

# Isotonic regression discontinuity designs\*

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## Abstract

In isotonic regression discontinuity designs, the average outcome and the treatment assignment probability are monotone in the running variable. We introduce novel nonparametric estimators for sharp and fuzzy designs based on the isotonic regression which is robust to the inference after the model selection problem. The large sample distributions of introduced estimators are driven by scaled Brownian motions originating from zero and moving in opposite directions. Since these distributions are not pivotal, we also introduce a novel trimmed wild bootstrap procedure, which does not require additional nonparametric smoothing, typically needed in such settings, and show its consistency. We illustrate our approach on the well-known dataset of Lee (2008), estimating the incumbency effect in the U.S. House elections.

**Keywords:** regression discontinuity design, monotonicity, shape constraints, isotonic regression at the boundary, boundary corrections, inference after model selection, bootstrapping isotonic regression, wild bootstrap.

**JEL classification:** C14, C31.

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# 1 Introduction

Regression discontinuity designs, see [Thistlethwaite and Campbell \(1960\)](#), are widely recognized as one of the most credible quasi-experimental strategies for identifying and estimating causal effects. In a nutshell, such designs exploit the discontinuity in the treatment assignment probability around a certain cut-off value of some covariate. The discontinuous treatment assignment probability frequently occurs due to laws and regulations governing economic and political life. A comprehensive list of empirical applications using regression discontinuity designs can be found in [Lee and Lemieux \(2010\)](#); see also [Imbens and Lemieux \(2008\)](#) for the methodological review, [Cook \(2008\)](#) for the historical perspective, and [Imbens and Wooldridge \(2009\)](#), [Abadie and Cattaneo \(2018\)](#) for their place in the program evaluation literature.

On the methodological side, in the seminal paper [Hahn, Todd and Van der Klaauw \(2001\)](#) translate regression discontinuity designs into the potential outcomes framework. They also establish the nonparametric identification and argue in favor for the nonparametric local polynomial estimator. Currently, the most popular practical implementation of discontinuity designs is based on this estimator and requires the appropriate choice of the bandwidth parameter. As a result, the magnitude of the estimated causal effect can be sensitive to this choice,<sup>1</sup> which is an important threat to the internal validity.

A popular form of sensitivity analysis is to report estimates for a range of bandwidth parameters; see, e.g., [Gertler et al. \(2016\)](#). Such robustness checks may lead to the bandwidth snooping problem, and require additional adjustments, as was pointed out recently in [Armstrong and Kolesár \(2017\)](#). At the same time, the literature has made significant progress on the bandwidth selection problem; see [Imbens and Kalyanaraman \(2012\)](#), [Calonico, Cattaneo and Titiunik \(2014\)](#), and [Armstrong and Kolesár \(2018\)](#). However, empirical practice rarely acknowledges that the data-driven bandwidth leads to the inference after the model selection problem. For instance, ([Frölich and Sperlich, 2019](#), p.82) describes this problem as follows: *You can interpret the bandwidth choice as the fine-tuning of model selection: you have avoided choosing a functional form but the question of smoothness is still open. Like in model selection, once you have chosen a bandwidth, it is taken as given for any further inference. This is standard practice even if it contradicts the philosophy of purely non-parametric analysis. The reason is that an account of any further inference for the randomness of data adaptively estimated bandwidths is often just too*

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<sup>1</sup>The local polynomial estimator also depends on the choice of the kernel and the degree of the polynomial. While these choices may look innocuous, they lead to different estimates in finite samples and the  $L_2$  norm of the kernel function appears in the asymptotic distribution.

*complex. It is actually not even clear whether valid inference is possible without the assumption of having the correct bandwidth.*

In this paper, we aim to develop nonparametric estimators for monotone designs that would be robust to the inference after the model selection problem. Monotonicity restricts the expected outcome and the treatment assignment probability to be non-decreasing or non-increasing in the running variable. Since shape-restricted estimators, see [Groeneboom and Jongbloed \(2014\)](#) and [Chetverikov, Santos and Shaikh \(2018\)](#), do not require to select the bandwidth parameter, they avoid the inference after the model selection problem.

Under the local monotonicity, we obtain new identifying conditions for sharp designs that turn out to be both necessary and in a certain sense sufficient. We introduce isotonic regression discontinuity design (iRDD) estimators building on the isotonic regression; see [Brunk \(1970\)](#). These estimators exploit the global monotonicity. To the best of our knowledge, the isotonic regression has not been previously considered in the RDD setting; see [Armstrong \(2015\)](#) for the optimal adaptive one-sided test under monotonicity based on the k-nearest neighborhood estimator.

Monotone regression discontinuity designs appear frequently in empirical practice. For instance, development and educational programs are often prescribed based on poverty or achievement scores that are monotonically related to average outcomes. As an example, when evaluating the effect of subsidies for fertilizers on yields, the yield per acre is expected to be non-decreasing in the size of the farm due to the increasing returns to scale. Alternatively, when evaluating the effectiveness of the cash transfers program on the households food expenditures, we expect that more affluent households spend, on average, more on food, since food is a normal good; see [Appendix A.3](#) for a sample of other examples drawn from the empirical research.

Roughly speaking, there are two approaches to monotone regressions: direct constrained estimation with the bandwidth-free isotonic regression, and unconstrained estimation, e.g., with kernel smoothing, and ex-post monotonicization. We focus on the first approach; see [Chen et al. \(2019\)](#) for recent advances in the second approach. The isotonic regression estimator has a relatively long history in statistics, originating from the work of [Ayer et al. \(1955\)](#), [Brunk \(1956\)](#), and [van Eeden \(1958\)](#). [Brunk \(1970\)](#) derives the asymptotic distribution of the isotonic regression estimator at the *interior point* under restrictive assumptions that the regressor is deterministic and regression errors are homoskedastic. His treatment builds upon the ideas of [Rao \(1969\)](#), who derived the asymptotic distribution of the monotone density estimator, also known as the Grenander estimator, see [Grenander \(1956\)](#). [Wright \(1981\)](#) provides the final characterization of the large sample distribution for the interior point when the regressor is random and regression errors are heteroskedastic.

However, to the best of our knowledge, little is known about the behavior of the isotonic regression at the *boundary point*, which is a building block of our iRDD estimators. This situation contrasts strikingly with the local polynomial estimator, whose boundary behavior is well-understood; see [Fan and Gijbels \(1992\)](#). Most of the existing results for isotonic estimators at the boundary are available only for the Grenander estimator; see [Woodroffe and Sun \(1993\)](#), [Kulikov and Lopuhaä \(2006\)](#), and [Balabdaoui et al. \(2011\)](#). More precisely, we know that the Grenander estimator is inconsistent at the boundary of the support and that the consistent estimator can be obtained with additional boundary correction or penalization. At the same time, some isotonic estimators, e.g., in the current status model, are consistent at the boundary without corrections; see [Durot and Lopuhaä \(2018\)](#). [Anevski and Hössjer \(2002\)](#) discuss the inconsistency of the isotonic regression at the discontinuity point with deterministic equally spaced covariate and homoskedasticity. However, [Anevski and Hössjer \(2002\)](#) do not discuss whether the isotonic regression with random covariate is consistent at the boundary of its support and do not provide a consistent estimator even in the restrictive equally spaced fixed design case.

In this paper, we aim to understand the behavior of the isotonic regression with a random regressor at the boundary of its support. We show that when the regression errors conditionally on the regressor can take negative values, the isotonic regression estimator is inconsistent. The inconsistency is related to the extreme-value behavior of the closest to the boundary observation. We introduce boundary-corrected estimators and derive large sample approximations to corresponding distributions. The major technical difficulty when deriving asymptotic distributions in this setting is to establish the tightness of the maximizer of a certain empirical process. This condition is typically needed in order to apply the argmax continuous mapping theorem of [Kim and Pollard \(1990\)](#). The difficulty stems from the fact that conventional tightness results of [Kim and Pollard \(1990\)](#) and [van der Vaart and Wellner \(2000\)](#) do not always apply. For the Grenander estimator, [Kulikov and Lopuhaä \(2006\)](#) suggest a solution to this problem based on the Komlós-Major-Tusnády strong approximation. In our setting, this approach entails the strong approximation to the general empirical process, see [Koltchinskii \(1994\)](#) and [Chernozhukov, Newey and Santos \(2015\)](#), which is more problematic to apply due to slower convergence rates. Consequently, we provide the alternative generic proof which does not rely on the strong approximation and which might be applied to other boundary-corrected shape-constrained estimators.

Since the asymptotic distribution is not pivotal, we introduce a novel trimmed wild bootstrap procedure and establish its consistency. The procedure consists of trimming values of the estimated regression function that are very close to the bound-

ary when simulating wild bootstrap samples. Somewhat unexpectedly, we discover that the trimming and the appropriate boundary correction restores the consistency of the wild bootstrap without additional nonparametric smoothing or subsampling, which is typically needed in such settings. In contrast, the bootstrap typically fails at the interior point; see [Kosorok \(2008a\)](#), [Sen, Banerjee and Woodroffe \(2010\)](#), [Guntuboyina and Sen \(2018\)](#), and [Patra, Seijo and Sen \(2018\)](#) for the discussion of this problem and various case-specific remedies, and [Cattaneo, Jansson and Nagasawa \(2019\)](#) for generic solutions that apply to all cube-root consistent estimators.

The paper is organized as follows. In [Section 2](#), we look at identifying content of the monotonicity in regression discontinuity designs. [Section 3](#) describes the large sample distribution of our iRDD estimators and the trimmed wild bootstrap estimator. These results follow from a more comprehensive investigation of the large sample behavior of the isotonic regression at the boundary in [Section 4](#). In [Section 5](#), we study the finite sample performance of the iRDD estimator. [Section 6](#) estimates the effect of incumbency using the sharp iRDD on the data of [Lee \(2008\)](#). [Section 7](#) concludes. In [Appendix A.1](#), we show the inconsistency of the isotonic regression with random regressor at the boundary of the support. In [Appendix A.2](#), we collect proofs of all results discussed in the main text. Finally, in [Appendix A.3](#), we compile a list of empirical papers with monotone discontinuity designs.

## 2 Identification

Following [Hahn, Todd and Van der Klaauw \(2001\)](#), we focus on the potential outcomes framework

$$Y = Y_1D + Y_0(1 - D),$$

where  $D \in \{0, 1\}$  is a binary treatment indicator (1 if treated and 0 otherwise),  $Y_1, Y_0 \in \mathbf{R}$  are unobserved potential outcomes of treated and untreated units, and  $Y$  is the actual observed outcome.

The causal parameter of interest is the average treatment effect at the cut-off  $c$  of some running variable  $X \in \mathbf{R}$ , denoted

$$\theta = \mathbb{E}[Y_1 - Y_0 | X = c].$$

Without further assumptions,  $\theta$  is not identified in the sense that it depends on the distribution of unobserved potential outcomes  $(Y_0, Y_1)$ ; see [Holland \(1986\)](#).  $\theta$  is identified if there exists a mapping from the distribution of observed data  $(Y, D, X)$  to  $\theta$ . This mapping is given in [Eq. 1](#) below and is well-known in the literature.

In this section, we wish to see whether monotonicity has any implications for the identification in regression discontinuity designs.

The regression discontinuity design postulates that the probability of receiving the treatment changes discontinuously at the cut-off. In the iRDD, we also assume that the expected outcome and the probability of receiving the treatment are both monotone in the running variable. We introduce several assumptions below.

**Assumption (M1).** *The following functions are monotone in some neighborhood of  $c$  (i)  $x \mapsto \mathbb{E}[Y_1|X = x]$  and  $x \mapsto \mathbb{E}[Y_0|X = x]$ ; (ii)  $x \mapsto \Pr(D = 1|X = x)$ .*

**Assumption (M2).**  *$\mathbb{E}[Y_1|X = c] \geq \mathbb{E}[Y_0|X = c]$  in the non-decreasing case or  $\mathbb{E}[Y_1|X = c] \leq \mathbb{E}[Y_0|X = c]$  in the non-increasing case.*

**Assumption (RD).** *Suppose that*

$$\lim_{x \downarrow c} \Pr(D = 1|X = x) \neq \lim_{x \uparrow c} \Pr(D = 1|X = x).$$

**Assumption (OC).** *Under Assumption (M1), suppose that  $x \mapsto \mathbb{E}[Y_1|X = x]$  is right-continuous and  $x \mapsto \mathbb{E}[Y_0|X = x]$  is left-continuous at  $c$ .*

In the particular case of the sharp regression discontinuity design, all individuals with values of the running variable exceeding the cut-off  $c$  receive the treatment, while all individuals below the cut-off do not. In other words,  $D = \mathbf{1}\{X \geq c\}$ , and, whence  $x \mapsto \Pr(D = 1|X = x)$  trivially satisfies (M1), (ii). Assumption (RD) is also trivially satisfied for the sharp design. (M2) states the local responsiveness to the treatment at the cut-off. It is not necessary for the identification, but as we shall see below, it allows us to characterize in some sense both necessary and sufficient conditions. (OC) is weaker than the full continuity at the cut-off. For more general fuzzy designs, we need additionally the conditional independence assumption.

**Assumption (CI).** *Suppose that  $D \perp\!\!\!\perp (Y_1, Y_0)|X = x$  for all  $x$  in some neighborhood of  $c$ .*

**Proposition 2.1.** *Under Assumptions (M1) and (OC), in the sharp design*

$$\lim_{x \downarrow c} \mathbb{E}[Y|X = x] - \lim_{x \uparrow c} \mathbb{E}[Y|X = x] \tag{1}$$

*exists and equals to  $\theta$ . Moreover, under (M1) and (M2), if  $\theta$  equals to the expression in Eq. 1, then (OC) is satisfied.*

If additionally, Assumptions (RD) and (CI) are satisfied, and instead of (OC), we assume that  $x \mapsto \mathbb{E}[Y_0|X = x]$  and  $x \mapsto \mathbb{E}[Y_1 - Y_0|X = x]$  are continuous at  $c$ , then

$$\frac{\lim_{x \downarrow c} \mathbb{E}[Y|X = x] - \lim_{x \uparrow c} \mathbb{E}[Y|X = x]}{\lim_{x \downarrow c} \Pr(D = 1|X = x) - \lim_{x \uparrow c} \Pr(D = 1|X = x)} \quad (2)$$

exists and equals to  $\theta$ .

Proposition 2.1 shows that for sharp designs,  $\theta$  is identified for a slightly larger class of distributions than are typically discussed in the literature. It shows that the continuity at the cut-off of both conditional mean functions is not needed and that under monotonicity conditions (M1) and (M2), the one-sided continuity<sup>2</sup> turns out to be both necessary and sufficient for the identification. We illustrate this point in Figure 1. Panel (a) shows that the causal effect  $\theta$  can be identified without full continuity. Panel (b) shows that it may happen that the expression in Eq. (1) coincides with  $\theta$ , yet the two conditional mean functions do not satisfy (OC). Such counterexamples are ruled out by (M2). Inspection of the proof of the Proposition 2.1 reveals that monotonicity can be easily relaxed if we assume instead that all limits in Eq. (1) and Eq. (2) exist, in which case we recover the result of Hahn, Todd and Van der Klaauw (2001) under weaker (OC) condition for the sharp design. For the fuzzy designs, we still need the full continuity and it is an open question whether it is possible to characterize both necessary and sufficient identifying conditions.

It is also worth mentioning that for the fuzzy design, the local monotonicity of the treatment in the running variable allows to identify the causal effect for local compliers; see (Hahn, Todd and Van der Klaauw, 2001, Theorem 3).

## 3 Nonparametric inference for monotone designs

### 3.1 iRDD estimators

We are interested in estimating the average causal effect  $\theta$  of a binary treatment on some outcome. For unit  $i$ , with  $i = 1, \dots, n$ , we observe  $(Y_i, D_i, X_i)$ . Assuming that  $\theta$  is identified from the distribution of  $(Y, D, X)$  according to Eq. 1, the estimator of  $\theta$  is obtained by plugging-in corresponding estimators of conditional mean functions before and after the cut-off.

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<sup>2</sup>It is likely that the manipulation in the running variable may invalidate the left continuity of  $x \mapsto \mathbb{E}[Y_0|X = x]$ . We leave the investigation of this important problem and whether our weaker identifying conditions lead to sharper testable implications than currently known in the literature for future research.

