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All-Pay Auctions with Ties

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Abstract We study the two-player, complete information all-pay auction in which a tie ensues if neither player outbids the other by more than a given amount. In the event of a tie, each player receives an identical fraction of the winning prize. Thus players engage in two margins of competition: losing versus tying, and tying versus winning. Two pertinent parameters are the margin required for victory and the value of tying relative to winning. We fully characterize the set of Nash equilibria for the entire parameter space. For much of the parameter space, there is a unique Nash equilibrium which is also symmetric. Equilibria typically involve randomizing over multiple disjoint intervals, so that in essence players randomize between attempting to tie and attempting to win. In equilibrium, expected bids and payoffs are non-monotonic in both the margin required for victory and the relative value of tying.

Keywords All-pay auction · contest · ties · draws · bid differential

JEL Classification C72 · D44 · D72 · D74

1 Introduction

Eighteen seasons into his playing career, baseball great Frank Robinson famously stated, "Close don't count in baseball. Close only counts in horseshoes and hand grenades" (*Time* magazine, 31 July 1973). Although being close but coming up short does not count for much in baseball, it still has value in many other contexts. Ties in the business arena can take the form of multi-source or split-award procurement contracts. A firm that clearly outshines the competition may receive the full contract; but when competitors fail to adequately distinguish themselves, the contract may be split between them.[1] Political gridlock connotes a tie in which the status quo is perpetuated

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[1] For example, in the first great engine war of the mid-1980's, the U.S. Air Force split the award of a $10 billion engine contract between General Electric and Pratt & Whitney. A key feature of that split-award decision was

instead of either party achieving its ideal policy. Even so, for a party that has fallen behind in numbers, achieving gridlock is a partial victory. In military conflicts, instead of one side dominating the other, more or less equally matched forces may tie in the sense of a stalemate or a standoff. Simply having the larger force, of itself, is not enough to win. Winning rather entails far and away surpassing the opponent by some critical degree.

Contests such as these where players compete by making sunk resource expenditures have long been modeled with the classic all-pay auction (Baye et al. 1996; Hillman and Riley 1989).[2] In its standard formulation, the prize is awarded to the player with the highest resource expenditure, no matter how large or small the difference in the players' expenditures may be. Ties are therefore a knife-edge event. For many contest settings, however, the magnitude of the expenditure difference matters, and a tie becomes a viable third outcome.[3] This paper extends the two-player, complete information all-pay auction by allowing a tie to occur when the difference in expenditures falls below a specified threshold. We refer to this threshold as the *tie margin*. The introduction of a tie margin creates two distinct margins of competition: losing versus tying, and tying versus winning. As such, players are concerned with both the tie margin and the *tie prize*, or the prize value that each player receives in the event of a tie. Under the assumption that winning is preferred to tying and that tying is preferred to losing, this paper completely characterizes the set of equilibria for the entire parameter space of tie margins and tie prizes.

As in the classic all-pay auction (with a tie margin of zero), equilibrium in the all-pay auction with a tie prize and a (strictly positive) tie margin is in mixed strategies. In any mixed strategy equilibrium of the all-pay auction with ties, each bid (in the support of an equilibrium mixed strategy) faces either a losing-versus-tying margin of competition or a tying-versus-winning margin of competition. For example, any bid between zero and the size of the tie margin can at best tie an opponent's low bid. Thus, losing-versus-tying is the relevant margin of competition for bids in this range. For bids above the size of the tie margin, in any equilibrium, players' mixed strategies randomize over a set of intervals of bids and systematic gaps in such a way that each bid faces a single margin of competition: losing-versus-tying or tying-versus-winning. Introducing a strictly positive tie margin therefore results in a mixed-strategy equilibrium featuring the randomization of bids across disjoint intervals. These intervals are then further divided into sub-intervals which have one of two distinct density rates, corresponding to either the losing-versus-tying margin of competition or the tying-versus-winning margin of competition. We find that there exists a unique symmetric equilibrium for nearly all parameter configurations. For a range of parameters in which the tie prize is less than or equal to half of the winning prize, there also exist asymmetric equilibria.

In the unique symmetric mixed-strategy equilibrium, the number of disjoint intervals in the support—as well as the measure of each interval—is dependent on both the tie margin and the tie prize. For a given tie prize, the number of disjoint intervals in the support of the symmetric equilibrium (weakly) increases as the tie margin decreases. In the limit, as the tie margin

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the strategic uncertainty as to whether a single proposal would sufficiently dominate the competition and win the contract outright or whether the proposals would be relatively close and result in a split contract. More recently, the second great engine war over the contract to supply engines for the F-35 joint strike fighter currently features a single winner (or supplier), Pratt & Whitney. See Drewes (1987) and Amick (2005) for further details.

[2] The list of applications is widespread and includes lobbying, litigation, R&D competitions, college admissions, election campaigns, warfare, etc. Konrad (2009) and Dechenaux et al. (2015) respectively survey the theoretical and experimental literature.

[3] Again, the notion of a tie here is that player's expenditures are close enough for an intermediate prize outcome to occur (such as political gridlock or a battlefield stalemate). Just as expenditures may differ in reaching such an outcome, players' net payoffs (the value of the intermediate prize less the expenditure) may likewise differ.

approaches zero, the number of disjoint intervals becomes arbitrarily large and converges to the equilibrium of the classic all-pay auction (with a tie margin of zero), in which players continuously randomize over the entire interval of bids from zero to the value of the winning prize. As a result of the discrete jumps between the disjoint intervals, we find that for sufficiently high tie prizes, expected bids are non-monotonic in the tie-margin. A parallel result is that expected bids are also non-monotonic in the tie prize for a range of sufficiently low tie margins.

Multiple margins of competition—along the lines of losing-versus-tying and tying-versus-winning—arise in several applications. An area that is rich in examples is Bertrand-like price competition when there is some theshold that segments different blocks of customers:

- **Tariffs.** Customers will purchase the domestic product unless the price of the foreign good undercuts the domestic good's price by more than the size of the tariff (Fisher and Wilson 1995). The domestic firm ties by maintaining its home market, it wins by capturing the foreign market, and the tariff is the tie margin.
- **Transportation costs.** Customers purchase from the nearest firm unless a competitor's price is low enough to outweigh the added transportation cost (Shilony 1977). A firm ties by maintaining the customers in its immediate vicinity, it wins by capturing further customer segments, and the transportation cost is the tie margin.
- **Consumer rebates.** In the simplest case, each firm offer a homogeneous rebate to a distinct subset of customers (e.g., loyalty customers), and each customer receives a rebate from exactly one firm (Klemperer 1987; Szech and Weinschenk 2013). A firm ties by maintaining the customers that it offered rebates to, it wins by capturing the customers a competitor sent rebates to, and the rebates form a tie margin.

Another related setting is split-award (procurement) auctions, which feature explicit rules for how awards are split conditional on the profile of bids received. Recent examples include Chaturvedi et al. (2014) and Gong et al. (2012).

Ties have been studied in the context of the all-pay auction in terms of bidding caps, incomplete information, and discrete strategy spaces.[4] For example, Szech (2015) extends Che and Gale's (1998) model of the all-pay auction with a common bidding cap to examine the issue of asymmetric tie-breaking rules.[5] The tie-margin is zero in that formulation, but because of the presence of a bidding cap, the choice of a tie-breaking rule (which is equivalent to a tie prize under risk neutrality) is an important determinant of equilibrium behavior. Stong (2014) identifies preliminary results for the all-pay auction with ties under incomplete information. Focusing specifically on a case where the tie margins are relatively large, he likewise identifies that players randomize their bids over disjoint intervals in equilibrium.[6] In the case of an all-pay auction in which the strategy space is discrete, ties may arise with positive probability (e.g. Bouckaert et al.

[4] Ties have also been studied in the logit-type contests of Tullock (1980), as well as in the rank-order (difference-form) tournament of Lazear and Rosen (1981). Examples of the former include Blavatskyy (2010), Jia (2012), and Yildizparlak (2017); while examples of the latter include Nalebuff and Stiglitz (1983), Eden (2006), and Imhof and Kräkel (2014, 2015). The possibility of ties under these contest success functions is discussed in more detail in Gelder et al. (2015).

[5] Che and Gale show that if bidders in an all-pay auction have asymmetric valuations for the winning prize, then an auction-designer can increase expected revenue by introducing a bidding cap that levels the playing field by reducing the stronger player's ability to outbid the weaker player. Szech (2015) then goes on to show that the auction designer can do even better, with regards to equilibrium expected expenditure, by introducing an asymmetric tie-breaking rule that favors the weaker player (instead of using a symmetric tie-breaking rule where each player wins the prize with equal probability in the event of a tie).

[6] Stong focuses on equilibria where the upper bound of the support is no more than twice the size of the tie margin (in our map of the parameter space, this corresponds to region II of Figure 5). His model has incomplete

1992; Baye et al. 1994; Cohen and Sela 2007; Cohen and Schwartz 2013). With a discrete strategy space the gap between feasible bids creates similar losing-verses-tying and tying-verses-winning tradeoffs as our tie margin. Cohen and Sela (2007), in particular, argue that the size of the prize in the event of a tie does not affect the players' efforts. As we show in this paper, the discrete strategy space game creates qualitatively different incentives than the continuous strategy space version of the game.

2 Model

The all-pay auction with ties, which we label:

$$APT\{\delta, \beta, v\}$$

is the one-shot, complete-information game in which two players each privately submit a non-refundable bid $x_i \geq 0$. The difference between the bids determines the outcome. A player wins a prize of $v > 0$ if his bid exceeds his opponent's by strictly more than $\delta \geq 0$. If the two bids are within δ of each other, each player receives βv where $\beta \in [0,\ 1)$. A player who is outbid by more than δ receives no prize. Thus, player i's payoff is as follows:

$$u_i(x_i,\ x_{-i}) = \begin{cases} v - x_i & \text{if} \quad x_i > x_{-i} + \delta \\ \beta v - x_i & \text{if} \quad x_{-i} - \delta \ \leq \ x_i \ \leq \ x_{-i} + \delta \\ -x_i & \text{if} \quad x_i < x_{-i} - \delta \end{cases} \tag{1}$$

For $\delta = 0$, this becomes the standard all pay auction, where in the unique Nash equilibrium, players uniformly randomize their bids between zero and the value of the prize:

$$G_i(x) = \begin{cases} x/v & x \in [0,\ v] \\ 1 & x > v \end{cases}$$

On the other extreme, when $\delta \geq (1-\beta)v$, the unique Nash equilibrium is for players to bid 0 with certainty. Winning, rather than tying, involves a bidding cost greater than δ; but such a cost is greater than the difference between the winning prize v and the prize for tying βv. So players are content to tie at zero.

Equilibria progressively transition between these two extremes as the tie margin δ moves through the intervening values of $(0,\ (1-\beta)v)$. As δ decreases, bids of zero become less and less frequent. Bids instead disperse over greater portions of the interval $[0,\ v]$.

For much of the parameter space, equilibria are in non-degenerate mixed strategies. Letting G_i be player i's bid distribution, and denoting a mass point at x within G_i as $\alpha_i(x) \in [0,1]$, we can write player i's expected utility for a bid of x as:

$$\begin{aligned} u_i(x,\ G_{-i}) &= [G_{-i}(x-\delta) - \alpha_{-i}(x-\delta)]\, v\, + \\ &\qquad [G_{-i}(x+\delta) - G_{-i}(x-\delta) + \alpha_{-i}(x-\delta)]\, \beta v - x \\ &= G_{-i}(x+\delta)\, \beta v + [G_{-i}(x-\delta) - \alpha_{-i}(x-\delta)]\, (1-\beta)v - x \end{aligned} \tag{2}$$

The last line of Equation 2 highlights the two relevant margins of competition. A bid of x narrowly ties a bid of $x+\delta$ and is rewarded with βv, the marginal benefit for tying relative to losing. Similarly, x beats bids below $x - \delta$, and since these bids otherwise would be tied, the marginal benefit is $(1-\beta)v$.

information in that players have privately known valuations of the winning prize and privately known bidding costs. The ratio of the tie prize to the winning prize is, however, common across players.

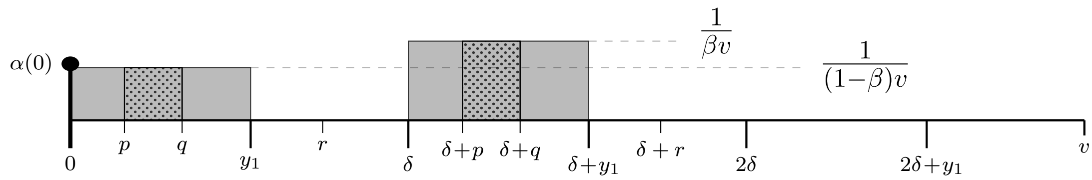


Fig. 1: Symmetric equilibrium strategy for an intermediate tie margin.

3 Intermediate Tie Margins

An initial case of intermediate tie margins is instructive in establishing patterns and intuition for equilibria that exist throughout the parameter space. Unlike smaller values of the tie margin δ which require separate analysis depending on whether the tie prize is less than, equal to, or greater than one-half of the winning prize, the case of intermediate tie margins spans all three. For the straightforward scenario where the tie prize is half the winning prize, this case encompasses tie margins of one-quarter to one-half the winning prize: $\beta = 1/2$ and $\delta \in [v/4,\ v/2)$. For $\beta \in (0,\ 1/2)$, it includes $\delta \in [(1-\beta)v/2,\ (1-\beta)v)$; and for $\beta \in (1/2,\ 1)$, it covers $\delta \in [(1-\beta)\beta v,\ (1-\beta)v)$.

For any tie prize and tie margin within these bounds, the unique symmetric equilibrium is in mixed strategies, consisting of a mass point at zero and randomization over two disjoint intervals. As depicted in Figure 1, the lower interval stretches over $[0,\ y_1]$ at the uniform density rate of $1/[(1-\beta)v]$, while the upper interval spans $[\delta,\ \delta + y_1]$ at a density rate of $1/(\beta v)$. The size of the mass point $\alpha(0)$ and the length of each interval y_1 are endogenously defined as:

$$\alpha(0) = \frac{\delta - (1-\beta)\beta v}{(1-\beta)^2 v}; \qquad y_1 = \beta v - \left(\frac{\delta\beta}{1-\beta}\right)$$

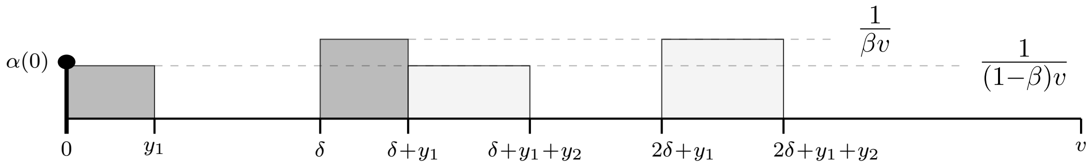
Figure 1 also includes markings for a handful of bids to help in clarifying intuition. A bid of $p \in (0,\ y_1)$ will never win since it is less than δ. But it will tie all bids (weakly) less than $\delta + p$. For the equilibrium to hold, a risk neutral player would need to be indifferent between bidding p and bidding $q \in (p,\ y_1]$. The marginal increase in the bidding cost, $q - p$, is exactly offset by the expected value of tying a bid in $(\delta + p,\ \delta + q]$. Specifically,

$$q - p = \left[\frac{q-p}{\beta v}\right] \times \beta v$$

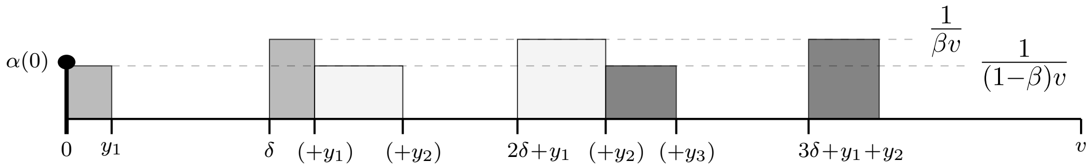
The same principal holds for bids of $\delta + p$ and $\delta + q$. A bid of $\delta + p$ will beat all bids below p and tie the remaining bids in the distribution. The marginal cost of increasing to a bid of $\delta + q$ is then balanced by the expected value of beating (instead of tying) bids in $[p,\ q)$:

$$q - p = \left[\frac{q-p}{(1-\beta)v}\right] \times (v - \beta v)$$

A bid of $r \in (y_1,\ \delta)$ between the two intervals is not profitable. It will tie all bids in the distribution, but y_1 will as well and at a lower cost. Crossing the gap between y_1 and δ requires some compensation if players are to be indifferent between bids in the lower and upper intervals. The mass point at zero $\alpha(0)$ does just that, providing a large enough winning probability to counter



(a) Two pairs of intervals, plus a mass point at zero.



(b) Three pairs of intervals, plus a mass point at zero.

Fig. 2: Density plots of the unique symmetric equilibrium strategies for small δ when $\beta < 1/2$.

the higher bidding cost.[7] This permits the structure of players randomizing between attempting to tie (with bids in the lower interval) and attempting to win (bids in the upper interval).

4 Small Tie Margins

4.1 Tie prize less than one-half

Continuing with Figure 1, like the barren region between y_1 and δ, it is also unprofitable to place bids in $(\delta+y_1,\ 2\delta]$. There are simply no new bids to tie or beat (the result is the same as at $\delta+y_1$, but at a higher cost). It is only for bids above 2δ that there are new bids to beat. In particular, beating bids in the interval $[\delta,\ \delta+y_1]$ at a density rate of $1/(\beta v)$ more than compensates for the marginal bidding cost when $\beta < 1/2$. A player bidding above the distribution would thus have the highest expected payoff at $2\delta + y_1$ where winning is certain. For a fixed tie prize $\beta < 1/2$, as the tie margin δ gets smaller, the expected payoff at this point is increasing, and for $\delta < (1-\beta)v/2$, it would actually be a profitable deviation. The equilibrium construct therefore needs to be modified for smaller tie margins.

The solution is to add another pair of intervals immediately above $\delta + y_1$ (as shown in the top panel of Figure 2). This second interval pair has the same properties as the first: the lower interval has a density rate of $1/[(1-\beta)v]$, while the upper has a rate of $1/(\beta v)$; their lower bound are spaced δ apart; and they have the same length (in this case y_2, which can differ from y_1). For even smaller tie margins, it will once again become profitable to outbid this distribution by δ and win with certainty. Three interval pairs will then be needed (as in the bottom panel of Figure 2).

This pattern of adding another interval pair once the tie margin decreases below some threshold holds more generally. For a given tie prize βv and tie margin δ, the exact number of interval pairs is implicitly defined through a system of equations subject to a constraint. The system solves for

[7] A bid of precisely δ is an exception. Tying instead of beating $\alpha(0) > 0$ leads to a strictly lower payoff at δ compared to bids arbitrarily close to δ from above. However, since distributional supports are necessarily closed sets, a bid of δ is permitted to be in the support as the endpoint of an interval. Even still, equilibrium behavior is invariant to imposing a special tie breaking rule where a bid of δ beats a bid of zero—a bid of δ occurs with zero probability either way. Other examples of select points within equilibrium supports having lower payoffs due the the presence of mass points include Osborne and Pitchik (1986) and Deneckere and Kovenock (1996).

the size of the mass point at zero and the lengths of the intervals in each pair–all of which must be strictly positive. Moreover, the system itself (wherein each interval pair is represented by an equation) must be the largest system for which the positivity constraint holds.[8]

Letting x_j be an element of the lower interval of the j^{th} interval pair, the following must hold for $j \in \{1, \ldots, k\}$, where k is the number of interval pairs:

$$u_i(x_j,\ G_{-i}) = u_i(x_j + \delta,\ G_{-i}) \tag{3}$$

We denote the length of each interval in the j^{th} pair as y_j. With just one interval pair ($k=1$), as in Figure 1, the expected utilities for bids of x_1 and $x_1 + \delta$ are:

$$u_i(x_1,\ G_{-i}) = \left[\alpha(0) + \frac{y_1}{(1-\beta)v} + \frac{x_1}{\beta v}\right]\beta v - x_1$$

$$u_i(x_1+\delta,\ G_{-i}) = \left[\alpha(0) + \frac{x_1}{(1-\beta)v}\right]v + \left[\frac{y_1 - x_1}{(1-\beta)v} + \frac{y_1}{\beta v}\right]\beta v - x_1 - \delta$$

So $u_i(x_1,\ G_{-i}) = u_i(x_1 + \delta,\ G_{-i})$ implies that:

$$\delta = \alpha(0)(1-\beta)v + y_1 \tag{4}$$

If there are two or more interval pairs ($k \geq 2$), then $u_i(x_1 + \delta,\ G_{-i})$ has the additional term of $y_2\,[\beta/(1-\beta)]$ since a bid of $x_1 + \delta$ also ties the lower interval of the second interval pair. In which case, $u_i(x_1,\ G_{-i}) = u_i(x_1 + \delta,\ G_{-i})$ becomes:

$$\delta = \alpha(0)(1-\beta)v + y_1 + y_2\left(\frac{\beta}{1-\beta}\right) \tag{5}$$

Equation 3 yields similar expressions for x_j when $j \geq 2$. The bids of x_j and $x_j + \delta$ both beat any mass in the first $j-2$ interval pairs, as well as mass in the lower interval of the $(j-1)^{th}$ pair. The only relevant intervals are the upper interval of the $(j-1)^{th}$ pair, the lower interval of the $(j+1)^{th}$ pair (if $j < k$), and both intervals in the j^{th} pair. For $j \in \{2, \ldots, k-1\}$, Equation 3 implies:

$$\delta = y_{j-1}\left(\frac{1-\beta}{\beta}\right) + y_j + y_{j+1}\left(\frac{\beta}{1-\beta}\right) \tag{6}$$

Finally, for $j = k$, we have:

$$\delta = y_{k-1}\left(\frac{1-\beta}{\beta}\right) + y_k \tag{7}$$

The set of Equations 5, 6, and 7 (or only Equation 4 for case where $k = 1$) form a system of k equation with $k+1$ unknowns: $(\alpha(0),\ y_1, \ldots,\ y_k)$. We close this system by requiring that the total mass in the distribution sum to one:

$$1 = \alpha(0) + \sum_{j=1}^{k} \frac{y_j}{(1-\beta)\beta v} \tag{8}$$

Together with equation 8, we refer to this system of $k+1$ equations collectively as System (☆). As shown in the appendix, for any $k \geq 1$, this system uniquely defines values for $\alpha(0)$ and $y_1, \ldots,\ y_k$ (see Proposition 1). In a symmetric equilibrium, each of these values must be strictly positive (if one of the values were zero, then players could not be indifferent between the upper and lower intervals of an adjoining pair). Payoff equivalence within the support is given by System (☆), and

[8] Although the exact number of interval pairs k is implicitly defined, there is a finite set of possible values it can take. The upper bound of that set $\lfloor v/\delta \rfloor$, where $\lfloor \cdot \rfloor$ is the floor function, is based on the fact that intervals within a pair have lower bounds that are spaced δ apart, and a bid of zero strictly dominates bids greater than v.

potential deviations are ruled out with arguments similar to those used with Figures 1 and 2. The necessary conditions for establishing uniqueness are presented in the appendix as a series of lemmata, each successively pinning down the precise form that an equilibrium strategy must take. We explicitly characterize the unique symmetric equilibrium below.

