A Theory of Stability in Dynamic Matching Markets

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Abstract

I study dynamic matching markets where matching opportunities arrive over time, and matching is one-to-one and irreversible. The proposed stability notion, dynamic stability, incorporates a backward induction notion to an otherwise cooperative model, which takes into account the time at which the arriving agents can form binding agreements. Dynamically stable matchings may fail to exist in two-sided economies (e.g., adoption markets), and in the allocation of objects with priorities (e.g., public housing). However, dynamically stable matchings always exist in one-sided economies (e.g., deceased-donor organ allocation). The non-existence result reveals a new form of unraveling in matching markets: agents wish to delay the time at which they are matched so as to improve their matching prospects. These findings rationalize why clearing houses in different markets adopt very different rules to deal with the event in which agents reject a current offer to wait for a better match. In particular, in two-sided markets and in the allocation of objects with priorities, to guarantee that efficiency is achieved, the central clearing house needs to restrict agents' option to wait for a better match.

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Matching markets such as adoption markets, the allocation of public housing, and deceased-donor organ allocation share the following key features: (i) matching is one-to-one, (ii) matching opportunities arrive over time, and (iii) matching is irreversible. For example, in the case of deceased-donor kidney allocation (i) each patient receives one kidney, (ii) patients and kidneys arrive into the system at various points in time, and (iii) once a patient is transplanted a kidney, she permanently exits the market. Moreover, for each of these markets, a central clearing house exists: adoption agencies manage adoptions, the Public Housing Authority (PHA) manages the allocation of public housing, and the United Network for Organ Sharing (UNOS) manages the allocation of deceased-donor organs.

Since matching opportunities arrive over time, and matching is irreversible, in these markets the option of waiting for a better match is valuable. For instance, suppose UNOS offers Alex, a 20-year-old patient, a kidney from a 65-year-old deceased-donor. On dyalisis alone, Alex could maintain her current quality of life for up to 4 years (see Schold et al. [44]). Because matching opportunities arrive over time, Alex understands that within the next 4 years, a kidney from a younger deceased-donor may arrive. Such kidneys are more desirable than the currently offered kidney. Therefore, Alex faces the following dynamic problem. On the one hand, she could take now the 65-year-old deceased-donor kidney. If she does this, because matching is irreversible, she quits the market and misses out on the option of receiving the younger kidney within the next 4 years. On the other hand, at the cost of giving up her current offer, she may want to keep her options open, and wait for a younger deceased-donor organ to enter the system. Alex's ultimate decision will depend on how she weighs the pros and cons of waiting versus matching immediately.

Despite their similarities, which imply that in all these markets the option of waiting for a better match is valuable, the clearing houses through which matches occur differ greatly in how they deal with the event in which an agent is proposed a match, but turns it down to wait for a better prospect. In adoption markets, once a match between an adoptive parent and a birth mother is proposed, adoption agencies allow birth mothers to costlessly renege on the match. In contrast, depending on the agency, adoptive parents face different restrictions after reneging on the match: for example, restarting the process within the agency, having to proceed with adoption with a different agency, or being placed at the bottom of the agency's wait list. The PHA [35] establishes that if an applicant rejects an offer to match with a compatible building today, then she is either removed from the waiting list, and has to restart her application, or she is placed at the bottom of the waiting list with a new application date. In contrast, if a patient refuses to match with a compatible kidney, UNOS does not penalize the patient in any way.

This paper provides a theory of stability in dynamic matching markets that satisfy properties (i)-(iii) above. Stability notions are used to predict which arrangements agents reach in a laissez-faire situation. Since stable outcomes are Pareto efficient, whenever they exist, even in the absence of central clearing houses, one expects exchanges to be efficient. Thus, while clearing houses may exist to facilitate other aspects of the matching market, they need not regulate the actions that agents can take for the purpose of achieving efficiency: agents themselves will naturally reach a stable, and thus efficient, allocation.

This paper makes two contributions. First, it provides a framework that rationalizes why, across the above three markets, the institutional rules that regulate what happens when agents wait for a better match are so different, even though similar forces drive the incentive to wait for a better match in all three. Second, I use the framework to understand when, to guarantee efficiency, a central clearing house is needed to regulate the actions agents can take.