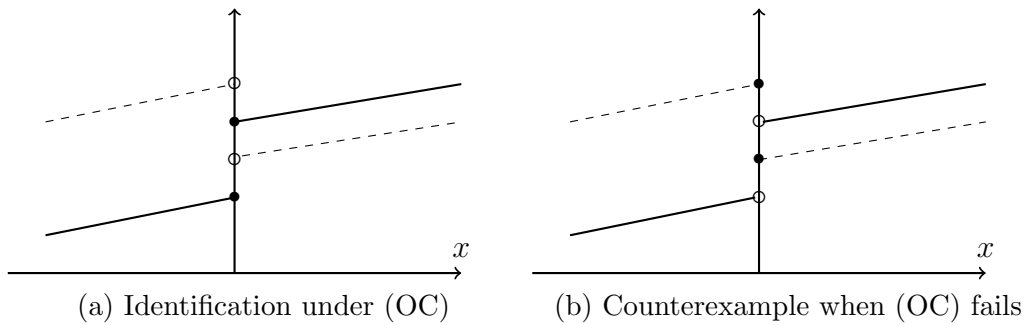


Figure 1: Identification in the sharp RDD. The thick line represents  $\mathbb{E}[Y_0|X = x], x < 0$  and  $\mathbb{E}[Y_1|X = x], x \geq 0$  while the dashed line represents  $\mathbb{E}[Y_1|X = x], x < 0$  and  $\mathbb{E}[Y_0|X = x], x \geq 0$ . The thick line coincides with  $x \mapsto \mathbb{E}[Y|X = x]$ .

iRDD estimators exploit the monotonicity of the expected outcome  $x \mapsto \mathbb{E}[Y|X = x]$  and the treatment assignment probability  $x \mapsto \Pr(D = 1|X = x)$ . For concreteness, assume that both functions are non-decreasing. We also assume that  $X$  has compact support  $[-1, 1]$  and normalize the cut-off to  $c = 0$ . This restriction is without loss of generality up to the monotone transformation, and bounded supports can also be relaxed if needed. Let

$$\mathcal{M}[a, b] = \{m : [a, b] \rightarrow \mathbf{R} : m(x_1) \leq m(x_2), \forall x_1 \leq x_2\}$$

be the set of non-decreasing functions on  $[a, b]$ . The sharp iRDD estimator consists of fitting two isotonic regressions

$$\hat{m}_-(.) = \arg \min_{m \in \mathcal{M}[-1, 0]} \sum_{i \in I_-} (Y_i - m(X_i))^2$$

and

$$\hat{m}_+(.) = \arg \min_{m \in \mathcal{M}[0, 1]} \sum_{i \in I_+} (Y_i - m(X_i))^2,$$

where  $I_-$  and  $I_+$  are sets of indices corresponding to negative and positive values of observations of the running variable, and we denote the estimated regression functions before and after the cut-off as  $\hat{m}_-(.)$  and  $\hat{m}_+(.)$ . Interestingly, in the absence of the shape constraint, the solution to the least-squares problem would just interpolate the data. The monotonicity constraint alone is powerful enough to regularized the problem and allows to obtain tuning-free non-parametric estimators.

An efficient way to compute  $\hat{m}_-(.)$  and  $\hat{m}_+(.)$  is via the pool adjacent violators algorithm; see [Ayer et al. \(1955\)](#). Although the isotonic regression features the



number of estimated parameters of the same magnitude as the sample size, the pool adjacent violators algorithm is typically computationally cheaper than nonparametric kernel estimators, and its computational complexity is closer to that of the OLS estimator.

The natural sharp iRDD estimator is

$$\hat{m}_+(X_{(k+1)}) - \hat{m}_-(X_{(k)}),$$

where  $X_{(k)}$  is the largest observation of  $X$  before the cut-off and  $X_{(k+1)}$  is the smallest observation of  $X$  after the cut-off. Unfortunately, it follows from Theorem A.1 that this estimator is inconsistent for the causal effect  $\theta$  in a substantial and natural class of distributions. The inconsistency occurs because  $X_{(k)}$  and  $X_{(k+1)}$  converge to zero too fast according to laws of the extreme value theory. Therefore, we need to take the values of  $\hat{m}_+$  and  $\hat{m}_-$  at points close to the cut-off, but not too close to offset the extreme-value behavior. We focus on the following boundary-corrected sharp iRDD estimator

$$\hat{\theta} = \hat{m}_+(n^{-1/3}) - \hat{m}_-(-n^{-1/3})$$

and show its consistency in the following section.

For fuzzy designs, we also need to estimate treatment assignment probabilities before the cut-off

$$\hat{p}_-(\cdot) = \arg \min_{p \in \mathcal{M}[-1,0]} \sum_{i \in I_-} (D_i - p(X_i))^2$$

and after the cut-off

$$\hat{p}_+(\cdot) = \arg \min_{p \in \mathcal{M}[0,1]} \sum_{i \in I_+} (D_i - p(X_i))^2.$$

The fuzzy iRDD estimator is

$$\hat{\theta}^F = \frac{\hat{m}_+(n^{-1/3}) - \hat{m}_-(-n^{-1/3})}{\hat{p}^+(n^{-1/3}) - \hat{p}_-(-n^{-1/3})}.$$

### 3.2 Large sample distribution

Put  $m(x) = \mathbb{E}[Y|X = x]$ ,  $\sigma^2(x) = \text{Var}(Y|X = x)$ ,  $p(x) = \Pr(D = 1|X = x)$ , and let  $f$  be the Lebesgue density of  $X$ . For a function  $g : [-1, 1] \rightarrow \mathbf{R}$ , with some abuse of notation, define  $g_+ = \lim_{x \downarrow 0} g(x)$  and  $g_- = \lim_{x \uparrow 0} g(x)$ . Put also  $\varepsilon = Y - m(X)$ .

**Assumption 3.1.**  $(Y_i, D_i, X_i)_{i=1}^n$  is an *i.i.d.* sample of  $(Y, D, X)$  such that (i)  $\mathbb{E}[|\varepsilon|^{2+\delta}|X] \leq C < \infty$  for some  $\delta > 0$  and  $m$  is uniformly bounded; (ii) the distribution of  $X$  has Lebesgue density  $f$ , uniformly bounded away from zero and infinity,

and such that  $f_-$  and  $f_+$  exist; (iii)  $\sigma^2$  is uniformly bounded on  $[-1, 1]$  and  $\sigma_+^2$  and  $\sigma_-^2$  exist; (iv)  $m$  is continuously differentiable in the right and left neighborhoods of zero with  $m'_-, m'_+ > 0$ .

This assumption is comparable to assumptions typically used in the RDD literature, e.g., see (Hahn, Todd and Van der Klaauw, 2001, Theorem 4) with the difference that we do not need to select the kernel and the bandwidth and to make appropriate assumption on those. We are also agnostic about the smoothness of the marginal density of  $X$ , and only assume the existence of one-sided derivatives of conditional means. The differentiability of  $m$  will be relaxed to the Hölder continuity when it comes to inference. Under the stated assumption, the large sample distribution of the sharp iRDD estimator can be approximated by the difference of slopes of the greatest convex minorants<sup>3</sup> of two scaled independent Brownian motions plus the parabola originating from zero and running in opposite directions. For a function  $g : A \rightarrow \mathbf{R}$  at  $x$  with  $A \subset \mathbf{R}$ , let  $D_A^L(g)(x)$  denote the left derivative of its greatest convex minorant.

**Theorem 3.1.** *Under Assumption 3.1*

$$n^{1/3}(\hat{\theta} - \theta) \xrightarrow{d} D_{[0, \infty)}^L \left( \sqrt{\frac{\sigma_+^2}{f_+}} W_t^+ + \frac{t^2}{2} m'_+ \right) (1) - D_{(-\infty, 0]}^L \left( \sqrt{\frac{\sigma_-^2}{f_-}} W_t^- + \frac{t^2}{2} m'_- \right) (-1),$$

where  $W_t^+$  and  $W_t^-$  are two independent standard Brownian motions originating from zero and running in opposite directions.

This result is a consequence of a more general result for the isotonic regression estimator at the boundary  $\hat{m}(cn^{-1/3})$  with arbitrary constant  $c > 0$  in the following section. We argue that it is natural to set  $c = 1$ . Indeed, with this choice the asymptotic distribution at the boundary point coincides with a one-sided version of the asymptotic distribution of the isotonic regression at the interior point, see Cattaneo, Jansson and Nagasawa (2019). It is worth stressing that this choice is robust to the scale of the data, since we normalize  $X$  to belong to  $[-1, 1]$ . Interestingly, as was pointed out in Kulikov and Lopuhaä (2006), for the boundary point we can try to improve upon  $c = 1$ , which is not possible for the interior point.

We deliberately, suggest that tuning the constant  $c$  should be avoided in practice, since any data-driven choice of  $c$  will lead to the inference after the model selection problem that we try to avoid in the first place. It is also worth mentioning that the local polynomial estimator involves the choice of the kernel function, the choice of

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<sup>3</sup>The greatest convex minorant of a function  $g$  is the largest convex function dominated by  $g$ .

the degree of the polynomial, and the choice of the bandwidth parameter and that all these choices have important implications for the finite-sample performance of the estimator. The data-driven choice of the bandwidth parameter may allow us to adapt better to specific features like the heteroskedasticity or the density near the cut-off at the cost of introducing the random tuning parameter which may invalidate inference<sup>4</sup>. It is also not clear if some of the popular data-driven bandwidth selectors, e.g., the cross-validation actually adapt to such features of the underlying DGP.

To state the large sample distribution for the fuzzy iRDD, define additionally the covariance function  $\rho(x) = \mathbb{E}[\varepsilon(D - p(X))|X = x]$ .

**Theorem 3.2.** *Suppose that Assumption 3.1 is satisfied. Suppose additionally that  $p$  is continuously differentiable in the right and left neighborhoods of zero with  $p'_-, p'_+ > 0$ , and  $p_+, p_- \in (0, 1)$ . Then*

$$n^{1/3}(\hat{\theta}^F - \theta) \xrightarrow{d} \frac{1}{p_+ - p_-} \xi_1 - \frac{m_+ - m_-}{(p_+ - p_-)^2} \xi_2$$

with

$$\begin{aligned} \xi_1 &= D_{[0, \infty)}^L \left( \sqrt{\frac{\sigma_+^2}{f_+}} W_t^+ + \frac{t^2}{2} m'_+ \right) (1) - D_{(-\infty, 0]}^L \left( \sqrt{\frac{\sigma_-^2}{f_-}} W_t^- + \frac{t^2}{2} m'_- \right) (-1) \\ \xi_2 &= D_{[0, \infty)}^L \left( \sqrt{\frac{p_+(1-p_+)}{f_+}} B_t^+ + \frac{t^2}{2} p'_+ \right) (1) - D_{(-\infty, 0]}^L \left( \sqrt{\frac{p_-(1-p_-)}{f_-}} B_t^- + \frac{t^2}{2} p'_- \right) (-1), \end{aligned}$$

where  $W_t^+, W_t^-, B_t^+$ , and  $B_t^-$  are standard Brownian motions such that any two processes with different signs are independent, and

$$\begin{aligned} \text{Cov}(W_t^+, B_s^+) &= \frac{\rho_+}{\sqrt{\sigma_+^2 p_+(1-p_+)}} t \wedge s, \\ \text{Cov}(W_t^-, B_s^-) &= \frac{\rho_-}{\sqrt{\sigma_-^2 p_-(1-p_-)}} t \wedge s. \end{aligned}$$

Both results follow from the CLT for the boundary-corrected isotonic regression estimator obtained in Theorem 4.1. A consequence of Theorems 3.1 and 3.2 is that the boundary-corrected iRDD estimators  $\hat{\theta}$  and  $\hat{\theta}^F$  are consistent in probability for the causal effect parameter  $\theta$  and provide valid point estimates. However, using the asymptotic distribution for inference is problematic because it depends on quantities

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<sup>4</sup>In Section 5, we investigate the finite-sample behavior of the iRDD estimator with different designs featuring heteroskedasticity and low density near the cut-off.

that should be estimated nonparametrically. While the monotonicity allows us to obtain bandwidth-free estimates of  $m_+$  and  $m_-$ , the nonparametric estimation of other features of the asymptotic distribution may involve additional tuning parameters.<sup>5</sup> Somewhat more troubling is that appearance of derivatives in the asymptotic distribution leads to the inconsistency of the bootstrap.

In the following section, in Theorem 3.3, we show that the valid trimmed wild bootstrap confidence intervals can be obtained when we use slightly different boundary correction. Whether one should use Theorem 3.1 or Theorem 3.3 depends on the objective of the researcher. If one is interested in point estimates, then one should use the estimator in Theorem 3.1. On the other hand, if one is interested in constructing confidence intervals robust to the inference after model selection or in point estimates under weaker smoothness restrictions than differentiability, then we recommend using the estimator from Theorem 3.3.

### 3.3 Trimmed wild bootstrap

In this section, we introduce a novel trimmed wild bootstrap inferential procedure. The procedure is as follows. First, we construct the trimmed estimator

$$\tilde{m}(x) = \begin{cases} \hat{m}_-(x), & x \in (-1, -n^{-1/2}) \\ \hat{m}_-(-n^{-1/2}), & x \in [-n^{-1/2}, 0] \\ \hat{m}_+(n^{-1/2}), & x \in (0, n^{-1/2}] \\ \hat{m}_+(x), & x \in (n^{-1/2}, 1). \end{cases}$$

Second, we simulate the wild bootstrap samples as follows

$$Y_i^* = \tilde{m}(X_i) + \eta_i^* \tilde{\varepsilon}_i, \quad i = 1, \dots, n,$$

where  $(\eta_i^*)_{i=1}^n$  are i.i.d. multipliers, independent of  $(Y_i, D_i, X_i)_{i=1}^n$ , and  $\tilde{\varepsilon}_i = Y_i - \tilde{m}(X_i)$ . We call this procedure trimming since it trims the estimator close to boundaries when we generate bootstrap samples. Trimming is needed in addition to the boundary correction of the iRDD estimator

$$\hat{\theta} = \hat{m}_+(n^{-1/2}) - \hat{m}_-(-n^{-1/2})$$

and its bootstrapped counterpart

$$\hat{\theta}^* = \hat{m}_+^*(n^{-1/2}) - \hat{m}_-^*(-n^{-1/2}),$$

---

<sup>5</sup>Tuning-free estimates might be obtained under additional shape restrictions, but, e.g., the monotonicity of marginal densities is obviously questionable in most applications. Additionally, we would need to discretize and to truncate the time.

where  $\hat{m}_+^*$  and  $\hat{m}_-^*$  are isotonic estimators computed from the bootstrapped sample  $(Y_i^*, D_i, X_i)_{i=1}^n$  similarly to  $\hat{m}_-$  and  $\hat{m}_+$ . For the trimmed wild bootstrap scheme, we will operate in a different asymptotic regime where we relax the assumption that  $m$  is continuously differentiable. We say that  $m$  is  $\gamma$ -Hölder continuous in the left or right neighborhood of zero if there exists a positive constant  $C < \infty$  such that

$$|m(x) - m_+| \leq C|x|^\gamma, x > 0 \quad \text{and} \quad |m(x) - m_-| \leq C|x|^\gamma, x < 0.$$

The Hölder continuity restricts how wiggly the regression function is in the neighborhood of zero.

**Theorem 3.3.** *Suppose that Assumptions 3.1 (ii)-(iii) are satisfied and that  $m$  is  $\gamma$ -Hölder continuous in the left and the right neighborhoods of zero with  $\gamma > 1/2$ . Suppose also that  $m$  is continuous on  $[-1, 0)$  and  $(0, 1]$ , and that  $\mathbb{E}[\varepsilon^4|X] \leq C < \infty$ . If multipliers  $(\eta_i^*)_{i=1}^n$  are such that  $\mathbb{E}\eta_i^* = 0$ ,  $\text{Var}(\eta_i^*) = 1$ , and  $\mathbb{E}|\eta_i^*|^{2+\delta} < \infty$  for some  $\delta > 0$ , then for every  $u \in \mathbf{R}$*

$$\left| \Pr^* \left( n^{1/4}(\hat{\theta}^* - \hat{\theta}) \leq u \right) - \Pr \left( n^{1/4}(\hat{\theta} - \theta) \leq u \right) \right| \xrightarrow{P} 0,$$

where  $\Pr^*(\cdot) = \Pr(\cdot | (X_i, Y_i)_{i=1}^\infty)$ .

The bootstrapped estimator converges at a slightly slower than the cube-root rate, which is a consequence of using less stringent  $\gamma$ -Hölder continuity assumption with smoothness index  $\gamma > 1/2$ , instead of the full differentiability. For the comparison, the "irregular" trajectory of the Brownian motion is  $\gamma$ -Hölder smooth for every  $\gamma < 1/2$ . Consequently, we could not find any good reasons to relax the smoothness constraint to  $\gamma < 1/2$ , since in the vast majority of economic applications, average outcomes are expected to be smoother than the Brownian motion. While the Theorem 3.3 allows us to do inference under the minimal smoothness assumption on the conditional mean function, the main attractiveness of this asymptotic regime comes from the fact that the bootstrap works without additional nonparametric smoothing. However, if the researcher is interested in point estimates only, and is willing to assume that  $m$  is differentiable, we recommend to use the estimator and the asymptotic regime described in the Theorem 3.1.

For fuzzy designs, the bootstrapped estimator is

$$\hat{\theta}^{F*} = \frac{\hat{m}_+^*(n^{-1/2}) - \hat{m}_-^*(-n^{-1/2})}{\hat{p}_+(n^{-1/2}) - \hat{p}_-(-n^{-1/2})}.$$

The proof of the bootstrap consistency is similar to the proof of Theorems 3.2 and 3.3, so we omit it.

## 4 Boundary-corrected isotonic regression

### 4.1 Estimator

We focus on the generic nonparametric regression model

$$Y = m(X) + \varepsilon, \quad \mathbb{E}[\varepsilon|X] = 0$$

with  $m : [0, 1] \rightarrow \mathbf{R}$ . Let  $f$  denote the Lebesgue density of  $X$  and  $\sigma^2(x) = \text{Var}(Y|X = x)$  be the conditional variance. We assume that the conditional mean function  $m(x) = \mathbb{E}[Y|X = x]$  belongs to the set of non-decreasing functions  $\mathcal{M}[0, 1]$ .