Theorem 1 *For $\delta \in (0,\ (1-\beta)v)$ and $\beta \in (0,\ 1/2)$, the game $APT\{\delta, \beta, v\}$ has a unique symmetric equilibrium. Let $\boldsymbol{x}^\star = (\alpha, y_1, \ldots, y_z, \ldots, y_k)$ be the unique solution to System (☆), where $k \in \{1, \ldots, \lfloor v/\delta \rfloor\}$ is implicitly defined as the largest integer for which all elements of $\boldsymbol{x}^\star$ are strictly positive. Each player's bid distribution G_i is then defined as follows:*

$$G_i(x) = \begin{cases} \alpha + \dfrac{x}{(1-\beta)v} & x \in [0,\ y_1) \\ \alpha + \dfrac{y_1}{(1-\beta)v} & x \in [y_1,\ \delta) \\ \alpha + \dfrac{y_1}{(1-\beta)v} + \dfrac{x-\delta}{\beta v} & x \in [\delta,\ \delta + y_1) \\ \alpha + \dfrac{y_1}{(1-\beta)\beta v} + \dfrac{x-\delta-y_1}{(1-\beta)v} & x \in [\delta + y_1,\ \delta + y_1 + y_2) \\ \alpha + \dfrac{y_1}{(1-\beta)\beta v} + \dfrac{y_2}{(1-\beta)v} & x \in [\delta + y_1 + y_2,\ 2\delta + y_1) \\ \alpha + \dfrac{y_1}{(1-\beta)\beta v} + \dfrac{y_2}{(1-\beta)v} + \dfrac{x - 2\delta - y_1}{\beta v} & x \in [2\delta + y_1,\ 2\delta + y_1 + y_2) \\ \vdots & \vdots \\ \alpha + \dfrac{\sum_{j=1}^{z-1} y_j}{(1-\beta)\beta v} + \dfrac{x - (z-1)\delta - \sum_{j=1}^{z-1} y_j}{(1-\beta)v} & x \in [(z-1)\delta + \sum_{j=1}^{z-1} y_j, \\ & \quad (z-1)\delta + \sum_{j=1}^{z} y_j) \\ \alpha + \dfrac{\sum_{j=1}^{z-1} y_j}{(1-\beta)\beta v} + \dfrac{y_z}{(1-\beta)v} & x \in [(z-1)\delta + \sum_{j=1}^{z} y_j, \\ & \quad z\delta + \sum_{j=1}^{z-1} y_j) \\ \alpha + \dfrac{\sum_{j=1}^{z-1} y_j}{(1-\beta)\beta v} + \dfrac{y_z}{(1-\beta)v} + \dfrac{x - z\delta - \sum_{j=1}^{z-1} y_j}{\beta v} & x \in [z\delta + \sum_{j=1}^{z-1} y_j, \\ & \quad z\delta + \sum_{j=1}^{z} y_j) \\ \vdots & \vdots \\ 1 & x > k\delta + \sum_{j=1}^{k} y_j \end{cases}$$

4.2 Tie prize equal to one-half

The case of $\beta = 1/2$ is a seemingly natural one since it represents the winning prize being evenly split in the event of a tie. That simplicity also leads to some peculiar challenges. For other values of the tie prize, equilibria are typified by the density rates of $1/\beta v$ and $1/[(1-\beta)v]$, but these are one and the same when $\beta = 1/2$. Consequently, equilibria are frequently not unique and can take on more amorphous forms. There is, however, a periodic pattern that provides a governing structure.

To illustrate this, Figure 3 shows the topmost portion of player i's distribution—from the upper bound $\overline{x}_i$ down to 5δ below that. In every equilibria, the interval $[\overline{x}_i - 2\delta,\ \overline{x}_i - \delta]$ will have a density rate of $2/v$, and $[\overline{x}_i - 4\delta,\ \overline{x}_i - 3\delta]$ will have no mass. There is considerable freedom

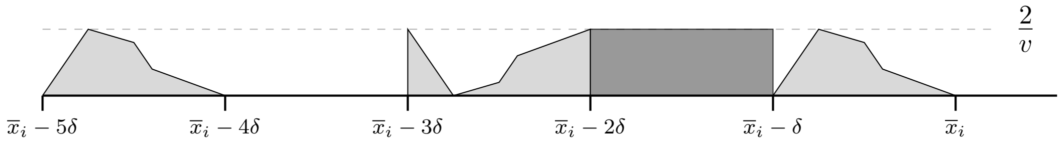


Fig. 3: Illustration of Property $\mathscr{P}$.

with regard to the distribution over $[\overline{x}_i - 3\delta,\ \overline{x}_i - 2\delta]$ and $[\overline{x}_i - \delta,\ \overline{x}_i]$. The only stipulation is that the mass be continuously distributed with the density rates summing to $2/v$ for any two points that are 2δ apart. Formally, $g_i(x) + g_i(x - 2\delta) = 2/v$ for any $x \in [\overline{x}_i - \delta,\ \overline{x}_i]$ where $g_i(x)$ is the density rate at x. Periodicity comes into play because the full pattern over the length-4δ interval from $\overline{x}_i$ down to $\overline{x}_i - 4\delta$ repeats over the next 4δ down. And then again over the next. Specifically, $g_i(x) = g_i(x - 4\delta)$ for each x in the length-4δ pattern. Each repetition uses up $4\delta/v$ worth of the distribution's mass, and the repetitions continue until there is less than $4\delta/v$ left in the total distribution (i.e. the length-4δ pattern occurs $p = \lfloor v/4\delta \rfloor$ times). We refer to these rules collectively as Property $\mathscr{P}$.

Although there tends to be a rich multiplicity of equilibria, the bottom of the distribution gives further structure to $\mathscr{P}$. The remaining mass of $1 - (4\delta p/v)$ either becomes an isolated mass point at zero or one that is accompanied by a pair of intervals with lower bounds at zero and δ. So except for the case where $\delta = v/4p$, the set of equilibria for any given δ are payoff equivalent. The isolated mass point occurs when $\delta \in [v/(4p + 2),\ v/4p)$, with an expected payoff of $(v/2) - 2\delta p$ (there is no mass between zero and δ). This expected payoff, together with $\mathscr{P}$, then dictates the amount of mass that must be placed between δ and 2δ, between 2δ and 3δ, and each δ-interval thereafter.[9] When $\delta \in (v/(4p + 4),\ v/(4p + 2))$, the mass point at zero is accompanied by an interval pair, and δ is also the expected payoff.[10]

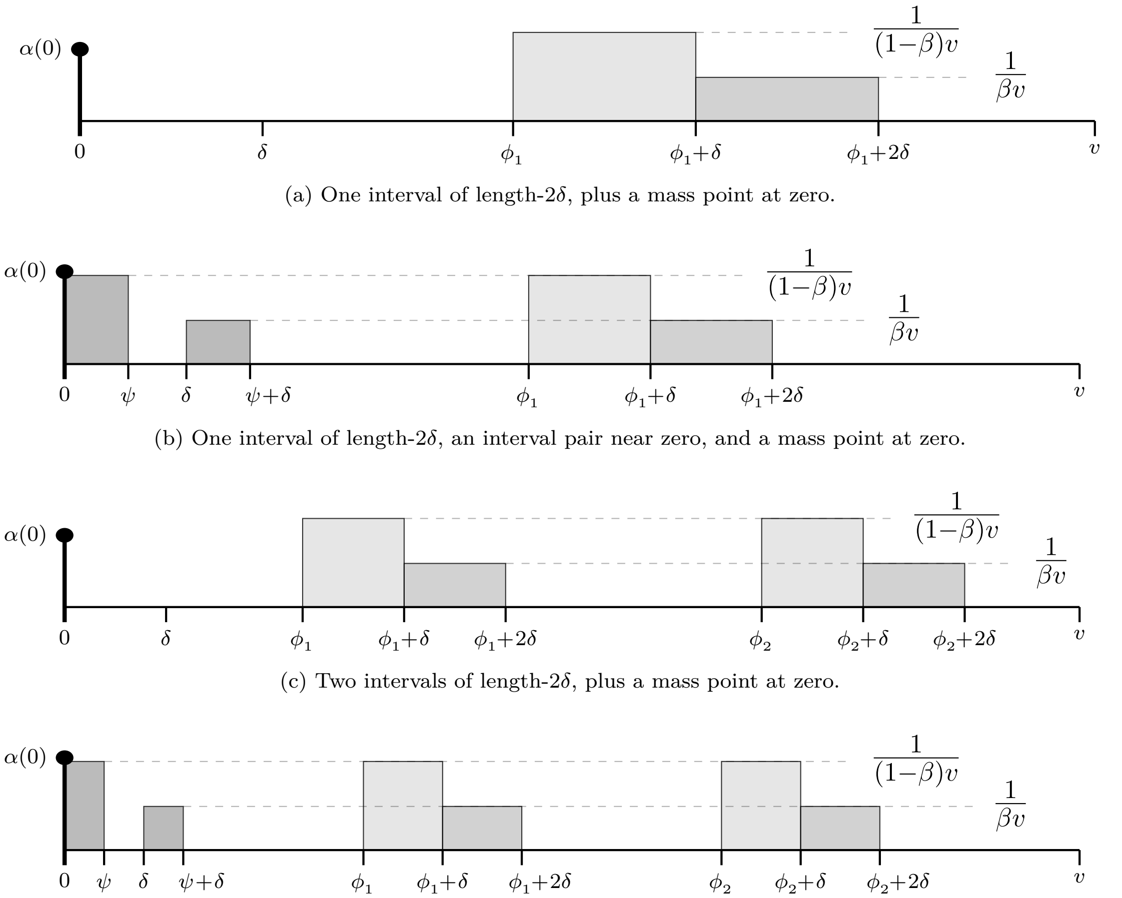
4.3 Tie prize greater than one-half

It may seem odd to think of tie prizes that are greater than one-half the winning prize—after all, when a tie occurs, the sum of prizes across players is greater than when one player is the sole winner.[11] Yet this is not an unnatural situation. As an extreme example, there is arguably little difference in the accolades and honors that are bestowed upon individuals that split a Nobel Prize versus individuals that win one outright (an actual monetary prize is split, but the honors multiply). In a procurement competition, where tying entails dividing a production run between two firms, the relative size of the tie prize to the winning prize will depend on the economies (or diseconomies) of scale associated with the full and divided production runs. The marginal profitability of the divided run (tying versus losing) may indeed be higher than the marginal

[9] For instance, if $\delta = v/5$, there is a mass point at zero of $v/5$. Another $v/5$ is distributed over $[\delta,\ 2\delta]$; $2v/5$ is distributed over $[2\delta,\ 3\delta]$; and $v/5$ is distributed over $[3\delta,\ 4\delta]$. Note that there is no unique upper bound in this case as a multitude of distributions over $[\delta,\ 2\delta]$ and $[3\delta,\ 4\delta]$ can satisfy $\mathscr{P}$. When $\delta = v/(4p + 2)$, such as when $\delta = v/10$, there is no mass in $[\delta,\ 2\delta]$, so $\mathscr{P}$ can only be satisfied when $[2\delta,\ 4\delta]$ has a density rate of $2/v$. This is a rare case of uniqueness.

[10] The intervals span $[0,\ v/2 - \delta(2p+1)]$ and $[\delta,\ v/2 - 2\delta p]$. The lower bound of the bottommost length-4δ pattern of $\mathscr{P}$ begins promptly at $v/2 - 2\delta p$. This implies that the distribution has a unique upper bound.

[11] In terms of rent seeking, a tie prize of $\beta > 1/2$ leads to rent creation in the event of a tie, whereas $\beta < 1/2$ leads to rent destruction. The full rent of the game is endogenous and varies based on whether there is a tie or a clear winner.



(a) One interval of length-2δ, plus a mass point at zero.

(b) One interval of length-2δ, an interval pair near zero, and a mass point at zero.

(c) Two intervals of length-2δ, plus a mass point at zero.

(d) Two intervals of length-2δ, an interval pair near zero, and a mass point at zero.

Fig. 4: Density plots of symmetric equilibrium strategies for various δ when $\beta > 1/2$. Equilibria contain intervals of length-2δ, a mass point at zero, and, periodically, an interval pair near zero.

profitability of the second part of the larger run (winning versus tying). For instance, there may be space or time constraints that could make the larger run unwieldy.

Equilibria with $\beta > 1/2$ are primarily built of intervals that are 2δ in length. As the tie margin δ decreases in size, these equilibria are composed of successively more intervals of length-2δ. Equilibria also have a mass point at zero, and an occasional pair of intervals with lower bounds at zero and δ. Figure 4 plots four examples of the actual equilibrium strategies. Each length-2δ interval has a density rate of $1/[(1-\beta)v]$ over the bottom half and $1/(\beta v)$ over the top half. Figure 4 also shows that these density rates apply to the pair of intervals at zero and δ. As with $\beta < 1/2$, the tandem density rates of $1/[(1-\beta)v]$ and $1/(\beta v)$ allow players to remain indifferent over bids within each interval. Consecutive intervals of length-2δ are spaced so that their lower bounds, ϕ_j and ϕ_{j+1}, are always $\delta/[(1-\beta)\beta]$ apart. This spacing satisfies $u_i(\phi_j,\ G_{-i}) = u_i(\phi_{j+1},\ G_{-i})$ and also ensures that bids between the length-2δ intervals are strictly dominated.[12] The lower

[12] Expected payoffs monotonically decrease over $(\phi_j + 2\delta,\ \phi_{j+1} - \delta]$ since there is no additional mass to tie, and beating mass that was previously tied cannot compensate for the cost of bidding (i.e. $(1-\beta)/\beta < 1$). Expected payoffs then rise over $[\phi_{j+1} - \delta,\ \phi_{j+1})$ since tying mass at a rate of $1/[(1-\beta)v]$ more than covers the bidding cost (i.e. $\beta/(1-\beta) > 1$). Similar arguments rule out deviations in $[\phi_1 - \delta,\ \phi_1)$ or $(\phi_p + 2\delta,\ \phi_p + 3\delta]$, where p is the total number of length-2δ intervals. The high value of tying also precludes the incentive to win with certainty by outbidding the entire distribution by δ.

bound of the first length-2δ interval ϕ_1 is likewise defined to satisfy the indifference condition $u_i(0,\ G_{-i}) = u_i(\phi_1,\ G_{-i})$.[13]

Transitioning from the absence to the presence of the interval pair at zero and δ stems from monitoring a particularly critical potential deviation. If $2\delta < \phi_1$, it may become profitable to bid immediately above δ and beat the mass point at zero. Denoting the supremum of player i's expected utility as δ is approached from above by $u_i(\overline{\delta},\ G_{-i})$, in equilibrium, we must have $u_i(0,\ G_{-i}) \geq u_i(\overline{\delta},\ G_{-i})$.[14] Where p is the number of length-2δ intervals, this equates to:

$$\delta \geq \alpha(0)(1-\beta)v \quad \Rightarrow \quad \delta \geq \frac{(1-\beta)\beta v}{p+\beta} \tag{9}$$

For a given p, as soon as Equation 9 fails to hold, the interval pair at zero and δ becomes part of the equilibrium. With the presence of the interval pair, $u_i(0,\ G_{-i}) = u_i(\phi_1,\ G_{-i})$ implies:

$$\phi_1 = \alpha(0)(1-\beta)v + \psi\left(\frac{1+\beta}{\beta}\right) + \delta\left(\frac{\beta}{1-\beta}\right)$$

We can fully specify ϕ_1, $\alpha(0)$, and ψ with the use of two additional equations. The first is the constraint that the total mass must sum to one:

$$1 = \alpha(0) + \frac{\psi}{(1-\beta)\beta v} + \frac{\delta p}{(1-\beta)\beta v}$$

The second, $\delta = \alpha(0)(1-\beta)v + \psi$, comes from equating expected utilities across the intervals at δ and zero.[15] Combining these yields the following:

$$\phi_1 = v - \frac{\delta p}{(1-\beta)\beta}; \quad \alpha(0) = \frac{\delta(1+p) - (1-\beta)\beta v}{(1-\beta)^2 v}; \quad \psi = \beta v - \delta\left(\frac{p+\beta}{1-\beta}\right) \tag{10}$$

The positivity of ψ is based on reversing the inequality in Equation 9. Additionally, the constraint that $\alpha(0) \geq 0$ coincides with the constraint that $\psi \leq \delta$. Each is satisfied so long as:

$$\delta \geq \frac{(1-\beta)\beta v}{1+p} \tag{11}$$

When Equation 11 holds with equality, $\alpha(0) = 0$ and $\psi = \delta$, so the interval pair at zero and δ becomes yet another length-2δ interval. Without a mass point at zero to anchor expected payoffs, there are a multiplicity of symmetric equilibria, and the lower bound of the support may assume any value in $[0,\ \beta^2 v/(1+p)]$ (here, p does not include the newly formed length-2δ interval). Expected payoffs for these equilibria fall from $\delta\beta/(1-\beta)$ to 0 as the lower bound increases from 0 to $\beta^2 v/(1+p)$. We formally characterize equilibria for tie prizes greater than one-half as follows.

Theorem 2 *Let $\delta \in (0,\ (1-\beta)v)$ and $\beta \in (1/2,\ 1)$. Also, define $p = \lfloor (1-\beta)\beta v/\delta \rfloor$ where $\lfloor \cdot \rfloor$ is the floor function. If $\delta \neq (1-\beta)\beta v/(1+p)$, there exists a unique Nash equilibrium which is*

[13] Expected utilities at 0 and ϕ_1 are contingent on the presence or absence of an interval pair at zero and δ. When absent, $u_i(0,\ G_{-i}) = \alpha(0)\beta v$ and $u_i(\phi_1,\ G_{-i}) = \alpha(0)v + [\delta/(1-\beta)v]\,\beta v - \phi_1$, so $\phi_1 = \alpha(0)(1-\beta)v + [\delta\beta/(1-\beta)]$. The size of the mass point at zero is then simply the remainder after each of the length-2δ intervals. Specifically, for p intervals of length-2δ, $\alpha(0) = 1 - [\delta p/(1-\beta)\beta v]$.

[14] A bid of δ technically ties a bid of zero, but a deviating player would want to bid on the extreme low end of $(\delta,\ \phi_1 - \delta)$ to reduce bidding costs. The notation $u_i(\overline{\delta},\ G_{-i})$ is repeatedly used in the proofs in the appendix.

[15] Since a bid of $\psi + \delta$ does not tie ϕ_1, this is is equivalent to Equation 4 with y_1 replaced by ψ.

also symmetric. The characterization is as follows. If $p > 1$*, let* $\phi_z - \phi_{z-1} = \delta/(1-\beta)\beta$ *for* $z \in \{2, \ldots, p\}$*. For* $p = 0$*, define* $H(x) = 1$*. Otherwise, define* $H : [\phi_1,\ \infty) \to [\xi,\ 1]$ *as:*

$$H(x) = \begin{cases} \xi + \dfrac{x-\phi_1}{(1-\beta)v} & x \in [\phi_1,\ \phi_1+\delta) \\ \xi + \dfrac{\delta}{(1-\beta)v} + \dfrac{x-\phi_1-\delta}{\beta v} & x \in [\phi_1+\delta,\ \phi_1+2\delta) \\ \xi + \dfrac{\delta}{(1-\beta)\beta v} & x \in [\phi_1+2\delta,\ \phi_2) \\ \vdots & \vdots \\ \xi + \dfrac{(z-1)\delta}{(1-\beta)\beta v} + \dfrac{x-\phi_z}{(1-\beta)v} & x \in [\phi_z,\ \phi_z+\delta) \\ \xi + \dfrac{(z-1)\delta}{(1-\beta)\beta v} + \dfrac{\delta}{(1-\beta)v} + \dfrac{x-\phi_z-\delta}{\beta v} & x \in [\phi_z+\delta,\ \phi_z+2\delta) \\ \xi + \dfrac{z\delta}{(1-\beta)\beta v} & x \in [\phi_z+2\delta,\ \phi_{z+1}) \\ \vdots & \vdots \\ 1 & x > \phi_p + 2\delta \end{cases}$$

For $p \geq 1$*, if* $\delta \in [\,(1-\beta)\beta v/(p+\beta),\ (1-\beta)\beta v/p\,)$*, let* $\phi_1 = \alpha(0)(1-\beta)v + [\delta\beta/(1-\beta)]$ *and* $\alpha(0) = \xi = 1 - [\delta p/(1-\beta)\beta v]$*. Each player's equilibrium strategy is:*

$$G_i(x) = \begin{cases} \alpha(0) & x \in [0,\ \phi_1) \\ H(x) & x \geq \phi_1 \end{cases} \tag{12}$$

If $\delta \in (\,(1-\beta)\beta v/(1+p),\ (1-\beta)\beta v/(p+\beta)\,)$*, then* ϕ_1*,* $\alpha(0)$*, and* ψ *are defined by Equation* 10*. With* $\xi = \alpha(0) + [\psi/(1-\beta)\beta v]$*, each player has the following equilibrium strategy:*

$$G_i(x) = \begin{cases} \alpha(0) + \dfrac{x}{(1-\beta)v} & x \in [0,\ \psi) \\ \alpha(0) + \dfrac{\psi}{(1-\beta)v} & x \in [\psi,\ \delta) \\ \alpha(0) + \dfrac{\psi}{(1-\beta)v} + \dfrac{x-\delta}{\beta v} & x \in [\delta,\ \delta+\psi) \\ \alpha(0) + \dfrac{\psi}{(1-\beta)\beta v} & x \in [\delta+\psi,\ \phi_1) \\ H(x) & x \geq \phi_1 \end{cases} \tag{13}$$

Finally, for $p \geq 1$*, if* $\delta = (1-\beta)\beta v/p$*, there is a continuum of symmetric Nash equilibrium (which also constitutes the full set of Nash equilibrium). Let* $\ell \in [0,\ \beta^2 v/(1+p)]$*;* $\phi_1 = \ell + [\delta/(1-\beta)\beta]$*; and* $\xi = \delta/(1-\beta)\beta v$*. The complete set of symmetric equilibria is characterized by the following strategy:*

$$G_i(x) = \begin{cases} \dfrac{x-\ell}{(1-\beta)v} & x \in [\ell,\ \ell+\delta) \\ \dfrac{\delta}{(1-\beta)v} + \dfrac{x-\ell-\delta}{\beta v} & x \in [\ell+\delta,\ \ell+2\delta) \\ \dfrac{\delta}{(1-\beta)\beta v} & x \in [\ell+2\delta,\ \phi_1) \\ H(x) & x \geq \phi_1 \end{cases}$$

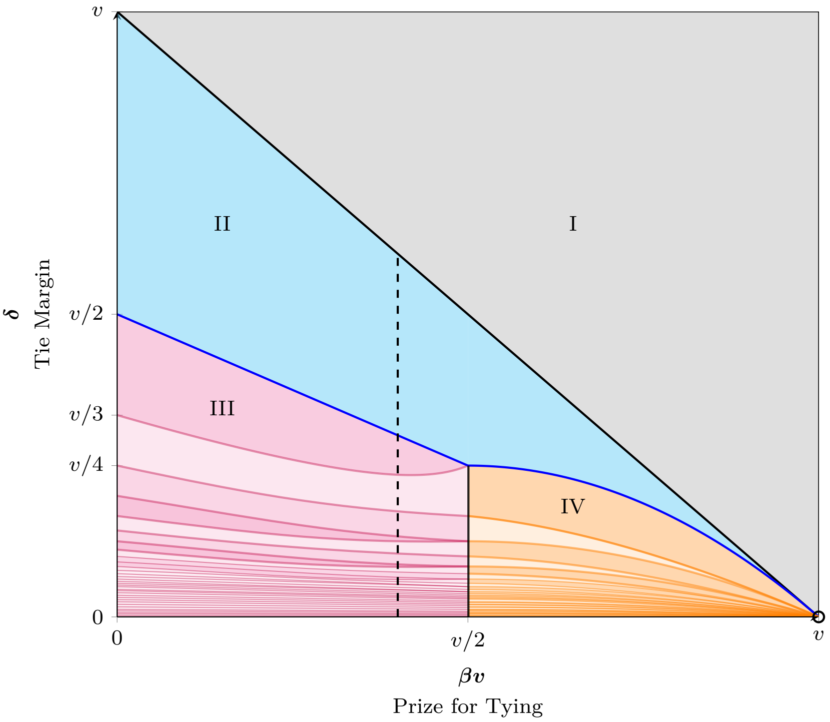


Fig. 5: Symmetric equilibrium regions in the parameter space.