In order to do so, I proceed in three steps. First, while markets with properties (i)-(iii) above have been studied in the literature, none of the existing papers provide stability notions. Therefore, I define a stability notion which is natural for such markets. I denote it *dynamic stability*. Second, I show that for markets such as adoption markets and public housing, dynamically stable matchings may fail to exist. This non-existence problem arises precisely because agents have the option to wait for better matching opportunities. In contrast, in markets such as the allocation of deceased-donor organs, dynamically stable matchings always exist. It follows from these two observations that, in the first two markets, a central clearing house which regulates agents' actions after they decide to reject a current offer to match is needed to guarantee that the allocation is efficient. Third, in markets such as adoption markets and public housing, I provide sufficient conditions under which dynamically stable matchings exist. Hence, under these conditions, a clearing house need not restrict agents' actions in order to guarantee efficiency.

1.1 Overview of model

In this section, I present the four key components of the environment. First, I briefly describe the model. Second, I define matchings in a dynamic setting. Third, I discuss how different assumptions on the timing at which agents can form binding agreements determine what a feasible block is. Finally, I use the definitions of a matching and a feasible block to define the two stability notions I consider in this paper.

I consider a stylized model that captures features (i)-(iii) mentioned before. The economy lasts for two periods, and there are two sides, A (*a*doptive parents in adoption markets, *a*pplicants in public housing, *a*iling patients in the allocation of organs), and B (resp., *b*irth mothers, *b*uildings, *b*ody parts). Agents on side A arrive in period 1, while agents on side B arrive (stochastically) over time. Matching is one-to-one, and irreversible. Agents on side A are discounted expected utility maximizers. For agents on side B, I consider three different assumptions to accommodate the different applications of the model: (1) in the *two-sided economy*, agents on side B are also discounted expected utility maximizers, (2) in the *allocation of objects with priorities*, agents on side B are endowed with strict atemporal rankings over agents on side A, and (3) in the *one-sided economy*, agents on side B are objects with no preferences over agents on side A. (The nomenclature follows the one in Abdulkadiroğlu and Sönmez [2].)

In these dynamic environments, a matching should be defined as a complete contingent plan (henceforth, contingent matching). It should specify (1) what matches occur in period 1, and (2) for every set of agents that are unmatched at the beginning of period 2, what matches occur in period 2. In particular, if an agent is supposed to be matched in period 1, the matching also prescribes what her period 2 outcome is, if she decides instead to remain unmatched. This information is relevant for her decision to carry through or not the matching prescribed in period 1.

Because the environment is dynamic, one has to be careful when defining what a feasible block is. Consider the following example. Ariel, Anna, and Bob are present in period 1. Brad arrives in the market in period 2. Suppose Anna is prescribed to match with Bob in period 1. The following is an a priori feasible agreement between Anna and Brad. (Notice that I am only dealing with feasibility, regardless of whether this is optimal for the agents.) In period 1, Anna and Brad can agree, in advance, that Anna waits for Brad to arrive, and they match in period 2, i.e., they agree that they will match in period 2, regardless of Brad's possibility of matching with Ariel in period 2. In some applications, assuming that Anna and Brad can make this agreement in period 1, when Brad is not yet in the market, might be a realistic assumption, while in other applications it might not. Depending on the stance one takes with respect to the feasibility of such agreements, one obtains different notions of what a feasible blocking coalition is. In turn, different notions of what a feasible blocking coalition is determine different stability notions. In the stability notions I present below, I consider the two polar cases: either agents can form binding agreements regardless of when they enter in the market, or agents are only allowed to form binding agreements with their contemporaries.

The first stability notion, the *core*, allows agents to make agreements amongst themselves regardless of when they enter the market (Definition 4.10). A contingent matching is in the core if it satisfies the following. There is no subset of agents, regardless of whether they are contemporary or not, that can propose an alternative matching amongst themselves with the property that every agent within the subset is better off under the alternative matching than under the original one. Since this is the natural extension of the core from static settings to the dynamic setting, core matchings are used as benchmarks throughout the paper.