The isotonic regression, see [Brunk \(1970\)](#), solves the restricted least-squares problem

$$\hat{m}(\cdot) = \arg \min_{m \in \mathcal{M}[0,1]} \frac{1}{n} \sum_{i=1}^n (Y_i - m(X_i))^2$$

The estimator is uniquely determined at data points and is conventionally interpolated as the piecewise constant function with jumps at data points elsewhere (more general polynomial interpolation is also possible). Alternatively, the isotonic regression estimator solves

$$\arg \min_{\phi_1 \leq \phi_2 \leq \dots \leq \phi_n} \sum_{i=1}^n (Y_{(i)} - \phi_i)^2,$$

where  $Y_{(1)}, Y_{(2)}, \dots, Y_{(n)}$  are values of  $Y$  corresponding to the sample ordered according to values of the regressor  $X_{(1)} < X_{(2)} < \dots < X_{(n)}$ . In this section, we provide a comprehensive description of the asymptotic behavior of boundary-corrected estimators  $\hat{m}(cn^{-a})$  with  $c > 0$  and  $a \in (0, 1)$ .

### 4.2 Large sample distribution

There is a voluminous literature on the isotonic regression. [Wright \(1981\)](#) derives the large sample approximation to the distribution of the isotonic regression estimator with random regressor and heteroskedasticity at the *interior point*. To the best of our knowledge, the behavior of the isotonic regression estimator at the *boundary of the support* is not known. [Theorem A.1](#) shows that the isotonic regression estimator is inconsistent at the boundary for a large and natural class of distributions. In this section, we provide a complete description of the large sample behavior of the boundary-corrected isotonic regression estimator.

We are interested in estimating the value of the regression function at the boundary of its support  $[0, 1]$ . For simplicity, we focus on the regression function at the left boundary, denoted  $m(0) = \lim_{x \downarrow 0} m(x)$ .

**Assumption 4.1.**  $(Y_i, X_i)_{i=1}^n$  is an i.i.d. sample of  $(Y, X)$  such that (i)  $\mathbb{E}[|\varepsilon|^{2+\delta}|X] \leq C < \infty$  for some  $\delta > 0$  and  $m$  is uniformly bounded; (ii)  $F$  has Lebesgue density  $f$ , uniformly bounded away from zero and infinity, and  $f(0) = \lim_{x \downarrow 0} f(x)$  exists; (iii)  $\sigma^2$  is uniformly bounded on  $[0, 1]$  and  $\sigma^2(0) = \lim_{x \downarrow 0} \sigma^2(x)$  exists; (iv)  $m$  is continuously differentiable in the neighborhood of zero with  $m'(0) = \lim_{x \downarrow 0} m'(x) > 0$ .

Let  $D_A^L(Z_t)(s)$  denote the left derivative of the greatest convex minorant of  $t \mapsto Z_t$  on  $A \subset \mathbf{R}$  at a point  $t = s$ . We say that  $m$  is  $\gamma$ -Hölder continuous in the neighborhood of zero if there exists a constant  $C < \infty$  such that for all  $x > 0$  in this neighborhood

$$|m(x) - m(0)| \leq C|x|^\gamma.$$

**Theorem 4.1.** *Suppose that Assumption 4.1 is satisfied and let  $c > 0$ . Then*

(i) For  $a \in (0, 1/3)$

$$n^{1/3} (\hat{m}(cn^{-a}) - m(0)) \xrightarrow{d} \left| \frac{4m'(0)\sigma^2(0)}{f(0)} \right|^{1/3} \arg \max_{t \in \mathbf{R}} \{W_t - t^2\}.$$

(ii) For  $a \in [1/3, 1)$

$$n^{(1-a)/2} (\hat{m}(cn^{-a}) - m(0)) \xrightarrow{d} D_{[0, \infty)}^L \left( \sqrt{\frac{\sigma^2(0)}{cf(0)}} W_t + \frac{t^2 c}{2} m'(0) \mathbb{1}_{a=1/3} \right) (1),$$

where for  $a \in (1/3, 1)$ , we can replace Assumption 4.1 (iv) by the  $\gamma$ -Hölder continuity with  $\gamma > (1 - a)/2a$ .

The proof of the Theorem can be found in the appendix. The most challenging part of the proof is establishing tightness when  $a \in (1/3, 1)$ . The difficulty comes from the fact that in this regime, the quadratic term vanishes and the standard tightness results for isotonic estimators, e.g., see [Kim and Pollard \(1990\)](#) and [van der Vaart and Wellner, 2000](#), Theorem 3.2.5) do not apply. For the Grenander estimator, [Kulikov and Lopuhaä \(2006\)](#) suggest a solution to this problem based on the Komlós-Major-Tusnády strong approximation. In our case, we would need to apply the strong approximation to the general empirical process, see [Koltchinskii \(1994\)](#), which gives suboptimal results due to slower convergence rates and uniform boundedness restrictions.

For "slow" boundary corrections with  $a \in (0, 1/3)$ , the distribution is similar to the asymptotic distribution at the interior point, c.f., [Wright \(1981\)](#). However, such

estimators typically have large finite-sample bias compared to estimators with correction  $a \in [1/3, 1)$ . For "fast" boundary corrections with  $a \in [1/3, 1)$ , the distribution is different, and when  $a > 1/3$ , the convergence rate is slower than the cube-root due to less stringent smoothness assumptions. For instance, when  $a = 1/2$ , we only need the Hölder smoothness with  $\gamma > 1/2$ , instead of assuming that  $m'$  exists, in which case we obtain the convergence rate  $n^{-1/4}$ . This case is the most interesting when it comes to inference, because as we shall show in the following section, in this regime the bootstrap works without additional nonparametric smoothing.

As a consequence of Theorem 4.1, we obtain bandwidth-free rate-optimal point estimates, setting  $a = 1/3$  and  $c = 1$ , in which case

$$n^{1/3}(\hat{m}(n^{-1/3}) - m(0)) \xrightarrow{d} D_{[0, \infty)}^L \left( \sqrt{\frac{\sigma^2(0)}{f(0)}} W_t + \frac{t^2}{2} m'(0) \right) (1).$$

Consequently,  $\hat{m}(n^{-1/3})$  is a consistent estimator of  $m(0)$ . One can try to use the data-driven constant  $c$ , e.g., optimizing the asymptotic MSE, but this entails estimating unknown features of the asymptotic distribution, and creates the inference after the model selection, which we want to avoid in the first place.

**Remark 4.1.** *It is possible to show that for the non-decreasing function  $m : [-1, 0] \rightarrow \mathbf{R}$  and  $a \in (0, 1/3)$  the asymptotic distribution of  $\hat{m}(-cn^{-a})$  is the same while for  $a \in [1/3, 1)$*

$$n^{(1-a)/2} (\hat{m}(-cn^{-a}) - m(0)) \xrightarrow{d} D_{(-\infty, 0]}^L \left( \sqrt{\frac{\sigma^2(0)}{cf(0)}} W_t + \frac{t^2 c}{2} m'(0) \mathbb{1}_{a=1/3} \right) (-1),$$

where  $f(0), m(0)$ , and  $\sigma^2(0)$  are limits from the left assuming they satisfy Assumption 4.1.

**Remark 4.2.** *It is easy to see that the distribution remains the same if  $m : [a, b] \rightarrow \mathbf{R}$  for arbitrary  $a < b$ .*

### 4.3 Trimmed wild bootstrap

It is well-known that the bootstrap fails for various isotonic estimators at the interior point<sup>6</sup>; see Kosorok (2008a), Sen, Banerjee and Woodroffe (2010), and Cattaneo,

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<sup>6</sup>More generally, we know that bootstrap fails for estimators with Chernoff (1964) limiting distribution; see also Delgado, Rodríguez-Poo and Wolf (2001), Léger and MacGibbon (2006), Abrevaya and Huang (2005) for early evidences.



Jansson and Nagasawa (2019). Several resampling schemes are available in the literature, including the smoothed nonparametric or m-out-of-n bootstrap, see Sen, Banerjee and Woodroffe (2010) and Patra, Seijo and Sen (2018); reshaping the objective function, see Cattaneo, Jansson and Nagasawa (2019); and smoothed residual bootstrap, see Guntuboyina and Sen (2018). Interestingly, as we shall show below, for the boundary point, an appropriate boundary correction restores the consistency of the bootstrap. Consequently, we focus on more conventional bootstrap inferences.

The wild bootstrap, see Wu (1986) and Liu (1988), is arguably the most natural resampling scheme for the nonparametric regression. Unlike the naive nonparametric bootstrap, the wild bootstrap imposes the structure of the nonparametric regression model in the bootstrap world, so we may expect it to work better in finite samples than resampling methods that do not incorporate such information. At the same time, unlike the residual bootstrap, it allows for higher-order dependence between regressors and regression errors, such as heteroskedasticity.

The bootstrap procedure is as follows. First, we obtain the isotonic regression estimator  $\hat{m}$ , construct the trimmed estimator

$$\tilde{m}(x) = \begin{cases} \hat{m}(x), & x \in (n^{-a}, 1) \\ \hat{m}(cn^{-a}), & x \in [0, n^{-a}], \end{cases}$$

and compute residuals  $\tilde{\varepsilon}_i = Y_i - \tilde{m}(X_i)$ . Second, we construct the wild bootstrap samples as follows:

$$Y_i^* = \tilde{m}(X_i) + \eta_i^* \tilde{\varepsilon}_i, \quad i = 1, \dots, n,$$

where  $(\eta_i^*)_{i=1}^n$  are i.i.d., independent of  $(Y_i, X_i)_{i=1}^n$ , random variables such that  $\mathbb{E}\eta_i^* = 0$ ,  $\text{Var}(\eta_i^*) = 1$ , and  $\mathbb{E}|\eta_i^*|^{2+\delta} < \infty$ .

Let  $\text{Pr}^*(\cdot) = \text{Pr}(\cdot|\mathbf{X})$  denote the bootstrap probability conditionally on the data  $\mathbf{X} = (Y_i, X_i)_{i=1}^\infty$ , and let  $\hat{m}^*$  be the isotonic regression estimator computed from the bootstrapped sample  $(Y_i^*, X_i)_{i=1}^n$ .

**Theorem 4.2.** *Suppose that Assumptions 4.1 (ii) and (iii) are satisfied. Suppose additionally that (i)  $\mathbb{E}[\varepsilon^4|X] \leq C < \infty$  and  $m$  is uniformly continuous; (iv)  $m$  is  $\gamma$ -Hölder continuous with  $\gamma > 1/2$ . Then for every  $u < 0$  and  $a \in (1/3, 1)$*

$$\text{Pr}^* \left( n^{(1-a)/2} (\hat{m}^*(cn^{-a}) - \hat{m}(cn^{-a})) \leq u \right) \xrightarrow{P} \Pr \left( D_{[0, \infty)}^L \left( \sqrt{\frac{\sigma^2(0)}{cf(0)}} W_t \right) (1) \leq u \right)$$

and

$$\text{Pr}^* \left( n^{(1-a)/2} (\hat{m}^*(cn^{-a}) - \hat{m}(cn^{-a})) \geq 0 \right) \xrightarrow{P} 0.$$

In practice we recommend to use  $c = 1$  and  $a = 1/2$ , which ensures that the quadratic term disappears sufficiently fast from the asymptotic distribution and leads to the estimator that converges at a reasonable  $n^{1/4}$  convergence rate<sup>7</sup>. Therefore, this gives us a natural way to avoid the inference after the model selection problem.

**Remark 4.3.** *It is possible to show that for the non-decreasing function  $m : [-1, 0] \rightarrow \mathbf{R}$  and  $a \in (1/3, 1)$*

$$\Pr^* \left( n^{(1-a)/2} (\hat{m}^*(-cn^{-a}) - \hat{m}(-cn^{-a})) \leq u \right) \xrightarrow{P} \Pr \left( D_{(-\infty, 0]}^L \left( \sqrt{\frac{\sigma^2(0)}{cf(0)}} W_t \right) (-1) \leq u \right)$$

and

$$\Pr^* \left( n^{(1-a)/2} (\hat{m}^*(-cn^{-a}) - \hat{m}(-cn^{-a})) \geq 0 \right) \xrightarrow{P} 0.$$

where  $f(0)$ ,  $m(0)$ , and  $\sigma^2(0)$  are now left-sided limits, assuming that they exist.

## 5 Monte Carlo experiments

In this section, we study the finite-sample performance of our iRDD estimator. We simulate 5,000 samples of size  $n \in \{100, 500, 1000\}$  as follows:

$$Y = m(X) + \theta \mathbb{1}_{[0,1]}(X) + \sigma(X)\varepsilon,$$

where  $\varepsilon \sim N(0, 1)$  and  $\varepsilon \perp\!\!\!\perp X$ .

In our baseline DGP, we set  $m(x) = x^3 + 0.25x$ ,  $\theta = 1$ ,  $X \sim 2 \times \text{Beta}(2, 2) - 1$ , and  $\sigma(x) = 1$  (homoskedasticity), and  $\sigma(x) = \sqrt{x + 1}$  (heteroskedasticity). We compute the boundary-corrected estimator using the pool adjacent violators algorithm. Figure 2 shows the piecewise-constant interpolation of fitted isotonic regression functions before and after the cut-off, the population regression function, and the data. Our data-generating process (DGP) features a low signal-to-noise ratio, and the magnitude of the jump discontinuity is not detectable from visual inspection of the scatter plot.

Figure 3 illustrates the finite sample distribution of the boundary-corrected iRDD estimator for samples of different sizes. The exact finite-sample distribution is centered around the population value of the parameter and concentrates around the population parameter as the sample size increases.

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<sup>7</sup>This is also supported by our Monte Carlo experience, where we can see that gains from optimizing the constant are quite modest and that  $a = 1/2$  works the best.

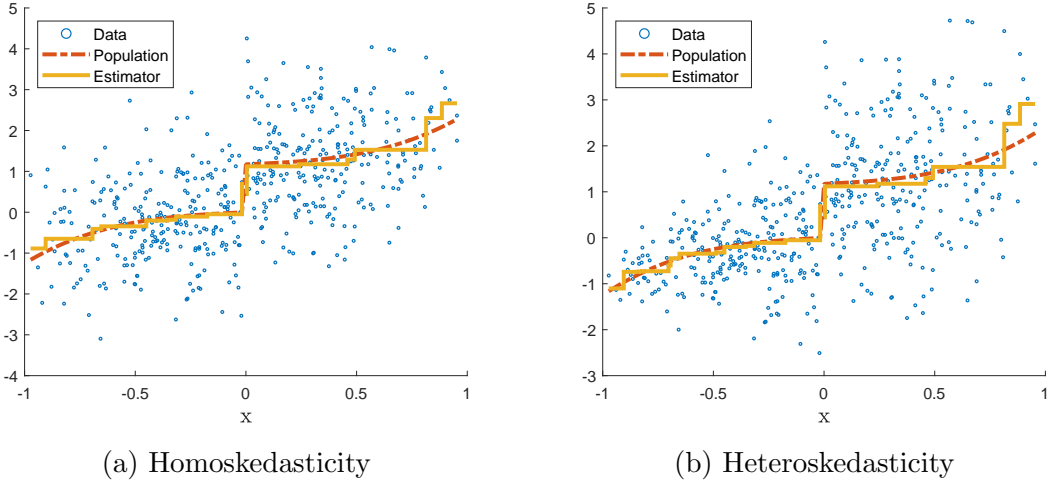


Figure 2: Single MC experiment,  $n = 500$ .

Table 1 reports results of more comprehensive Monte Carlo experiments for several data-generating processes and shows the exact finite-sample bias, variance, and MSE of our iRDD estimator. We consider the following variations of the baseline DGP with two different functional forms and different amount of density near the cut-off:

1. DGP 1 sets  $X \sim 2 \times \text{Beta}(2, 2) - 1$  and  $m(x) = \exp(0.25x)$ ;
2. DGP 2 sets  $X \sim 2 \times \text{Beta}(0.5, 0.5) - 1$  (low density) and  $m(x) = \exp(0.25x)$ ;
3. DGP 3 sets  $X \sim 2 \times \text{Beta}(2, 2) - 1$  and  $m(x) = x^3 + 0.25x$ ;
4. DGP 4 sets  $X \sim 2 \times \text{Beta}(0.5, 0.5) - 1$  (low density) and  $m(x) = x^3 + 0.25x$ ;

Results of our experiments are consistent with the asymptotic theory. The bias, the variance, and the MSE reduce dramatically with the sample size. As expected, the MSE is higher when the density near the cut-off is lower. The heteroskedasticity does not have a noticeable impact on the performance. We also investigated the sensitivity to our default choice of the boundary-correcting sequence  $n^{-1/3}$  and find that choices  $n^{-1/6}$  and  $n^{-5/12}$  are inferior in terms of the MSE. Results of these sensitivity checks are available upon request.