5 Convergence and the Equilibrium Manifold

As we have seen, the qualitative nature of equilibria varies throughout different regions of the parameter space. Figure 5 provides a concise picture of how these regions fit together. The prize for tying βv is plotted along the horizontal axis, and the tie margin δ is on the vertical axis. The top-right triangle (marked with the Roman numeral I) represents the region where the cost of tying exceeds the difference between the tie prize and the winning prize: $\delta \geq (1-\beta)v$. Once again, the equilibrium here is for players to trivially tie at zero. The region marked by II corresponds to the intermediate tie margins in Section 3. Here, the marginal cost of winning relative to tying has decreased enough so that players attempt to win on occasion. But each bid that attempts to win is balanced by a bid δ below it attempting to tie. We thus get the equilibrium pattern of a pair of intervals extending up from 0 and δ, anchored by a mass point at zero.

Along the vertical axis of Figure 5, winning requires outbidding the other player by a strictly positive tie margin. Yet the result of tying is no different from losing. The unique equilibrium for $\delta \in (0,\ v)$ and $\beta = 0$ then entails a set of mass points, starting at 0 and placed at multiples of δ thereafter.[16] With the possible exception of the top mass point, each contains just enough mass (δ/v) so that beating it by δ exactly compensates for the cost of bidding.[17] Thus here, as in the standard all-pay auction, the expected equilibrium payoff is zero.

[16] The existence of this equilibrium requires a slight alteration of Equation 1: $u_i(x_i,\, x_{-i}) = v - x_i$ if $x_i - x_{-i} = \delta$ (otherwise, there is no minimum distance greater than δ to space the mass points by). A similar equilibrium with mass points spaced at regular intervals arises in the difference-form contest of Che and Gale (2000). There, when players' bids are relatively close, there is some probability that the player with the lower bid may win (the rest is made up by luck). Winning is only assured when the bidding difference is above a given threshold. In equilibrium, the mass points are spaced according to that threshold, just as they are here.

[17] If $v/\delta = \lfloor v/\delta \rfloor$, the topmost mass point is at $(\lfloor v/\delta \rfloor - 1)\,\delta$ and has a mass of (δ/v) like the other mass points. Otherwise, it is at $\lfloor v/\delta \rfloor\,\delta$ and has the remaining mass of $1 - \lfloor v/\delta \rfloor\,(\delta/v)$.

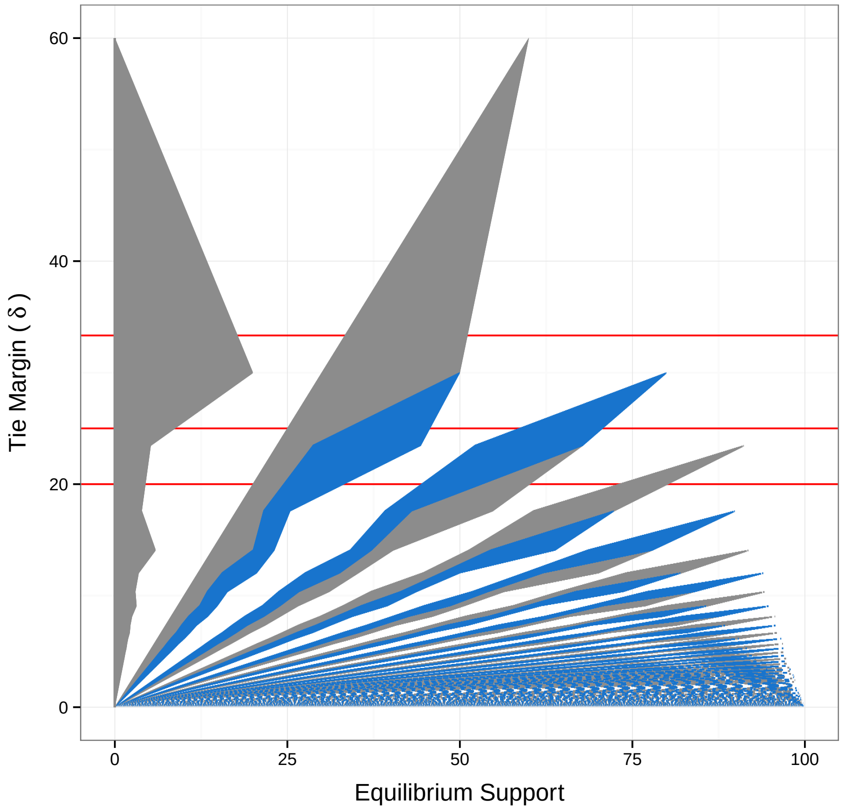


Fig. 6: Support of the unique symmetric equilibrium: $\beta = 0.4$, $v = 100$, and $\delta \in (0,\ (1-\beta)v)$. The three horizontal red lines correspond to the equilibria in Figures 1 and 2 for $\delta \in \{v/3,\ v/4,\ v/5\}$.

Region III of Figure 5, which is composed of many sub-regions, covers tie prizes less than half the winning prize, $\beta \in (0,\ 1/2)$, and includes all $\delta \in (0,\ (1-\beta)v/2)$. The topmost sub-region (extending from $v/2$ to $v/3$ along the vertical axis and marked with the III) represents the portion of the parameter space where the symmetric equilibrium has two pairs of intervals, plus a mass point at zero. In the next sub-region (from $v/3$ to $v/4$ along the vertical axis), there are three interval pairs instead of two. Each successive sub-region down adds another interval pair in the unique symmetric equilibrium (in other words, k successively increases by one in Theorem 1). Region IV is similarly divided into numerous sub-regions. This region spans $\beta \in (1/2,\ 1)$ for $\delta \in (0,\ (1-\beta)\beta v)$. The first four sub-regions (starting with the one containing the Roman numeral IV) correspond to the equilibrium formats in panels (a) through (d) of Figure 4: a new sub-region beginning each time a length-2δ interval or an interval pair at 0 and δ is added. Sub-regions occur with increasing rapidly as the tie margin δ decreases and approaches zero.

To illustrate the full transition from a mass point at zero in region I to uniform randomization over $[0,\ v]$ when $\delta = 0$ (the standard all-pay auction), we can take a cross-section of the parameter space. Specifically, for a given tie prize, we can examine equilibria for all δ between 0 and $(1-\beta)v$. The dashed line in Figure 5 represents one such cross-section. This dashed line is at $\beta = 0.4$, and setting $v = 100$, Figure 6 plots the support of the unique symmetric equilibrium for all δ between 0 and 60. For tie margins above 60 (i.e. $(1-\beta)v$), zero is the only point in the support. Immediately below 60, the equilibrium support is expanded to include points slightly above zero and near 60. These are the intervals with lower bounds of 0 and δ. As δ gets smaller, the intervals grow wider, but the overall upper bound of the support wanes. There comes a point when the upper bound has decreased enough that if it decreased any further it would become profitable to outbid the distribution by δ and win with certainty. To prevent this from happening, a second pair of intervals appears. In Figure 6, this happens right below a tie margin of 30: the second interval pair is shaded blue instead of gray. A third interval pair (shaded gray) begins at roughly $\delta = 23.478$. The three horizontal red lines represent examples of equilibria with one, two, and three interval pairs. In fact, the equilibria depicted in Figures 1 and 2 are drawn to scale so that correspond with the equilibrium supports depicted by these red lines.[18]

Successive interval pairs in Figure 6 arise at the point where it would otherwise become profitable to outbid the upper bound of the support by δ. In the uppermost pair, the width of the intervals always increases as δ decreases. The reverse is true for the second highest pair. Thereafter the effect is non-monotonic—the width sometimes increasing, sometimes decreasing. The net effect, however, is that as δ decreases, the intervals and their corresponding gaps become increasingly fine and gradually fill the full range of bids between zero and the winning prize v. As the tie margin δ approach zero, the equilibrium converges to that of the standard all pay auction in which players uniformly randomize between zero and v at the rate of $1/v$. Although for any strictly positive δ there are gaps in the support, the average density rate over any measurable subset of $[0,\ v]$ converges to $1/v$. This is true for all values of the tie prize β.[19]

6 Comparative Statics

The characterization of equilibrium naturally leads to predictions about how aggressively players compete on average. We can particularly address the question of whether players tend to be more or less aggressive for various fluctuations in the size of the tie prize and the tie margin. From the perspective of a policy maker or contest organizer who is trying to achieve some overall level of competition—be it high or low—this issue is paramount. Following the equilibrium characterization in Theorems 1 and 2, Figure 7 plots a player's expected bid (vertical axis) in terms of both tie margins (horizontal axis) and tie prizes (different lines within each graph). Fixing the prize for winning again at $v = 100$, the left panel shows six values of $\beta < 1/2$, while five values of $\beta > 1/2$ are represented in the right panel (axes have differing scales across panels). Several features here are worth highlighting, the first of which we state as a formal result.

Theorem 3 *For sufficiently low tie margins (roughly $\delta < 0.2625 \times v$), expected equilibrium bids are non-monotonic in the tie prize. Likewise, for sufficiently high tie prizes (roughly $\beta > 0.3347$), expected equilibrium bids are non-monotonic in the tie margin.*

Cutoffs here are based on numerical calculations. In terms of Figure 7, the first part of the theorem refers to lines that cross, while the second part addresses non-monotonicity within a given line. Not only is there non-monotonicity in each dimension, but the magnitudes vary considerably. Fixing β, the largest oscillation is always the first. Thereafter, oscillations occur more rapidly as δ continues to decrease, but the crest-to-trough distances becomes successively smaller. Increasing

[18] The top red line at $\delta = 33.33$ (or $v/3$) involves one pair of intervals: $[0,\ 17.78]$ and $[33.33,\ 51.11]$, with respective density rates of $1/60$ and $1/40$ (alternatively, $1/[(1-\beta)v]$ and $1/\beta v$). The second red line at $\delta = 25$ ($v/4$) has two pairs of intervals, the first pair in gray ($[0,\ 8.67]$ and $[25,\ 33.67]$), and the second in blue ($[33.67,\ 45.67]$ and $[58.67,\ 70.67]$). Then at $\delta = 20$ ($v/5$) there are three interval pairs, marked gray, blue, and gray again ($[0,\ 4.44]$ and $[20,\ 24.44]$; $[24.44,\ 33.14]$ and $[44.44,\ 53.14]$; $[53.14,\ 60.10]$ and $[73.14,\ 80.10]$).

[19] To illustrate this, and to show the nature of the equilibrium support throughout the parameter space, the online appendix contains eight panels similar to Figure 6 that show the equilibrium support for $\delta \in (0,\ (1-\beta)v)$ when $\beta \in \{0.1, 0.3, 0.45, 0.49999, 0.50001, 0.55, 0.7, 0.9\}$ and $v = 100$.

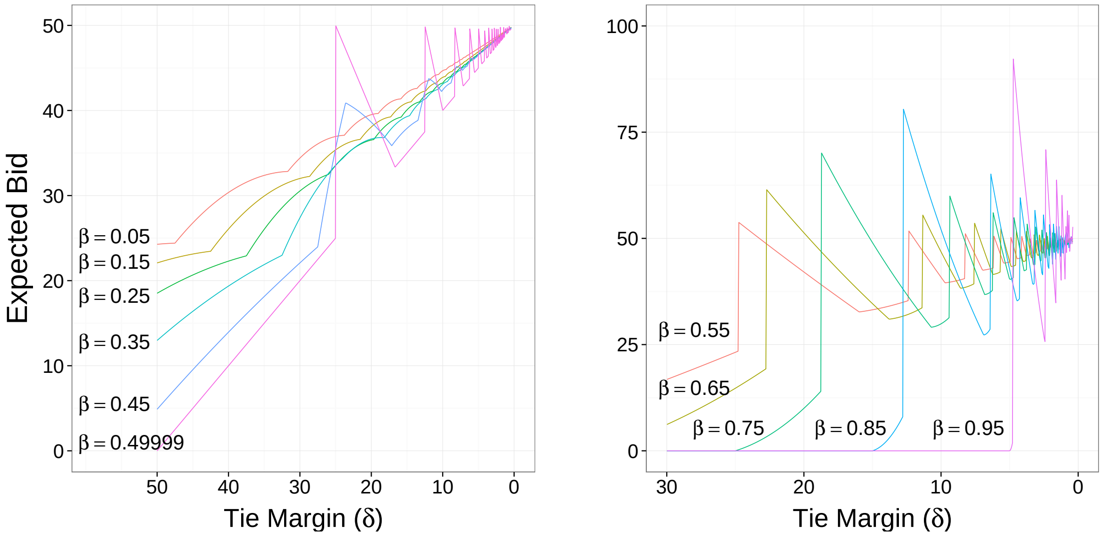


Fig. 7: Expected equilibrium bids by tie margin and tie prize; $v = 100$.

β leads to more volatile oscillations in the sense that the crest-to-trough distances are greater over even shorter wavelengths. The angle of ascent also grows so that each oscillation begins with a shear vertical jump once $\beta \geq 1/2$. These jumps correspond to the parameter values for which there is a continuum of equilibria, and every expected bid within the jump is attainable by some equilibrium. Another thing to note is the convergence of expected bids to $v/2$, the expected bid in the standard all-pay auction, as the tie margin decreases to zero. Although the paths to convergence differ considerably between the two panels. On the left, with $\beta < 1/2$, convergence is entirely from below; whereas on the right, with $\beta > 1/2$, the oscillations extend both far above and below $v/2$, so that $v/2$ only becomes a focal point as the oscillations vanish in size.

7 Asymmetric Equilibria

Asymmetric equilibria arise in limited portions of the parameter space and never when $\beta > 1/2$.[20] Lemmata 1 through 5 form a set of necessary conditions which must hold in any equilibrium, either symmetric or asymmetric, when $\beta \in (0, 1/2)$ and $\delta \in (0, (1-\beta)v)$. Taken together, these lemmata specify that an equilibrium must have the form depicted in Figure 8. Namely, mass points at zero $\alpha_w(0)$, $\alpha_y(0) \in [0, 1)$, followed by interval pairs with the familiar density rates of $1/[(1-\beta)v]$ and $1/(\beta v)$: the length of the lower interval in one player's distribution matching the length of the upper interval in the other player's distribution. We label the length of the successive $1/[(1-\beta)v]$ segments for player w as $w_1, w_2, \ldots, w_k \geq 0$, and likewise for player y as $y_1, y_2, \ldots, y_k \geq 0$. When each player's mass point does indeed have positive mass and when each of the interval pairs has a positive length, then the equilibrium is necessarily symmetric (see Proposition 1). Asymmetric equilibria arise when an interval pair has a length of zero or a mass point has no mass.

Zeroing out an interval pair or a mass point is not without consequence. Each segment with a density rate of $1/[(1-\beta)v]$ must be immediately preceded by some other mass (see Lemma 5 and the discussion following its proof in Appendix B). So if one player's mass point does not have any mass, their first interval must have a length of zero (i.e. if $\alpha_w(0) = 0$ then $w_1 = 0$). Similarly,

[20] For $\beta = 1/2$, asymmetric equilibria occur along parts of the line where $\delta \in (0, v/4]$. Such equilibria are governed by Property $\mathscr{P}$. Our focus in this section is $\beta < 1/2$.

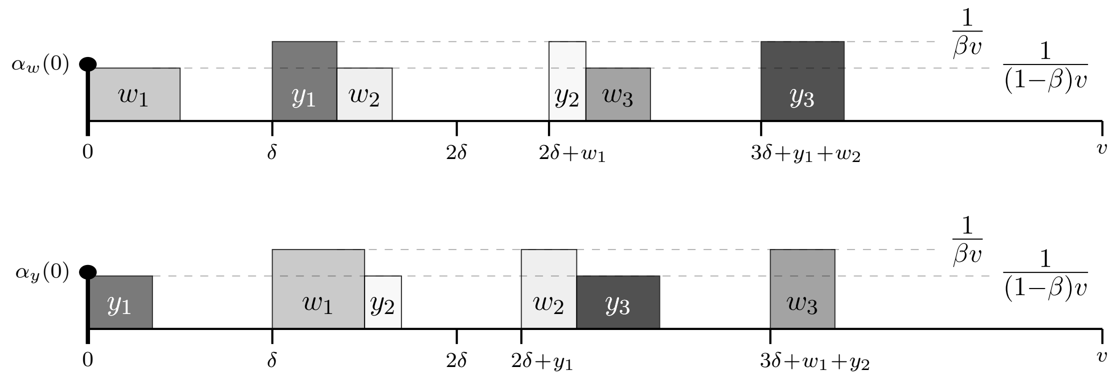


Fig. 8: Density plot of strategies for player w (Top) and player y (Bottom) when $\beta < 1/2$. Each interval's length is labeled as y_i or w_i.

when one interval pair has a length of zero, the next adjoining one must be zeroed out as well (so if $w_j = 0$ then $y_{j+1} = 0$, and if $y_j = 0$ then $w_{j+1} = 0$ for all $j \geq 1$). Exhaustively checking for all forms of asymmetric equilibria is therefore an exercise in varying the combination of interval pairs which are set to zero (potentially including one player's mass point). This process is described in Algorithm 1 in Appendix E. The online appendix provides specific examples and also maps out the regions of the parameter space where asymmetric equilibria arise.

8 Conclusion

We theoretically characterize equilibrium behavior in the two-player, complete information, all-pay auction with ties. Unlike the typical knife-edge formulation of players tying if they submit precisely the same bid, a tie occurs in our environment if neither player outbids the other by more than a predetermined amount. This expands the win-loss paradigm so that ties become a viable third outcome. Such an expansion is vital for studying contests where the distinction of winning may not be granted to any player; and when it is granted, it requires a quantum difference in performance (as is the case in military standoffs, political gridlock, and split-award procurement contracts to name a few). A pervasive feature of equilibrium strategies is that players randomize over multiple disjoint intervals. Even when asymmetric equilibria exist, this feature persists. Intuitively, players randomize their costly bids between attempting to win and attempting to tie—the spacing between the disjoint intervals reflecting the required threshold for winning.

There are several natural extensions to this study of ties. Just as the standard all-pay auction has been used as a building block for studying a variety of dynamic contests (including elimination tournaments, "best-of" multi-battle competitions, tug-of-wars, and hybrids of these), the all-pay auction with ties can similarly be used as a basis for more expansive dynamic architectures. A pertinent caveat to our model is that it is fundamentally symmetric. Yet players may have asymmetric valuations over either of their two margins of competition: the relative value of winning compared to tying, or the relative value of tying compared to losing. A further caveat is that the threshold for winning remains constant over all ranges of bids. Winning requirements could, however, be much more nuanced based on the intensity and level of competition. The competition's very nature may even endogenously determine the size of the threshold. In addition to these, the all-pay auction with ties may be extended along any of the numerous dimensions in which the standard all-pay auction has been studied.

Appendices

A General Necessary Conditions

Here we present the necessary conditions for equilibrium. Our first result limits the occurrence of a mass point to a bid of zero. If a mass point were to occur elsewhere, the opponent could profit by either barely tying it from below or slightly beating it from above, which in turn provides an incentive for moving the mass point. A mass point at zero is different, however, because it cannot be undercut.

Lemma 1 *Let $\delta > 0$ and $\beta > 0$. A positive mass point may only occur at zero in any equilibrium.*

Proof Suppose that G_{-i} includes a mass point at $x^* > 0$ of size $\alpha_{-i}(x^*) \in (0,1)$. For any $k \in (0,\delta)$, unless G_i either has some mass in $[x^*+\delta-k,\ x^*+\delta]$ which x^* can tie or some mass in $[x^*-\delta-k,\ x^*-\delta)$ which x^* can beat, then it is profitable for player $-i$ to move x^* down since this reduces the cost of bidding but maintains the probability of a tie and a win. Suppose that G_i does indeed contain mass in $[x^*+\delta-k,\ x^*+\delta]$ or $[x^*-\delta-k,\ x^*-\delta)$ for any arbitrarily small k. We will show that there exists a k such that placing mass in either interval is not optimal for player i. In which case, a mass point at x^* cannot be a best response for player $-i$. We begin by showing that for sufficiently small λ, player i strictly prefers a bid of $x^*+\delta+\lambda$ to a bid of $x^*+\delta-\mu$, where $\lambda > \mu \geq 0$. From Equation 2, we have:

$$\begin{aligned} u_i(x^*{+}\delta{+}\lambda,\ G_{-i}) - u_i(x^*{+}\delta{-}\mu,\ G_{-i}) &= [G_{-i}(x^*{+}2\delta{+}\lambda) - G_{-i}(x^*{+}2\delta{-}\mu)]\beta v \\ &+ [G_{-i}(x^*{+}\lambda) - G_{-i}(x^*{-}\mu) - \alpha_{-i}(x^*{+}\lambda) + \alpha_{-i}(x^*{-}\mu)](1-\beta)v - \lambda - \mu \end{aligned}$$

By definition, $G_{-i}(x^*{+}\lambda) - G_{-i}(x^*{-}\mu) \geq \alpha_{-i}(x^*)$. Let y^* be the next mass point in G_{-i}, if such a mass point exists. That is, $y^* = \min\{x \in \text{supp}(G_{-i}) \mid x > x^* \text{ and } \alpha(x) > 0\}$. If y^* does not exist, then $\alpha_{-i}(x^*{+}\lambda) = 0$. Otherwise, if $x^*{+}\lambda < y^*$, then we still have $\alpha_{-i}(x^*{+}\lambda) = 0$. Hence, for sufficiently small λ:

$$u_i(x^*{+}\delta{+}\lambda,\ G_{-i}) - u_i(x^*{+}\delta{-}\mu,\ G_{-i}) \ \geq\ \alpha_{-i}(x^*)(1-\beta)v - \lambda - \mu$$

Since $\lambda > \mu \geq 0$, then $u_i(x^*{+}\delta{+}\lambda,\ G_{-i}) - u_i(x^*{+}\delta{-}\mu,\ G_{-i}) > 0$ holds whenever $\lambda + \mu < \alpha_{-i}(x^*)(1-\beta)v$; or rather, whenever $\lambda < \alpha_{-i}(x^*)(1-\beta)(v/2)$ (and with the further restriction that $\lambda < y^* - x^*$ if y^* exists). So for any k that is less than such a λ, player i strictly prefers a bid of $x^*+\delta+\lambda$ to any bid in $[x^*+\delta-k,\ x^*+\delta]$. Since bids below zero are not possible, this completes the proof for $x^* \in (0,\ \delta]$. For $x^* > \delta$, we must also rule out the possibility of mass in $[x^*-\delta-k,\ x^*-\delta)$. Similar to the previous argument, we will show that player i strictly prefers a bid of $x^*-\delta$ to a bid of $x^*-\delta-\gamma$ where $\gamma > 0$:

$$\begin{aligned} u_i(x^*{-}\delta,\ G_{-i}) - u_i(x^*{-}\delta{-}\gamma,\ G_{-i}) &= [G_{-i}(x^*) - G_{-i}(x^*{-}\gamma)]\beta v \\ &+ [G_{-i}(x^*{-}2\delta) - G_{-i}(x^*{-}2\delta{-}\gamma) - \alpha_{-i}(x^*{-}2\delta) + \alpha_{-i}(x^*{-}2\delta{-}\gamma)](1-\beta)v - \gamma \\ &\geq\ \alpha_{-i}(x^*)\beta v - \alpha_{-i}(x^*-2\delta)(1-\beta)v - \gamma \end{aligned}$$

