\ &= \frac{1}{\left(\beta \sum_j \frac{p_j}{D_j A_j^\sigma}\right)^{1/\sigma}} k_t + \frac{1}{\left(\beta \sum_j \frac{p_j}{D_j A_j^\sigma}\right)^{1/\sigma}} \sum_j p_j \left(\frac{1/(D_j A_j^\sigma)}{\sum_l p_l/(D_l A_l^\sigma)}\right) \frac{B_j}{A_j} - \delta(k_t) \end{aligned}$$

or

$$k_t = \frac{1}{1 + \left(\beta \sum_j \frac{p_j}{D_j A_j^\sigma}\right)^{1/\sigma}} \left(\left(\beta \sum_j \frac{p_j}{D_j A_j^\sigma}\right)^{1/\sigma} x_t - \sum_j p_j \left(\frac{1/(D_j A_j^\sigma)}{\sum_l p_l/(D_l A_l^\sigma)}\right) \frac{B_j}{A_j} + \delta(k_t)\right)$$

Recalling our choice of notation D_j and A_j, we see that this can now be written as

$$k_t = \frac{1}{1+Q_t}\left(Q_t x_t - \mathbb{E}_t\left(\frac{M_{t+1}}{\mathbb{E}_t M_{t+1}}\right)\frac{B_{t+1}}{A_{t+1}} + \delta(k_t)\right)$$

We note that the definitions of B_j and A_j give

$$\frac{B_j}{A_j} = \left(w_j + \mathbb{E}_{t+1}\left(\sum_{s=t+2}^{T}\left(\prod_{r=t+2}^{s}\frac{M_r}{\mathbb{E}_{r-1}M_r}\right)\frac{W_s\ell_s}{\prod_{r=t+2}^{s}(1-\delta+R_r)}\right)\right)$$

so this further reduces to

$$k_t = \frac{1}{1+Q_t}\left(Q_t x_t - \mathbb{E}_t\sum_{s=t+1}^{T}\left(\prod_{r=t+1}^{s}\frac{M_r}{\mathbb{E}_{r-1}M_r}\right)\frac{W_s\ell_s}{\prod_{r=t+1}^{s}(1-\delta+R_r)} + \delta(k_t)\right)$$

As in previous arguments, we define

$$\epsilon^{(t)}(x_t,\mathcal{L}_t,\mathcal{F}_t) := \delta\left(k^{(t)}(x_t,\mathcal{L}_t,\mathcal{F}_t)\right)$$

which gives us the desired form of the savings function.

The limits of the nonlinear error follow as usual: the limit at the borrowing constraint from rearranging the identity

$$\underline{k}_t(\mathcal{L}_t,\mathcal{F}_t) = \frac{1}{1+Q_t}\Bigg(Q_t\underline{k}_t(\mathcal{L}_t,\mathcal{F}_t) - \mathbb{E}_t\sum_{s=t+1}^{T}\left(\prod_{r=t+1}^{s}\frac{M_r}{\mathbb{E}_{r-1}M_r}\right)\frac{W_s\ell_s}{\prod_{r=t+1}^{s}(1-\delta+R_r)} + \lim_{x_t\to\underline{k}_t(\mathcal{L}_t,\mathcal{F}_t)}\epsilon^{(t)}(x_t,\mathcal{L}_t,\mathcal{F}_t)\Bigg)$$

and the limit at ∞ from the definition of $\epsilon^{(t)}$ above, the behavior of $k^{(t)}$ at ∞, and the definition of δ.