The situation changes dramatically once we shift our attention from point estimates to the inference. As we know from Theorem 4.2, the trimmed wild bootstrap works for fast boundary corrections with  $a > 1/3$ . Theorem 3.3 shows bootstrap

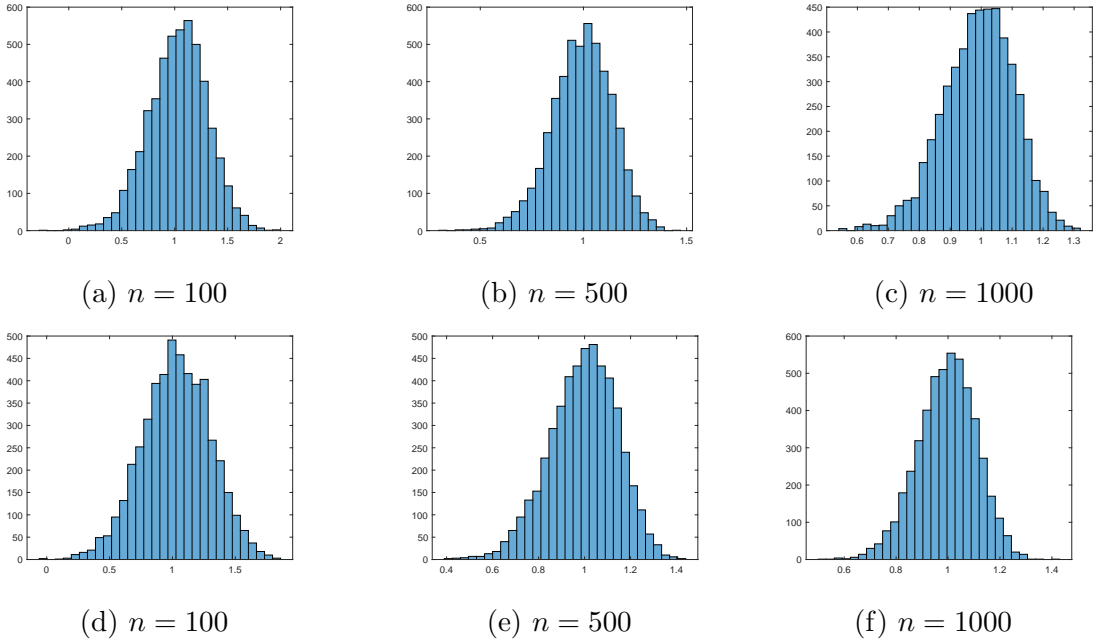


Figure 3: Finite sample distribution of the iRDD estimator: homoskedastic design in (a)-(c) and heteroskedastic in (d)-(f). Results based on 5,000 MC experiments.

consistency for the correction  $a = 1/2$ . With this choice, we have decent convergence rate, and at the same time, the quadratic term vanishes, making the bootstrap consistent.

In Figure 4, we plot the exact distribution  $n^{1/4}(\hat{\theta} - \theta)$  and the bootstrap distribution  $n^{1/4}(\hat{\theta}^* - \hat{\theta})$  for samples of size  $n \in \{100, 1000\}$ . As we can see from panels (b) and (e), the naive wild bootstrap without trimming and boundary correction does not work. On the other hand, the trimmed wild bootstrap mimics the finite-sample distribution. In our simulations, we use Rademacher multipliers for the bootstrap in all our experiments, i.e.,  $\eta_i \in \{-1, 1\}$  with equal probabilities.

## 6 Empirical illustration

Do incumbents have any electoral advantage? An extensive literature, going back at least to [Cummings Jr. \(1967\)](#), aims to answer this question. Estimating the causal effect is elusive because incumbents, by definition, are more successful politicians. Using the regression discontinuity design, [Lee \(2008\)](#), documents that for the U.S.

Table 1: MC experiments

$n$	Homoskedasticity			Heteroskedasticity		
	Bias	Var	MSE	Bias	Var	MSE
<b>DGP 1</b>						
100	0.020	0.077	0.077	0.027	0.077	0.078
500	-0.008	0.022	0.022	-0.006	0.022	0.022
1000	-0.006	0.013	0.013	-0.005	0.013	0.013
<b>DGP 2</b>						
100	-0.153	0.137	0.160	-0.138	0.141	0.160
500	-0.081	0.044	0.050	-0.077	0.045	0.050
1000	-0.063	0.027	0.031	-0.060	0.027	0.031
<b>DGP 3</b>						
100	0.093	0.089	0.097	0.098	0.090	0.099
500	0.017	0.024	0.024	0.018	0.024	0.024
1000	0.006	0.015	0.015	0.007	0.015	0.015
<b>DGP 4</b>						
100	-0.021	0.184	0.184	-0.014	0.186	0.186
500	-0.038	0.050	0.051	-0.036	0.050	0.052
1000	-0.036	0.031	0.032	-0.035	0.031	0.032

Exact finite-sample bias, variance, and MSE of  $\hat{\theta}$ . 5,000 experiments.

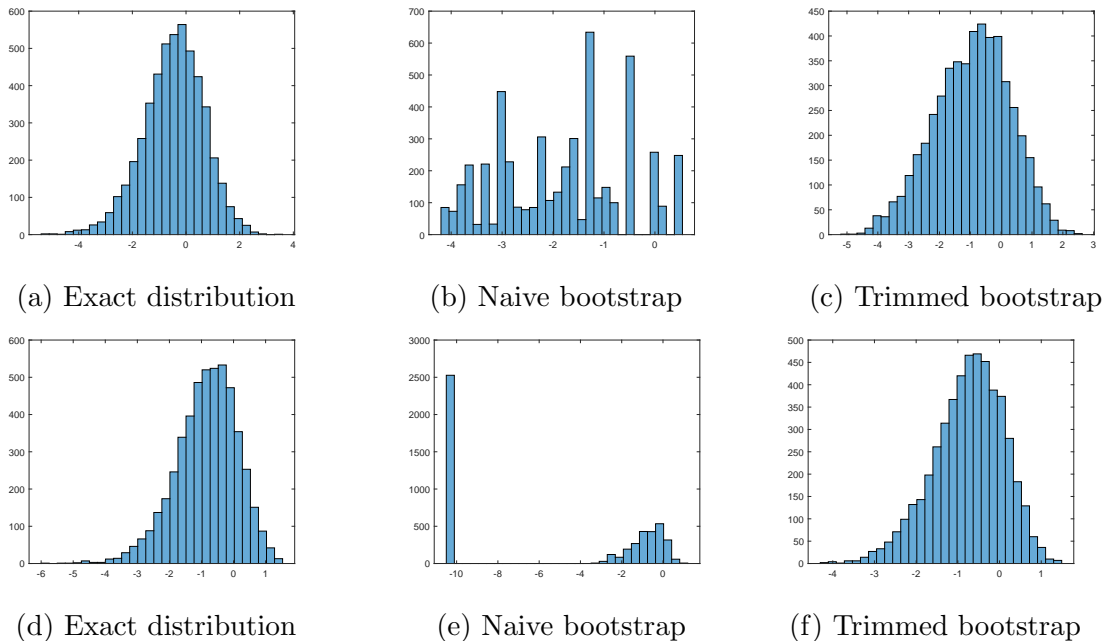


Figure 4: Exact finite sample distribution and the bootstrap distribution. Sample size:  $n = 100$  in panels (a)-(c) and  $n = 1,000$  in panels (d)-(f). 5,000 MC experiments.

Congressional elections during 1946-1998, the incumbency advantage is 7.7% of the votes share on the next election. The design is plausibly monotone, since we do not expect that candidates with a larger margin should have a smaller vote share on the next election, at least on average. The unconstrained regression estimates presented by Lee (2008) also support the monotonicity empirically.

We revisit the main finding of Lee (2008) with our sharp iRDD estimator. The dataset is publicly available as a companion for the book by Angrist and Pischke (2008). Figure 5 presents the isotonic regression estimates<sup>8</sup> of the average vote share for the Democratic party at next elections as a function of the vote share margin at the previous election (left panel). There is a pronounced jump in average outcomes for Democrats who barely win the election, compared to the results for the penultimate election (right panel). We find the point estimate of 13.8% with the 95% confidence interval [6.6%, 26.5%]. While we reject the hypothesis that the incumbency advantage did not exist, our confidence intervals give a wider range of

<sup>8</sup>We use the piecewise-constant interpolation, but a higher-order polynomial interpolation is another alternative that would produce visually more appealing estimates.

estimates. Our confidence interval may be conservative if the underlying regression is two times differentiable, however, it is robust to the failure of this assumption as well as to the inference after the model selection issues.

Of course, different approaches work differently in finite samples and it is hard to have a definite comparison. Lee (2008) estimates the causal effect by fitting parametric regressions with the global fourth-degree polynomial, which might be unstable at the boundary; see Gelman and Imbens (2019). We, on the other hand, rely on the nonparametric boundary-corrected isotonic regression. We also compute iRDD estimates using isotonic regressions without the boundary correction, and evaluating the regression function at the most extreme to the boundary observations. With this approach, we obtain point estimates of 6.6%. However, as we have shown, the naive iRDD estimator without the boundary corrections is inconsistent, so the 6.6% point estimate is most probably wrong. It is difficult to say by how much, since we expect that without corrections the limit is a stochastic quantity.

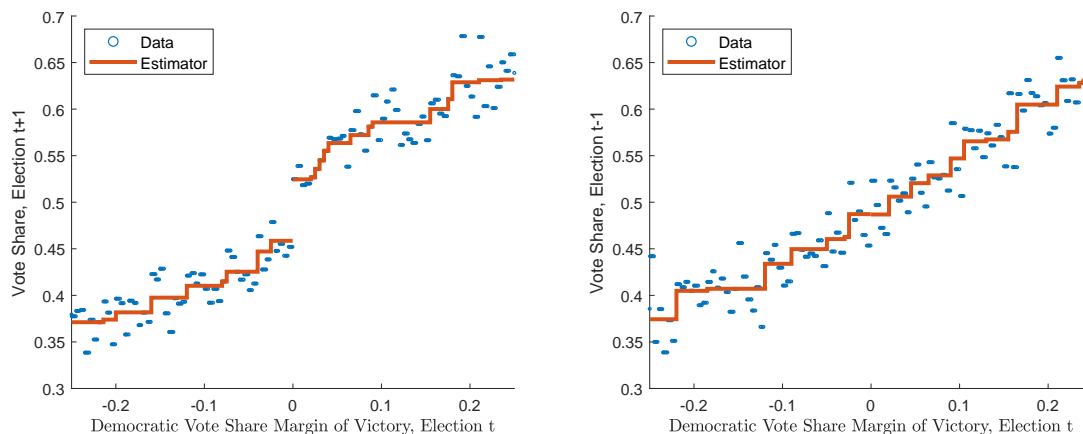


Figure 5: Incumbency advantage. Sample size: 6,559 observations with 3,819 observations below the cut-off.

## 7 Conclusion

This paper offers a new perspective on monotone regression discontinuity designs and contributes more broadly to the growing literature on nonparametric identification, estimation, and inference under shape restrictions. The first contribution of the paper is to introduce novel iRDD estimators based on the boundary-corrected isotonic

regression. An attractive feature of these estimators is that they do not depend on the bandwidth parameter, the kernel function, the degree of the polynomial, and do not suffer from the inference after the model selection problem. In this respect, they have the attractiveness of matching estimators with a fixed number of matches, c.f., [Abadie and Imbens \(2006\)](#). We show that the large sample distribution of iRDD estimators is driven by Brownian motions originating from zero and moving in opposite directions. Second, our work is the first to consider the isotonic regression estimator at the boundary of the support and to provide a comprehensive description of the asymptotic behavior of the boundary-corrected estimator. These results are of independent interest for nonparametric econometrics and statistics. A third significant contribution of our paper is to introduce a novel trimmed wild bootstrap procedure and to prove its consistency. Our procedure does not rely on nonparametric smoothing or subsampling, which in our setting constitutes an advantage relative to other bootstrap methods proposed in the literature.

The paper opens several directions for future research. For instance, it could be interesting to investigate how monotonicity can be used when estimating quantile treatment effects, see [Frandsen, Frölich and Melly \(2012\)](#), when the running variable is discrete, when variables are measured with errors, or when additional covariates are available, see [Escanciano and Cattaneo \(2017\)](#) for the review various extensions of sharp and fuzzy designs. In some applications, other shape restrictions, e.g., convexity, might be more plausible than monotonicity. Finally, in the large-sample approximations that we use for the inference, we do not assume the existence of derivatives and rely instead on the weaker one-sided Hölder continuity condition. This indicates that our results might be honest to the relevant Hölder class, but additional study is needed to confirm this conjecture.

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# APPENDIX

## A.1 Inconsistency at the boundary

Put

$$F_n(t) = \frac{1}{n} \sum_{i=1}^n \mathbb{1}\{X_i \leq t\} \quad \text{and} \quad M_n(t) = \frac{1}{n} \sum_{i=1}^n Y_i \mathbb{1}\{X_i \leq t\}.$$

By (Barlow et al., 1972, Theorem 1.1),  $\hat{m}(x)$  is the left derivative of the greatest convex minorant of the cumulative sum diagram

$$t \mapsto (F_n(t), M_n(t)), \quad t \in [0, 1]$$

at  $t = x$ ; see Figure A.1. The natural estimator of  $m(0)$  is  $\hat{m}(X_{(1)})$ , which corresponds to the slope of the first-segment

$$\hat{m}(X_{(1)}) = \min_{1 \leq i \leq n} \frac{1}{i} \sum_{j=1}^i Y_{(j)},$$

where  $X_{(j)}$  is the  $j^{\text{th}}$  order statistics and  $Y_{(j)}$  is the corresponding observation of the dependent variable.

The following result shows the isotonic regression estimator is inconsistent at the boundary whenever the regression error  $\varepsilon$  is allowed to take negative values.

**Theorem A.1.** *Suppose that  $x \mapsto \Pr(Y \leq y | X = x)$  is continuous for every  $y$  and that  $F_{\varepsilon|X=0}(-\epsilon) > 0$  for some  $\epsilon > 0$ . Then*

$$\liminf_{n \rightarrow \infty} \Pr(|\hat{m}(X_{(1)}) - m(0)| > \epsilon) > 0.$$

*Proof.* For any  $\epsilon > 0$

$$\begin{aligned} \Pr(|\hat{m}(X_{(1)}) - m(0)| > \epsilon) &\geq \Pr\left(\min_{1 \leq i \leq n} \frac{1}{i} \sum_{j=1}^i Y_{(j)} < m(0) - \epsilon\right) \\ &\geq \Pr(Y_{(1)} < m(0) - \epsilon) \\ &= \int \Pr(Y \leq m(0) - \epsilon | X = x) dF_{X_{(1)}}(x) \\ &\rightarrow \Pr(Y \leq m(0) - \epsilon | X = 0) \\ &= F_{\varepsilon|X=0}(-\epsilon), \end{aligned}$$

where we use the fact that  $X_{(1)} \xrightarrow{d} 0$ . □

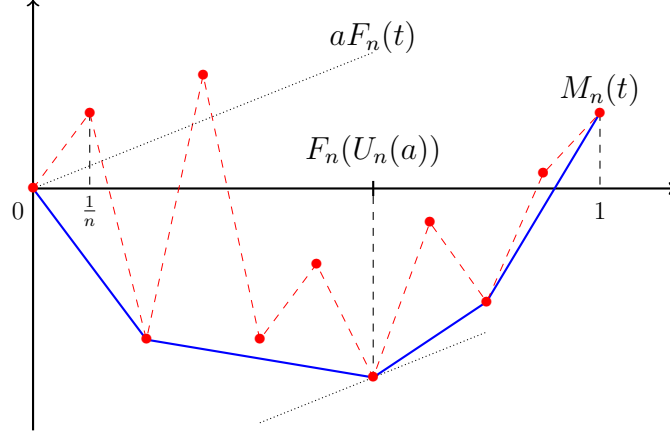


Figure A.1: If  $\hat{m}(x) \leq a$ , then the largest distance between the line with slope  $a$  going through the origin and the greatest convex minorant (brocken blue line) of the cumulative sum diagram  $t \mapsto (F_n(t), M_n(t))$  (red dots) will be achieved at some point to the right of  $F_n(x)$ .

## A.2 Proofs of main results

*Proof of Proposition 2.1.* Since treatment of non-decreasing and non-increasing cases is similar, we focus only on the former. Under (M1), (Rudin, 1976, Theorem 4.29) ensures that all limits in Eq. (1) and Eq. (2) exist. In the sharp design

$$\begin{aligned}
 \theta &= \mathbb{E}[Y_1 - Y_0 | X = c] \\
 &= \lim_{x \downarrow c} (\mathbb{E}[Y_1 | X = x] - \mathbb{E}[Y_0 | X = -x]) \\
 &= \lim_{x \downarrow c} \mathbb{E}[Y | X = x] - \lim_{x \uparrow c} \mathbb{E}[Y | X = x],
 \end{aligned}$$

where the second line follows under Assumption (OC), and the third since for any  $x > 0$

$$\mathbb{E}[Y | X = x] = \mathbb{E}[Y_1 | X = x] \quad \text{and} \quad \mathbb{E}[Y | X = -x] = \mathbb{E}[Y_0 | X = -x],$$

which itself is a consequence of  $Y = DY_1 + (1 - D)Y_0$  and  $D = \mathbb{1}\{X \geq c\}$ .