The inequality follows from observing that $G_{-i}(x^*) - G_{-i}(x^*{-}\gamma) \geq \alpha_{-i}(x^*)$ and that the omitted terms are weakly positive. Then if $\gamma < \alpha_{-i}(x^*)\beta v - \alpha_{-i}(x^*-2\delta)(1-\beta)v$, player i profits from moving any mass in $[x^*-\delta-\gamma,\ x^*-\delta)$ up to $x^*-\delta$. Moreover, if $\alpha_{-i}(x^*)\beta > \alpha_{-i}(x^*-2\delta)(1-\beta)$, then such a γ exists, which in turn implies that there is a k that meets our requirements (in particular, any $k < \min\{\gamma,\ \lambda\}$ where γ and λ satisfy the bounds specified above). Since $\alpha_{-i}(x^*-2\delta)$ necessarily equals zero for $x^* \in (0,\ 2\delta)$, mass points in $(0,\ 2\delta)$ are not a best response. Now suppose that $\alpha_{-i}(x^*)\beta \leq \alpha_{-i}(x^*-2\delta)(1-\beta)$ where $x^* \geq 2\delta$. Following the above argument, $x^*-2\delta$ can only be sustained as a mass point in equilibrium if $\alpha_{-i}(x^*-2\delta)\beta \leq \alpha_{-i}(x^*-4\delta)(1-\beta)$. More generally, a mass point in equilibrium at $x^*-2q\delta$ requires that $\alpha_{-i}(x^*-2q\delta)\beta \leq \alpha_{-i}(x^*-2[q+1]\delta)(1-\beta)$ for $q \in \{0, \ldots, \lfloor x^*/2\delta \rfloor - 1\}$. However, since there are no mass points in $(0,\ 2\delta)$, then provided that x^* is not evenly divisible by 2δ, there cannot be a mass point at $x^*-2q\delta$. The final case to consider is a sequence of mass points at 0, 2δ, 4δ, etc. It suffices to show that a mass point at 2δ is not a best response; any successive mass points would then fail to hold in equilibrium. For $\alpha_{-i}(2\delta) > 0$ to be sustained in equilibrium, G_i must contain mass in a neighborhood immediately below δ. This, however, cannot be. Since $\alpha_{-i}(2\delta) > 0$ requires that $\alpha_{-i}(0) > 0$, player i strictly prefers a bid slightly above δ to a bid of $\delta - c$ where $c < \alpha_{-i}(0)(1-\beta)v$. Specifically, $u_i(\overline{\delta},\ G_{-i}) - u_i(\delta - c,\ G_{-i}) = [G_{-i}(2\delta) - G_{-i}(2\delta - c)]\beta v + \alpha_{-i}(0)(1-\beta)v - c > 0$ for $c < \alpha_{-i}(0)(1-\beta)v$.[21] Therefore, $k < \min\{c,\ \lambda\}$ with c and λ meeting their respective bounds satisfies our requirements, so a mass point at 2δ is not optimal. □

With mass points limited to zero, the next result stems from the indifference condition that must hold when players are randomizing between multiple bids. That is, $u_i(x,\ G_{-i}) = u_i(y,\ G_{-i})$ for $x, y \in \text{supp}(G_i)$ where $x > y$. Using Lemma 1 and Equation 2, we can restate this indifference condition as:

$$[G_{-i}(x+\delta) - G_{-i}(y+\delta)]\,\beta v + [G_{-i}(x-\delta) - G_{-i}(y-\delta)]\,(1-\beta)v = x - y \tag{14}$$

The added cost of the higher bid must either be compensated by tying mass in $[y+\delta,\ x+\delta]$ or beating mass in $[y-\delta,\ x-\delta]$. Notably, the absence of mass in either of these intervals pins down the necessary distribution over the other. This principle is formalized in the following lemma.

Lemma 2 *For $\delta \in (0,\ (1-\beta)v)$ and $\beta > 0$, let G_i and G_{-i} be equilibrium distributions for players i and $-i$.*

- **A.** *Let $b \geq 0$ satisfy $G_{-i}(b) - \alpha_{-i}(b) = G_{-i}(b-c) - \alpha_{-i}(b-c)$ for $c \in (0,\ \delta]$. If the subset $(\underline{a},\ \overline{a}] \subseteq (b+\delta-c,\ b+\delta]$ is in the support of G_i, then $(\underline{a}+\delta,\ \overline{a}+\delta]$ is in the support of G_{-i}. Moreover, the distribution over $(\underline{a}+\delta,\ \overline{a}+\delta]$ in G_{-i} is uniform at the rate of $1/(\beta v)$.*
- **B.** *Let $b > \delta$ satisfy $G_{-i}(b) = G_{-i}(b+c)$ for $c \in (0,\ \delta]$. If the subset $(\underline{a},\ \overline{a}] \subseteq (b-\delta,\ b-\delta+c]$ is in the support of G_i, then $(\underline{a}-\delta,\ \overline{a}-\delta]$ is in the support of G_{-i}. Additionally, the distribution over $(\underline{a}-\delta,\ \overline{a}-\delta]$ in G_{-i} is uniform at the rate of $1/[(1-\beta)v]$.*

Proof Part A. To show the first claim, suppose that the subset $(\underline{a}+\delta,\ \overline{a}+\delta]$ is not in the support of G_{-i}. Since G_{-i} also has no mass in $[b-c,\ b)$, player i strictly prefers a bid of $\underline{a}$ to any $k \in (\underline{a},\ \overline{a}]$. This is because the probability of winning or tying remains unchanged but the cost of bidding is higher (i.e. $u_i(\underline{a},\ G_{-i}) - u_i(k,\ G_{-i}) = k - \underline{a} > 0$). Hence, $(\underline{a},\ \overline{a}]$ cannot be in the support of G_i. For the second claim, suppose that $(\underline{a},\ \overline{a}]$ is in the support of G_i. Consequently, $(\underline{a}+\delta,\ \overline{a}+\delta]$ is then in the support of G_{-i}. Let $x, y \in (\underline{a},\ \overline{a}]$ where $x > y$. By payoff equivalence, $u_i(x,\ G_{-i}) = u_i(y,\ G_{-i})$, which then implies that $G_{-i}(x+\delta) - G_{-i}(y+\delta) = (1/\beta v)(x-y)$. Since this equation holds for any x and y, including values which are arbitrarily close, the result then follows.

Part B. If $(\underline{a}-\delta,\ \overline{a}-\delta]$ is not in the support of G_{-i}, then since G_{-i} has no mass in $(b,\ b+c]$, player i strictly prefers a bid of $\underline{a}$ to any $k \in (\underline{a},\ \overline{a}]$ (i.e. the probability of a win and a tie

[21] The notation $u_i(\overline{\delta},\ G_{-i})$ is defined on p. 11 as the limit of player i's expected utility as δ is approached from above.

remains the same, but k has a higher bidding cost than $\underline{a}$). So $(\underline{a},\ \overline{a}]$ is not in the support of G_i. For the next part, suppose that $(\underline{a},\ \overline{a}]$ is in the support of G_i. By payoff equivalence, we have $u_i(x,\ G_{-i}) = u_i(y,\ G_{-i})$ for any $x, y \in (\underline{a},\ \overline{a}]$ such that $x > y$. From this equality, $G_{-i}(x-\delta) - G_{-i}(y-\delta) - \alpha_{-i}(x-\delta) + \alpha_{-i}(y-\delta) = (x-y)/[(1-\beta)v]$. Since mass points may only occur at zero in equilibrium (see Lemma 1), and since this equation holds for any x and y which are arbitrarily close, the result then follows. □

Lemma 2 is particularly applicable at the upper and lower bound of a distribution. If $\overline{x}_i$ is the upper bound of G_i, then any mass in G_{-i} over $[\overline{x}_i - \delta,\ \overline{x}_i]$ must be balanced by mass in G_i, shifted down by δ in $[\overline{x}_i - 2\delta,\ \overline{x}_i - \delta]$, with a density rate of $1/[(1-\beta)v]$. Precisely δ below that, if there is more mass in G_{-i}, Equation 14 can again be used to identify the necessary density rate for mass in G_i over $[\overline{x}_i - 4\delta,\ \overline{x}_i - 3\delta]$. This process continues—iteratively moving down the distribution—and a similar process holds for moving up the distribution. However, the necessary density rates can only hold in both directions if $\beta = 1/2$. We therefore obtain the following result.

Lemma 3 *Let $\delta \in (0,\ (1-\beta)v\,)$ and $\beta > 0$. In any equilibrium, if $\beta \neq 1/2$, then all continuously distributed mass must be uniform at a rate of either $1/(\beta v)$ or $1/[(1-\beta)v]$.*

Proof Let $[\underline{s}_0,\ \overline{s}_0] \in \text{supp}(G_i)$ be given such that $G_{-i}(\underline{s}_0 - \delta) = G_{-i}(\overline{s}_0 - \delta)$ if $\overline{s}_0 > \delta$. By Lemma 2.A, $[\underline{s}_0 + \delta,\ \overline{s}_0 + \delta] \in \text{supp}(G_{-i})$ with uniformly distributed mass at the rate of $1/(\beta v)$. Likewise, by Lemma 2.B, if $G_i(\underline{s}_0 + 2\delta) = G_i(\overline{s}_0 + 2\delta)$, then the distribution over $[\underline{s}_0,\ \overline{s}_0]$ in G_i is uniform at the rate of $1/[(1-\beta)v]$. Suppose instead that $G_i(\underline{s}_0 + 2\delta) < G_i(\overline{s}_0 + 2\delta)$. Let $[\underline{s}_1,\ \overline{s}_1] \subseteq [\underline{s}_0 + 2\delta,\ \overline{s}_0 + 2\delta]$ such that $[\underline{s}_1,\ \overline{s}_1] \in \text{supp}(G_i)$ (by Lemma 1, such an interval must exist since mass points may only occur at zero). We first show that for $\beta < 1/2$ that $[\underline{s}_1,\ \overline{s}_1] \notin \text{supp}(G_i)$. For any $x, y \in [\underline{s}_1,\ \overline{s}_1]$, since $G_{-i}(x-\delta) - G_{-i}(y-\delta) = (x-y)/\beta v$, Equation 14 implies that:

$$G_{-i}(x+\delta) - G_{-i}(y+\delta) = (x-y)\left(\frac{2\beta - 1}{\beta^2 v}\right) \tag{15}$$

Since this holds for x and y which are arbitrarily close, the distribution over $[\underline{s}_1 + \delta,\ \overline{s}_1 + \delta]$ must be uniform at a rate of $(2\beta - 1)/(\beta^2 v)$. For $\beta < 1/2$, this contradicts the monotonicity of G_{-i}, and so $[\underline{s}_1,\ \overline{s}_1] \notin \text{supp}(G_i)$. This completes the proof for $\beta < 1/2$. For $\beta = 1/2$, there is simply no mass in $[\underline{s}_1 + \delta,\ \overline{s}_1 + \delta]$. We therefore turn to the case where $\beta > 1/2$.

Suppose for contradiction that $G_i(\underline{s}_1 + 2\delta) = G_i(\overline{s}_1 + 2\delta)$. By Lemma 2.B, mass over the interval $[\underline{s}_1,\ \overline{s}_1]$ in G_i must be uniform with a density rate of $1/[(1-\beta)v]$. Using Equation 14, the indifference condition $u_{-i}(x,\ G_i) = u_{-i}(y,\ G_i)$, with $x, y \in [\underline{s}_1 - \delta,\ \overline{s}_1 - \delta] \subseteq [\underline{s}_0 + \delta,\ \overline{s}_0 + \delta]$ and $x > y$ implies:[22]

$$G_i(x-\delta) - G_i(y-\delta) = (x-y)\left(\frac{1-2\beta}{(1-\beta)^2 v}\right)$$

This is a contradiction for $\beta > 1/2$, and it is reached if any subset of $[\underline{s}_1 + 2\delta,\ \overline{s}_1 + 2\delta]$ is not in the support of G_i. Thus, if $[\underline{s}_1 + \delta,\ \overline{s}_1 + \delta] \in \text{supp}(G_{-i})$, it must be that $[\underline{s}_1 + 2\delta,\ \overline{s}_1 + 2\delta] \in \text{supp}(G_i)$. This argument holds more generally. Letting $[\underline{s}_1 + (\ell-2)\delta,\ \overline{s}_1 + (\ell-2)\delta]$ and $[\underline{s}_1 + \ell\delta,\ \overline{s}_1 + \ell\delta] \in \text{supp}(G_{-i})$ for any $\ell \in \mathbb{N}$, and supposing that $[\underline{s}_1 + (\ell+1)\delta,\ \overline{s}_1 + (\ell+1)\delta] \notin \text{supp}(G_i)$, then the density rates over $[\underline{s}_1 + (\ell-1)\delta,\ \overline{s}_1 + (\ell-1)\delta]$ and $[\underline{s}_1 + (\ell-3)\delta,\ \overline{s}_1 + (\ell-3)\delta]$ in G_i are $1/[(1-\beta)v]$ and $(1-2\beta)/[(1-\beta)^2 v]$. A similar statement holds if $[\underline{c},\ \overline{c}] \notin \text{supp}(G_i)$ for any $[\underline{c},\ \overline{c}] \subseteq [\underline{s}_1 + (\ell+1)\delta,\ \overline{s}_1 + (\ell+1)\delta]$. Hence, by contradiction, $[\underline{s}_1 + (\ell+1)\delta,\ \overline{s}_1 + (\ell+1)\delta] \in \text{supp}(G_i)$. Mass over intervals that are 2δ apart

[22] Recall that Equation 14 is written in terms of player i's indifference: $u_i(x,\ G_{-i}) = u_i(y,\ G_{-i})$. So the corresponding version of Equation 14 for player $-i$ replaces each G_{-i} with G_i.

in G_{-i} requires that G_i has mass in an even higher interval. However, as we will show next, this in turn requires that G_{-i} has mass in yet a higher interval.

Again drawing on Equation 14, in order for player i to be indifferent between $x, y \in [\underline{s}_1 + (\ell - 1)\delta,\ \overline{s}_1 + (\ell - 1)\delta]$, where $x > y$, then:

$$G_{-i}(x+\delta) - G_{-i}(y+\delta) = \left(\frac{x-y}{\beta v}\right) - [G_{-i}(x-\delta) - G_{-i}(y-\delta)]\left(\frac{1-\beta}{\beta}\right)$$

For $\ell = 1$, the specific value of $G_{-i}(x+\delta) - G_{-i}(y+\delta)$ is given by Equation 15. This value then becomes $G_{-i}(x-\delta) - G_{-i}(y-\delta)$ for $\ell = 2$. Iterating, we obtain the following general form for $\ell \geq 1$:[23]

$$G_{-i}(x+\delta) - G_{-i}(y+\delta) = (x-y)\left(\frac{1}{\beta^{\ell+1} v}\right)\sum_{j=0}^{\ell}(-1)^{\ell-j}\beta^j(1-\beta)^{\ell-j} \qquad (16)$$

The positivity of Equation 16 for $\beta > 1/2$ can be seen by doing a pairwise summation of right-hand side terms (i.e. sum $j = \ell$ with $j = \ell - 1$; $j = \ell - 2$ with $j = \ell - 3$; etc.). Thus, we have:

$$\sum_{j=0}^{\ell}(-1)^{\ell-j}\beta^j(1-\beta)^{\ell-j} = (1-\beta)^{\ell}\,\mathcal{I}(\ell) + (2\beta - 1)\sum_{j=0}^{\lfloor(\ell-1)/2\rfloor}\beta^{\ell-1-2j}(1-\beta)^{2j} > 0$$

where $\mathcal{I}(\ell)$ is an indicator function equal to 1 if ℓ is even and 0 otherwise, and $\lfloor\cdot\rfloor$ is the floor function. Since Equation 16 is strictly positive for any $x, y \in [\underline{s}_1 + (\ell-1)\delta,\ \overline{s}_1 + (\ell-1)\delta]$, then $[\underline{s}_1 + \ell\delta,\ \overline{s}_1 + \ell\delta] \in \text{supp}(G_{-i})$.

The escalating supports of G_i and G_{-i} ultimately rise above v where bids are strictly dominated, contradicting the initial supposition that G_{-i} have mass over intervals that are 2δ apart (i.e. there is no pair of intervals $[\underline{s}_1 + (\ell-2)\delta,\ \overline{s}_1 + (\ell-2)\delta]$ and $[\underline{s}_1 + \ell\delta,\ \overline{s}_1 + \ell\delta]$ that are both in the support of G_{-i} for any $\ell \in \mathbb{N}$). In particular, $[\underline{s}_1 + \delta,\ \overline{s}_1 + \delta] \notin \text{supp}(G_{-i})$, so $[\underline{s}_1,\ \overline{s}_1] \notin \text{supp}(G_i)$. Therefore, $[\underline{s}_0,\ \overline{s}_0]$ has a density rate of $1/[(1-\beta)v]$ in G_i. □

The density rates of $1/(\beta v)$ and $1/[(1-\beta)v]$ have intuitive appeal since βv is the marginal value of tying relative to losing and $(1-\beta)v$ is the marginal value of winning relative to tying. In isolating these density rates, we also derive the following corollary result:

Corollary 1 *Let $\delta \in (0,\ (1-\beta)v)$, $\beta > 0$, and $\beta \neq 1/2$. For any $\overline{z} > \underline{z} \geq 0$ such that $\overline{z} - \underline{z} \leq \delta$, in equilibrium the interval $[\underline{z},\ \overline{z}]$ has a density rate of $1/[(1-\beta)v]$ in G_i if and only if $[\underline{z}+\delta,\ \overline{z}+\delta]$ has a density rate of $1/(\beta v)$ in G_{-i}. In which case, $G_{-i}(\underline{z}-\delta) = G_{-i}(\overline{z}-\delta)$, and $G_i(\underline{z}+2\delta) = G_i(\overline{z}+2\delta)$.*

B Proofs Specific to $\beta < 1/2$

Lemma 4 *Let $\delta \in (0,\ (1-\beta)v)$ and $\beta < 1/2$. In any equilibrium, any continuously distributed mass over $[0,\ \delta]$ must be connected, have a lower bound of zero and a density rate of $1/[(1-\beta)v]$. Similarly, if $p, q \in \text{supp}(G_{-i})$ such that $p < q$ and $G_{-i}(p) = G_{-i}(q)$, then any continuously distributed mass in G_i over $[p+\delta,\ q+\delta]$ must also be connected, have a lower bound of $p+\delta$, and have a density rate of $1/[(1-\beta)v]$.*

[23] This pattern is easier to see by writing $2\beta - 1$ as $\beta - (1-\beta)$ for $\ell = 1$; and then $\beta^2 - \beta(1-\beta) + (1-\beta)^2$ for $\ell = 2$, etc.

Proof For $z \in (0,\ \delta)$, suppose there exists $[z,\ z+b] \in \text{supp}(G_i)$ such that $G_i(z) = G_i(z-c)$, where $b > 0$ and $c \in (0,\ z]$. In equilibrium, $u_i(z,\ G_{-i}) \geq u_i(z-c,\ G_{-i})$, and so:

$$[G_{-i}(z+\delta) - G_{-i}(z+\delta-c)]\beta v \geq c$$

By Corollary 1, since $G_i(z) = G_i(z-c)$, then any mass in G_{-i} over $[z+\delta-c,\ z+\delta]$ must have a density rate of $1/[(1-\beta)v]$. So for some $s \in [0,\ c]$, we have:

$$[G_{-i}(z+\delta) - G_{-i}(z+\delta-c)]\beta v = s\beta/(1-\beta) \geq c$$

However, this cannot hold for $\beta < 1/2$. The same argument holds for $z \in [p+\delta,\ q+\delta]$ and $c \in (p+\delta,\ z]$, where p and q are defined as in the statement of the lemma. □

Lemma 5 *Let* $\delta \in (0,\ (1-\beta)v\,)$ *and* $\beta < 1/2$. *In equilibrium, there does not exist* $\underline{z} \geq 0$ *such that* $G_i(\underline{z}-\delta) = G_i(\underline{z})$ *and* $[\underline{z},\ \overline{z}]$ *has a density rate of* $1/[(1-\beta)v]$ *in* G_i *for* $\overline{z} > \underline{z}$.

Proof Suppose to the contrary that in equilibrium there exists a $\underline{z}$ such that $G_i(\underline{z}-\delta) = G_i(\underline{z})$ and $[\underline{z},\ \overline{z}]$ has a density rate of $1/[(1-\beta)v]$ in G_i. By Corollary 1, $[\underline{z}+\delta,\ \overline{z}+\delta]$ has a density rate of $1/(\beta v)$ in G_{-i}. In equilibrium, $u_{-i}(\underline{z}+\delta,\ G_i) \geq u_{-i}(\underline{z},\ G_i)$, which can only be satisfied if $[\underline{z}+\delta,\ \underline{z}+2\delta]$ has a density rate of $1/(\beta v)$ in G_i (since $G_i(\underline{z}-\delta) = G_i(\underline{z})$ and $\beta < 1/2$). So now $\underline{z}+2\delta \in \text{supp}(G_i)$. We reach a contradiction in that equilibrium requires $u_i(\underline{z}+2\delta,\ G_{-i}) \geq u_i(\overline{z}+2\delta,\ G_{-i})$, but $u_i(\underline{z}+2\delta,\ G_{-i}) - u_i(\overline{z}+2\delta,\ G_{-i}) \leq (\overline{z}-\underline{z}) - (\overline{z}-\underline{z})[(1-\beta)v/\beta v] < 0$. □

Since every $1/[(1-\beta)v]$ segment must be preceded by some other mass, equilibrium requires that at least one player must have a mass point at zero. Moreover, that same player must also have some mass in $[0,\ \delta]$ that is connected, with a density rate of $1/[(1-\beta)v]$, and a lower bound of zero (i.e. the properties in Lemma 4). If neither player's distribution began this way, Corollary 1, Lemma 4, and Lemma 5 would prohibit the placing of any continuously distributed mass in either player's distribution. Assuming that at least one player's distribution does indeed comply, these results dictate the pattern for placing any further mass. Labeling the distributions G_w and G_y, suppose that G_w has a mass point $\alpha_w(0) \in (0,\ 1)$ and a $1/[(1-\beta)v]$ segment in $[0,\ \delta]$ of length $w_1 > 0$. So by Corollary 1, G_y has a $1/(\beta v)$ segment of length w_1 spanning $[\delta,\ \delta+w_1]$. Drawing on Lemma 4, the gap in G_w following w_1 implies that any mass in G_y in the region above $\delta+w_1$ must be connected, with a density rate of $1/[(1-\beta)v]$, and a lower bound of $\delta+w_1$. If G_y does indeed have a $1/[(1-\beta)v]$ segment here, say of length y_2, then δ above that, G_w has a $1/(\beta v)$ segment of length y_2. Every $1/[(1-\beta)v]$ segment must follow in the immediate wake of a mass point at zero or a $1/(\beta v)$ segment, and the occurrence of $1/(\beta v)$ segments is wholly determined by $1/[(1-\beta)v]$ segments. Corollary 1 further limits the combined length of adjoining $1/(\beta v)$ and $1/[(1-\beta)v]$ segments to no more than δ. Any equilibrium must therefore be of the form depicted in Figure 8. Mass points $\alpha_w(0)$ and $\alpha_y(0) \in [0,1)$ are followed by alternating $1/[(1-\beta)v]$ and $1/(\beta v)$ segments. We again label the length of the successive $1/[(1-\beta)v]$ segments for player w as $w_1, w_2, \ldots, w_k \geq 0$, and likewise for player y as $y_1, y_2, \ldots, y_k \geq 0$. If indeed some $w_j = 0$ then $y_{j+1} = 0$, and like dominoes, $w_{j+2} = 0$, $y_{j+3} = 0$, etc. The principle again being that a $1/[(1-\beta)v]$ segment can only follow a $1/(\beta v)$ segment or a mass point at zero. Symmetric equilibria are therefore restricted to the case where all y_j and w_j are strictly positive (and, by implication, $\alpha_w(0),\ \alpha_y(0) > 0$). As Proposition 1 states, there is a unique equilibrium in which these are all positive and that equilibrium is symmetric.