The second stability notion, *dynamic stability*, only allows agents to form agreements with their contemporaries (Definition 4.4). In words, a contingent matching is dynamically stable if, for each possible outcome in period 1, the matching among the remaining agents and the new entrants is a stable matching, and, taking as given the outcomes expected for period 2, there is no group of agents in period 1 who finds it profitable to change the period 1 matching, either by waiting to match in period 2, by changing who they are matched to in period 1, or both. In the example above, if Anna prefers Brad to Bob, and Brad prefers Anna to Ariel, then any dynamically stable contingent matching should specify that, if no agent matches in period 1, then Anna and Brad should match in period 2. Hence, a contingent matching that matches Anna and Bob in period 1 cannot be dynamically stable: Anna can change the period 1 outcome by deciding to remain unmatched in period 1, and match with Brad in period 2 as prescribed by the contingent matching. In this case, the prediction made by the core and dynamic stability coincide. While the matching outcome is ultimately the same, the blocks that facilitate this matching are different: in the core, Anna and Brad agree in period 1 to match in period 2; in dynamic stability, Anna waits to be matched in period 2, and any stable matching in period 2 matches Anna and Brad together.

The two stability notions, the core and dynamic stability, defined above for two-sided economies, are extended to the allocation of objects with priorities (Definition 4.6), and the one-sided economy (Definition 4.9).

1.2 Overview of results

In two-sided economies, absent any restrictions on agents' preferences, dynamically stable matchings may fail to exist because agents who match in period 1 may improve their matching outcome by waiting to be matched in period 2, once certain agents leave the market (Propositions 5.1 and A.1). An agent's ability to know she can improve her matching outcome by waiting depends, in part, on the observability of everyone else's matching outcomes. It follows from the connection between two-sided markets and the allocation of objects with priorities, explained in Section 4.1, that the non-existence results translate to this case as well. In contrast to the above results, dynamically stable contingent matchings always exist in one-sided economies. Propositions 5.2-9.1 show that, in one-sided economies, any Pareto efficient matching is part of a dynamically stable matching. (Section 5 discusses the intuition behind the results.)

The observation that dynamically stable matchings may fail to exist, absent any restrictions on agents' preferences, motivates the sufficient conditions for existence. If agents are sufficiently *patient* dynamically stable matchings exist (Proposition 6.2 in the case of deterministic arrivals, and 10.3 in the case of stochastic). Alternatively, if preferences over matchings across different agents are sufficiently *aligned*, then dynamically stable matchings also exist (Propositions 6.3-6.4 when arrivals are deterministic, and Proposition 10.4 when they are stochastic). Each of these conditions imply that there are core matchings which are dynamically stable. Thus, under these conditions, core matchings can be obtained as part of a stability notion that makes less stringent assumptions on the binding agreements available to the agents. It follows from the connection between two-sided markets and the allocation of objects with priorities that similar, but weaker, conditions guarantee existence (see Sections 6.3-10.3).

The second observation (the possibility of improving one's outcome by waiting when observing everyone else's matches) motivates the definition of a weaker stability notion, correlated dynamic stability (Section 11). In correlated dynamic stability, in any given period, each agent is only informed of whom she is matched with that period, or if she is single, but not of next period's outcome. Since an agent is not informed of other agents' matches, unraveling may be precluded by introducing uncertainty about the benefits of waiting to be matched. Correlated dynamically stable matchings may exist even when dynamically stable do not (Proposition 11.1). However, that correlated dynamically stable matchings may not exist shows that the observability of matching outcomes is not the only force at play in the non-existence results (Proposition 11.2).

Finally, I also show that when arrivals are stochastic and agents discount the future, the core may be empty. This is in contrast to the case of deterministic arrivals in which it is always non-empty (Sections 6.1-10.1). When arrivals are stochastic, from an ex-ante point of view, agents are matched to a set of agents on the other side of the market (potentially one for each arrival realization). If agents discount the future, preferences may exhibit complementarities: how valuable matching with an agent at one particular realization is depends on who else is available to match at other realizations. It is well-known in static many-to-many matching that complementarities preclude existence (Kelso and Crawford [25]); however, the observation that discounting coupled with the stochasticity of the arrivals is a source of complementarities is new. To keep the discussion focused on the issues pertaining the dynamics of the problem, the full analysis of the core when arrivals are stochastic is provided in a companion paper [15].