Now suppose that (M1) and (M2) are satisfied and that  $\theta$  coincides with the expression in Eq. 1. Then under (M1)

$$\mathbb{E}[Y_1 | X = c] \leq \lim_{x \downarrow c} \mathbb{E}[Y_1 | X = x] = \lim_{x \downarrow c} \mathbb{E}[Y | X = x]$$

and

$$\lim_{x \uparrow c} \mathbb{E}[Y|X = x] = \lim_{x \uparrow c} \mathbb{E}[Y_0|X = x] \leq \mathbb{E}[Y_0|X = c].$$

Combining the two inequalities under (M2)

$$\lim_{x \uparrow c} \mathbb{E}[Y|X = x] \leq \mathbb{E}[Y_0|X = c] \leq \mathbb{E}[Y_1|X = c] \leq \lim_{x \downarrow c} \mathbb{E}[Y|X = x].$$

Finally,  $\theta$  is defined as the difference of inner quantities and also equals to the difference of outer quantities by assumption, which is possible only if (OC) holds, i.e.,

$$\lim_{x \uparrow c} \mathbb{E}[Y|X = x] = \mathbb{E}[Y_0|X = c] \leq \mathbb{E}[Y_1|X = c] = \lim_{x \downarrow c} \mathbb{E}[Y|X = x].$$

The proof for the fuzzy design is similar to the proof of (Hahn, Todd and Van der Klaauw, 2001, Theorem 2) with the only difference that monotonicity ensures existence of limits, so their Assumption (RD), (i) can be dropped.  $\square$

*Proof of Theorem 4.1.* By (Barlow et al., 1972, Theorem 1.1),  $\hat{m}(x)$  is the left derivative of the greatest convex minorant of the cumulative sum diagram

$$t \mapsto (F_n(t), M_n(t)), \quad t \in [0, 1]$$

at  $t = x$ . This corresponds to the piecewise-constant left-continuous interpolation. Put

$$U_n(a) = \arg \max_{s \in [0, 1]} \{aF_n(s) - M_n(s)\}.$$

Then for any<sup>9</sup>  $x \in (0, 1)$  and  $a \in \mathbf{R}$

$$\begin{aligned} \hat{m}(x) \leq a &\iff F_n(U_n(a)) \geq F_n(x) \\ &\iff U_n(a) \geq x, \end{aligned} \tag{A.1}$$

as can be seen from Figure A.1, see also (Groeneboom and Jongbloed, 2014, Lemma 3.2).

---

<sup>9</sup>For a continuous function  $\Phi : [0, 1] \rightarrow \mathbf{R}$ , we define  $\arg \max_{t \in [0, 1]} \Phi(t) = \max\{s \in [0, 1] : \Phi(s) = \max_{t \in [0, 1]} \Phi(t)\}$  to accommodate non-unique maximizers. Recall that continuous function attains its maximum on compact intervals and its argmax is a closed set with a well-defined maximal element.



**Case (i):**  $a \in (0, 1/3)$ . For every  $u \in \mathbf{R}$

$$\begin{aligned}
& \Pr \left( n^{1/3} \left( \hat{m}(cn^{-a}) - m(0) \right) \leq u \right) \\
&= \Pr \left( \hat{m}(cn^{-a}) \leq n^{-1/3}u + m(0) \right) \\
&= \Pr \left( \arg \max_{s \in [0,1]} \left\{ (n^{-1/3}u + m(0)) F_n(s) - M_n(s) \right\} \geq cn^{-a} \right) \\
&= \Pr \left( \arg \max_{t \in [-cn^{1/3-a}, (1-cn^{-a})n^{1/3}]} \left\{ (n^{-1/3}u + m(0)) F_n(tn^{-1/3} + cn^{-a}) - M_n(tn^{-1/3} + cn^{-a}) \right\} \geq 0 \right),
\end{aligned}$$

where the second equality follows by the switching relation in Eq. A.1 and the last by the change of variables  $s \mapsto tn^{-1/3} + cn^{-a}$ .

The location of the argmax is the same as the location of the argmax of the following process

$$Z_{n1}(t) \triangleq I_{n1}(t) + II_{n1}(t) + III_{n1}(t)$$

due to scale and shift invariance with

$$\begin{aligned}
I_{n1}(t) &= \sqrt{n}(P_n - P)g_{n,t}, \quad g_{n,t} \in \mathcal{G}_{n1} \\
II_{n1}(t) &= n^{2/3} \mathbb{E} \left[ (m(0) - Y) \left( \mathbf{1}_{[0, tn^{-1/3} + cn^{-a}]}(X) - \mathbf{1}_{[0, cn^{-a}]}(X) \right) \right] \\
III_{n1}(t) &= n^{1/3} u [F_n(tn^{-1/3} + cn^{-a}) - F_n(cn^{-a})],
\end{aligned}$$

where

$$\mathcal{G}_{n1} = \left\{ g_{n,t}(y, x) = n^{1/6}(m(0) - y) \left( \mathbf{1}_{[0, tn^{-1/3} + cn^{-a}]}(x) - \mathbf{1}_{[0, cn^{-a}]}(x) \right) : t \in [-K, K] \right\}.$$

We will show that the process  $Z_{n1}$  converges weakly to a non-degenerate Gaussian process in  $l^\infty[-K, K]$  for every  $K < \infty$ .

Under Assumption 4.1 (ii)-(iii) the covariance structure of the process  $I_{n1}$  converges pointwise to the one of the two-sided scaled Brownian motion (two independent Brownian motions starting from zero and running in the opposite directions). Indeed, when  $s, t \geq 0$

$$\begin{aligned}
\text{Cov}(g_{n,t}, g_{n,s}) &= n^{1/3} \mathbb{E} \left[ |Y - m(0)|^2 \mathbf{1}_{[cn^{-a}, cn^{-a} + n^{-1/3}(t \wedge s)]}(X) \right] + O(n^{-1/3}) \\
&= n^{1/3} \mathbb{E} \left[ (\varepsilon^2 + |m(X) - m(0)|^2) \mathbf{1}_{[cn^{-a}, cn^{-a} + n^{-1/3}(t \wedge s)]}(X) \right] + o(1) \\
&= n^{1/3} \int_{cn^{-a}}^{cn^{-a} + n^{-1/3}(t \wedge s)} \left( \sigma^2(x) + |m(x) - m(0)|^2 \right) f(x) dx + o(1) \\
&= \left( \sigma^2(\xi_n) + |m(0) - m(\xi_n)|^2 \right) f(\xi_n)(t \wedge s) + o(1) \\
&= \sigma^2(0) f(0)(s \wedge t) + o(1),
\end{aligned}$$

where we use the mean-value theorem for some  $\xi_n$  between  $cn^{-a}$  and  $cn^{-a} + n^{-1/3}(t \wedge s)$ . Similarly, it can be shown that

$$\text{Cov}(g_{n,t}, g_{n,s}) = \begin{cases} \sigma^2(0)f(0)(|s| \wedge |t|) + o(1) & s, t \leq 0 \\ o(1) & \text{sign}(s) \neq \text{sign}(t). \end{cases}$$

The class  $\mathcal{G}_{n1}$  is VC subgraph with VC index 2 and the envelop

$$G_{n1}(y, x) = n^{1/6}|y - m(0)|\mathbf{1}_{[cn^{-a}-n^{-1/3}K, cn^{-a}+n^{-1/3}K]}(x),$$

which is square integrable

$$\begin{aligned} \mathbb{E}G_{n1}^2(Y, X) &= n^{1/3}\mathbb{E}[|Y - m(0)|^2\mathbf{1}_{[cn^{-a}-n^{-1/3}K, cn^{-a}+n^{-1/3}K]}(X)] \\ &= n^{1/3}\mathbb{E}[(\varepsilon^2 + |m(X) - m(0)|^2)\mathbf{1}_{[cn^{-a}-n^{-1/3}K, cn^{-a}+n^{-1/3}K]}(X)] \\ &= n^{1/3}\int_{cn^{-a}-n^{-1/3}K}^{cn^{-a}+n^{-1/3}K}(\sigma^2(x) + |m(x) - m(0)|^2)f(x)dx \\ &= O(1). \end{aligned}$$

Next, we verify the Lindeberg's condition under Assumption 4.1 (i)

$$\begin{aligned} \mathbb{E}G_{n1}^2\mathbf{1}\{G_n > \eta\sqrt{n}\} &\leq \frac{\mathbb{E}G_n^{2+\delta}}{\eta^\delta n^{\delta/2}} \\ &= \frac{n^{(2+\delta)/6}}{\eta^\delta n^{\delta/2}}\mathbb{E}[|Y - m(0)|^{2+\delta}\mathbf{1}_{[cn^{-a}-n^{-1/3}K, cn^{-a}+n^{-1/3}K]}(X)] \\ &= \frac{n^{(2+\delta)/6}}{\eta^\delta n^{\delta/2}}O(n^{-1/3}) \\ &= o(1). \end{aligned}$$

Lastly, under Assumption 4.1 (iii), for every  $\delta_n \rightarrow 0$

$$\begin{aligned} \sup_{|t-s| \leq \delta_n} \mathbb{E}|g_{n,t} - g_{n,s}|^2 &= n^{1/3} \sup_{|t-s| \leq \delta_n} \mathbb{E}[|Y - m(0)|^2\mathbf{1}_{[cn^{-a}+n^{-1/3}t, cn^{-a}+n^{-1/3}s]}(X)] \\ &= n^{1/3} \sup_{|t-s| \leq \delta_n} \mathbb{E}[(\varepsilon^2 + |m(X) - m(0)|^2)\mathbf{1}_{[cn^{-a}+n^{-1/3}t, cn^{-a}+n^{-1/3}s]}(X)] \\ &= O(\delta_n) \\ &= o(1). \end{aligned}$$

Therefore, by (van der Vaart and Wellner, 2000, Theorem 2.11.22)

$$I_{n1}(t) \rightsquigarrow \sqrt{\sigma^2(0)f(0)}W_t \quad \text{in} \quad l^\infty[-K, K].$$

Under Assumption 4.1 (ii) and (iv) by Taylor's theorem

$$\begin{aligned}
II_{n1}(t) &= n^{2/3} \mathbb{E} \left[ (m(0) - Y) (\mathbb{1}_{[0, cn^{-a} + tn^{-1/3}]}(X) - \mathbb{1}_{[0, cn^{-a}]}(X)) \right] \\
&= n^{2/3} \int_{F(cn^{-a})}^{F(cn^{-a} + tn^{-1/3})} (m(0) - m(F^{-1}(u))) \, du \\
&= n^{2/3} (m(0) - m(cn^{-a})) [F(cn^{-a} + tn^{-1/3}) - F(cn^{-a})] \\
&\quad - n^{2/3} \frac{m'(0)}{2f(0)} (1 + o(1)) [F(cn^{-a} + tn^{-1/3}) - F(cn^{-a})]^2 \\
&= -n^{1/3-a} t c m'(0) f(0) (1 + o(1)) - \frac{t^2 m'(0)}{2f(0)} [f(0)]^2 (1 + o(1)) \\
&= -\frac{t^2}{2} m'(0) f(0) + o(1)
\end{aligned}$$

uniformly over  $[-K, K]$ . Lastly, by the uniform law of the iterated logarithm

$$\begin{aligned}
III_{n1}(t) &= n^{1/3} u [F(tn^{-1/3} + cn^{-a}) - F(cn^{-a})] + o(1) \\
&= ut f(0) + o(1)
\end{aligned}$$

uniformly over  $t \in [-K, K]$ . Therefore, for every  $K < \infty$

$$Z_{n1}(t) \rightsquigarrow ut f(0) - \sqrt{\sigma^2(0) f(0)} W_t - \frac{t^2}{2} m'(0) f(0) \triangleq Z_1(t), \quad \text{in } l^\infty[-K, K]. \quad (\text{A.2})$$

Next, we verify conditions of the argmax continuous mapping theorem (Kim and Pollard, 1990, Theorem 2.7). First, note that since

$$\text{Var}(Z_1(s) - Z_1(t)) = \sigma^2(0) f(0) |t - s| \neq 0, \quad \forall t \neq s,$$

by (Kim and Pollard, 1990, Lemma 2.6), the process  $t \mapsto Z_1(t)$  achieves its maximum a.s. at a unique point. Second, by law of iterated logarithm for the Brownian motion

$$\limsup_{t \rightarrow \infty} \frac{W_t}{\sqrt{2t \log \log t}} = 1, \quad \text{a.s.}$$

which shows that the quadratic term dominates asymptotically, i.e.,  $Z_1(t) \rightarrow -\infty$  as  $|t| \rightarrow \infty$ . It follows that the maximizer of  $t \mapsto Z_1(t)$  is tight. Lastly, by Lemma A.2.1 the argmax of  $t \mapsto Z_{n1}(t)$  is uniformly tight. Therefore, by the argmax continuous

mapping theorem, see (Kim and Pollard, 1990, Theorem 2.7)

$$\begin{aligned}
& \Pr \left( n^{1/3} \left( \hat{m} \left( cn^{-a} \right) - m(0) \right) \leq u \right) \\
& \rightarrow \Pr \left( \arg \max_{t \in \mathbf{R}} Z_1(t) \geq 1 \right) \\
& = \Pr \left( \arg \max_{t \in \mathbf{R}} \left\{ ut - \sqrt{\frac{\sigma^2(0)}{f(0)}} W_t - \frac{t^2}{2} m'(0) \right\} \geq 1 \right).
\end{aligned}$$

By the change of variables  $t \mapsto \left(\frac{a}{b}\right)^{2/3} s + \frac{c}{2b}$ , scale invariance of the Brownian motion  $W_{\sigma^2 t - \mu} = \sigma W_t - W_\mu$ , and scale and shift invariance of the argmax

$$\arg \max_{t \in \mathbf{R}} \{ aW_t - bt^2 + ct \} = \left(\frac{a}{b}\right)^{2/3} \arg \max_{s \in \mathbf{R}} \{ W_s - s^2 \} + \frac{c}{2b}.$$

This allows us to simplify the limiting distribution as

$$\begin{aligned}
\Pr \left( n^{1/3} \left( \hat{m} \left( cn^{-a} \right) - m(0) \right) \leq u \right) & \rightarrow \Pr \left( \arg \max_{t \in \mathbf{R}} \left\{ ut - \sqrt{\frac{\sigma^2(0)}{f(0)}} W_t - \frac{t^2}{2} m'(0) \right\} \geq 1 \right) \\
& = \Pr \left( \left| \frac{4m'(0)\sigma^2(0)}{f(0)} \right|^{1/3} \arg \max_{s \in \mathbf{R}} \{ W_s - s^2 \} \leq u \right),
\end{aligned}$$

where we use symmetry of the objective function.

**Case (ii):**  $a \in [1/3, 1)$ . For every  $u \in \mathbf{R}$

$$\begin{aligned}
& \Pr \left( n^{(1-a)/2} \left( \hat{m} \left( cn^{-a} \right) - m(0) \right) \leq u \right) \\
& = \Pr \left( \hat{m} \left( cn^{-a} \right) \leq m(0) + n^{(a-1)/2} u \right) \\
& = \Pr \left( \arg \max_{s \in [0,1]} \left\{ (n^{(a-1)/2} u + m(0)) F_n(s) - M_n(s) \right\} \geq cn^{-a} \right) \\
& = \Pr \left( \arg \max_{t \in [0, n^a/c]} \left\{ (n^{(a-1)/2} u + m(0)) F_n(cn^{-a}t) - M_n(cn^{-a}t) \right\} \geq 1 \right),
\end{aligned}$$

where the second equality follows by the switching relation in Eq. A.1 and the last by the change of variables  $s \mapsto cn^{-a}t$ .

The location of the argmax is the same as the location of the argmax of the following process

$$Z_{n2}(t) \triangleq I_{n2}(t) + II_{n2}(t) + III_{n2}(t) + IV_{n2}(t)$$

with

$$\begin{aligned}
I_{n2}(t) &= \sqrt{n}(P_n - P)g_{n,t}, \quad g_{n,t} \in \mathcal{G}_{n2} \\
II_{n2}(t) &= n^{(a+1)/2}\mathbb{E}[(m(0) - Y)\mathbf{1}_{[0, cn^{-a}t]}(X)] \\
III_{n2}(t) &= n^a u(F_n(cn^{-a}t) - F(cn^{-a}t)) \\
IV_{n2}(t) &= n^a uF(cn^{-a}t),
\end{aligned}$$

where

$$\mathcal{G}_{n2} = \{g_{n,t}(y, x) = n^{a/2}(m(0) - y)\mathbf{1}_{[0, cn^{-a}t]}(x) : t \in [0, K]\}.$$

We will show that the process  $Z_{n2}$  converges weakly to a non-degenerate Gaussian process in  $l^\infty[0, K]$  for every  $K < \infty$ .