Proposition 1 *Let* $\delta \in (0,\ (1-\beta)v\,)$ *and* $\beta < 1/2$. *For any* $k \in \mathbb{N}$, *there is a unique equilibrium satisfying the constraint that* $y_1,\ w_1,\ \ldots,\ y_k,\ w_k$ *are all strictly positive. Moreover, that equilibrium is symmetric* (*i.e.* $y_1 = w_1,\ \ldots,\ y_k = w_k$ *and* $\alpha_y(0) = \alpha_w(0)$).

Proof In equilibrium, all points within the support must have the same expected payoff. This property must particularly hold at each break in each player's support. Given the constraint that $y_1,\ w_1,\ \ldots,\ y_k,\ w_k$ are all strictly positive, each player's support has k breaks. For player w:

$$\begin{aligned}
&u_w(\overline{\delta},\ G_y) = u_w(w_1,\ G_y)\\
&u_w(2\delta + w_1,\ G_y) = u_w(\delta + y_1 + w_2,\ G_y)\\
&u_w(3\delta + y_1 + w_2,\ G_y) = u_w(2\delta + w_1 + y_2 + w_3,\ G_y)\\
&u_w(4\delta + w_1 + y_2 + w_3,\ G_y) = u_w(3\delta + y_1 + w_2 + y_3 + w_4,\ G_y)
\end{aligned}$$

In general, for an even integer q:

$$u_w\left(q\delta + \sum_{j\geq 1,\,odd}^{q-1} w_j + \sum_{j\geq 2,\,even}^{q-2} y_j,\ \ G_y\right) = u_w\left((q-1)\delta + \sum_{j\geq 1,\,odd}^{q-1} y_j + \sum_{j\geq 2,\,even}^{q} w_j,\ \ G_y\right)$$

$$u_w\left((q+1)\delta + \sum_{j\geq 1,\,odd}^{q-1} y_j + \sum_{j\geq 2,\,even}^{q-2} w_j,\ \ G_y\right) = u_w\left(q\delta + \sum_{j\geq 1,\,odd}^{q-1} w_j + \sum_{j\geq 2,\,even}^{q} y_j,\ \ G_y\right)$$

Corresponding equations for player y merely reverse the roles of all w_j and y_j. Across players, this system has a total of $2k$ equations with $2k+2$ unknowns (i.e. $\alpha_y(0),\ \alpha_w(0),\ y_1,\ w_1,\ \ldots,\ y_k,\ w_k$). We close the system by requiring the total mass in each distribution to sum to one:

$$\begin{aligned}
1 &= \alpha_w(0) + \sum_{j=1}^{k} \frac{w_j}{(1-\beta)v} + \frac{y_j}{\beta v}\\
1 &= \alpha_y(0) + \sum_{j=1}^{k} \frac{y_j}{(1-\beta)v} + \frac{w_j}{\beta v}
\end{aligned}$$

These mass constraints and the system of indifference conditions can be written in matrix form, where each row in the matrix represents an equation. The matrix for the case of $k = 4$ is as follows:

$$\begin{pmatrix}
-1 & 1 & 0 & \frac{1}{(1-\beta)v} & \frac{1}{\beta v} & \frac{1}{(1-\beta)v} & \frac{1}{\beta v} & \frac{1}{(1-\beta)v} & \frac{1}{\beta v} & \frac{1}{(1-\beta)v} & \frac{1}{\beta v}\\
-1 & 0 & 1 & \frac{1}{\beta v} & \frac{1}{(1-\beta)v} & \frac{1}{\beta v} & \frac{1}{(1-\beta)v} & \frac{1}{\beta v} & \frac{1}{(1-\beta)v} & \frac{1}{\beta v} & \frac{1}{(1-\beta)v}\\
-\delta & (1-\beta)v & 0 & 0 & 1 & \frac{\beta}{1-\beta} & 0 & 0 & 0 & 0 & 0\\
-\delta & 0 & (1-\beta)v & 1 & 0 & 0 & \frac{\beta}{1-\beta} & 0 & 0 & 0 & 0\\
-\delta & 0 & 0 & 1 & \frac{1-2\beta}{\beta} & 0 & 1 & \frac{\beta}{1-\beta} & 0 & 0 & 0\\
-\delta & 0 & 0 & \frac{1-2\beta}{\beta} & 1 & 1 & 0 & 0 & \frac{\beta}{1-\beta} & 0 & 0\\
-\delta & 0 & 0 & -1 & 1 & 1 & \frac{1-2\beta}{\beta} & 0 & 1 & \frac{\beta}{1-\beta} & 0\\
-\delta & 0 & 0 & 1 & -1 & \frac{1-2\beta}{\beta} & 1 & 1 & 0 & 0 & \frac{\beta}{1-\beta}\\
-\delta & 0 & 0 & 1 & -1 & -1 & 1 & 1 & \frac{1-2\beta}{\beta} & 0 & 1\\
-\delta & 0 & 0 & -1 & 1 & 1 & -1 & \frac{1-2\beta}{\beta} & 1 & 1 & 0
\end{pmatrix}
\begin{pmatrix}
1\\ \alpha_y(0)\\ \alpha_w(0)\\ y_1\\ w_1\\ y_2\\ w_2\\ y_3\\ w_3\\ y_4\\ w_4
\end{pmatrix} = \mathbf{0}$$

The top two rows are the mass constraints for players y and w. The third and fourth rows correspond to $u_w(\overline{\delta},\ G_y) = u_w(w_1,\ G_y)$ and $u_y(\overline{\delta},\ G_w) = u_y(y_1,\ G_w)$; the fifth and sixth to $u_w(2\delta + w_1,\ G_y) = u_w(\delta + y_1 + w_2,\ G_y)$ and $u_y(2\delta + y_1,\ G_w) = u_y(\delta + w_1 + y_2,\ G_w)$. Each successive pair of rows correspond to the indifference conditions over the next jump in each player's support. The above matrix with $k = 4$ is also useful for visualizing the general form for an arbitrary k. We denote the system for a given k by $\boldsymbol{A_k x = 0}$. The matrices $\boldsymbol{A_1}$, $\boldsymbol{A_2}$, and $\boldsymbol{A_3}$ are all partitions of the above matrix $\boldsymbol{A_4}$. For $\boldsymbol{A_1}$, it is the partition formed by the first four rows and five columns of $\boldsymbol{A_4}$. The matrix $\boldsymbol{A_2}$ comprises the first six rows and seven columns, and the first eight rows and nine columns form the matrix $\boldsymbol{A_3}$. In general, $\boldsymbol{A_k}$ has $r_k = 2k+2$ rows and $c_k = 2k+3$ columns. The matrix $\boldsymbol{A_{k-1}}$ constitutes the first $r_k - 2$ rows and $c_k - 2$ columns of $\boldsymbol{A_k}$. Elements in the last two rows and columns of $\boldsymbol{A_k}$ fit a clearly defined pattern. For $k \geq 2$, the first three elements of rows $r_k - 1$ and r_k are $\{\delta, 0, 0\}$. The fourth and fifth elements are shown in the matrix for $k \leq 4$. For $k \geq 4$, the fourth and fifth element of row $r_k - 1$ is the fourth and fifth element of row $r_k - 2$; and for row r_k it is the fourth and fifth elements of row $r_k - 3$. Elements six through c_k of rows $r_k - 1$ and r_k are the same as elements four through $c_k - 2$ of rows $r_k - 3$ and $r_k - 2$. Besides rows $r_k - 1$ and r_k, the only nonzero elements of columns $c_k - 1$ and c_k are in rows 1, 2, $r_k - 3$, and $r_k - 2$. The last two elements of rows 1 and 2 follow the established pattern for the mass constraints: $\{1/[(1-\beta)v],\ 1/(\beta v)\}$ and $\{1/(\beta v),\ 1/[(1-\beta)v]\}$. For rows $r_k - 3$ and $r_k - 2$, the last two elements are $\{\beta/(1-\beta),\ 0\}$ and $\{0,\ \beta/(1-\beta)\}$.

We obtain our uniqueness result by showing that the r_k rows of $\boldsymbol{A_k}$ are linearly independent for any $k \in \mathbb{N}$. The r_k rows are linearly independent if and only if the system $\boldsymbol{A_k x = 0}$ has a unique solution. We proceed by induction. Table 2 shows the unique solution to the system when $k \in \{1, 2, 3\}$, and it is also relevant that the unique solution is symmetric in each case. Now suppose that for $k \geq 3$ that the r_k rows of $\boldsymbol{A_k}$ are linearly independent. In the matrix $\boldsymbol{A_{k+1}}$, rows 1 through $r_{k+1} - 4$ are still linearly independent. This follows because the last two columns of rows 3 through $r_{k+1} - 4$ only contain zeros, and rows 1 and 2 are always linearly independent of each other because of their second and third elements. Since rows 1, 2, $r_{k+1} - 3$, and r_{k+1} are the only rows with nonzero elements in column $c_{k+1} - 1$, and since rows 1, 2, $r_{k+1} - 2$, and $r_{k+1} - 1$ are the only rows with nonzero elements in column c_k, then if each of these groups of four rows are linearly independent, then all r_{k+1} rows are linearly independent. For the first group (rows 1, 2, $r_{k+1} - 3$, and r_{k+1}), linear independence can be seen by looking at columns 2, 3, $c_{k+1} - 2$, and $c_{k+1} - 1$:

$$\begin{pmatrix} 1 & 0 & \frac{1}{\beta v} & \frac{1}{(1-\beta)v} \\ 0 & 1 & \frac{1}{(1-\beta)v} & \frac{1}{\beta v} \\ 0 & 0 & 1 & \frac{\beta}{1-\beta} \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

Linear independence for the second group (rows 1, 2, $r_{k+1} - 2$, and $r_{k+1} - 1$) can be seen from columns 2, 3, $c_{k+1} - 3$, and c_{k+1}:

$$\begin{pmatrix} 1 & 0 & \frac{1}{(1-\beta)v} & \frac{1}{\beta v} \\ 0 & 1 & \frac{1}{\beta v} & \frac{1}{(1-\beta)v} \\ 0 & 0 & 1 & \frac{\beta}{1-\beta} \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

Thus the r_{k+1} rows of $\boldsymbol{A_{k+1}}$ are linearly independent, so $\boldsymbol{A_{k+1} x = 0}$ has a unique solution. We next demonstrate that the unique solution is symmetric. That is, $\alpha_y(0) = \alpha_w(0)$ and $y_1 = w_1, \ldots, y_k = w_k$. As we stated earlier, for $k \in \{1, 2, 3\}$, the unique solution to $\boldsymbol{A_k x = 0}$ is indeed

Table 2: Unique solution to the system $\boldsymbol{A_k x}=\mathbf{0}$ for $k \in \{1,\ 2,\ 3\}$.

$\boldsymbol{k=1}$	$\alpha_y(0)=\alpha_w(0)$	$[\delta-(1-\beta)\beta v]\ /\ [(1-\beta)^2 v]$
	$y_1=w_1$	$[(1-\beta)\beta v-\beta\delta]\ /\ (1-\beta)$
$\boldsymbol{k=2}$	$\alpha_y(0)=\alpha_w(0)$	$(1-2\beta)\delta\ /\ [(1-\beta)^2 v]$
	$y_1=w_1$	$\left[(1-2\beta^2)\beta\delta-(1-\beta)^2\beta^2 v\right]\ /\ [(1-\beta)(1-2\beta)]$
	$y_2=w_2$	$([1-\beta]v-2\delta)(1-\beta)\beta\ /\ (1-2\beta)$
$\boldsymbol{k=3}$	$\alpha_y(0)=\alpha_w(0)$	$\dfrac{\delta-(7\beta^2-7\beta+6)(1-\beta)\beta\delta+(1-\beta)^3\beta^3 v}{(1-\beta)^3(1-3\beta+2\beta^2+\beta^3)v}$
	$y_1=w_1$	$(1-2\beta)\beta\delta\ /\ (1-\beta)^2$
	$y_2=w_2$	$\left[([1-3\beta]\beta+1)\beta\delta-(1-\beta)^2\beta v\right]\ /\ [1-3\beta+2\beta^2+\beta^3]$
	$y_3=w_3$	$\left[([6-2\beta]\beta-3)\beta\delta+(1-\beta)^3\beta v\right]\ /\ [1-3\beta+2\beta^2+\beta^3]$

symmetric (see Table 2). With symmetry, the system $\boldsymbol{A_k x}=\mathbf{0}$ collapses to the following system, which we call $\boldsymbol{B_k x}=\mathbf{0}$:

$$\begin{pmatrix} -1 & 1 & \frac{1}{(1-\beta)\beta v} & \frac{1}{(1-\beta)\beta v} & \frac{1}{(1-\beta)\beta v} & \frac{1}{(1-\beta)\beta v} & \cdots & \frac{1}{(1-\beta)\beta v} \\ -\delta & (1-\beta)v & 1 & \frac{\beta}{1-\beta} & 0 & 0 & \cdots & 0 \\ -\delta & 0 & \frac{1-\beta}{\beta} & 1 & \frac{\beta}{1-\beta} & 0 & \cdots & 0 \\ -\delta & 0 & 0 & \frac{1-\beta}{\beta} & 1 & \frac{\beta}{1-\beta} & & 0 \\ \vdots & \vdots & \vdots & & \ddots & & \ddots & \vdots \\ -\delta & 0 & 0 & 0 & 0 & \frac{1-\beta}{\beta} & 1 & \frac{\beta}{1-\beta} \\ -\delta & 0 & 0 & 0 & 0 & 0 & \frac{1-\beta}{\beta} & 1 \end{pmatrix} \begin{pmatrix} 1 \\ \alpha_y(0) \\ y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_k \end{pmatrix} = \mathbf{0}$$

By construction, if $\boldsymbol{B_k x}=\mathbf{0}$ has a unique solution, then it must coincide with the unique solution of $\boldsymbol{A_k x}=\mathbf{0}$. The matrix $\boldsymbol{B_k}$ has $k+1$ rows and $k+2$ columns, and $\boldsymbol{B_{k-1}}$ makes up the first k rows and $k+1$ columns of $\boldsymbol{B_k}$. The last column only has three nonzero elements: $1/[(1-\beta)\beta v]$ in the first row, $\beta/(1-\beta)$ in the second to last row, and 1 in the last row. The only other nonzero elements of the last row are the first and second to last elements: $-\delta$ and $(1-\beta)/\beta$. Since the linear independence of $\boldsymbol{B_{k-1}}$ guarantees that the first k rows of $\boldsymbol{B_k}$ are still linearly independent, we just need to check that the last row is also linearly independent. Since the last row is only one of three rows with a nonzero element in the last column (the other two rows being the first and second to last), it suffices to check that these three rows are linearly independent. This can be seen from the above matrix. Therefore, $\boldsymbol{B_k}$ has a unique solution. Sufficient conditions for equilibria when $\delta \in (0,\ (1-\beta)v\,)$ and $\beta < 1/2$ are easily checked and roughly sketched in Sections 3 and 4.1.

C Proofs Specific to $\beta > 1/2$

Lemma 6 *Let $\delta \in (0,\ (1-\beta)v\,)$ and $\beta > 1/2$. In any equilibrium, the subset of G_i that has a density rate of $1/[(1-\beta)v]$ in $[z,\ z+2\delta]$ for $z \geq 0$ is connected.*

Proof Suppose to the contrary that for some $z \geq 0$ that the subset of G_i with a density rate of $1/[(1-\beta)v]$ in $[z,\ z+2\delta]$ is disconnected. That is, G_i contains at least two intervals, $[a_1,\ b_1]$,

$[a_2,\, b_2] \subset [z,\ z+2\delta]$ where $b_1 < a_2$, with density rates of $1/[(1-\beta)v]$; furthermore, the density rates in a neighborhood immediately above b_1 and immediately below a_2 differ from $1/[(1-\beta)v]$. By Corollary 1, the intervals $[a_1+\delta,\, b_1+\delta]$ and $[a_2+\delta,\, b_2+\delta]$ have a density rate of $1/(\beta v)$ in G_{-i}. So in equilibrium, $u_{-i}(b_1+\delta,\ G_i) = u_{-i}(a_2+\delta,\ G_i)$:

$$[G_i(a_2+2\delta) - G_i(b_1+2\delta)]\,\beta v + [G_i(a_2) - G_i(b_1)]\,(1-\beta)v = a_2 - b_1$$

This can be rewritten using Lemma 3 as:

$$\left[\frac{r}{(1-\beta)v} + \frac{s}{\beta v}\right]\beta v + \left[\frac{m}{(1-\beta)v} + \frac{n}{\beta v}\right](1-\beta)v = \\ r\left(\frac{\beta}{1-\beta}\right) + s + m + n\left(\frac{1-\beta}{\beta}\right) = a_2 - b_1 \tag{17}$$

Here, r and s denote the respective lengths of the support of G_i in $[b_1+2\delta,\ a_2+2\delta]$ with density rates of $1/[(1-\beta)v]$ and $1/(\beta v)$; m and n and defined similarly for $[b_1,\ a_2]$. Suppose for contradiction that $r = 0$. Using Corollary 1, $[b_1+\delta,\, a_2+\delta]$ in G_{-i} must contain a portion of length s with density rate $1/[(1-\beta)v]$, a portion of length m with density rate of $1/(\beta v)$, and a portion of length n with no mass. Thus,

$$s + m + n \leq a_2 - b_1 \tag{18}$$

With $r = 0$ and $\beta > 1/2$, Equations 17 and 18 can only be jointly satisfied if $n = 0$ and if $s+m = a_2 - b_1$. Our next step is to show that if an interval $[\underline{c},\, \overline{c}]$ has a density rate of $1/[(1-\beta)v]$ in G_i (where the density rates differ in neighborhoods immediately above $\overline{c}$ and immediately below $\underline{c}$), then G_i has no mass in the interval $(\,\overline{c} - (\overline{c}-\underline{c})\beta/(1-\beta),\ \overline{c}\,]$. This follows from observing that $u_{-i}(\overline{c}-\delta) > u_{-i}(x-\delta)$ for $x \in (\,\overline{c} - (\overline{c}-\underline{c})\beta/(1-\beta),\ \overline{c}\,]$ when $\beta > 1/2$. Specifically,

$$[G_i(\overline{c}) - G_i(x)]\,\beta v + [G_i(\overline{c}-2\delta) - G_i(x-2\delta)]\,(1-\beta)v - \overline{c} + x \\ \geq (\overline{c}-\underline{c})\left(\frac{\beta}{1-\beta}\right) - \overline{c} + x > 0$$

Thus, for an interval of length ℓ, at most $[(1-\beta)/\beta]\ell$ of the interval may have a density rate of $1/[(1-\beta)v]$; in which case, the bottom $[(2\beta-1)/\beta]\ell$ of the interval contains no mass. Hence, $m < (a_2-b_1)[(1-\beta)/\beta]$, with a strict inequality since there is a space with no mass below a_2. Likewise, $s < (a_2-b_1-m)[(1-\beta)/\beta]$, and so for $\beta > 1/2$:

$$m + s < \left(\frac{1-\beta}{\beta}\right)\left(\frac{3\beta-1}{\beta}\right)(a_2-b_1) < a_2 - b_1$$

Therefore, it must be that $r > 0$. So there exists at least one interval $[a_3,\, b_3] \subset [b_1+2\delta,\ a_2+2\delta]$ in G_i with a density rate of $1/[(1-\beta)v]$ (with different density rates immediately above b_3 and below a_3). Since $[a_2,\, b_2]$ and $[a_3,\, b_3]$ are within a 2δ interval (i.e. $b_3 - a_2 \leq 2\delta$), then the same argument we just used for $[a_1,\, b_1]$ and $[a_2,\, b_2]$ implies that there is an interval $[a_4,\, b_4] \subset [b_2+2\delta,\ a_3+2\delta]$ in G_i with a density rate of $1/[(1-\beta)v]$. In general, $[a_k,\, b_k] \subset [b_{k-2}+2\delta,\ a_{k-1}+2\delta]$ is in G_i and has a density rate of $1/[(1-\beta)v]$, where $k \in \{3, 4, \ldots\}$. The sequence of a_k is then unbounded ($a_{k+2} - a_k > 2\delta$), rising to bids that are strictly dominated. This contradicts our original supposition that the subset of G_i with a density rate of $1/[(1-\beta)v]$ in $[z,\ z+2\delta]$ is disconnected. □

We can now largely piece together what an equilibrium strategy must look like. Lemma 7 all but characterizes the nature of equilibria when $\beta > 1/2$. It identifies that if either player's support has a gap of at least δ, which is immediately followed by a density rate of $1/[(1-\beta)v]$, then the remainder of each player's distribution is made up of connected intervals of length-2δ, with a density rate of $1/[(1-\beta)v]$ over a lower portion and $1/(\beta v)$ over the remainder. The existence of such a gap in a player's support is clarified in Lemma 8, which also delineates how mass must be distributed below the gap. Symmetry is then finally established in Lemma 9.