The above results help us distinguish between two types of economies. First, economies where, for the purposes of guaranteeing efficient outcomes, a central planner needs to either limit the ability of an agent to wait for a better match, or enforce punishments that dissuade agents from waiting to match. This rationalizes why the PHA expels agents from the waiting list after rejecting an offer to match with a compatible house. Absent any assumptions on preferences, if the PHA did not impose these punishments the market could unravel. Second, in one-sided markets and in two-sided economies where the restrictions on preferences hold, dynamically stable matchings always exist. This acts as a rationale for why UNOS does not punish agents for rejecting a current match: in these economies, the clearing house need not restrict agents' actions in order to guarantee efficient allocations are achieved.[1]

The paper contributes mainly to two strands of literature, which are reviewed in Section 12. The first strand, like here, studies stability notions; however, they apply to markets where matching opportunities are fixed, and pairings can be revised over time (e.g., Damiano and Lam [13], Kurino [29]). The contribution relative to this strand is to provide stability notions for markets with features (i)-(iii). The second strand is the algorithm/mechanism design literature which studies, like here, markets with features (i)-(iii), but from the point of view of optimality, instead of stability (Ünver [48], Leshno [31], Akbarpour et al [3], Baccara et al [6]). The present study of stability is important because stability is considered a key property for the success of algorithms (Roth [39]), and because it identifies a new form of unraveling due to agents waiting for better matching opportunities.

The rest of the paper is organized as follows. Section 2 introduces the core and dynamic stability by means of an example which highlights the limitations of the core when thinking about dynamic matching markets; it also discusses the implications of the theory for list kidney exchange. The reader interested in the definitions may skip these, and go directly to Section 3, where the model and notation are introduced, and Section 4 where the stability notions are defined. For ease of exposition, Sections 3-6 deal with the case in which arrivals are deterministic. Section 5 contains the (non-) existence results. Section 6 analyzes sufficient conditions on preferences under which dynamically stable matchings exist in two-sided economies, and the allocation of ob-

[1] I address in Remark 10.3 how the allocation of deceased-donor kidneys can be modeled as the allocation of objects with priorities in a way such that existence is guaranteed. Hence, the core message of the paper remains true: UNOS need not limit agents' option to wait to achieve efficiency. Here, I follow the literature in modeling the allocation of kidneys as a one-sided economy.

jects with priorities. Sections 7-10 deal with the case of stochastic arrivals, mirroring the presentation in Sections 3-6. Section 11 defines correlated dynamic stability, and discusses how randomization may be used to partially restore existence. Section 12 discusses extensions, and the relevant literature. Section 13 concludes.

2 Example: Core vs. Dynamic Stability

Example 2.1 below highlights the differences between the core and dynamic stability:

Example 2.1. [Based on Roth and Sotomayor [42]] There are two sides, A and B. Side A consists of Anna, Alex, and Ariel; they arrive in $t = 1$. Agents on side B arrive over time: Bob arrives in $t = 1$, and Blake and Brad in $t = 2$. One way to approach the problem of finding stable outcomes is to ignore the dynamics, and treat it as a static matching model in which an agent on side A prefers Bob over Brad if, and only if, matching with Bob in $t = 1$ is preferred to matching with Brad in $t = 2$. Table 1 lists the agents' preferences. Below, if $(\text{B}\cdot, 1)$ appears before $(\text{Bob}, 0)$ in the ranking of any woman, then said woman prefers to wait 1 period to match with B· than to match immediately with Bob.[2]

Anna:	$(\text{Blake}, 1)$	$(\text{Brad}, 1)$	$(\text{Bob}, 0)$	Bob:	Ariel	Alex	Anna
Alex:	$(\text{Blake}, 1)$	$(\text{Brad}, 1)$	$(\text{Bob}, 0)$	Blake:	Ariel	Alex	Anna
Ariel:	$(\text{Brad}, 1)$	$(\text{Blake}, 1)$	$(\text{Bob}, 0)$	Brad:	Alex	Anna	Ariel

Table 1: Preference rankings in Example 2.1

A matching is in the *core* if there is no pair of agents that prefers each other to their partner in the matching. It is easy to check that the unique core matching is as follows:

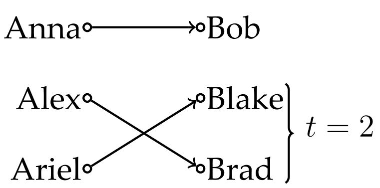


Figure 1: Core matching in Example 2.1

In what follows, I make three observations. First, note that the core matching does not specify what the outcome is if, say, Anna or Bob decided to wait to be matched in $t = 2$. In other words, a core matching is not a complete contingent plan stating for

[2] If women's intertemporal preferences correspond to discounted expected utility the $0, 1$ can be interpreted as exponents for the discount factor. For Bob assume that if $A\cdot$ is preferred to A$'\cdot$, then $(A\cdot, 1)$ comes before $(A'\cdot, 0)$ in his ranking.

each possible outcome in $t = 1$ what matching will ensue in $t = 2$. If waiting to be matched is an option for the agents, this should be specified.

Second, complete contingent plans should specify credible "off-path" outcomes. In turn, whether the "off-path" outcomes are credible or not depends on the binding agreements agents can form amongst themselves. Notice that, in $t = 2$, the remaining unmatched agents form a static matching market. Hence, in line with the rest of the matching literature, I assume agents can form binding agreements with their contemporaries. Hence, no agent should expect that a non-stable matching will arise in $t = 2$, making stable matchings the only reasonable "off-path" outcomes.

Third, the core implicitly makes an assumption about the time at which the agents can form a binding agreement. In particular, it assumes that a pair (α, β) can agree to a block at $t = 1$, even if not all of its members are present to carry out the block. To make this concrete, consider the following matching:

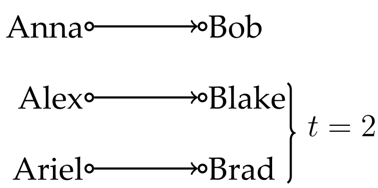


Figure 2: Non-core matching in Example 2.1

The above matching is not in the core because $\{\text{Anna}, \text{Brad}\}$ form a blocking coalition. The core rules out this matching because $\{\text{Anna}, \text{Brad}\}$ can form a binding agreement in $t = 1$ by which (i) Anna waits until Brad becomes available to match, and (ii) Brad matches with Anna in $t = 2$, regardless of the outcome that will ensue in the matching market in $t = 2$.

I now argue that the matching in Figure 2 is, indeed, a reasonable outcome when Brad cannot make binding agreements before he arrives, even though it is not in the core. Take the matching in Figure 2, and suppose Anna decides to block. To do so, she does not match with Bob, and waits for Brad to arrive to match together. Now, when no one matches in $t = 1$, the unique stable matching in $t = 2$ is the one in Figure 1. Hence, consistent with the first point made, any contingent plan has to specify this matching when no one matches in $t = 1$. Under this matching, Brad is matched to Alex whom he prefers to Anna. If Brad cannot commit to match to Anna after Anna carried out her part of the block, then Anna will not block by waiting to be matched in the first place.

Dynamic stability (Definition 4.1) builds on the 3 elements mentioned above: (i) a matching in a dynamic market is a complete contingent plan (contingent matching, henceforth), (ii) only stable matchings among remaining agents can arise in the last period, and (iii) agents can only "block" (that is, agree to match with an agent on the other side of the market) after they have arrived. In words, in a two-period economy, a contingent matching is dynamically stable if, for each possible outcome in $t = 1$, the matching among remaining agents and the new entrants is a stable matching, and, taking as given the outcomes expected for $t = 2$, there is no group of agents in $t = 1$ that can find it profitable to change the matching in $t = 1$, either by waiting to match in $t = 2$, by changing who they are matched to in $t = 1$, or both.

For the economy presented in the example, both matchings are dynamically stable.[3] In this economy, the set of dynamically stable matchings is a superset of the core. Hence, an outside observer may mistakenly rule out the matching in Figure 2 by using the wrong stability notion.

2.1 *Implications for list kidney exchange.*

Agents' incentives to wait to improve on their matching outcomes are particularly pervasive in deceased-donor organ allocation (Su and Zenios [46]). The following stylized example illustrates how list kidney exchange in the New England Kidney Exchange Program determines an agent's continuation matching after refusing to be transplanted an organ, and generates incentives for agents to wait, therefore undermining the mechanism's ability to implement some efficient matchings.