Under Assumption 4.1 (ii)-(iii) the covariance structure of the process  $I_{n2}$  converges pointwise to the one of the scaled Brownian motion

$$\begin{aligned}
\text{Cov}(g_{n,t}, g_{n,s}) &= n^a \mathbb{E} [|Y - m(0)|^2 \mathbf{1}_{[0, cn^{-a}(t \wedge s)]}(X)] + o(1) \\
&= n^a \mathbb{E} [\varepsilon^2 \mathbf{1}_{[0, cn^{-a}(t \wedge s)]}(X)] + n^a \mathbb{E} [|m(X) - m(0)|^2 \mathbf{1}_{[0, cn^{-a}(t \wedge s)]}(X)] + o(1) \\
&= n^a \int_0^{cn^{-a}(t \wedge s)} \sigma^2(x) dF(x) + n^a \int_0^{cn^{-a}(t \wedge s)} |m(x) - m(0)|^2 dF(x) + o(1) \\
&= \sigma^2(0)f(0)c(s \wedge t) + o(1).
\end{aligned}$$

The class  $\mathcal{G}_{n2}$  is VC subgraph with VC index 2 and envelop

$$G_{n2}(y, x) = n^{a/2}|y - m(0)|\mathbf{1}_{[0, cn^{-a}K]}(x),$$

which is square integrable

$$\begin{aligned}
PG_{n2}^2 &= n^a \mathbb{E}[|Y - m(0)|^2 \mathbf{1}_{[0, cn^{-a}K]}(X)] \\
&= n^a \mathbb{E}[\varepsilon^2 \mathbf{1}_{[0, cn^{-a}K]}(X)] + n^a \mathbb{E}[|m(X) - m(0)|^2 \mathbf{1}_{[0, cn^{-a}K]}(X)] \\
&= n^a \int_0^{n^{-a}K} \sigma^2(x) dF(x) + o(1) \\
&= O(1).
\end{aligned}$$

Next, we verify the Lindeberg's condition under Assumption 4.1 (i)

$$\begin{aligned}
\mathbb{E}G_{n2}^2 \mathbf{1}\{G_n > \eta\sqrt{n}\} &\leq \frac{\mathbb{E}G_{n2}^{2+\delta}}{\eta^\delta n^{\delta/2}} \\
&= \frac{n^{\frac{(2+\delta)a}{2}}}{\eta^\delta n^{\delta/2}} \mathbb{E}[|Y - m(0)|^{2+\delta} \mathbf{1}_{[0, cn^{-a}K]}(X)] \\
&= \frac{n^{\frac{(2+\delta)a}{2}}}{\eta^\delta n^{\delta/2}} O(n^{-a}) = O(n^{-\delta(1-a)/2}) \\
&= o(1).
\end{aligned}$$

Lastly, under Assumption 4.1 (iii), for every  $\delta_n \rightarrow 0$

$$\begin{aligned}
\sup_{|t-s| \leq \delta_n} \mathbb{E} |g_{n,t} - g_{n,s}|^2 &= n^a \sup_{|t-s| \leq \delta_n} \mathbb{E} [ |Y - m(0)|^2 \mathbf{1}_{[cn^{-a}t, cn^{-a}s]}(X) ] \\
&= n^a \sup_{|t-s| \leq \delta_n} \mathbb{E} [ \varepsilon^2 \mathbf{1}_{[cn^{-a}t, cn^{-a}s]}(X) ] + \mathbb{E} [ |m(X) - m(0)|^2 \mathbf{1}_{[cn^{-a}t, cn^{-a}s]}(X) ] \\
&= O(\delta_n) \\
&= o(1).
\end{aligned}$$

Therefore, by (van der Vaart and Wellner, 2000, Theorem 2.11.22)

$$I_{n2}(t) \rightsquigarrow \sqrt{\sigma^2(0)f(0)} cW_t \quad \text{in} \quad l^\infty[0, K].$$

Next,

$$II_{n2}(t) = n^{(a+1)/2} \int_0^{F(cn^{-a}t)} (m(0) - m(F^{-1}(u))) du.$$

For  $a = 1/3$ , under Assumption 4.1 (iv), by Taylor's theorem

$$\begin{aligned}
II_{n2}(t) &= -n^{(1-3a)/2} \frac{t^2 c^2}{2} m'(0) f(0) (1 + o(1)) \\
&= -\frac{t^2 c^2}{2} m'(0) f(0) + o(1),
\end{aligned}$$

while for  $a \in (1/3, 1)$  under  $\gamma$ -Hölder continuity of  $m$

$$\begin{aligned}
II_{n2}(t) &= n^{(a+1)/2} \int_0^{cn^{-a}t} (m(0) - m(x)) f(x) dx \\
&\lesssim n^{(a+1)/2} \int_0^{cn^{-a}t} |x|^\gamma dx \\
&= O\left(n^{\frac{1-2a\gamma-a}{2}}\right) \\
&= o(1),
\end{aligned}$$

since  $\gamma > (1-a)/2a$ , uniformly over  $t$  on compact sets. Next, by the maximal inequality (Kim and Pollard, 1990, p.199),

$$\mathbb{E} \left[ \sup_{t \in [0, K]} |F_n(cn^{-a}t) - F(cn^{-a}t)| \right] \leq n^{-1/2} J(1) \sqrt{PG_n^2}, \quad (\text{A.3})$$

where  $J(1) < \infty$  is the uniform entropy integral and

$$\begin{aligned} PG_n^2 &= F(cn^{-a}t) \\ &= f(0)(1 + o(1))cn^{-a}. \end{aligned}$$

Since  $a < 1$

$$III_{n2}(t) = O_P(n^{(a-1)/2}) = o_P(1).$$

uniformly over  $[0, K]$ .

Lastly,

$$\begin{aligned} IV_{n2}(t) &= n^a u F(cn^{-a}t) \\ &= utf(0)c + o(1) \end{aligned}$$

uniformly over  $t \in [0, K]$ . Therefore, for every  $K < \infty$

$$Z_{n2}(t) \rightsquigarrow utf(0)c + \sqrt{\sigma^2(0)f(0)c}W_t - \frac{t^2c^2}{2}m'(0)f(0)\mathbb{1}_{a=1/3} \triangleq Z_2(t) \quad \text{in } l^\infty[0, K]. \quad (\text{A.4})$$

Next, we extend processes  $Z_{n2}$  and  $Z_2$  to the entire real line as follows

$$\tilde{Z}_{n2}(t) = \begin{cases} Z_{n2}(t), & t \geq 0 \\ t, & t < 0 \end{cases}, \quad \tilde{Z}_2(t) = \begin{cases} Z_2(t), & t \geq 0 \\ 0, & t < 0. \end{cases}$$

We verify conditions of the argmax continuous mapping theorem (Kim and Pollard, 1990, Theorem 2.7). First, by the similar argument as before, the argmax of  $t \mapsto \tilde{Z}_2(t)$  is unique and tight. Second, by Lemma A.2.1, the argmax of  $\tilde{Z}_{n2}$  is uniformly tight for every  $u \in \mathbf{R}$  when  $a = 1/3$  and for every  $u < 0$  when  $a \in (1/3, 1)$ . Therefore, by the argmax continuous mapping theorem, see (Kim and Pollard, 1990, Theorem 2.7),

$$\begin{aligned} &\Pr \left( n^{(1-a)/2} (\hat{m}(cn^{-a}) - m(0)) \leq u \right) \\ &\rightarrow \Pr \left( \arg \max_{t \in [0, \infty)} Z_2(t) \geq 1 \right) \\ &= \Pr \left( \arg \max_{t \in [0, \infty)} \left\{ ut - \sqrt{\frac{\sigma^2(0)}{cf(0)}}W_t - \frac{t^2c}{2}m'(0)\mathbb{1}_{a=1/3} \right\} \geq 1 \right) \\ &= \Pr \left( D_{[0, \infty)}^L \left( \sqrt{\frac{\sigma^2(0)}{cf(0)}}W_t + \frac{t^2c}{2}m'(0)\mathbb{1}_{a=1/3} \right) (1) \leq u \right) \end{aligned}$$

where the last line follows by the switching relation similar to the one in Eq. A.1.

To conclude, it remains to show that when  $a \in (1/3, 1)$

$$\Pr \left( n^{(1-a)/2} (\hat{m}(cn^{-a}) - m(0)) \geq 0 \right) \rightarrow 0.$$

This follows since for every  $\epsilon > 0$

$$\begin{aligned} \Pr \left( n^{(1-a)/2} (\hat{m}(cn^{-a}) - m(0)) \geq 0 \right) &\leq \Pr \left( n^{(1-a)/2} (\hat{m}(cn^{-a}) - m(0)) \geq -\epsilon \right) \\ &\rightarrow \Pr \left( \arg \max_{t \in [0, \infty)} \left\{ -\epsilon t - \sqrt{\frac{\sigma^2(0)}{cf(0)}} W_t \right\} \leq c \right) \\ &= \Pr \left( \arg \max_{t \in [0, \infty)} \left\{ W_t - \sqrt{\frac{cf(0)}{\sigma^2(0)}} \epsilon t \right\} \leq c \right) \\ &= \Pr \left( \arg \max_{t \in [0, \infty)} \{ W_t - t \} \leq \frac{c^2 f(0)}{\sigma^2(0)} \epsilon \right), \end{aligned}$$

which tends to zero as  $\epsilon \downarrow 0$  as can be seen from

$$\limsup_{t \downarrow 0} \frac{W_t}{\sqrt{2t \log \log(1/t)}} = 1, \quad \text{a.s.}$$

□

The proof of Theorem 4.1 is based on the argmax continuous mapping theorem, [Kim and Pollard \(1990\)](#), one of the conditions of which is that the argmax is a uniformly tight sequence of random variables. In our setting it is sufficient to show that the argmax of

$$\mathbb{M}_{n1}(s) \triangleq (n^{-1/3}u + m(0)) [F_n(s+g) - F_n(g)] - [M_n(s+g) - M_n(g)], \quad s \in [0, 1]$$

is  $O_P(n^{-1/3})$  for  $a \in (0, 1/3)$ , where  $g > 0$  is arbitrary small, and that the argmax of

$$\mathbb{M}_{n2}(s) \triangleq (n^{(a-1)/2}u + m(0))F_n(s) - M_n(s), \quad s \in [0, 1]$$

is  $O_P(n^{-a})$  for  $a \in [1/3, 1)$ . The following lemma serves this purpose.

**Lemma A.2.1.** *Suppose that Assumption 4.1 is satisfied. Then*

(i) *For  $a \in (0, 1/3)$  and  $u \in \mathbf{R}$  and every  $g > 0$*

$$\arg \max_{s \in [-g, 1-g]} \mathbb{M}_{n1}(s) = O_P(n^{-1/3}).$$



(ii) For  $a = 1/3$  and  $u \in \mathbf{R}$

$$\arg \max_{s \in [0,1]} \mathbb{M}_{n2}(s) = O_P(n^{-1/3}).$$

(iii) For  $a \in (1/3, 1]$  and  $u < 0$

$$\arg \max_{s \in [0,1]} \mathbb{M}_{n2}(s) = O_P(n^{-a}).$$

*Proof.*

**Case (i):**  $a \in (0, 1/3)$ . Put  $\mathbb{M}_1(s) \triangleq m(g)[F(s+g) - F(g)] - [M(s+g) - M(g)]$  with  $M(s) = \int_0^{F(s)} m(F^{-1}(u))du$ . For  $s_0 = 0$ ,  $\mathbb{M}_1(s_0) = \mathbb{M}_{n1}(s_0) = 0$ . By Taylor's theorem, there exists  $\xi_{1s} \in (F(s), F(g+s))$  such that

$$\begin{aligned} M(g+s) - M(s) &= \int_{F(g)}^{F(g+s)} m(F^{-1}(z))dz \\ &= m(g)[F(g+s) - F(s)] + \frac{1}{2} \frac{m'(F^{-1}(\xi_{1s}))}{f(F^{-1}(\xi_{1s}))} (F(g+s) - F(s))^2 \\ &= m(g)[F(g+s) - F(s)] + \frac{1}{2} \frac{m'(F^{-1}(\xi_{1s}))}{f(F^{-1}(\xi_{1s}))} f^2(\xi_{2s})s^2, \end{aligned}$$

where the second line follows by the mean-value theorem for some  $\xi_{2s} \in (0, s)$ . Then for every  $s$  in the neighborhood of  $s_0$

$$\begin{aligned} \mathbb{M}_1(s) - \mathbb{M}_1(s_0) &= m(g)[F(g+s) - F(s)] - [M(g+s) - M(s)] \\ &= -\frac{1}{2} \frac{m'(F^{-1}(\xi_{1s}))}{f(F^{-1}(\xi_{1s}))} f^2(\xi_{2s})s^2 \\ &\lesssim -s^2 \end{aligned}$$

since under Assumption 4.1  $f$  is bounded away from zero and infinity and  $m'$  is finite in the neighborhood of zero. Next we will bound the modulus of continuity for some

$\delta > 0$

$$\begin{aligned}
\mathbb{E} \sup_{|s| \leq \delta} |\mathbb{M}_{n1}(s) - \mathbb{M}_1(s)| &\leq \mathbb{E} \left[ \sup_{|s| \leq \delta} |M_n(s+g) - M_n(g) - M(s+g) + M(g) \right. \\
&\quad \left. - [m(0) + n^{-1/3}u](F_n(s+g) - F_n(g) - F(s+g) + F(g)) \right] \\
&\quad + (n^{-1/3}|u| + |m(0) - m(g)|) |F(g+\delta) - F(g)| \\
&\leq \mathbb{E} \sup_{|s| \leq \delta} |(P_n - P)h_s| + \mathbb{E} \sup_{|s| \leq \delta} |R_n(s)| + O((n^{-1/3} + g)\delta),
\end{aligned} \tag{A.5}$$

where  $h_s \in \mathcal{H}_\delta = \{h_s(y, x) = (y - m(0))[\mathbb{1}_{[0, s+g]}(x) - \mathbb{1}_{[0, g]}(x)] : s \in [0, \delta]\}$  and

$$R_n(s) = n^{-1/3}u(F_n(s+g) - F_n(g) - F(s+g) + F(g)).$$

By the maximal inequality, (Kim and Pollard, 1990, p.199), the first term in the upper bound in Eq. A.5 is

$$\mathbb{E} \sup_{|s| \leq \delta} |(P_n - P)h_s| \leq n^{-1/2}J(1)\sqrt{PH_\delta^2},$$

where  $J(1)$  is the uniform entropy integral, which is finite since  $\mathcal{H}_\delta$  is a VC-subgraph class of functions with VC index 2,  $H_\delta(y, x) = |y - m(0)|\mathbb{1}_{[g, g+\delta]}(x)$  is the envelop of  $\mathcal{H}_\delta$ , and

$$\begin{aligned}
PH_\delta^2 &= \mathbb{E}[(\sigma^2(X) + |m(X) - m(0)|^2)\mathbb{1}_{[g, g+\delta]}(X)] \\
&= \int_g^{g+\delta} (\sigma^2(x) + |m(x) - m(0)|^2)dF(x) \\
&= O(\delta).
\end{aligned}$$

Next, by the maximal inequality

$$\begin{aligned}
\mathbb{E} \sup_{|s| \leq \delta} |R_n(s)| &\leq n^{-1/3}|u|\mathbb{E} \sup_{|s| \leq \delta} |F_n(s+g) - F_n(g) - F(s+g) + F(g)| \\
&\leq n^{-1/3}n^{-1/2}J(1)\sqrt{PH_\delta^2}|u| \\
&= O(n^{-5/6}\delta^{1/2}),
\end{aligned}$$

where  $J(1) < \infty$  is the uniform entropy integral and  $H_\delta(x) = \mathbb{1}_{[g, g+\delta]}(x)$  is the envelop.

Next, setting  $\delta = 1$ , we get

$$\sup_{s \in [-g, 1-g]} |\mathbb{M}_{n1}(s) - \mathbb{M}_1(s)| = o_P(1).$$

Since  $m(0) < m(x)$  and  $f(x) > 0$  for all  $x \in (0, 1]$ , the function  $s \mapsto \mathbb{M}_1(s)$  is strictly decreasing with a maximum achieved at  $-g$ , whence by (van der Vaart and Wellner, 2000, Corollary 3.2.3 (i))

$$\arg \max_{t \in [-g, 1-g]} \mathbb{M}_{n1}(t) = o_P(1).$$

Then  $\phi_n(\delta) = \delta^{1/2} + n^{1/6}\delta$  is a good modulus of continuity function for  $a = 3/2$  and  $r_n = n^{1/3}$ . Indeed, for this choice  $\delta \mapsto \phi_n(\delta)/\delta^a$  is decreasing and

$$r_n^2 \phi_n(r_n^{-1}) = O(n^{1/2})$$

Therefore, the result follows by (van der Vaart and Wellner, 2000, Theorem 3.2.5).