Lemma 7 *Let $\delta \in (0,\ (1-\beta)v\,)$ and $\beta > 1/2$. Let $\phi_1 \geq 0$ be the lowest bid such that $G_i(\phi_1 - \delta) = G_i(\phi_1)$ and $[\phi_1,\ \phi_1 + c]$ has a density rate of $1/[(1-\beta)v]$ in G_i for $c \in (0,\ \delta]$. If ϕ_1 exists, then in equilibrium, any portion of the support weakly greater than $\phi_1 + c - \delta$ in G_i and G_{-i} must have a distribution of the following form. For $j \in \{1, \ldots, k\}$, $k \in \mathbb{N}$, let ϕ_j and φ_j satisfy $\phi_{j+1} \geq \phi_j + \varphi_j + 3\delta$, $\varphi_{j+1} \in [\max\{\phi_j + 3\delta - \phi_{j+1},\ -\delta\},\ \delta]$, and $\varphi_1 \in [c - \delta,\ \delta]$. Then G_i has a density rate of $1/[(1-\beta)v]$ over $[\phi_j,\ \phi_j + \varphi_j + \delta]$ and $1/(\beta v)$ over $[\phi_j + \varphi_j + \delta,\ \phi_j + 2\delta]$; and G_{-i} has a density rate of $1/[(1-\beta)v]$ over $[\phi_j + \varphi_j,\ \phi_j + \delta]$ and $1/(\beta v)$ over $[\phi_j + \delta,\ \phi_j + \varphi_j + 2\delta]$. All other bids weakly greater than $\phi_1 + c - \delta$ are not in the support of G_i or G_{-i}.*

Proof Since $[\phi_1,\ \phi_1 + c]$ has a density rate of $1/[(1-\beta)v]$ in G_i, then by Corollary 1, $G_{-i}(\phi_1 - \delta) = G_{-i}(\phi_1 + c - \delta)$ and $[\phi_1 + \delta,\ \phi_1 + c + \delta]$ has a density rate of $1/(\beta v)$ in G_{-i}. Since $\phi_1 + \delta \in \text{supp}(G_{-i})$, then in equilibrium, $u_{-i}(\phi_1 + \delta,\ G_i) \geq u_{-i}(\phi_1,\ G_i)$. Using the density rates permitted by Lemma 3 and recalling that $G_i(\phi_1 - \delta) = G_i(\phi_1)$, we have:

$$\begin{aligned} u_{-i}(\phi_1 + \delta,\ G_i) - u_{-i}(\phi_1,\ G_i) &= [G_i(\phi_1 + 2\delta) - G_i(\phi_1 + \delta)]\,\beta v - \delta \\ &= \left[\frac{\tau}{\beta v} + \frac{\mu}{(1-\beta)v}\right]\beta v - \delta \geq 0 \end{aligned}$$

Here, τ and μ are the lengths of the support over $[\phi_1 + \delta,\ \phi_1 + 2\delta]$ in G_i that have density rates of $1/(\beta v)$ and $1/[(1-\beta)v]$, respectively. So $\tau + \mu \leq \delta$. Suppose $\mu = 0$. Then $\tau = \delta$ (or rather $[\phi_1 + \delta,\ \phi_1 + 2\delta]$ has a density rate of $1/(\beta v)$ in G_i), so Corollary 1 implies that $G_{-i}(\phi_1 + 2\delta) = G_{-i}(\phi_1 + 3\delta)$ and $[\phi_1,\ \phi_1 + \delta]$ has a density rate of $1/[(1-\beta)v]$ in G_{-i}. Since $\phi_1 + c$ and $\phi_1 + \delta \in \text{supp}(G_i)$, $u_i(\phi_1 + c,\ G_{-i}) = u_i(\phi_1 + \delta,\ G_{-i})$, so:

$$[G_{-i}(\phi_1 + 2\delta) - G_{-i}(\phi_1 + c + \delta)]\beta v + [G_{-i}(\phi_1) - G_{-i}(\phi_1 + c - \delta)](1-\beta)v = \delta - c$$

By Lemma 6, any mass in G_{-i} over $[\phi_1 + c + \delta,\ \phi_1 + 2\delta]$ must have a density rate of $1/(\beta v)$ (since any mass with a density rate of $1/[(1-\beta)v]$ would be disconnected from $[\phi_1,\ \phi_1 + \delta]$). The same is true for mass in G_i over $[\phi_1 + c - 2\delta,\ \phi_1 - \delta]$, so any mass in G_{-i} over $[\phi_1 + c - \delta,\ \phi_1]$ necessarily has a density rate of $1/[(1-\beta)v]$. Thus, for $q, r \in [0,\ \delta - c]$, we have:

$$\left[\frac{q}{\beta v}\right]\beta v + \left[\frac{r}{(1-\beta)v}\right](1-\beta)v = \delta - c$$

Lemma 6 is satisfied if G_{-i} has a density rate of $1/[(1-\beta)v]$ over $[\phi_1 - r,\ \phi_1 + \delta]$ and $1/(\beta v)$ over $[\phi_1 + \delta,\ \phi_1 + \delta + c + q]$. Substituting $q = \delta - c - r$, the second interval becomes $[\phi_1 + \delta,\ \phi_1 - r + 2\delta]$. Using Corollary 1, G_i has a density rate of $1/[(1-\beta)v]$ over $[\phi_1,\ \phi_1 - r + \delta]$ and $1/(\beta v)$ over $[\phi_1 - r + \delta,\ \phi_1 + 2\delta]$.

Now suppose instead that $\mu > 0$. Lemma 6 implies that the interval $[\phi_1,\ \phi_1 + \mu + \delta]$ has a density rate of $1/[(1-\beta)v]$ in G_i. Then, by Corollary 1, $[\phi_1 + \delta,\ \phi_1 + \mu + 2\delta]$ has a density rate of $1/(\beta v)$ in G_{-i}. Since $\phi_1 + \delta \in \text{supp}(G_{-i})$, $u_{-i}(\phi_1 + \delta,\ G_i) \geq u_{-i}(\phi_1 + \mu,\ G_i)$, or rather:

$$[G_i(\phi_1 + 2\delta) - G_i(\phi_1 + \delta + \mu)]\,\beta v \geq \delta - \mu$$

A density rate of $1/[(1-\beta)v]$ is not permitted in G_i over $[\phi_1 + \delta + \mu,\ \phi_1 + 2\delta]$, so the entire interval must instead have a density rate of $1/(\beta v)$. In which case, G_{-i} has a density rate of $1/[(1-\beta)v]$ over $[\phi_1 + \mu,\ \phi_1 + \delta]$. Combining the results for the cases where $\mu = 0$ and $\mu > 0$, G_i has a density rate of $1/[(1-\beta)v]$ over $[\phi_1,\ \phi_1+\varphi_1+\delta]$ and $1/(\beta v)$ over $[\phi_1+\varphi_1+\delta,\ \phi_1+2\delta]$; and G_{-i} has a density rate of $1/[(1-\beta)v]$ over $[\phi_1+\varphi_1,\ \phi_1+\delta]$ and $1/(\beta v)$ over $[\phi_1+\delta,\ \phi_1+\varphi_1+2\delta]$, where $\varphi_1 \in [c-\delta,\ \delta]$.

We next must show that this pattern holds for the remainder of the distribution. To do so, we first note that by Corollary 1, $G_i(\phi_1+2\delta) = G_i(\phi_1+\varphi_1+3\delta)$ and $G_{-i}(\phi_1+\varphi_1+2\delta) = G_{-i}(\phi_1+3\delta)$. Either $\phi_1+\varphi_1+3\delta-(\phi_1+2\delta) \geq \delta$ or $\phi_1+3\delta-(\phi_1+\varphi_1+2\delta) \geq \delta$. Let $\widehat{x}$ be the next element in either support. That is, $\widehat{x} = \min\{x \in \text{supp}(G_i)\bigcup\text{supp}(G_{-i}) \mid x \geq \min\{\phi_1+3\delta,\ \phi_1+\varphi_1+3\delta\}\}$. If $\widehat{x}$ does not exist, then we are done. Otherwise, $[\widehat{x},\ \widehat{x}+e]$ has a density rate of $1/[(1-\beta)v]$ in either G_i or G_{-i} for $e \in (0,\ \delta]$. If $\widehat{x} \geq \max\{\phi_1+3\delta,\ \phi_1+\varphi_1+3\delta\}$, then the conditions of the lemma are again satisfied, so the pattern continues (this includes the case where $\varphi = 0$). For the case where $\min\{\phi_1+3\delta,\ \phi_1+\varphi_1+3\delta\} \leq \widehat{x} < \max\{\phi_1+3\delta,\ \phi_1+\varphi_1+3\delta\}$, suppose without loss of generality that $\varphi_1 > 0$. Then $\widehat{x} \in \text{supp}(G_{-i})$. Since $u_{-i}(\widehat{x},\ G_i) = u_{-i}(\phi_1+\varphi_1+2\delta,\ G_i)$, then:

$$(\delta-\varphi_1)\left(\frac{1-\beta}{\beta}\right) + [G_i(\widehat{x}+\delta)-G_i(\widehat{x}-\delta)]\beta v \ = \ \widehat{x}-(\phi_1+\varphi_1+2\delta) \ \geq \ \delta-\varphi_1$$

Since $\beta > 1/2$, the equation can only be satisfied if $G_i(\widehat{x}+\delta)-G_i(\widehat{x}-\delta) > 0$. This mass must be in $[\phi_1+\varphi_1+3\delta,\ \widehat{x}+\delta]$ with a density rate of $1/[(1-\beta)v]$, and so the conditions of the lemma are satisfied yet again.[24] The pattern thus continues as long as there is any remaining mass to place in G_i or G_{-i}. □

Lemma 8 *Let* $\delta \in (0,\ (1-\beta)v\ )$ *and* $\beta > 1/2$. *If* $\max\{G_i(2\delta),\ G_{-i}(2\delta)\} < 1$ *in equilibrium, then* ϕ_1 *exists. Moreover, any mass in either player's distribution below* $\min\{\phi_1,\ \phi_1+\varphi_1\}$ *is limited to a mass point at zero, a lower interval at a density rate of* $1/[(1-\beta)v]$, *and an upper interval at a density rate of* $1/\beta v$. *The mass points and lower intervals may be distributed according to one of three forms, with the upper intervals following Corollary* 1*:*

A. $\alpha_i(0) > 0$ *and* $\alpha_{-i}(0) > 0$. *Lower intervals begin at zero and have length* $\psi_k \in [0,\ \delta)$ *for* $k \in \{i,\ -i\}$. *This is also the only possible equilibrium form when* ϕ_1 *does not exist.*
B. $\alpha_i(0) > 0$ *and* $\alpha_{-i}(0) \geq 0$. *Player i's lower interval begins at zero and has length* $\psi_i \in [0,\ 2\delta]$. *Player* $-i$ *has no lower interval.*
C. $\alpha_i(0) > 0$ *and* $\alpha_{-i}(0) = 0$. *Player i's lower interval begins at zero and has length* $\psi_i \in [\delta,\ 2\delta)$. *Player* $-i$*'s lower interval begins at* $\psi_i-\delta$ *and has length* $\psi_{-i} \in (0,\ 2\delta-\psi_i)$.

Proof The existence of ϕ_1 when $\max\{G_i(2\delta),\ G_{-i}(2\delta)\} < 1$ is trivial if $\alpha_i(0) = \alpha_{-i}(0) = 0$. It is likewise trivial if $G_i(\delta) = G_i(0)$ and $G_{-i}(\delta) = G_{-i}(0)$. We must therefore consider the cases where at least one player has a mass point at zero and at least one player has strictly positive mass in $(0,\ \delta]$. By Lemma 3 and Corollary 1, any mass in $(0,\ \delta]$ must have a density rate of $1/[(1-\beta)v]$. For $k \in \{i,\ -i\}$, let $[\mu_k,\ M_k]$ denote the lowest interval in G_k with a density rate of $1/[(1-\beta)v]$. Without loss of generality with respect to players, there are four cases.

Case 1: $\alpha_i(0) > 0$, $\alpha_{-i}(0) > 0$, $\mu_i < \delta$, $\mu_{-i} < \delta$. Since $\alpha_i(0),\ \alpha_{-i}(0) > 0$, then $M_i,\ M_{-i} < \delta$. Otherwise, elements of the support in $[\mu_k,\ \delta]$ would have a strictly lower expected payoff than elements in $(\delta,\ M_k]$. If $\mu_i > 0$, then to maintain payoff equivalence with zero, the other player's support must have mass over $[\delta,\ \mu_i+\delta]$ which can be tied. But by Lemma 6 and Corollary 1, neither player has mass in $[\delta,\ \mu_k+\delta]$. So $\mu_i = \mu_{-i} = 0$. They also imply that player i has no mass in $[M_i+\delta,\ M_{-i}+2\delta]$ and player $-i$ has no mass in $[M_{-i}+\delta,\ M_i+2\delta]$. The length of at

[24] Even if $\varphi_1 = \delta$, it is always the case that $\widehat{x}-(\phi_1+\varphi_1+2\delta) > 0$. This follows because, as we saw in the proof of Lemma 6, when $\beta > 1/2$, an interval with a density rate of $1/[(1-\beta)v]$ cannot immediately follow an interval with a density rate of $1/(\beta v)$.

least one of these intervals is weakly greater than δ. So if $G_i(\delta + M_{-i}) < 1$ or $G_{-i}(\delta + M_i) < 1$, then ϕ_1 exists, and any mass below $\min\{\phi_1,\ \phi_1 + \varphi_1\}$ is distributed according to the first form in Lemma 8.

Case 2: $\alpha_i(0) > 0$, $\alpha_{-i}(0) \geq 0$, $\mu_i < \delta$, $\mu_{-i} \geq \delta$. With $\mu_{-i} \geq \delta$, there are two subcases: $[\mu_{-i},\ M_{-i}] \subseteq [\delta,\ \mu_i + \delta]$ and $\mu_{-i} \geq M_i + \delta$. (Corollary 1 prohibits anything else.) Suppose first that $[\mu_{-i},\ M_{-i}] \subseteq [\delta,\ \mu_i + \delta]$. Therefore, $\mu_i > 0$. For player $-i$, $\mu_{-i} > \delta$ is strictly dominated by $\mu_{-i} = \delta$ since $G_i(0) = G_i(\mu_{-i} - \delta)$ and $G_i(2\delta) = G_i(\mu_{-i} + \delta)$ (see Lemma 6 and Corollary 1). In equilibrium, $u_i(0,\ G_{-i}) = u_i(\mu_i,\ G_{-i})$, so:

$$[G_{-i}(\mu_i + \delta) - G_{-i}(\delta)]\,\beta v = \mu_i \quad \Rightarrow \quad \left(\frac{M_{-i} - \mu_{-i}}{(1-\beta)v}\right)\beta v = \mu_i$$

Since $\beta > 1/2$, the last equality implies that $M_{-i} - \mu_{-i} < \mu_i$. However, with $\mu_{-i} = \delta$ and $M_{-i} < \mu_i + \delta$, player i could profitably deviate by shifting μ_i down to $M_{-i} - \delta$. So this is not an equilibrium. For the second subcase, suppose now that $\mu_{-i} \geq M_i + \delta$. By Lemma 6 and Corollary 1, $G_i(M_i + 2\delta) = G_i(M_i)$, so ϕ_1 exists if $G_i(M_i) < 1$. Also, $\mu_{-i} \geq M_i + \delta$ means that $G_{-i}(\delta) = G_{-i}(\mu_i + \delta)$, so we can only have $u_i(0,\ G_{-i}) = u_i(\mu_i,\ G_{-i})$ if $\mu_i = 0$. If $\alpha_{-i}(0) > 0$, then $M_i < \delta$; otherwise, Corollary 1 sets the upper bound of M_i at 2δ. Thus any mass below $\min\{\phi_1,\ \phi_1 + \varphi_1\}$ is distributed according to the second form in Lemma 8. Although it is possible for $M_i + \delta > 2\delta$ when $\alpha_{-i}(0) = 0$, we still have $G_{-i}(M_i + \delta) < 1$. A contradiction arises if $G_{-i}(M_i + \delta) = 1$, since $G_i(M_i)$ would also equal one by Corollary 1. But both cannot equal one since G_i has more mass:

$$\frac{M_i - \mu_i}{(1-\beta)v} + \alpha_i(0) > \frac{M_i - \mu_i}{\beta v}$$

Case 3: $\alpha_i(0) > 0$, $\alpha_{-i}(0) = 0$, $\mu_i < \delta$, $\mu_{-i} < \delta$. By Lemma 6 and Corollary 1, player i has no mass in $[M_{-i} + \delta,\ M_i + 2\delta]$, and player $-i$ has no mass in $[M_i + \delta,\ M_{-i} + 2\delta]$. At least one of these intervals has a length weakly greater than δ, so if $G_i(M_{-i} + \delta) < 1$ or $G_{-i}(M_i + \delta) < 1$, then ϕ_1 exists. Any mass below $\min\{\phi_1,\ \phi_1 + \varphi_1\}$ must be distributed according to the third form in Lemma 8. This is seen by demonstrating that $M_{-i} < \delta$, $M_i \geq \delta$, $\mu_i = 0$, $\mu_{-i} = M_i - \delta$, and $M_i \geq \delta$. Since $\alpha_i(0) > 0$ and $\mu_{-i} < \delta$, then $M_{-i} < \delta$. The result that $\mu_i = 0$ follows because $M_{-i} < \delta$ and $G_{-i}(\delta) = G_{-i}(\mu_i + \delta)$ (see Lemma 6 and Corollary 1). Without additional mass to tie, $\mu_i > 0$ is strictly dominated. Similarly, $\mu_{-i} = \max\{0,\ M_i - \delta\}$ since $G_i(M_i) = G_i(\mu_{-i} + \delta)$. We can pin down μ_{-i} further. Since $\alpha_{-i}(0) = 0$, unless $M_i \geq \delta$, player i would have a strictly lower payoff over $[\mu_{-i} + \delta,\ M_{-i} + \delta]$ than over $[0,\ M_i]$. Thus $\mu_{-i} = M_i - \delta$ and $M_i \geq \delta$. It remains to show that $G_{-i}(M_i + \delta) = 1$ is not an equilibrium. If indeed $G_{-i}(M_i + \delta) = 1$, then the total mass in G_i and G_{-i} is described as follows:

$$G_i : \qquad 1 = \alpha_i(0) + \frac{M_i}{(1-\beta)v} + \frac{M_{-i} - \mu_{-i}}{\beta v} \tag{19}$$

$$G_{-i} : \qquad 1 = \frac{M_{-i} - \mu_{-i}}{(1-\beta)v} + \frac{M_i}{\beta v} \tag{20}$$

Since $\delta,\ \mu_{-i} \in \text{supp}(G_{-i})$, and since $\mu_{-i} = M_i - \delta$, equilibrium requires that $u_{-i}(\overline{\delta},\ G_i) = u_{-i}(M_i - \delta,\ G_i)$. This can be written as:

$$\alpha_i(0)(1-\beta)v + (M_{-i} - \mu_{-i}) = 2\delta - M_i \tag{21}$$

Rearranging Equation 20 and combining Equations 19 and 21, we obtain the following:

$$M_i = \beta v \left[1 - \frac{M_{-i} - \mu_{-i}}{(1-\beta)v}\right] \quad \text{where} \quad M_{-i} - \mu_{-i} = \frac{[2\delta - (1-\beta)v]\beta}{2\beta - 1}$$

Using these expressions, $M_i > \delta$ is equivalent to $(1-\beta)\beta v > \delta$, which in turn implies that $M_{-i} - \mu_{-i} > (1-\beta)\beta v$. So Equation 21 can only be satisfied if $\alpha_i(0) < 0$, which obviously cannot hold in equilibrium.

Case 4: $\alpha_i(0) > 0$, $\alpha_{-i}(0) > 0$, $\mu_i \geq \delta$, $\mu_{-i} < \delta$. Since $\alpha_i(0) > 0$ and $\mu_{-i} < \delta$, then $M_{-i} < \delta$. In equilibrium, $u_i(0,\ G_{-i}) = u_i(\mu_{-i} + \delta,\ G_{-i})$, which can be rewritten as:

$$[G_{-i}(\mu_{-i} + 2\delta) - G_{-i}(\mu_{-i} + \delta)]\,\beta v = \mu_{-i} + \delta \tag{22}$$

With $\mu_i \geq \delta$ and $M_{-i} < \delta$, then by Lemma 6 and Corollary 1, player $-i$ can only have mass in $[\mu_{-i} + \delta,\ \mu_{-i} + 2\delta]$ if $[\mu_i,\ M_i] \subseteq [\delta,\ \mu_{-i} + \delta]$. Then Equation 22 becomes:

$$\left(\frac{M_i - \mu_i}{\beta v}\right)\beta v = \mu_{-i} + \delta \quad \Rightarrow \quad M_i - \mu_i = \mu_{-i} + \delta$$

However, given the bounds of $[\mu_i,\ M_i] \subseteq [\delta,\ \mu_{-i} + \delta]$, this is a contradiction.

When ϕ_1 does not exist: We will show that Parts B and C of Lemma 8 cannot hold if ϕ_1 does not exist. Beginning with Part B, in order for player i's total mass of $\alpha_i(0) + [\psi_i/(1-\beta)v]$ and player $-i$'s total mass of $\alpha_{-i}(0) + [\psi_i/(\beta v)]$ to each equal one, we must have $\alpha_{-i}(0) > 0$. With this positive mass point, preventing a jump in player i's expected payoff near δ requires that $\psi_i < 0$. The two mass constraints and player $-i$'s indifference condition between 0 and δ imply that $\alpha_i(0) = \delta/[(1-\beta)v]$; $\alpha_{-i}(0) = [\delta - (1-2\beta)v]/\beta v$ and $\psi_i = (1-\beta)v - \delta$. However, so long as $\delta < (1-\beta)v$, player i could profitably deviate with a bid of δ. For Part C, the two mass constraints and player $-i$'s indifference condition between $\psi_i - \delta$ and δ imply that $\alpha_i(0) = 2[\delta - (1-\beta)\beta v]/[(1-\beta)^2 v]$; $\psi_i = \delta + [(3\beta - 1)((1-\beta)\beta v - \delta)/(1-\beta)(2\beta - 1)]$; and $\psi_{-i} = [2\delta - (1-\beta)v]\beta/(2\beta - 1)$. Note, however, that the conditions in Part C for $\alpha_i(0) > 0$ and $\psi_i \in [\delta,\ 2\delta)$ cannot be jointly satisfied. (The closest case is $\delta = (1-\beta)\beta v$ so that $\alpha_i(0) = 0$ and $\psi_i = \psi_{-i} = \delta$, but then ϕ_1 would exist.) □

Lemma 9 *Let $\delta \in (0,\ (1-\beta)v\,)$ and $\beta > 1/2$. If $\max\{G_i(2\delta),\ G_{-i}(2\delta)\} < 1$, the equilibrium must be symmetric. In particular, $\alpha_i(0) = \alpha_{-i}(0)$, $\psi_i = \psi_{-i}$, and $\varphi_j = 0$ for $j \in \{1, \ldots, k\}$.*

Proof From Lemmata 7 and 8, mass in G_i and G_{-i} have the following forms:

$$1 = \alpha_i(0) + \frac{\psi_i}{(1-\beta)v} + \frac{\psi_{-i}}{\beta v} + \frac{\delta + \varphi_1}{(1-\beta)v} + \frac{\delta - \varphi_1}{\beta v} + \cdots + \frac{\delta + \varphi_k}{(1-\beta)v} + \frac{\delta - \varphi_k}{\beta v}$$

$$1 = \alpha_{-i}(0) + \frac{\psi_{-i}}{(1-\beta)v} + \frac{\psi_i}{\beta v} + \frac{\delta - \varphi_1}{(1-\beta)v} + \frac{\delta + \varphi_1}{\beta v} + \cdots + \frac{\delta - \varphi_k}{(1-\beta)v} + \frac{\delta + \varphi_k}{\beta v}$$

Combining these two equations yields:

$$\alpha_{-i}(0) - \alpha_i(0) = \left[\frac{4\beta - 2}{(1-\beta)\beta v}\right]\Big(\varphi_1 + \varphi_2 + \cdots + \varphi_k\Big) - \left[\frac{2\beta - 1}{(1-\beta)\beta v}\right](\psi_{-i} - \psi_i) \tag{23}$$

For $j \in \{1, \ldots, k-1\}$ for $k \geq 2$, $u_i(\phi_{j+1}) = u_i(\phi_j)$ and $u_{-i}(\phi_{j+1} + \varphi_{j+1}) = u_{-i}(\phi_j + \varphi_j)$ respectively imply the following:

$$\phi_{j+1} - \phi_j = \frac{\delta}{(1-\beta)\beta} + \left(\frac{1-\beta}{\beta}\right)\varphi_j - \left(\frac{\beta}{1-\beta}\right)\varphi_{j+1}$$

$$\phi_{j+1} - \phi_j = \frac{\delta}{(1-\beta)\beta} + \left(\frac{2\beta - 1}{\beta}\right)\varphi_j - \left(\frac{2\beta - 1}{1-\beta}\right)\varphi_{j+1}$$

We then obtain the following from the above two equations:

$$\varphi_{j+1} = \left[\frac{2 - 3\beta}{3\beta - 1}\right]\left(\frac{1-\beta}{\beta}\right)\varphi_j \tag{24}$$

Since Equation 24 takes the form $\varphi_{j+1} = H\varphi_j$, we can write all ϕ_j in terms of ϕ_1. Of particular note:

$$(\varphi_1 + \varphi_2 + \cdots + \varphi_k) \;=\; \varphi_1 \left[1 + H + H^2 + \cdots + H^{k-1}\right] \;=\; \varphi_1 \left[\frac{1-H^k}{1-H}\right]$$

So Equation 23 becomes:

$$[\alpha_{-i}(0) - \alpha_i(0)]\,(1-\beta)v = \left[\frac{1-H^k}{1-H}\right]\left[\frac{4\beta-2}{\beta}\right]\varphi_1 - \left[\frac{2\beta-1}{\beta}\right](\psi_{-i} - \psi_i) \tag{25}$$

If $\alpha_i(0) = \alpha_{-i}(0)$ and $\psi_i = \psi_{-i}$, then Equation 25 simplifies to:

$$0 = \left[\frac{1-H^k}{1-H}\right]\left[\frac{4\beta-2}{(1-\beta)\beta v}\right]\varphi_1$$

This can only be satisfied by $\varphi_1 = 0$ since $1 = H^k$ and $4\beta = 2$ both require $\beta = 1/2$. So by Equation 24, $\varphi_j = 0$ for $j \in \{1, \ldots, k\}$.