Example 2.2. There are three patients Anna, Ariel, and Alex. Anna and Ariel are in the list for deceased-donor transplants. Ariel and Alex are blood type B, Anna is AB. Alex has a donor with blood type AB, and hence cannot be transplanted her donor's kidney. This period one deceased-donor kidney of blood type B, β_1, is available, though it is from an older patient; next period another blood type B kidney will become available, β_2. Both Alex and Ariel prefer to wait for β_2, than to be matched this period to β_1, and prefer the latter to being unmatched. Assume deceased-donor kidneys expire: if they are not allocated the period they arrive, they lose their viability. The following allocation is Pareto efficient: Alex donates her donor's kidney to agent Anna, Alex receives β_1, Ariel receives β_2. Moreover, this is allowed for by list exchange: the incompatible patient takes the recipient's place in the waiting list, and obtains right of first refusal (ROFR) for the next ABO identical deceased-donor kidney, and keeps it until she accepts a transplant (see Delmonico et. al [14]).

After Anna receives Alex's donor's kidney, Alex is offered β_1. However, ROFR implies that, by refusing kidney β_1, she obtains the right to receive β_2 next period. If Alex

[3] After appropriately specifying the $t = 2$ matchings for the different $t = 1$ outcomes - see Appendix F for details

and Ariel are located far away from each other, then, after Alex refuses kidney β_1, it expires before reaching Ariel, who is left without a transplant. Therefore, ROFR together with the implied consensus that nobody can force a patient to accept a transplant (implicit in the fact that Alex will obtain β_2 if she refuses β_1) imply that the aforementioned Pareto efficient allocation can't be obtained, and as a consequence a patient is left without a transplant.

Notice that, by clearly specifying the "implied property rights" over kidneys, a Pareto efficient allocation that respects ROFR can be achieved: allocate Ariel kidney β_1, and Alex kidney β_2. Clearly specifying "property rights" over the objects is key to showing that Pareto efficient allocations are dynamically stable in one-sided economies (see Section 5.2).

3 Model

The model, definitions, and results are presented in a two-period economy since this suffices to introduce the main concepts and issues.[4] Below, to remove any ambiguity of what I mean by a stochastic arrival, I define the components of the model in the general case of stochastic arrivals. Section 3.1, then, defines matchings and preferences in the case of deterministic arrivals. Section 7.1 defines these objects when arrivals are stochastic.

The economy lasts for two periods. There are two sides. Side A consists of a finite set of agents, $\mathcal{A}$. Agents $a \in \mathcal{A}$ are referred to as α-agents. They stand for *a*doptive parents in adoption markets, *a*pplicants in public housing or job markets, *a*iling patients in the allocation of organs. Side B consists of β-agents, who arrive over time. $\beta-$agents have characteristics belonging to a finite set $\mathcal{B}$. They stand for *b*irth mothers in adoption markets, *b*uildings in public housing, *b*usinesses in job markets, *b*ody parts in the allocation of organs. A stochastic arrival is a distribution over pairs of subsets of $\mathcal{B}$, the first subset being the agents arriving in $t = 1$. More precisely, denote by G a distribution on $2^{\mathcal{B}} \times 2^{\mathcal{B}}$ such that supp $G \subseteq \{(B_1, B_2) \in 2^{\mathcal{B}} \times 2^{\mathcal{B}} : B_1 \cap B_2 = \emptyset\}$. This implies that no two β-agents of the same type arrive in different periods; this maintains the assumption of strict preferences typical of two-sided matching.[5]

The model encompasses the case of deterministic arrivals by setting supp $G = \{(B_1, B_2)\}$ for some $(B_1, B_2) \in 2^{\mathcal{B}} \times 2^{\mathcal{B}}$, $B_1 \cap B_2 = \emptyset$. The rest of this section focuses on this case.

[4] When relevant, I point out how to apply the conditions in Sections 6-10 for more than two periods.
[5] The assumption is not used in the one-sided economy.

I define below matchings for this economy. The definitions capture three ingredients. First, a matching specifies who an agent is matched to every period after she/he arrives. Second, matching is one-to-one and irreversible, so once α is matched to β at period t, she continues to be matched to β at all future periods. Third, the focus is on contingent matchings that specify, for each $t = 1$ outcome, the matching that ensues in $t = 2$.