**Case (ii):**  $a = 1/3$ . Put  $\mathbb{M}_2(s) \triangleq m(0)F(s) - M(s)$  with  $M(s) = \int_0^{F(s)} m(F^{-1}(u))du$ . For  $s_0 = 0$ ,  $\mathbb{M}_2(s_0) = \mathbb{M}_{n2}(s_0) = 0$ . By Taylor's theorem, there exists  $\xi_{1s} \in (0, F(s))$  such that

$$\begin{aligned} M(s) &= \int_0^{F(s)} m(F^{-1}(u))du \\ &= m(0)F(s) + \frac{1}{2} \frac{m'(F^{-1}(\xi_{1s}))}{f(F^{-1}(\xi_{1s}))} (F(s))^2 \\ &= m(0)F(s) + \frac{1}{2} \frac{m'(F^{-1}(\xi_{1s}))}{f(F^{-1}(\xi_{1s}))} f^2(\xi_{2s})s^2, \end{aligned}$$

where the second line follows by the mean-value theorem for some  $\xi_{2s} \in (0, s)$ . Then

$$\begin{aligned} \mathbb{M}_2(s) - \mathbb{M}_2(s_0) &= m(0)F(s) - M(s) \\ &= -\frac{1}{2} \frac{m'(F^{-1}(\xi_{1s}))}{f(F^{-1}(\xi_{1s}))} f^2(\xi_{2s})s^2 \\ &\lesssim -s^2. \end{aligned}$$

Next we will bound the modulus of continuity for some  $\delta > 0$

$$\begin{aligned} \mathbb{E} \sup_{|s| \leq \delta} |\mathbb{M}_{n2}(s) - \mathbb{M}_2(s)| &\leq \mathbb{E} \sup_{|s| \leq \delta} |M_n(s) - M(s) - (m(0) + n^{(a-1)/2}u)(F_n(s) - F(s))| \\ &\quad + n^{(a-1)/2}|u|F(\delta) \\ &\leq \mathbb{E} \sup_{|s| \leq \delta} |(P_n - P)g_s| + \mathbb{E} \sup_{|s| \leq \delta} |R_n(s)| + O(n^{(a-1)/2}\delta), \end{aligned} \tag{A.6}$$

where  $g_s \in \mathcal{G}_\delta = \{g_s(y, x) = (y - m(0))\mathbb{1}_{[0, s]}(x) : s \in [0, \delta]\}$  and

$$R_n(s) = n^{(a-1)/2}u(F_n(s) - F(s)).$$

By the maximal inequality (Kim and Pollard, 1990, p.199), the first term in the upper bound in Eq. A.6 is

$$\mathbb{E} \sup_{|s| \leq \delta} |(P_n - P)g_s| \leq n^{-1/2}J(1)\sqrt{PG_\delta^2},$$

where  $J(1)$  is the uniform entropy integral, which is finite since  $\mathcal{G}_\delta$  is a VC-subgraph class of functions with VC-index 2,  $G_\delta(y, x) = |y - m(0)|\mathbb{1}_{[0, \delta]}(x)$  is the envelop of  $\mathcal{G}_\delta$ , and

$$\begin{aligned} PG_\delta^2 &= \mathbb{E}[\sigma^2(X)\mathbb{1}_{[0, \delta]}(X)] + \mathbb{E}[|m(X) - m(0)|^2\mathbb{1}_{[0, \delta]}(X)] \\ &= \int_0^\delta (\sigma^2(x) + |m(x) - m(0)|^2)dF(x) \\ &= O(\delta). \end{aligned}$$

Next, by the maximal inequality

$$\begin{aligned} \mathbb{E} \sup_{|s| \leq \delta} |R_n(s)| &\leq n^{(a-1)/2}|u|\mathbb{E} \sup_{|s| \leq \delta} |F_n(s) - F(s)| \\ &\leq n^{(a-1)/2}n^{-1/2}J(1)\sqrt{PH_\delta^2}|u| \\ &= O(n^{(a-2)/2}\delta^{1/2}), \end{aligned}$$

where  $J(1) < \infty$  is the uniform entropy integral and  $H_\delta(x) = \mathbb{1}_{[0, \delta]}(x)$  is the envelop.

Next, setting  $\delta = 1$ , we get

$$\sup_{s \in [0, 1]} |\mathbb{M}_{n2}(s) - \mathbb{M}_2(s)| = o_P(1).$$

Since  $m(0) < m(x)$  and  $f(x) > 0$  for all  $x \in (0, 1]$ , the function  $s \mapsto \mathbb{M}_2(s)$  is strictly decreasing with maximum achieved at 0, whence by (van der Vaart and Wellner, 2000, Corollary 3.2.3 (i))

$$\arg \max_{t \in [0, 1]} \mathbb{M}_{n2}(t) = o_P(1).$$

Then the modulus of continuity is  $\phi_n(\delta) = \delta^{1/2} + n^{a/2}\delta$ . This is a good modulus of the continuity function for  $\alpha = 3/2$  and  $r_n = n^{1/3}$ . For this choice  $\delta \mapsto \phi_n(\delta)/\delta^\alpha$  is decreasing and  $r_n^2\phi_n(r_n^{-1}) = O(n^{1/2})$ . Therefore, the result follows by (van der Vaart and Wellner, 2000, Theorem 3.2.5).

**Case (iii):**  $a \in (1/3, 1]$  Here (van der Vaart and Wellner, 2000, Theorem 3.2.5) gives the order  $O_P(n^{-1/3})$  only, so we will show directly using the "peeling device" that after the change of variables

$$\arg \max_{s \in [0, n^a]} \{n^{(1-a)/2} (m(0)F_n(sn^{-a}) - M_n(sn^{-a})) + uF_n(sn^{-a})\} = O_P(1).$$

Denote the process inside of the argmax as

$$Z_{n2}(s) \triangleq n^{(1-a)/2} (m(0)F_n(sn^{-a}) - M_n(sn^{-a})) + n^a u F_n(sn^{-a}).$$

Decompose  $Z_{n2} = I_{n2} + II_{n2} + III_{n2} + IV_{n2}$  similarly as in the proof of Theorem 4.1 (with  $c = 1$ ). Next, partition the set  $[0, \infty)$  into intervals  $S_j = \{s : 2^{j-1} < s \leq 2^j\}$  with  $j$  ranging over integers. Then if the argmax exceeds  $2^K$ , it will be located in one of the intervals  $S_j$  with  $j \geq K$  and  $2^{j-1} \leq n^a$ . Therefore, using the fact that  $u < 0$ ,  $II_{n2} \leq 0$ , and  $Z_{n2}(0) = 0$

$$\begin{aligned} \Pr \left( \arg \max_{s \in [0, n^a]} Z_{n2}(s) > 2^K \right) &\leq \sum_{\substack{j \geq K \\ 2^{j-1} \leq n^a}} \Pr \left( \sup_{s \in S_j} Z_{n2}(s) \geq 0 \right) \\ &\leq \sum_{\substack{j \geq K \\ 2^{j-1} \leq n^a}} \Pr \left( \sup_{s \in S_j} |I_{n2}(s) + III_{n2}(s)| \geq -n^a u F(2^j n^{-a}) \right) \\ &\leq \sum_{\substack{j \geq K \\ 2^{j-1} \leq n^a}} \frac{1}{-un^a F(2^j n^{-a})} \mathbb{E} \left[ \sup_{s \in S_j} |I_{n2}(s) + III_{n2}(s)| \right] \\ &\lesssim \sum_{\substack{j \geq K \\ 2^{j-1} \leq n^a}} \frac{1}{-un^a F(2^j n^{-a})} \{2^{j/2} + n^{(a-1)/2} 2^{j/2}\} \\ &\lesssim \sum_{j \geq K} 2^{-j/2}, \end{aligned}$$

where the third line follows by Markov's inequality and the fourth by the maximal inequality as in the proof of Theorem 4.1. The last expression can be made arbitrarily small by the choice of  $K$ .  $\square$

*Proof of Theorem 4.2.* Put

$$M_n^*(t) = \frac{1}{n} \sum_{i=1}^n Y_i^* \mathbf{1}\{X_i \leq t\}$$

and note that now  $\hat{m}^*(x)$  is the left derivative of the greatest convex minorant of the cumulative sum diagram

$$t \mapsto (F_n(t), M_n^*(t)), \quad t \in [0, 1]$$

at  $t = x$ . By the argument similar to the one used in the proof of Theorem 4.1 for every  $u < 0$

$$\begin{aligned} & \Pr^* \left( n^{(1-a)/2} (\hat{m}^*(cn^{-a}) - \hat{m}(cn^{-a})) \leq u \right) \\ &= \Pr^* \left( \arg \max_{t \in [0, n^a/c]} \{ (n^{(a-1)/2} u + \hat{m}(cn^{-a})) F_n(cn^{-a}t) - M_n^*(cn^{-a}t) \} \geq 1 \right). \end{aligned}$$

The location of the argmax is the same as the location of the argmax of the following process

$$Z_n^*(t) \triangleq I_n^*(t) + II_n^*(t) + III_n^*(t) + IV_n^*(t)$$

with

$$\begin{aligned} I_n^*(t) &= -n^{(a-1)/2} \sum_{i=1}^n \eta_i^* \varepsilon_i \mathbb{1}_{[0, cn^{-a}t]}(X_i) \\ II_n^*(t) &= n^{(a-1)/2} \sum_{i=1}^n \eta_i^* (\tilde{m}(X_i) - m(X_i)) \mathbb{1}_{[0, cn^{-a}t]}(X_i) \\ III_n^*(t) &= n^{(1+a)/2} \int_0^{cn^{-a}t} (\hat{m}(cn^{-a}) - \tilde{m}(x)) dF_n(x) \\ IV_n^*(t) &= n^a u F_n(cn^{-a}t) \end{aligned}$$

The process  $I_n^*$  is the multiplier empirical process indexed by the class of functions

$$\mathcal{G}_n = \{ (\varepsilon, x) \mapsto -n^{a/2} \varepsilon \mathbb{1}_{[0, cn^{-a}t]}(x) : t \in [0, K] \}.$$

This class is of VC subgraph type with VC index 2 and envelop  $G_n(\varepsilon, x) = n^{a/2} |\varepsilon| \mathbb{1}_{[0, cn^{-a}K]}(x)$ , which is square-integrable

$$PG_n^2 = n^a \int_0^{cn^{-a}K} \sigma^2(x) dF(x) = O(1).$$

This envelop satisfies Lindeberg's condition for every  $\eta > 0$

$$\begin{aligned} \mathbb{E} G_n^2 \{ G_n > \eta \sqrt{n} \} &\leq \frac{n^{\frac{\alpha(2+\delta)}{2}} \mathbb{E} [ |\varepsilon|^{2+\delta} \mathbb{1}_{[0, cn^{-a}K]}(X) ]}{\eta^\delta n^{\delta/2}} \\ &= O(n^{\delta(a-1)/2}) \\ &= o(1) \end{aligned}$$

and for every  $g_{n,t}, g_{n,s} \in \mathcal{G}_n$  and  $\delta_n \rightarrow 0$

$$\begin{aligned} \sup_{|t-s| \leq \delta_n} \mathbb{E}|g_{n,t} - g_{n,s}|^2 &= n^a \sup_{|t-s| \leq \delta_n} \mathbb{E} [\varepsilon^2 \mathbf{1}_{[cn^{-a}s, cn^{-a}t]}(X)] \\ &= O(\delta_n) \\ &= o(1). \end{aligned}$$

Next, we show that the covariance structure is

$$\begin{aligned} \mathbb{E}^*[I_n^*(t)I_n^*(s)] &= n^{a-1} \sum_{i=1}^n \varepsilon_i^2 \mathbf{1}_{[0, cn^{-a}(t \wedge s)]}(X_i) \\ &= n^a \mathbb{E}[\varepsilon^2 \mathbf{1}_{[0, cn^{-a}(t \wedge s)]}(X_i)] + R_n(t, s) \end{aligned}$$

with

$$R_n(t, s) = n^{a-1} \sum_{i=1}^n \varepsilon_i^2 \mathbf{1}_{[0, cn^{-a}(t \wedge s)]}(X_i) - n^a \mathbb{E} [\varepsilon^2 \mathbf{1}_{[0, cn^{-a}(t \wedge s)]}(X_i)]$$

Since  $\mathbb{E}[\varepsilon^4|X] \leq C$ , the variance of  $R_n$  tends to zero

$$\begin{aligned} \text{Var}(R_n(t, s)) &= n^{2a-1} \text{Var}(\varepsilon^2 \mathbf{1}_{[0, cn^{-a}(t \wedge s)]}(X)) \\ &\leq n^{2a-1} \mathbb{E}[\varepsilon^4 \mathbf{1}_{[0, cn^{-a}(t \wedge s)]}(X)] \\ &\leq C n^{2a-1} F(cn^{-a}(t \wedge s)) \\ &= O(n^{a-1}) \\ &= o(1), \end{aligned}$$

whence by Chebyshev's inequality  $R_n(t, s) = o_P(1)$ . Therefore, the covariance structure converges pointwise to the one of the scaled Brownian motion

$$\begin{aligned} \mathbb{E}^*[I_n^*(t)I_n^*(s)] &= n^a \int_0^{cn^{-a}(t \wedge s)} \sigma^2(x) dF(x) + o_P(1) \\ &= \sigma^2(0) f(0) c(t \wedge s) + o_P(1), \end{aligned}$$

where  $\mathbb{E}^*[\cdot] = \mathbb{E}[\cdot | (Y_i, X_i)_{i=1}^n]$ . By (van der Vaart and Wellner, 2000, Theorem 2.11.22), the class  $\mathcal{G}_n$  is Donsker, whence by the multiplier central limit theorem, (van der Vaart and Wellner, 2000, Theorem 2.9.6)

$$\sup_{h \in BL_1(l^\infty[0, K])} \left| \mathbb{E}^* h(I_n^*) - \mathbb{E} h \left( \sqrt{\sigma^2(0) f(0) c} W_t \right) \right| \xrightarrow{P} 0.$$

Next,  $II_n^*$  is a multiplier empirical process indexed by the degenerate class of functions

$$\mathcal{H}_n = \{x \mapsto n^{a/2} (\tilde{m}(x) - m(x)) \mathbb{1}_{[0, n^{-a}t]}(x) : t \in [0, K]\}.$$

Since this class is of VC subgraph type with VC index 2, by the maximal inequality

$$\begin{aligned} \mathbb{E}^* \left[ \sup_{t \in [0, K]} |II_n^*(t)| \right] &\lesssim \sqrt{n^a \int_0^{cn^{-a}K} |\tilde{m}(x) - m(x)|^2 dF_n(x)} \\ &= \sqrt{n^a \int_0^K |\tilde{m}(cn^{-a}y) - m(cn^{-a}y)|^2 dF_n(cn^{-a}y)} \\ &= \sqrt{o_P(1) n^a F_n(cn^{-a}K)} \\ &= o_P(1), \end{aligned}$$

where we apply Proposition A.2.1.

Next, changing variables  $x \mapsto cn^{-a}y$  and using the fact that  $\hat{m}$  is non-decreasing

$$\begin{aligned} III_n^*(t) &= n^{(1+a)/2} \int_0^t (\hat{m}(cn^{-a}) - \tilde{m}(cn^{-a}y)) dF_n(cn^{-a}y) \\ &\leq n^{(1+a)/2} \int_0^1 (\hat{m}(cn^{-a}) - \tilde{m}(cn^{-a}y)) dF_n(cn^{-a}y) \\ &\leq n^{(1+a)/2} \sup_{y \in [0, 1]} |\hat{m}(cn^{-a}) - \tilde{m}(cn^{-a}y)| F_n(cn^{-a}) \\ &= o_P(1) (n^{(1+a)/2} (F_n(cn^{-a}) - F(cn^{-a})) + n^{(1+a)/2} F(cn^{-a})) \\ &= o_P(1) (O_P(1) + O(n^{(1-a)/2})) \\ &= o_P(1), \end{aligned}$$

where the fourth line follows by Proposition A.2.1 and Theorem 4.1 (ii), and the fifth since the variance of the term inside is  $O(1)$ .

Next

$$IV_n^*(t) = utf(0)c + o_P(1)$$

in the same way we treat  $III_{n2} + IV_{n2}$  in the proof of Theorem 4.1.