Now suppose that either $\alpha_i(0) \neq \alpha_{-i}(0)$ or $\psi_i \neq \psi_{-i}$. Based on Lemma 8, if there is any mass below ϕ_1, then at least one player has a strictly positive mass point at zero. Without loss of generality, assume this is player i. Then $u_i(0,\ G_{-i}) = u_i(\phi_1,\ G_i)$. For player $-i$, we have $u_{-i}(0,\ G_{-i}) \leq u_{-i}(\phi_1{+}\varphi_1,\ G_i)$, which holds with strict equality in equilibrium whenever $\alpha_{-i}(0) > 0$. These imply the following:

$$\begin{aligned}
\phi_1 \;&=\; \alpha_{-i}(0)(1-\beta)v + \psi_{-i} + \left(\frac{\psi_i}{\beta}\right) + (\delta - \varphi_1)\left(\frac{\beta}{1-\beta}\right) \\
\phi_1 \;&\leq\; \alpha_i(0)(1-\beta)v + \psi_i + \left(\frac{\psi_{-i}}{\beta}\right) + (\delta + \varphi_1)\left(\frac{\beta}{1-\beta}\right) - \varphi_1
\end{aligned}$$

Combined, we have:

$$[\alpha_{-i}(0) - \alpha_i(0)]\,(1-\beta)v \;\leq\; (\psi_{-i} - \psi_i)\left(\frac{1-\beta}{\beta}\right) + \left(\frac{3\beta-1}{1-\beta}\right)\varphi_1 \tag{26}$$

Together, Equations 25 and 26 imply the following, which holds with strict equality whenever $\alpha_{-i}(0) > 0$:

$$(\psi_{-i} - \psi_i) \geq (3\beta - 1)\left[(1-H^k) - \left(\frac{1}{1-\beta}\right)\right]\varphi_1 \tag{27}$$

Based on the three forms of mass below ϕ_1 in Lemma 8, and with $\alpha_i(0) > 0$ and $\alpha_{-i}(0) \geq 0$, we have $\psi_i > 0$ and $\psi_{-i} \geq 0$. So $u_{-i}(\overline{\delta},\ G_i) = u_{-i}(\phi_1 + \varphi_1,\ G_i)$ and $u_i(\overline{\delta},\ G_{-i}) \leq u_i(\phi_1,\ G_{-i})$; the latter holds with strict equality if $\psi_{-i} > 0$. These can be rewritten as follows:

$$\begin{aligned}
\phi_1 \;&=\; \psi_i + \psi_{-i}\left(\frac{1-\beta}{\beta}\right) + \delta\left(\frac{1}{1-\beta}\right) + \varphi_1\left(\frac{2\beta-1}{1-\beta}\right) \\
\phi_1 \;&\leq\; \psi_{-i} + \psi_i\left(\frac{1-\beta}{\beta}\right) + \delta\left(\frac{1}{1-\beta}\right) - \varphi_1\left(\frac{\beta}{1-\beta}\right)
\end{aligned}$$

Combining them yields:

$$(\psi_{-i} - \psi_i) \geq \left(\frac{3\beta-1}{2\beta-1}\right)\left(\frac{\beta}{1-\beta}\right)\varphi_1 \tag{28}$$

We will show that Equations 27 and 28 can only jointly hold if $\psi_{-i} = \psi_i$ and $\varphi_1 = 0$. Suppose instead that $\psi_{-i} \neq \psi_i$, and without loss of generality, suppose that $\psi_{-i} < \psi_i$. With $\beta \in (1/2,\ 1)$, the coefficient on φ_1 in Equation 28 is strictly positive. So since the left-hand side of Equation 28 is negative, it must be that $\varphi_1 < 0$. Next note that the coefficient on φ_1 in Equation 27 is strictly negative for $\beta \in (1/2,\ 1)$. Showing that this coefficient is negative is equivalent to showing that $H^k(1-\beta) + \beta > 0$, which holds for $\beta \in (1/2,\ 1)$ since the minimal value of H^k over this range is

$4\sqrt{3}-7 \approx -0.0718$ (the minimum is obtained at $\beta = (3+\sqrt{3})/6 \approx 0.7887$ and $k = 1$). With a left-hand side that is negative, and a coefficient on φ_1 that is also negative, Equation 28 can only hold if $\varphi_1 > 0$. Hence, we have a contradiction. Since the coefficients on φ_1 in Equations 27 and 28 are nonzero for $\beta \in (1/2,\ 1)$, these equations can only hold simultaneously if $\psi_{-i} = \psi_i$ and $\varphi_1 = 0$. If this is the case, then by Equation 24, $\varphi_j = 0$ for $j \in \{1, \dots, k\}$, and by Equation 25, $\alpha_i(0) = \alpha_{-i}(0)$. The equilibrium must therefore be symmetric. □

D Proofs Specific to $\beta = 1/2$

Lemma 10 *Let $\beta = 1/2$ and $\delta \in (0,\ v/4]$. If G_i and G_{-i} are equilibrium strategies for players i and $-i$, then $u_i(a,\ G_{-i}) = u_i(a-2\delta,\ G_{-i})$ for all $a \in \mathrm{supp}(G_{-i})$ such that $a \geq 2\delta$. Equivalently, $[G_{-i}(a+\delta) - G_{-i}(\max\{a-3\delta,\ 0\})] = 4\delta/v$.*

Proof We begin by showing that for any $\kappa > 0$, $[G_{-i}(\kappa+4\delta) - G_{-i}(\kappa)] \leq 4\delta/v$. This is done by construction. Suppose that any mass in G_{-i} over $[\kappa,\ \kappa+4\delta]$ has the maximal density rate of $2/v$ (any higher density rate would violate Equation 14). This is possible if $G_{-i}(\kappa+4\delta) = G_{-i}(\kappa+6\delta)$ and $G_{-i}(\max\{\kappa-2\delta,\ 0\}) = G_{-i}(\kappa)$ so that any mass in G_i over $[\kappa-\delta,\ \kappa+\delta]$ and $[\kappa+3\delta,\ \kappa+5\delta]$ is entirely balanced by the mass in G_{-i} over $[\kappa,\ \kappa+4\delta]$. To allow for the largest amount of $[\kappa,\ \kappa+4\delta]$ to be covered at the density rate of $2/v$, we further suppose that $G_i(\kappa+\delta) = G_i(\kappa+3\delta)$, since any mass in G_i over $[\kappa+\delta,\ \kappa+3\delta]$ would necessitate a lower (perhaps zero) density rate in G_{-i} over some portion of $[\kappa,\ \kappa+4\delta]$. In equilibrium, it must be that $\kappa+\delta$ and $\kappa+3\delta \in \mathrm{supp}(G_i)$. Otherwise, if $e-d > 2\delta$ where $d = \max\{\mathrm{supp}(G_i) \cap [\kappa-\delta,\ \kappa+\delta]\}$ and $e = \min\{\mathrm{supp}(G_i) \cap [\kappa+3\delta,\ \kappa+5\delta]\}$, then we would have $d+\delta$ and $e-\delta \in \mathrm{supp}(G_{-i})$, but $u_{-i}(d+\delta,\ G_i) > u_{-i}(e-\delta,\ G_i)$ (that is, the winning and tying probability would remain the same for bids of $d+\delta$ and $e-\delta$, but the cost of effort would differ). From Equation 14, $u_i(\kappa+\delta,\ G_{-i}) = u_i(\kappa+3\delta,\ G_{-i})$ implies that $[G_{-i}(\kappa+4\delta) - G_{-i}(\kappa)] = 4\delta/v$, which is the desired upper bound. With this property in hand, the main result follows quickly. Since G_i and G_{-i} are equilibrium strategies, and since $a \in \mathrm{supp}(G_{-i})$, it cannot be the case that $u_i(a,\ G_{-i}) < u_i(a-2\delta,\ G_{-i})$. For the purpose of contradiction, suppose that $u_i(a,\ G_{-i}) > u_i(a-2\delta,\ G_{-i})$ for some a. Since $u_i(a,\ G_{-i}) = [G_{-i}(a+\delta) + G_{-i}(a-\delta)](v/2) - a$ and $u_i(a-2\delta,\ G_{-i}) = [G_{-i}(a) + G_{-i}(a-3\delta)](v/2) - (a-2\delta)$, then $u_i(a,\ G_{-i}) > u_i(a-2\delta,\ G_{-i})$ implies that $[G_{-i}(a+\delta) - G_{-i}(a-3\delta)] > 4\delta/v$. This, however, is a contradiction, and so $u_i(a,\ G_{-i}) = u_i(a-2\delta,\ G_{-i})$. □

Lemma 11 *Let $\beta = 1/2$ and $\delta \in (0,\ v/4]$. Property $\mathscr{P}$ must hold in any equilibrium.*

Proof For added clarity, we will refer to G_i and G_{-i} and G_w and G_y. We will also denote $\overline{w}_1 = \max\{x \in \mathrm{supp}(G_w)\}$ and $\overline{y}_1 = \max\{x \in \mathrm{supp}(G_y)\}$. Without loss of generality, assume that $\overline{w}_1 \geq \overline{y}_1$; so $\overline{y}_1 \in [\overline{w}_1 - \delta,\ \overline{w}_1]$. Our first step is to show that $G_w(\overline{w}_1 - 3\delta) = G_w(\overline{w}_1 - 4\delta)$ and that $G_y(\overline{w}_1 - 3\delta) = G_y(\overline{w}_1 - 4\delta)$. By Lemma 10, since $\overline{w}_1 \in \mathrm{supp}(G_w)$ and $G_y(\overline{w}_1 + \delta) = G_y(\overline{y}_1)$, then:

$$[G_y(\overline{y}_1) - G_y(\overline{w}_1 - 3\delta)] = 4\delta/v$$

Also, since $\overline{y}_1 - \delta \in \mathrm{supp}(G_w)$, Lemma 10 implies that:

$$[G_y(\overline{y}_1) - G_y(\overline{y}_1 - 4\delta)] = 4\delta/v$$

Hence, $G_y(\overline{w}_1 - 3\delta) = G_y(\overline{y}_1 - 4\delta)$, and by a similar argument, $G_w(\overline{y}_1 - 3\delta) = G_w(\overline{w}_1 - 4\delta)$. If $\overline{w}_1 = \overline{y}_1$, then we are done. Otherwise, for $\overline{w}_1 > \overline{y}_1$, we must still show that $G_w(\overline{w}_1 - 3\delta) = G_w(\overline{y}_1 - 3\delta)$. Suppose instead that $G_w(\overline{w}_1 - 3\delta) > G_w(\overline{y}_1 - 3\delta)$. Denote $\underline{w} = \min\{x \in \mathrm{supp}(G_w) \mid x \geq \overline{y}_1 - 3\delta\}$.

Since $\overline{w}_1 - \delta \in \text{supp}(G_y)$, then by Lemma 10, $u_y(\overline{w}_1 - 3\delta,\ G_w) = u_y(\overline{w}_1 - \delta,\ G_w)$. Hence, in equilibrium, $u_y(\overline{w}_1 - 3\delta) \geq u_w(\underline{w})$, or rather:

$$[G_w(\overline{w}_1 - 2\delta) - G_w(\underline{w} + \delta)]v/2 \ + \ [G_w(\overline{w}_1 - 4\delta) - G_w(\underline{w} - \delta)]v/2 \ \geq \ \overline{w}_1 - 3\delta - \underline{w}$$

Since G_w has at least some mass immediately below $\overline{w}_1$ in $[\underline{w} + 3\delta,\ \overline{w}_1]$, any mass 2δ below that in G_w necessarily has a density rate less than $2/v$ (that is, $[G_w(\overline{w}_1 - 2\delta) - G_w(\underline{w} + \delta)]v/2 \ < \ \overline{w}_1 - 3\delta - \underline{w}$; otherwise Equation 14 is not satisfied for all mass in G_y over $[\underline{w} + 2\delta,\ \overline{w}_1 - \delta]$). Hence, it must be that $G_w(\overline{w}_1 - 4\delta) - G_w(\underline{w} - \delta) > 0$. For any $\ell \in \text{supp}(G_w) \cap [\underline{w} - \delta,\ \overline{w}_1 - 4\delta]$, we have $\ell - \delta \in \text{supp}(G_y)$, and so $u_y(\ell - 3\delta,\ G_y) = u_y(\ell - \delta,\ G_y)$ (see Lemmata 2 and 10). Moreover, to satisfy Equation 14 for all $\ell - \delta$, the density rates in G_w over $[\underline{w} - \delta,\ \overline{w}_1 - 4\delta]$ and those 2δ below it must sum to $2/v$. At least a portion of the mass in G_w over $[\underline{w} - 3\delta,\ \overline{w}_1 - 6\delta]$ must therefore have a density rate less than $2/v$. However, for $u_y(\ell - 3\delta,\ G_y) = u_y(\ell - \delta,\ G_y)$ to hold, G_w must either have a density rate of $2/v$ over the entirety of $[\underline{w} - 3\delta,\ \overline{w}_1 - 6\delta]$ or there must be a positive density rate over the $\ell - 4\delta$ in G_w. That is, $G_w(\overline{w}_1 - 8\delta) - G_w(\underline{w} - 4\delta) > 0$. Since the density rate over all the $\ell - \delta$ in G_y is $2/v$, the density rate over the $\ell - 3\delta$ in G_y is 0, and so $\ell - 5\delta \in \text{supp}(G_y)$. The argument then repeats. Ultimately, however, there is a contradiction: at some point the bottom of the distribution is reached, so payoffs can no longer be sustained by additional mass δ below. We therefore have the desired result that $G_w(\overline{w}_1 - 3\delta) = G_w(\overline{w}_1 - 4\delta)$ and $G_y(\overline{w}_1 - 3\delta) = G_y(\overline{w}_1 - 4\delta)$.

Since G_w and G_y each have a gap of at least δ, then this argument also applies to mass below these gaps. Let $\overline{w}_2 = \max\{x \in \text{supp}(G_w) \mid x \leq \overline{w}_1 - 4\delta\}$, $\overline{y}_2 = \max\{x \in \text{supp}(G_y) \mid x \leq \overline{y}_1 - 4\delta\}$, and $m_2 = \max\{\overline{w}_2,\ \overline{y}_2\}$. Using the same argument, $G_w(m_2 - 3\delta) = G_w(m_2 - 4\delta)$ and $G_y(m_2 - 3\delta) = G_y(m_2 - 4\delta)$. Or more generally, for $\overline{w}_z = \max\{x \in \text{supp}(G_w) \mid x \leq \overline{w}_{z-1} - 4\delta\}$, $\overline{y}_z = \max\{x \in \text{supp}(G_y) \mid x \leq \overline{y}_{z-1} - 4\delta\}$, and $m_z = \max\{\overline{w}_z,\ \overline{y}_z\}$, where $z \in \{2, 3, \ldots\}$, then $G_w(m_z - 3\delta) = G_w(m_z - 4\delta)$ and $G_y(m_z - 3\delta) = G_y(m_z - 4\delta)$. Moreover,

$$[G_w(m_z) - G_w(m_z - 3\delta)] = [G_y(m_z) - G_y(m_z - 3\delta)] = 4\delta/v$$

To satisfy the constraint that all mass must sum to one, G_w and G_y each have $p \equiv \lfloor v/4\delta \rfloor$ such intervals. That is, p intervals of length 4δ, each having a total mass of $4\delta/v$, none of which is in the bottom δ (the remaining $1 - [4\delta p/v]$ is then at the bottom of the distribution; more on this later). Placing a mass of $4\delta/v$ within 3δ, with no mass δ above or below, requires that G_w and G_y each have $2\delta/v$ over $[m_z - 2\delta,\ m_z - \delta]$ and $2\delta/v$ over $[m_z - 3\delta,\ m_z - 2\delta] \bigcup [m_z - \delta,\ m_z]$ for $z \in \{1, \ldots, p\}$. Mass over $[m_z - 2\delta,\ m_z - \delta]$ must be at a density rate of $2/v$, while the density rates at x and $x - 2\delta$ for $x \in [m_z - \delta,\ m_z]$ must sum to $2/v$ (see Lemmata 2 and Equation 14).

We can also state how successive length 4δ intervals fit together. Since $[m_z - 2\delta,\ m_z - \delta]$ is in the support of G_w and G_y, then by Lemma 10, players are indifferent between any bid in $[m_z - 2\delta,\ m_z - \delta]$ and any bid in $[m_z - 3\delta,\ m_z - 4\delta]$. In particular, $u_y(m_z - 3\delta,\ G_w) = u_y(m_z - 4\delta,\ G_w)$ and $u_w(m_z - 3\delta,\ G_y) = u_w(m_z - 4\delta,\ G_y)$ respectively imply:

$$\begin{aligned} [G_w(m_z - 2\delta) - G_w(m_z - 5\delta)] &= 2\delta/v \\ [G_y(m_z - 2\delta) - G_y(m_z - 5\delta)] &= 2\delta/v \end{aligned} \tag{29}$$

So G_w and G_y each have $2\delta/v$ over $[m_z - 5\delta,\ m_z - 3\delta] \bigcup [m_z - 5\delta,\ m_z - 3\delta]$, and the density rates at $x \in [m_z - 3\delta,\ m_z - 5\delta]$ and $x - 2\delta$ must sum to $2/v$ to support the expected payoffs in $[m_z - 3\delta,\ m_z - 4\delta]$. Consequently, players are also indifferent between bids in $[m_z - 3\delta,\ m_z - 4\delta]$ and bids in $[m_z - 4\delta,\ m_z - 5\delta]$ (if $x \in [m_z - 3\delta,\ m_z - 5\delta]$ is in the player's support, then the indifference comes from Lemma 10; if not, then the indifference comes from $x-2\delta$ being in the player's support). We therefore have $u_y(m_z-3\delta,\ G_w) = u_y(m_z-5\delta,\ G_w)$ and $u_w(m_z-3\delta,\ G_y) = u_w(m_z-5\delta,\ G_y)$, and so $[G_w(m_z-2\delta)-G_w(m_z-6\delta)] = 4\delta/v$ and $[G_y(m_z-2\delta)-G_y(m_z-6\delta)] = 4\delta/v$. Combined with Equation 29, $[G_w(m_z-5\delta)-G_w(m_z-6\delta)] = 2\delta/v$ and $[G_y(m_z-5\delta)-G_y(m_z-6\delta)] = 2\delta/v$, which can only hold if the mass is distributed at a rate of $2/v$. Finally, $u_y(m_z-4\delta,\ G_w) = u_y(m_z-6\delta,\ G_w)$ and $u_y(m_z-5\delta,\ G_w) = u_y(m_z-7\delta,\ G_w)$ give us $G_w(m_z-7\delta) = G_w(m_z-8\delta)$; the corresponding equations for u_w yield $G_y(m_z-7\delta) = G_y(m_z-8\delta)$. □

Lemma 12 *Let* $\beta = 1/2$ *and* $\delta \in (0,\ v/4]$. *Also, let* $p = \lfloor v/4\delta \rfloor$ *be the number of length-*4δ *intervals with the properties specified by* $\mathscr{P}$. *In any equilibrium, the top of the* p^{th} *such interval (or bottommost interval) is in* $[2\delta,\ 3\delta]$ *if* $\delta = v/4p$*; in* $(3\delta,\ 4\delta]$ *if* $\delta \in [v/(4p+2),\ v/4p)$*; and at* $(v/2)-\delta(2p-4)$ *if* $\delta \in (v/(4p+4),\ v/(4p+2))$. *Below the* p^{th} *interval, a total mass of* $1-[4\delta p/v]$ *is distributed as follows:*

- **A.** *If* $\delta \in [v/(4p+2),\ v/4p)$*, the remaining* $1-[4\delta p/v]$ *is at zero, neither player has mass in* $(0,\ \delta)$*, and all equilibria have an expected payoff of* $(v/2)-2\delta p$. *For* $\delta = v/(4p+2)$*, the equilibrium is unique: there is no mass in* $(0,\ 2\delta)$ *and the top of the* p^{th} *interval is at* 4δ.
- **B.** *If* $\delta \in (v/(4p+4),\ v/(4p+2))$*, each player has a mass point at zero of* $[4\delta(p+1)/v]-1$*, a uniform density rate of* $2/v$ *over the intervals* $[0,\ v/2-\delta(2p+1)]$ *and* $[\delta,\ v/2-2\delta p]$*, and an expected payoff of* δ. *This also holds for* $\delta \in (v/4,\ v/2)$ *(i.e.* $p=0$*).*

Proof Following the notation from the proof of Lemma 11, let $m_p = m_1 - 4\delta(p-1)$, where $m_1 = \max\{\overline{w}_1,\ \overline{y}_1\}$. That is, m_p is the top of the p^{th} length-4δ interval (specifically for the player whose support contains the highest element; alternatively, $m_p = \max\{\overline{w}_p,\ \overline{y}_p\}$). By Lemma 11, these p intervals satisfy $\mathscr{P}$, and so the remaining mass of $1-[4\delta p/v]$ must be distributed below them at the bottom of the distribution. There are two bounds that we can quickly place on m_p. First, $m_p \geq 2\delta$; otherwise, with a maximal density rate of $2/v$ it is not possible to have $4\delta/v$ of continuously distributed mass. Second, $m_p < 6\delta$, or it would be possible to have $4\delta/v$ of continuously distributed mass below $m_p - 4\delta$. We will show that $m_p \in [2\delta,\ 3\delta]$ when $\delta = v/4p$; $m_p \in (3\delta,\ 4\delta]$ when $\delta \in [v/(4p+2),\ v/4p)$; and $m_p \in (5\delta,\ 6\delta)$ when $\delta \in (v/(4p+4),\ v/(4p+2))$. This covers the complete range of δ for any given p. Furthermore, we will show how the remaining $1-[4\delta p/v]$ is distributed, as well as a uniqueness result for $\delta = v/(4p+2)$.