Definition 3.1. A *period t matching* is an injective map $m_t : \mathcal{A} \cup \bigcup_{\tau=1}^{t} B_\tau \mapsto \mathcal{A} \cup \bigcup_{\tau=1}^{t} B_\tau$ such that $(\forall a \in \mathcal{A}) m_t(a) \in \{a\} \cup \bigcup_{\tau=1}^{t} B_\tau$, and $(\forall b \in \bigcup_{\tau=1}^{t} B_\tau) m_t(b) \in \mathcal{A} \cup \{b\}$. M_t denotes the set of all period t matchings.

Definition 3.2. A pair $(m_1, m_2) \in M_1 \times M_2$ is *feasible* if $(\forall a \in \mathcal{A}) m_1(a) \neq a \Rightarrow m_2(a) = m_1(a)$.[6] M denotes the set of feasible matchings.

Having defined what a static matching is, and what are the feasible matchings in the economy, I define below a contingent matching. A contingent matching μ selects a period 1 matching, $\mu(\emptyset)$, and, for each possible matching in period 1, m_1, it selects a period 2 matching, $\mu(m_1) \in M_2$, such that $(m_1, \mu(m_1))$ is feasible. Formally,

Definition 3.3. A *contingent matching* μ is a map:

$$\begin{aligned}&\mu : \{\emptyset\} \cup M_1 \mapsto M_1 \cup M_2 \\ &\quad \text{s.t. } \mu(\emptyset) \in M_1 \text{ and } (\forall m_1 \in M_1) \mu(m_1) \in M_2 \wedge (m_1, \mu(m_1)) \text{ is feasible.}\end{aligned}$$

Let $\mathcal{M}$ denote the set of contingent matchings.

Any contingent matching defines how agents match "on-path", i.e. what matching ensues when all agents match according to what is prescribed by μ:

Definition 3.4. Given $\mu \in \mathcal{M}$, the *on-path matching for* μ is the feasible matching $m_\mu = (m_{1,\mu}, m_{2,\mu}) \in M$ defined as follows:

$$\begin{aligned}m_{1,\mu} &= \mu(\emptyset), \\ m_{2,\mu} &= \mu(\mu(\emptyset)).\end{aligned}$$

To complete the definition of the economy, I define each agent's preferences over (on-path) matchings. I assume agents on side A are discounted utility maximizers. That is, for each $a \in \mathcal{A}$, assume there exists a Bernoulli utility function $u(a, \cdot) : \mathcal{B} \mapsto \mathbb{R}$, and a discount factor $\delta_a \in [0, 1]$, such that a's utility of matching with b at time t is given by $\delta_a^t u(a, b)$. Assume $u(a, a) = 0$. In general, given an on-path matching $m \in M$, a's utility

[6] Since m_2 is injective, this automatically implies that $m_2(m_1(a)) = a$.

from matching m is given by:

$$U(a,m) = \delta_a^{\mathbf{1}[m_1(a)=a]} u(a, m_2(a)).$$

Since matching is irreversible, a's partner under m is given by $m_2(a)$. If a is unmatched in $t = 1$, then a obtains the discounted payoff from matching with $m_2(a)$ in $t = 2$.

For two-sided economies, I assume agents on side B are also discounted utility maximizers. That is, for each $b \in \mathcal{B}$, there exists a Bernoulli utility function $v(\cdot, b) : \mathcal{A} \mapsto \mathbb{R}$, and a discount factor $\delta_b \in [0,1]$. Assume $v(b,b) = 0$. For $b \in B_1$, his utility from matching $m \in M$ is given by:

$$V_1(b,m) = \delta_b^{\mathbf{1}[m_1(b)=b]} v(m_2(b), b),$$

and, for $b \in B_2$, his utility from matching m is given by:

$$V_2(b,m) = v(m_2(b), b).$$

For the allocation of objects with priorities, I assume (*b*uildings) $b \in \mathcal{B}$ have a complete and transitive binary relation $\rhd_b \subset \mathcal{A} \times \mathcal{A}$. I extend $\rhd$ to an (incomplete) order over matchings as follows. Fix $b \in B_1$. In what follows, say $a \unrhd_b a'$ if $a = a'$, or $a \rhd_b a'$. Let $m, m' \in M, m \neq m'$. If $m_1(b) \neq b$, then b prefers m to m' if, and only if, $m_1(b) \unrhd_b m_2'(b)$. That is, b prefers to match with $m_1(b)$ if either $m_1(b)$ is at least as good as $m_2'(b)$.