By a similar argument to previous cases, monotonicity of the error term will follow from showing that the derivative of

$$\left(\sum_{j=1}^{J}\frac{p_j}{D_j(A_j k_t + C_j(k_t))^{\sigma}}\right)^{-1/\sigma}$$

with respect to k_t is larger than

$$\frac{1}{\left(\sum_j p_j/(D_j A_j^\sigma)\right)^{1/\sigma}}$$

Taking the derivative, this amounts to showing that

$$\left(\sum_{j=1}^{J} \frac{p_j}{D_j(A_j k_t + C_j(k_t))^\sigma}\right)^{-\frac{1}{\sigma}-1} \left(\sum_{j=1}^{J} \frac{p_j(A_j + \frac{\partial C_j}{\partial k_t})}{D_j(A_j k_t + C_j(k_t))^{\sigma+1}}\right) > \frac{1}{\left(\sum_j p_j/(D_j A_j^\sigma)\right)^{1/\sigma}}$$

Noting that

$$\frac{\partial C_j}{\partial k_t} = -(1 - \delta + R_j)\frac{\partial \epsilon^{(t+1)}}{\partial x_{t+1}} > 0$$

it is sufficient to show that

$$\left(\sum_{j=1}^{J} \frac{p_j}{D_j(A_j k_t + C_j(k_t))^\sigma}\right)^{\frac{1}{\sigma}+1} \leq \left(\sum_{j=1}^{J} \frac{p_j A_j}{D_j(A_j k_t + C_j(k_t))^{\sigma+1}}\right) \left(\sum_j p_j/(D_j A_j^\sigma)\right)^{1/\sigma}$$

which is implied by showing

$$\sum_{j=1}^{J} \frac{p_j}{D_j A_j^\sigma (k_t + C_j(k_t)/A_j)^\sigma} \leq \left(\sum_{j=1}^{J} \frac{p_j}{D_j A_j^\sigma (k_t + C_j(k_t)/A_j)^{\sigma+1}}\right)^{\frac{\sigma}{\sigma+1}} \left(\sum_j p_j/(D_j A_j^\sigma)\right)^{\frac{1}{1+\sigma}}$$

This last inequality is true by Hölder's inequality.

To prove convexity, we once again observe that the linear-plus-error structure implies that

$$\frac{\partial^2 k^{(t)}}{\partial x_t^2} = \frac{\partial^2 \epsilon^{(t)}}{\partial x_t^2}$$

so that, exactly as in previous instances, it suffices to show that δ is convex. Letting

$$h(k_t) := \sum_{j=1}^{J} \frac{p_j}{D_j(A_j k_t + C_j(k_t))^\sigma}$$
$$g(k_t) := h(k_t)^{-1/\sigma}$$

we can do the familiar calculation from previous cases to conclude that it suffices to demonstrate that

$$\left(\frac{1}{\sigma}+1\right)\left(\frac{\partial h}{\partial k_{T-1}}\right)^2 < h(k_{T-1})\frac{\partial^2 h}{\partial k_{T-1}^2} \tag{36}$$

To show this we calculate

$$\frac{\partial h}{\partial k_t} = -\sigma \sum_{j=1}^{J} \frac{p_j(A_j + \frac{\partial C_j}{\partial k_t})}{D_j(A_j k_t + C_j(k_t))^{\sigma+1}}$$

$$\frac{\partial^2 h}{\partial k_t^2} = \sigma(\sigma+1)\sum_{j=1}^{J} \frac{p_j(A_j + \frac{\partial C_j}{\partial k_t})^2}{D_j(A_j k_t + C_j(k_t))^{\sigma+2}} - \sigma \sum_{j=1}^{J} \frac{p_j \frac{\partial^2 C_j}{\partial k_t^2}}{D_j(A_j k_t + C_j(k_t))^{\sigma+1}}$$

Since $\epsilon^{(t+1)}$ is strictly convex in resources, it follows that the second sum here is strictly positive. Consequently, it's contribution is to make the right side of (36) larger, and hence to complete the proof we must only show that

$$\left[\sum_{j=1}^{J} \frac{p_j(A_j + \frac{\partial C_j}{\partial k_t})}{D_j(A_j k_t + C_j(k_t))^{\sigma+1}}\right]^2 \le \left[\sum_{j=1}^{J} \frac{p_j}{D_j(A_j k_t + C_j(k_t))^{\sigma}}\right]\left[\sum_{j=1}^{J} \frac{p_j(A_j + \frac{\partial C_j}{\partial k_t})^2}{D_j(A_j k_t + C_j(k_t))^{\sigma+2}}\right]$$

This inequality is yet again a consequence of the Cauchy-Schwartz inequality, proving convexity.