If $m_p \in [2\delta,\ 4\delta]$, then since $4\delta/v$ is distributed over $(0,\ m_p]$, Lemma 10 requires that the remaining mass of $1-[4\delta p/v]$ be at zero. If $m_p \in [2\delta,\ 3\delta)$, then each player's support contains a neighborhood above and below δ with a density rate of $2/v$. However, if the opponent has a strictly positive mass point at zero, placing mass immediately below δ cannot hold in equilibrium. So for $m_p \in [2\delta,\ 3\delta)$, we must have $1-[4\delta p/v] = 0$, or equivalently, $\delta = v/4p$. We can also quickly show that $m_p \notin (3\delta,\ 4\delta]$ when $\delta = v/4p$. Without a mass point at zero, each player's distribution must have a density rate of $2/v$ over $[\delta,\ 2\delta]$; otherwise, players could obtain a higher expected payoff by bidding zero. Maintaining even the minimum expected payoff of zero entails randomizing at the rate of $2/v$ over $[\delta,\ 3\delta]$, but then the mass of $4\delta/v$ is used up. So $m_p \notin (3\delta,\ 4\delta]$ when $\delta = v/4p$.

Therefore, if $m_p \in (3\delta,\ 4\delta]$, there must be a strictly positive mass point at zero of $1-[4\delta p/v]$. And if one player is granted that privilege, they both must be. So $\min\{\overline{w}_p,\ \overline{y}_p\} \in (3\delta,\ 4\delta]$, and 2δ is in each player's support at a density rate of $2/v$. Equating $u_w(2\delta,\ G_y) = u_w(0,\ G_y)$ and $u_y(2\delta,\ G_w) = u_y(0,\ G_w)$, we obtain:

$$G_w(3\delta) = G_y(3\delta) = 4\delta/v \tag{30}$$

At least $2\delta/v$ of this $4\delta/v$ is distributed at a density rate of $2/v$ over $[\overline{w}_p - 2\delta,\ \overline{w}_p - \delta]$ or $[\overline{y}_p - 2\delta,\ \overline{y}_p - \delta]$. So the mass point of $1 - [4\delta p/v]$ must be weakly less than $2\delta/v$. This implies that $\delta \geq v/(4p+2)$. We already showed that this range of m_p does not hold for $\delta = v/4p$, and so $m_p \in (3\delta,\ 4\delta]$ implies that $\delta \in [v/(4p+2),\ v/4p)$. Equation 30 also rules out the possibility of a strictly positive mass point when $m_p = 3\delta$ (there is already $4\delta/v$ in $(0,\ 3\delta]$, so there is no room for a mass point).

Next suppose that $m_p \in (4\delta,\ 5\delta]$. We will show that this cannot hold in equilibrium. Without loss of generality, assume that $m_p = \overline{w}_p$. As a property of $\mathscr{P}$, there is no mass in G_w over $[\overline{w}_p - 4\delta,\ \overline{w}_p - 3\delta]$ and no mass in G_y over $(\max\{\overline{y}_p - 4\delta,\ 0\},\ \overline{y}_p - 3\delta]$. We begin by showing that G_y has no mass in $(0,\ \delta]$. If $\overline{y}_p \in (\overline{w}_p - \delta,\ 4\delta]$, then the remaining mass of $1 - [4\delta p/v]$ is at zero and G_y has no mass in $(0,\ \delta]$. If instead $\overline{y}_p \in (4\delta,\ \overline{w}_p]$, then by Lemma 2, since G_w has no mass in $(\delta,\ \overline{w}_p - 3\delta]$, then G_y has no mass in $(0,\ \overline{w}_p - 4\delta]$. There is also no mass in G_y over $[\overline{w}_p - 4\delta,\ \delta]$ since this is a subset of $[\overline{y}_p - 4\delta,\ \overline{y}_p - 3\delta]$. Thus, G_y has no mass in $(0,\ \delta]$. We next note that $u_w(0,\ G_y) = u_w(\overline{\delta},\ G_y)$. Since 0 is in the support of G_w, we must have $u_w(0,\ G_y) \geq u_w(\overline{\delta},\ G_y)$. But if $u_w(0,\ G_y) > u_w(\overline{\delta},\ G_y)$, then we would also have $u_w(0,\ G_y) > u_w(x',\ G_y)$ where $x' = \min\{x \in \text{supp}(G_w) \mid x \geq \overline{w}_p - 3\delta\}$. With no mass in G_y over $(0,\ \delta]$, the maximal density rate of $2/v$ over $[2\delta,\ x' + \delta]$ can compensate for the added bidding cost between δ and x', but no more. Thus, since we must have $u_w(0,\ G_y) = u_w(\overline{\delta},\ G_y)$ and since $\overline{w}_p - 3\delta > \delta$, then we must also have a density rate of $2/v$ in G_w over $[2\delta,\ \overline{w}_p - 2\delta]$. However, this implies that there is no mass in G_w over $[4\delta,\ \overline{w}_p]$, which contradicts $\overline{w}_p \in (4\delta,\ 5\delta]$.

Finally, if $m_p \in (5\delta,\ 6\delta)$, then $m_p - 4\delta \in (\delta,\ 2\delta)$. We again assume that $m_p = \overline{w}_p$. Following $\mathscr{P}$, there is no mass in G_w over $[\overline{w}_p - 4\delta,\ \overline{w}_p - 3\delta]$, and consequently, by Lemma 2, no mass in G_y over $[\overline{w}_p - 5\delta,\ \delta]$. Likewise, there is no mass in G_y over $[\overline{y}_p - 4\delta,\ \overline{y}_p - 3\delta]$, and no mass in G_w over $[\overline{y}_p - 5\delta,\ \delta]$.[25] Applying parts A and B of Lemma 2, any continuously distributed mass over $[0,\ \overline{y}_p - 5\delta] \bigcup [\delta,\ \overline{w}_p - 4\delta]$ in G_w and over $[0,\ \overline{w}_p - 5\delta] \bigcup [\delta,\ \overline{y}_p - 4\delta]$ in G_y must have a density rate of $2/v$. It follows then that G_w and G_y have the same amount of continuously distributed mass, and so $\alpha_w(0) = \alpha_y(0) \equiv \alpha(0)$. To compensate for the respective gaps in G_w and G_y over $[\overline{y}_p - 5\delta,\ \delta]$ and $[\overline{w}_p - 5\delta,\ \delta]$, we need $\alpha(0) > 0$.[26] Since a bid of $\overline{w}_p - 4\delta$ for player w and of $\overline{y}_p - 4\delta$ for player y have the same expected payoffs as a bid 2δ above that or any other bid in their support (see Lemmata 10 and 11), it must be that $u_w(\overline{w}_p - 4\delta,\ G_y) \geq u_w(\overline{\delta},\ G_y)$ and $u_y(\overline{y}_p - 4\delta,\ G_w) \geq u_y(\overline{\delta},\ G_w)$. These can only be satisfied if G_y has a density rate of $2/v$ over $[0, \overline{w}_p - 5\delta]$ and if G_w has a density rate of $2/v$ over $[0, \overline{y}_p - 5\delta]$. By Lemma 2, these in turn imply a density rate of $2/v$ over $[\delta,\ \overline{w}_p - 4\delta]$ in G_w and $[\delta,\ \overline{y}_p - 4\delta]$ in G_y. With a common mass point, each player will only be indifferent between these two intervals if $\overline{w}_p = \overline{y}_p$. From $u_w(\overline{w}_p - 4\delta,\ G_y) = u_w(0,\ G_y)$, we can identify $\overline{w}_p = (v/2) - 2\delta(p-2)$, which pins down $\alpha(0) = [4\delta(p+1)/v] - 1$.[27] We can establish bounds on δ from $u_w(\overline{w}_p - 4\delta,\ G_y) = u_w(\overline{w}_p - 3\delta,\ G_y)$, since this implies $G_y(\overline{w}_p - 5\delta) = 2\delta/v$. Thus, the remaining mass of $1 - (4\delta/v) \in (2\delta/v,\ 4\delta/v)$. Equivalently, $\delta \in (v/(4p+4),\ v/(4p+2))$. This argument also applies to the case of $\delta \in (v/4,\ v/2)$.

[25] The claim that G_w has no mass in $[\overline{y}_p - 5\delta,\ \delta]$ is contingent on $\overline{y}_p \in (5\delta,\ \overline{w}_p]$. We can quickly rule out the possibility of $\overline{y}_p \in (\overline{w}_p - \delta,\ 5\delta]$: the lack of mass in G_y over $[\overline{y}_p - 4\delta,\ \overline{y}_p - 3\delta]$ would preclude mass in G_w over $(0,\ \overline{y}_p - 4\delta)$ (see Lemma 2). Since there is also no mass in G_w over $[2\delta,\ \overline{w}_p - 3\delta]$, there would be no way for player y to recover the bidding cost between δ and $\overline{w}_p - 4\delta$ (i.e. the next element in G_y above $\overline{w}_p - 4\delta$ would need to be compensated by more than the maximal density rate of $2/v$).

[26] If $\alpha(0) = 0$, at most one player could have continuously distributed mass below δ, but that player would then have a sizable gap before there was any more mass in the other player's distribution to tie or beat. So that cannot be an equilibrium.

[27] The mass point is the remainder of $1 - (4\delta p/v)$ after subtracting $[(v/2) - 2\delta(p-2) - 5\delta] \times (2/v) \times 2$.

Algorithm 1 Asymmetric Equilibria: $\beta \in (0,\, 1/2)$, $\delta \in (0,\, (1-\beta)v)$

for all $z \in \{0,\, 1,\, 2,\, \ldots,\, \lfloor v/\delta \rfloor - 1\}$ **do** ▷ Vary first zeroed element
    Set to zero: $w_z,\, y_{z+1},\, w_{z+2},\, y_{z+3}, \ldots,\, w_{\lfloor v/\delta \rfloor - 1},\, y_{\lfloor v/\delta \rfloor}$ (if $z = 0$, set to zero: $a_w(0),\, w_1,\, y_2,\, \ldots,\, w_{\lfloor v/\delta \rfloor -1},$ $y_{\lfloor v/\delta \rfloor}$).
    for all $p \in \{\max\{1,\, z\}, \ldots,\, \lfloor v/\delta \rfloor\}$ **do** ▷ Vary upper bound
        Set to zero: $w_j,\, y_j$ for all $j > p$.
        Solve system of equations $\boldsymbol{Ax} = \boldsymbol{b}$ (one indifference equation for each jump in each player's distribution, plus constraints for mass summing to one; $\boldsymbol{x}$: sizes of nonzero mass points and lengths of nonzero $w_i,\, y_i$).
        verify whether $\boldsymbol{x}^* = \boldsymbol{A} \backslash \boldsymbol{b}$ is an equilibrium:
        • Not an equilibrium if any elements of $\boldsymbol{x}^*$ are not strictly positive.
        • If $z = 0$, not an equilibrium if $u_w(0,\, G_y) > u_w(\delta,\, G_y)$.
        • Not an equilibrium if $u_y(\zeta + \delta,\, G_w) > u_y(0,\, G_w)$ where ζ is where the $1/[(1-\beta)]v$ segment of w_z would have begun ($\zeta = 0$ if $z = 0$).
        • Not an equilibrium if the player with the smaller upper bound can do better by bidding δ above the other player's upper bound.
        • Otherwise, $\boldsymbol{x}^*$ constitutes an equilibrium.

Having established the various bounds for δ for each potential value of m_p, we conclude with a uniqueness result at $\delta = v/(4p+2)$. As we have already shown, there is a mass point at zero of $1 - (4\delta p/v)$ and there is no mass in $(0,\ \delta)$. Also, as we argued earlier in the paragraph covering $m_p \in (4\delta,\ 5\delta]$, we must also have $u_i(0,\ G_{-i}) = u_i(\overline{\delta},\ G_{-i})$ (or else the expected payoff of elements in the support above δ would be less than the expected payoff at zero). From this equality we obtain $G_{-i}(2\delta) - G_{-i}(\delta) = [\delta(4p+2)/v] - 1$, which equals zero when $\delta = v/(4p+2)$. Hence, there is no mass in $(0,\ 2\delta)$, so the $4\delta/v$ in the p^{th} interval must be distributed over $[2\delta,\ 4\delta]$ at a rate of $2/v$. The rest of the distribution follows from $\mathscr{P}$. □

E Asymmetric Equilibria

For $\delta \in (0,\ (1-\beta)v)$ and $\beta \in (0,\ 1/2)$, Algorithm 1 identifies the complete set of asymmetric equilibria for the game $APT\{\delta, \beta, v\}$ (the labels are the same as in Figure 8). With players arbitrarily assigned as player w or player y, the algorithm systematically varies the first $1/[(1-\beta)v]$ segment in player w's distribution to omit, as well as the uppermost interval pair in the two distributions. Then for each combination of omitted interval pairs, there are at most four conditions that must be checked to verify the existence of an asymmetric equilibrium. First, the system of equations formed from the indifference conditions between the intervals in each player's distribution needs to produce strictly positive lengths for each of the non-omitted interval pairs and strictly positive mass for the non-excluded mass points (the system of equations also includes two equations which specify that the mass in each player's distribution must sum to one). Second, if player w's mass point was excluded, player w cannot profitably deviate by bidding zero.

All other profitable deviations are captured by the third and fourth conditions. Third, bidding δ above the first omitted $1/[(1-\beta)v]$ segment in player w's distribution cannot be profitable for player y. Below this point, bids within gaps in either player's distribution can be ruled out by arguments similar to those for the symmetric case (see the paragraphs leading up to Theorem 1). Above this point, the gaps are so large that bidding within a gap does not adequately increase the amount of mass a player is tying or beating. Precisely at this point, however, player y beats all of the $1/(\beta v)$ segment that is δ below it, and so the expected payoff rises to a peak—the only peak in this gap. It therefore suffices to check that this peak is not too high. The fourth condition similarly pertains to a peak. As in the symmetric case, it merely specifies that outbidding the opponent's distribution by δ cannot be profitable. Here, however, one player's upper bound is a $1/[(1-\beta)v]$ segment while the other's is a $1/(\beta v)$ segment, which is already δ above the first. So we simply need to verify that the player with the smaller upper bound cannot profit by outbidding their opponent's distribution and winning with certainty.

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Online Appendix for "All-Pay Auctions with Ties" (Gelder, Kovenock, Roberson)

Two topics are covered in this online appendix:

1. An extended discussion of asymmetric equilibria, including a map of where they occur in the parameter space and the forms that they take.
2. For symmetric equilibria, the nature of the equilibrium support throughout the parameter space. This is depicted in Figure 10 with eight panels (similar to Figure 6 in the main body of the paper) that show the equilibrium support for $\delta \in (0, (1-\beta)v)$ when $\beta \in \{0.1, 0.3, 0.45, 0.49999, 0.50001, 0.55, 0.7, 0.9\}$ and $v = 100$.

Asymmetric Equilibria

As discussed in the main body of the paper, asymmetric equilibria arise when an interval pair has a length of zero or a mass point has no mass. Omitting a certain combination of interval pairs may form an equilibrium in one part of the parameter space, while another part of the parameter space may require a different combination. Figure 9 portrays the portions of the parameter space (with $\beta < 1/2$) where any form of asymmetric equilibrium exists. The changing colors represent the transition from one form of equilibrium to another between adjoining regions that each have asymmetric equilibria. For instance, in the triangle formed by $(0, v/2)$, $(0, v)$, and $(v/3, v/3)$ (i.e. by $\delta \in [(1-\beta)v/2,\ (1-2\beta)v]$ for $\beta \in (0,\ 1/3]$), there is only one positive mass point and one interval pair. Player y specifically has a mass point $\alpha_y(0) > 0$ and randomizes over $[0, y_1]$, while player w solely randomizes over $[\delta, \delta + y_1]$ (the assignment of players to the roles of y and w is of course arbitrary and the roles and could be switched).[28]

The adjoining triangle to the right, spanning $(v/4, v/2)$, $(v/3, v/2)$, and $(v/3, v/3)$, is similar except that, in addition to $\alpha_y(0) > 0$ and $y_1 > 0$, player w also has a strictly positive mass point at zero: $\alpha_w(0) > 0$. Immediately below the first triangle, there are also two crescent-shaped regions (with endpoints of $(0, v/2)$ and $(v/3, v/3)$). Each of these regions involve two interval pairs with positive lengths (y_1, $w_2 > 0$). Player y thus randomizes between the antipodes of either very low or very high bids, while player w targets bids in the middle.[29] The difference between these two crescents is that player w has a mass point at zero in the lower crescent, but not in the upper one.

Following suit with the symmetric case, as the tie margin δ decreases, the number of interval pairs included in each asymmetric equilibrium progressively increases.[30] Most notably, this leads again to a convergence result in which the average density rate over any measurable subset of $[0, v]$ approaches $1/v$ (the standard all-pay auction rate) as δ goes to zero. It is also worth mentioning that for $\delta < v/4$, the regions distinguishing one form of asymmetric equilibrium from another occasionally overlap so that a given point may have multiple distinct forms of asymmetric equilibria.

[28] Solving for this equilibrium, $y_1 = \beta v$ and $\alpha_y(0) = 1 - [\beta/(1-\beta)]$.

[29] Player y randomizes over $[0, \delta]$ at $1/[(1-\beta)v]$ and over $[2\delta + y_1, 2\delta + y_1 + w_2]$ at $1/(\beta v)$. Player w randomizes over $[\delta, \delta + y_1]$ at $1/(\beta v)$ and over $[\delta + y_1, \delta + y_1 + w_2]$ at the rate of $1/[(1-\beta)v]$.

[30] Three additional examples from Figure 9 will suffice. Of the two crescents with endpoints of $(v/3, v/3)$ and $(v/2, v/4)$, the upper one has positive values for $\alpha_y(0)$, $\alpha_w(0)$, y_1, w_1, and y_2, while the lower crescent also has a positive value for w_3. Another crescent between $(v/3, v/4)$ and $(v/2, v/4)$ has positive values for $\alpha_y(0)$, $\alpha_w(0)$, y_1, w_1, y_2, w_2, and y_3 (omitting w_3).

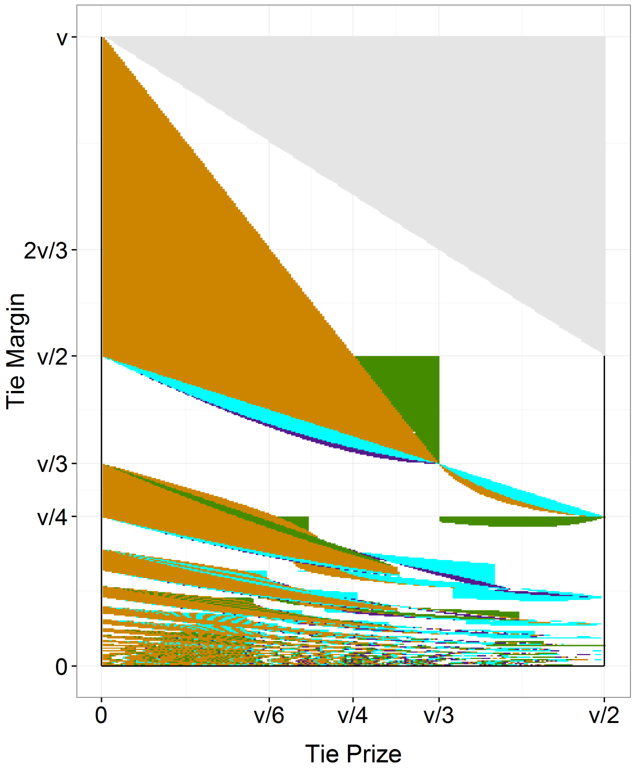

Fig. 9: Regions where asymmetric equilibria exist for $\beta \in (0,\ 1/2)$ and $\delta \in (0,\ (1-\beta)v)$. Adjoining regions with different forms of asymmetric equilibria are colored differently.

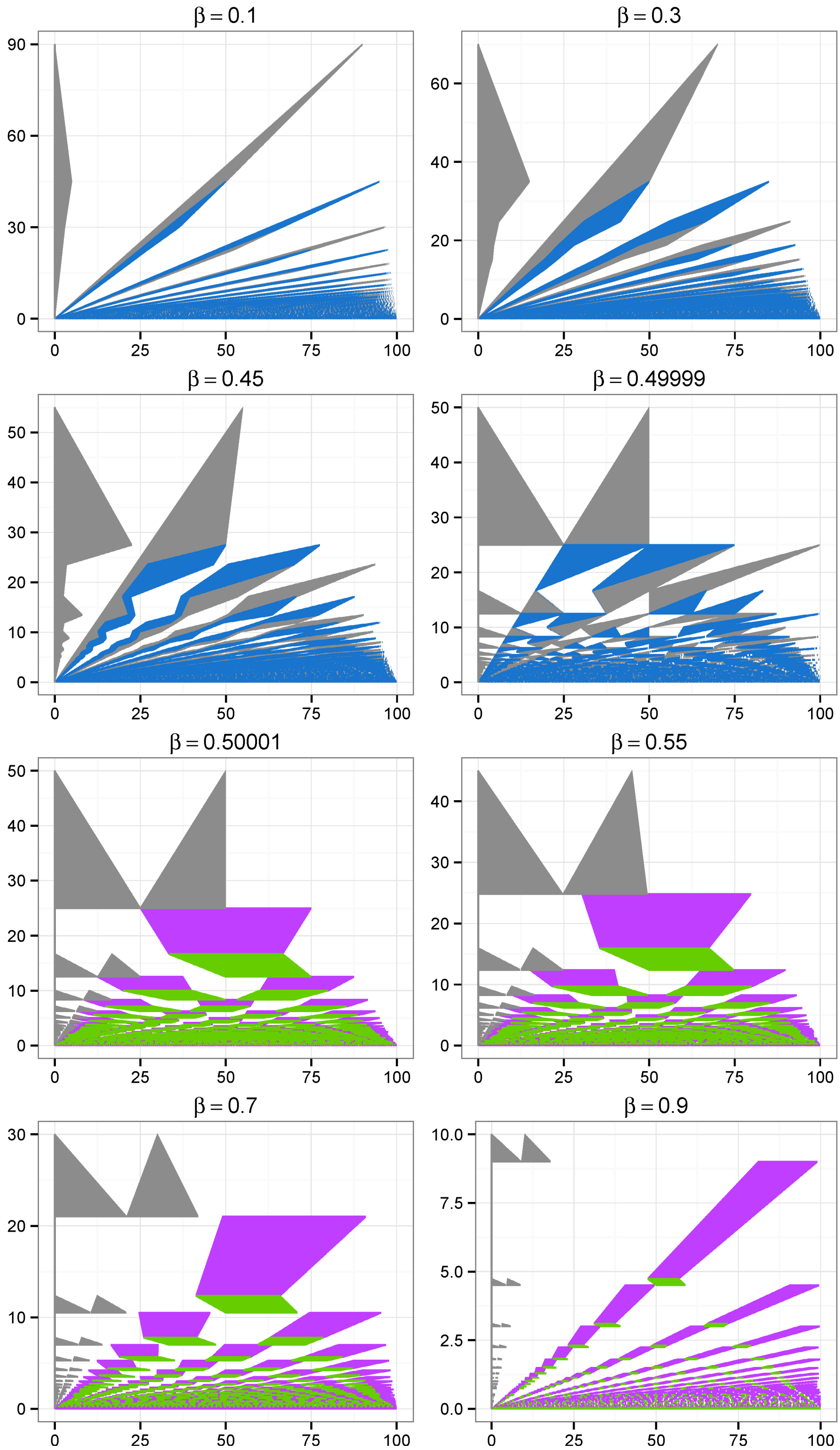

Fig. 10: Support of the unique symmetric equilibrium for various β when $v = 100$. The vertical axis plots $\delta \in (0, (1-\beta)v)$. Shaded regions on the horizontal axis for a given δ mark the support.