Combining all estimates obtained above together

$$\begin{aligned}
& \sup_{h \in BL_1(l^\infty[0,K])} \left| \mathbb{E}^* h(Z_n^*) - \mathbb{E} h \left( utf(0)c + \sqrt{\sigma^2(0)f(0)c}W_t \right) \right| \\
&= \sup_{h \in BL_1(l^\infty[0,K])} \left| \mathbb{E}^* h(Z_n^*) - \mathbb{E}^* h(utf(0)c + I_n^*) \right| \\
&\quad + \sup_{h \in BL_1(l^\infty[0,K])} \left| \mathbb{E}^* h(utf(0)c + I_n^*) - \mathbb{E} h \left( utf(0)c + \sqrt{\sigma^2(0)f(0)c}W_t \right) \right| \\
&\leq \sup_{t \in [0,K]} |III_n^*(t) + IIII_n^*(t) + IV_n^*(t)| + o_P(1) \\
&= o_P(1).
\end{aligned}$$

By Lemma A.2.2, the argmax of  $Z_n^*(t)$  is uniformly tight, so by Lemma A.2.3

$$\Pr^* \left( n^{(1-a)/2} (\hat{m}^*(cn^{-a}) - \hat{m}(cn^{-a})) \leq u \right) \xrightarrow{P} \Pr \left( D_{[0,\infty)}^L \left( \sqrt{\frac{\sigma^2(0)}{cf(0)}} W_t \right) (1) \leq u \right).$$

To conclude, it remains to show that when  $a \in (1/3, 1)$

$$\Pr^* \left( n^{(1-a)/2} (\hat{m}^*(cn^{-a}) - \hat{m}(cn^{-a})) \geq 0 \right) \rightarrow 0.$$

This follows since for every  $\epsilon > 0$

$$\begin{aligned}
\Pr^* \left( n^{(1-a)/2} (\hat{m}^*(cn^{-a}) - \hat{m}(cn^{-a})) \geq 0 \right) &\leq \Pr^* \left( n^{(1-a)/2} (\hat{m}^*(cn^{-a}) - \hat{m}(cn^{-a})) \geq -\epsilon \right) \\
&\xrightarrow{P} \Pr \left( \arg \max_{t \in [0,\infty)} \left\{ -\epsilon t - \sqrt{\frac{\sigma^2(0)}{cf(0)}} W_t \right\} \leq c \right) \\
&= \Pr \left( \arg \max_{t \in [0,\infty)} \{W_t - t\} \leq c^2 \frac{f(0)}{\sigma^2(0)} \epsilon \right),
\end{aligned}$$

which tends to zero as  $\epsilon \downarrow 0$  similarly to the proof of Theorem 4.1.  $\square$

The following lemma shows tightness of the argmax of the bootstrapped process:

$$\mathbb{M}_n^*(s) \triangleq n^{(1-a)/2} (\hat{m}(n^{-a})F_n(sn^{-a}) - M_n^*(sn^{-a})) + n^a u F_n(sn^{-a}), \quad s \in [0, n^a].$$

**Lemma A.2.2.** *Suppose that assumptions of Theorem 4.2 are satisfied. Then for every  $a \in (1/3, 1]$  and  $u < 0$*

$$\arg \max_{s \in [0, n^a]} \mathbb{M}_n^*(s) = O_P(1).$$

*Proof.* Decompose  $M_n^* = I_n^* + II_n^* + III_n^* + IV_n^*$  similarly to the proof of Theorem 4.2 (with  $c = 1$ ). Next, partition the set  $[0, \infty)$  into intervals  $S_j = \{s : 2^{j-1} < s \leq 2^j\}$  with  $j$  ranging over integers. Let  $\|\cdot\|_{S_j}$  be the supremum norm over  $S_j$ . Then if the argmax exceeds  $2^K$ , it will be located in one of the intervals  $S_j$  with  $j \geq K$  and  $2^{j-1} \leq n^a$ . Therefore, using the fact that  $u < 0$ ,  $II_{n2} \leq 0$ , and  $M_n^*(0) = 0$

$$\begin{aligned}
& \Pr^* \left( \arg \max_{s \in [0, n^a]} M_n^*(s) > 2^K \right) \\
& \leq \sum_{\substack{j \geq K \\ 2^{j-1} \leq n^a}} \Pr^* \left( \sup_{s \in S_j} M_n^*(s) \geq 0 \right) \\
& \leq \sum_{\substack{j \geq K \\ 2^{j-1} \leq n^a}} \Pr^* \left( \|I_n^* + II_n^* + III_n^* + IV_n^* - n^a u F(\cdot n^{-a})\|_{S_j} \geq -n^a u F(2^j n^{-a}) \right) \\
& \leq \sum_{\substack{j \geq K \\ 2^{j-1} \leq n^a}} \frac{1}{-u n^a F(2^j n^{-a})} \mathbb{E}^* \|I_n^* + II_n^* + III_n^* + IV_n^* - n^a u F(\cdot n^{-a})\|_{S_j} \\
& \lesssim \sum_{\substack{j \geq K \\ 2^{j-1} \leq n^a}} \frac{1}{-u n^a F(2^j n^{-a})} 2^{j/2} O_P(1) \\
& \lesssim \sum_{j \geq K} 2^{-j/2} O_P(1),
\end{aligned}$$

where the third line follows by Markov's inequality and the fourth by computations below. The last expression is  $o_P(1)$  for every  $K = K_n \rightarrow \infty$ . To see that all terms above are controlled as was claimed, first note that the process  $I_n^*$  is a multiplier empirical process indexed by the class of functions

$$\mathcal{G}_n = \{(\epsilon, x) \mapsto -n^{a/2} \epsilon \mathbf{1}_{[0, n^{-a}t]}(x) : t \in S_j\}.$$

This class is of VC subgraph type with VC index 2, whence by the maximal inequality

$$\begin{aligned}
\mathbb{E}^* \left[ \sup_{s \in S_j} |I_n^*(t)| \right] & \lesssim n^{a/2} \sqrt{\frac{1}{n} \sum_{i=1}^n \epsilon_i^2 \mathbf{1}_{[0, n^{-a}2^j]}(X_i)} \\
& = \sqrt{n^a \mathbb{E}[\epsilon^2 \mathbf{1}_{[0, n^{-a}2^j]}(X)]} + o_P(1) \\
& = O_P(2^{j/2}),
\end{aligned}$$

where the second line follows since  $\mathbb{E}[\varepsilon^4|X] \leq C$ . Next, it follows from the proof of Theorem 4.2 (replacing  $K$  by  $2^j$ ) that

$$\mathbb{E}^* \left[ \sup_{s \in S_j} |II_n^*(t)| \right] = o_P(2^{j/2})$$

and that  $\|III_n^*\|_{S_j} = o_P(1)$ . Lastly, by the maximal inequality

$$\sup_{s \in S_j} |IV_n^*(s) - n^a uF(n^{-a}s)| = O_P(n^{(a-1)/2} 2^{j/2}).$$

□

The following lemma is a conditional argmax continuous mapping theorem for bootstrapped processes.

**Lemma A.2.3.** *Suppose that for every  $K < \infty$*

(i)

$$\sup_{h \in BL_1(l^\infty[0, K])} |\mathbb{E}^* h(Z_n^*) - \mathbb{E} h(Z)| \xrightarrow{P} 0,$$

(ii)

$$\limsup_{n \rightarrow \infty} \Pr^* \left( \arg \max_{t \in [0, n^a]} Z_n^*(t) > K \right) = o_P(1), \quad K \rightarrow \infty.$$

(iii)  $t \mapsto Z(t)$  has unique maximizer on  $[0, \infty)$ , which is a tight random variable.

Then

$$\Pr^* \left( \arg \max_{t \in [0, n^a]} Z_n^*(t) \geq z \right) \xrightarrow{P} \Pr \left( \arg \max_{t \in [0, \infty)} Z(t) \geq z \right), \quad \forall z > 0.$$

*Proof.* For every  $K$

$$\Pr^* \left( \arg \max_{t \in [0, n^a]} Z_n^*(t) \geq z \right) = \Pr^* \left( \arg \max_{t \in [0, K]} Z_n^*(t) \geq z \right) + R_{n, K},$$

where by (ii)

$$\begin{aligned}
\limsup_{n \rightarrow \infty} R_{n,K} &= \limsup_{n \rightarrow \infty} \Pr^* \left( \arg \max_{t \in [0,K]} Z_n^*(t) < z, \arg \max_{t \in [K,n^a]} Z_n^*(t) \geq z \right) \\
&\leq \limsup_{n \rightarrow \infty} \Pr^* \left( \arg \max_{t \in [0,n^a]} Z_n^*(t) > K \right) \\
&= o_P(1), \quad K \rightarrow \infty,
\end{aligned}$$

by (i) and (iii)

$$\begin{aligned}
\Pr^* \left( \arg \max_{t \in [0,K]} Z_n^*(t) \geq z \right) &= \Pr \left( \arg \max_{t \in [0,K]} Z(t) \geq z \right) + o_P(1) \\
&= \Pr \left( \arg \max_{t \in [0,\infty)} Z(t) \geq z \right) + o_P(1), \quad K \rightarrow \infty.
\end{aligned}$$

More precisely, we used the continuous mapping theorem for the bootstrapped process (Kosorok, 2008b, Proposition 10.7):

$$\begin{aligned}
\Pr^* \left( \arg \max_{t \in [0,K]} Z_n^*(t) \geq z \right) &= \Pr^* \left( \sup_{t \in [z,K]} Z_n^*(t) \geq \sup_{t \in [0,K]} Z_n^*(t) \right) \\
&\xrightarrow{P} \Pr \left( \sup_{t \in [z,K]} Z(t) \geq \sup_{t \in [0,K]} Z(t) \right) \\
&= \Pr \left( \arg \max_{t \in [0,K]} Z(t) \geq z \right),
\end{aligned}$$

where the convergence is actually uniform over  $z$  in arbitrary closed subset of the set of continuity points of  $z \mapsto \Pr \left( \arg \max_{t \in [0,K]} Z(t) \geq z \right)$ ; see (Kosorok, 2008b, Lemma 10.11).  $\square$

The following result is a probabilistic statement of the fact that for monotone functions converging pointwise to a continuous limit we also have the uniform convergence.

**Proposition A.2.1.** *Suppose that assumptions of Theorem 4.1 are satisfied. If  $m$  is continuous on  $[0, 1]$ , then*

$$\sup_{y \in [0,1]} |\tilde{m}(cn^{-a}y) - m(0)| \xrightarrow{P} 0.$$

*Proof.* For every  $y \in [0, 1]$ , by Theorem 4.1

$$|\tilde{m}(cn^{-a}y) - m(0)| \xrightarrow{P} 0.$$

Since  $m$  is uniformly continuous, one can find  $0 \leq y_1 \leq \dots \leq y_p \leq 1$  such that  $|m(cn^{-a}y_j) - m(cn^{-a}y_{j-1})| < \epsilon/2$  for all  $j = 2, \dots, p$ . Then on the event  $\{|\tilde{m}(cn^{-a}y_j) - m(cn^{-a}y_j)| \leq \epsilon/2, \forall j = 1, \dots, p\}$  by the monotonicity of  $\tilde{m}$ , for every  $x$ , there exists  $j = 2, \dots, p$  such that

$$m(cn^{-a}x) - \epsilon \leq \tilde{m}(cn^{-a}y_{j-1}) \leq \tilde{m}(cn^{-a}x) \leq \tilde{m}(cn^{-a}y_j) \leq m(cn^{-a}x) + \epsilon,$$

whence

$$\Pr(|\tilde{m}(cn^{-a}y) - m(cn^{-a}y)| \leq \epsilon, \forall y \in [0, 1]) \geq 1 - \sum_{j=1}^p \Pr(|\tilde{m}(cn^{-a}y_j) - m(cn^{-a}y_j)| > \epsilon/2).$$

Since  $p$  is fixed, the sum of probabilities tends to zero by the pointwise consistency of  $\tilde{m}$ , which gives the result as  $\epsilon > 0$  is arbitrary.  $\square$

*Proof of Theorem 3.1.* Since

$$n^{1/3}(\hat{\theta} - \theta) = n^{1/3}(\hat{m}_+(n^{-1/3}) - m_+) - n^{1/3}(\hat{m}_-(-n^{-1/3}) - m_-),$$

the proof is similar to the proof of Theorem 4.1 and Remark 4.1 with  $c = 1$  and  $a = 1/3$ . Strictly speaking, the proof of Theorem 4.1 and Remark 4.1 change a little. Now  $F(0) \neq 0$  and we will have  $\tilde{F}(x) = F(x) - F(0)$  and  $\tilde{F}_n(x) = F_n(x) - F_n(0)$  instead of  $F(x)$  and  $F_n(x)$  everywhere in the proof of Theorem 4.1, which will allow us to proceed as before. In the proof of Remark 4.1, we will have  $\tilde{F}(x) = F(0) - F(x)$  and  $\tilde{F}_n(x) = F_n(0) - F_n(x)$  instead of  $F(x)$  and  $F_n(x)$ . The independence of  $W_t^+$  and  $W_t^-$  follows from the independence of two samples.  $\square$

*Proof of Theorem 3.2.* Put  $\hat{g} = \hat{m}_+(n^{-1/3}) - \hat{m}_-(-n^{-1/3})$ ,  $\hat{h} = \hat{p}_+(n^{-1/3}) - \hat{p}_-(-n^{-1/3})$ ,  $g = m_+ - m_-$ , and  $h = p_+ - p_-$ . By a similar argument as in the proof of Theorem 4.1 and Theorem 3.1

$$n^{1/3}(\hat{g} - g) \xrightarrow{d} \xi_1 \quad \text{and} \quad n^{1/3}(\hat{h} - h) \xrightarrow{d} \xi_2.$$

Consequently,

$$\begin{aligned} n^{1/3}(\hat{\theta}^F - \theta) &= n^{1/3} \left( \frac{\hat{g}}{\hat{h}} - \frac{g}{h} \right) \\ &= \frac{n^{1/3}(\hat{g} - g)h - n^{1/3}(\hat{h} - h)g}{h\hat{h}} \\ &\xrightarrow{d} \frac{1}{h}\xi_1 - \frac{g}{h^2}\xi_2. \end{aligned}$$

The independence between processes with different signs follows from the independence of two samples, while the covariance structure for processes with the same sign follows from the proof of Theorem 4.1 and

$$\begin{aligned} & \text{Cov} \left( n^{a/2} \varepsilon_i \mathbb{1}_{[0, cn^{-a}t]}(X_i), n^{a/2} (D_i - p(X_i)) \mathbb{1}_{[0, cn^{-a}s]}(X_i) \right) \\ &= n^a \int_0^{cn^{-a}(t \wedge s)} \mathbb{E} [\varepsilon(D - p(X)) | X = x] f(x) dx \\ &\rightarrow \rho_+ c f_+(t \wedge s) \end{aligned}$$

and similar computations for negative observations.  $\square$

*Proof of Theorem 3.3.* For every  $u < 0$ , Theorem 4.1 (ii) with  $c = 1$  and  $a = 1/2$  gives

$$\Pr \left( n^{1/4} (\hat{\theta} - \theta) \leq u \right) \rightarrow \Pr \left( D_{[0, \infty)}^L \left( \sqrt{\frac{\sigma_+^2}{f_+}} W_t^+ \right) (1) - D_{(-\infty, 0]}^L \left( \sqrt{\frac{\sigma_-^2}{f_-}} W_t^- \right) (-1) \leq u \right)$$

and Theorem 4.2 gives

$$\Pr^* \left( n^{1/4} (\hat{\theta}^* - \hat{\theta}) \leq u \right) \xrightarrow{P} \Pr \left( D_{[0, \infty)}^L \left( \sqrt{\frac{\sigma_+^2}{f_+}} W_t^+ \right) (1) - D_{(-\infty, 0]}^L \left( \sqrt{\frac{\sigma_-^2}{f_-}} W_t^- \right) (-1) \leq u \right),$$

whence the result.  $\square$

*Proof of Remark 4.1.* We sketch only the most important differences below:

$$\begin{aligned} & \Pr \left( n^{(1-a)/2} (\hat{m}(-cn^{-a}) - m(0)) \leq u \right) \\ &= \Pr \left( \hat{m}(-cn^{-a}) \leq un^{(a-1)/2} + m(0) \right) \\ &= \Pr \left( \arg \max_{s \in [-1, 0]} \{ (n^{(a-1)/2} u + m(0)) F_n(s) - M_n(s) \} \geq -cn^{-a} \right) \\ &= \Pr \left( \arg \max_{t \in [-n^a/c, 0]} \{ (n^{(a-1)/2} u + m(0)) F_n(cn^{-a}t) - M_n(cn^{-a}t) \} \geq -1 \right) \\ &\rightarrow \Pr \left( \arg \max_{t \in (-\infty, 0]} \left\{ ut - \sqrt{\frac{\sigma^2(0)}{cf(0)}} W_t - \frac{t^2 c}{2} m'(0) \mathbb{1}_{a=1/3} \right\} \geq -1 \right) \\ &= \Pr \left( D_{(-\infty, 0]}^L \left( \sqrt{\frac{\sigma^2(0)}{cf(0)}} W_t + \frac{t^2 c}{2} m'(0) \mathbb{1}_{a=1/3} \right) (-1) \leq u \right) \end{aligned}$$

$\square$

### A.3 Examples of monotone discontinuity designs

In Table A.1, we collect a list of empirical papers with monotone regression discontinuity designs. We focus only on papers where the global monotonicity is economically plausible and is empirically supported. It is worth stressing that monotonicity restricts only how the average outcome changes with the running variable and that in some references monotonicity appears due to the restricted set of values of the running variable, e.g., elderly people. However, we do not include papers where we might have global piecewise monotonicity with known change points, so the scope of the empirical applicability is probably larger.

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Table A.1: Monotone regression discontinuity designs

Study	Outcome(s)	Treatment(s)	Running variable
Lee (2008)	Votes share in next election	Incumbency	Initial votes share
Dufo, Dupas and Kremer (2011)	Endline scores	Higher-achieving peers	Initial attainment
Abdulkadiröglü, Angrist and Pathak (2014)	Standardized test scores	Attending elite school	Admission scores
Lucas and Mbiti (2014)	Probability of graduation	Attending elite secondary school	Admission scores
Hoekstra (2009)	Earnings	Attending flagship state university	SAT score
Clark (2010)	Test scores, university enrollment	Attending selective high school	Assignment test
Kaniel and Parham (2017)	Net capital flow	Appearance in the WSJ ranking	Returns
Schmieder, Von Wachter and Bender (2012)	Unemployment duration	Unemployment benefits	Age
Card, Dobkin, and Maestas (2008)	Health care utilization	Coverage under Medicare	Age
Shigeoka (2014)	Outpatient visits	Cost-sharing policy	Age
Carpenter and Dobkin (2009)	Alcohol-related mortality	Ability to drink legally	Age
Jacob and Lefgren (2004)	Academic achievements	Summer school, grade retention	Test scores
Baum-Snow and Marion (2009)	Income, property value	Tax credit program	Fraction of eligible
Buettner (2006)	Business tax rate	Fiscal equalization transfers	Tax base
Card, Chetty, and Weber (2007)	Job finding hazard	Severance pay	Job tenure
Chiang (2009)	Medium run test scores	Sanctions threat	School performance
Ferreira (2010)	Probability to move to a new house	Ability to transfer tax benefits	Age
Lalive (2007)	Unemployment duration	Unemployment benefits	Age
Litschig and Morrison (2013)	Education, literacy, poverty	Government transfers	Size of municipality
Ludwig and Miller (2007)	Mortality, educational attainment	Head Start funding	County poverty rate
Matsudaira (2008)	Test scores	Summer school	Test scores
Chay and Greenstone (2005)	Housing prices	Regulatory status	Pollution levels
Greenstone and Gallagher (2012)	Housing prices	Superfund clean-up status	Ranking of hazard



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