This closes the induction and completes the proof. □

Finally, we fill in the algebra motivating our algorithm for computing the linear part of the household savings function in the case of logarithmic utility, stated in the main text as Proposition 3.

Proposition 3: Let $k_{\text{lin}}^{(t)}$ denote the linear component of the savings function in Theorem 3. Then $k_{\text{lin}}^{(t)}$ solves the series of intertemporal equations

$$\frac{1}{x_t - k_{\text{lin}}^{(t)}} = \frac{\beta}{k_{\text{lin}}^{(t)} + \mathbb{E}_t\left[\left(\frac{1}{(1-\delta+R_{t+1})}\right)\left(W_{t+1}\ell_{t+1} - k_{\text{lin}}^{(t+1)}\right)\right]}$$

for $t = 1, ..., T-1$ with $k_{\text{lin}}^{(T)} \equiv 0$.

Proof. Once again, this follows an induction. Taking $t = T-1$ we have

$$\frac{1}{x_{T-1} - k_{T-1}} = \frac{\beta}{k_{T-1} + \mathbb{E}_{T-1}\left[\left(\frac{1}{(1-\delta+R_T)}\right) W_T \ell_T\right]}$$

Rearranging gives

$$\begin{aligned} k_{T-1} &= \frac{1}{1+\beta}\left(x_{T-1} - \mathbb{E}_{T-1}\frac{W_T\ell_T}{(1-\delta+R_T)}\right) \\ &= k_{\text{lin}}^{(T-1)} \end{aligned}$$

Then, supposing that the proposition is true for $t+1$, we write

$$\frac{1}{x_t - k_t} = \frac{\beta}{k_t + \mathbb{E}_t\left[\left(\frac{1}{(1-\delta+R_{t+1})}\right)\left(W_{t+1}\ell_{t+1} - k_{\text{lin}}^{(t+1)}\right)\right]}$$

We let $Q_{t+1} = \beta + ... + \beta^{T-t-1}$. Then we have

$$\begin{aligned} &k_t + \mathbb{E}_t\left[\left(\frac{1}{(1-\delta+R_{t+1})}\right)\left(W_{t+1}\ell_{t+1} - k_{\text{lin}}^{(t+1)}\right)\right] \\ &\quad = k_t + \mathbb{E}_t\left[\left(\frac{1}{(1-\delta+R_{t+1})}\right)W_{t+1}\ell_{t+1} - \frac{Q_{t+1}}{1+Q_{t+1}}\left(k_t + \frac{W_{t+1}\ell_{t+1}}{1-\delta+R_{t+1}}\right.\right. \\ &\qquad\qquad \left.\left. + \frac{1}{1+Q_{t+1}}\frac{1}{1-\delta+R_{t+1}}\mathbb{E}_{t+1}\left[\sum_{s=t+2}^{T}\left(\frac{W_s\ell_s}{\prod_{r=t+2}s(1-\delta+R_r)}\right)\right]\right)\right] \\ &\quad = \frac{1}{1+Q_{t+1}}k_t + \frac{1}{1+Q_{t+1}}\mathbb{E}_t\sum_{s=t+1}^{T}\left(\frac{W_s\ell_s}{\prod_{r=t+1}s(1-\delta+R_r)}\right) \end{aligned}$$

We therefore solve

$$\frac{1}{x_t - k_t} = \frac{\beta(1+Q_{t+1})}{k_t + \mathbb{E}_t\sum_{s=t+1}^{T}\left(\frac{W_s\ell_s}{\prod_{r=t+1}s(1-\delta+R_r)}\right)}$$

for k_t, getting $k_t = k_{\text{lin}}^{(t)}$. □