Finally, in the one-sided economy, to make the presentation of the definitions in Section 4.1 homogeneous, one can specify "trivial" preferences for $b \in \mathcal{B}$, so that b prefers to be matched over being unmatched. Fix $B_1 \subseteq \mathcal{B}$. For $b \in B_1$, and $m \in M(B_1)$, his utility from matching m is given by:

$$V_1(b,m) = \begin{cases} 0 & \text{if } m_2(b) = b \\ 1 & \text{otherwise} \end{cases},$$

and, for $b \in B_2$, his utility from matching m is given by:

$$V_2(b,m) = \begin{cases} 0 & \text{if } m_2(b) = b \\ 1 & \text{otherwise} \end{cases}.$$

The tuple $\mathcal{E} = \langle \mathcal{A}, \mathcal{B}, G, \{\delta_a, u(a,b)\}_{a \in \mathcal{A}}, \{\delta_b, v(a,b)\}_{b \in \mathcal{B}}, \mathcal{M} \rangle$ defines the two-sided economy, $\mathcal{E}_{\mathcal{B},\rhd} = \langle \mathcal{A}, \mathcal{B}, G, \{\delta_a, u(a,b)\}_{a \in \mathcal{A}}, \{\rhd_b\}_{b \in \mathcal{B}}, \mathcal{M} \rangle$ defines the allocation of objects with priorities, and $\mathcal{E}_{\mathcal{B}} = \langle \mathcal{A}, \mathcal{B}, G, \{\delta_a, u(a,b)\}_{a \in \mathcal{A}}, \mathcal{M} \rangle$ defines the one-sided economy.

4.1 *Dynamic Stability*

In this section, I define dynamic stability. I do so first, for two-sided economies, then, for the allocation of objects with priorities, and, finally, for one-sided economies.

Fix a matching $\mu \in \mathcal{M}$, and a period 1 matching, m_1. Let

$$\mathcal{A}(m_1) = \{a \in \mathcal{A} : m_1(a) = a\},$$
$$\mathcal{B}(m_1) = \{b \in B_1 : m_1(b) = b\} \cup B_2,$$

denote the unmatched agents at the beginning of period 2, when in period 1 agents matched according to m_1. For all purposes $\langle \mathcal{A}(m_1), \mathcal{B}(m_1)\rangle$ is a static matching market. Hence, the definition of a block at m_1 (Definition 4.2) coincides with the standard notion in static matching markets. Definition 4.2 states that only agents in $\mathcal{A}(m_1)\cup\mathcal{B}(m_1)$ may block $\mu(m_1)$ (all other agents have matched in $t = 1$), and these blocks are subject to two restrictions (Definition 4.1): (1) any new matches have to be amongst members of the coalition, and (2) if a match is broken as a consequence of the block, one of the agents involved in the broken match is part of the block.

Definition 4.1. Fix a (static) matching $m : A\cup B \mapsto A\cup B$. Coalition $\emptyset \neq A'\cup B' \subseteq A\cup B$ can *enforce* m' over m if $(\forall k \in A' \cup B')$:

1. $m'(k) \notin \{k, m(k)\} \Rightarrow \{k, m'(k)\} \subset A' \cup B'$,
2. $m'(k) = k \neq m(k) \Rightarrow \{k, m(k)\} \cap (A' \cup B') \neq \emptyset$.

Definition 4.2. Fix $\mu \in \mathcal{M}$, and a period 1 matching $m_1 \in M_1$. $\langle A', B', m'\rangle$ is a *block of μ at m_1* if:

1. $A' \cup B' \subset \mathcal{A}(m_1) \cup \mathcal{B}(m_1)$ can enforce m' over $\mu(m_1)$,
2. $(\forall a \in A')u(a, m'(a)) > u(a, \mu(m_1)(a))$, and
3. $(\forall b \in B')v(m'(b), b) > v(\mu(m_1)(b), b)$.

Definition 4.3 states what a feasible blocking coalition in period 1 is. First, it can only consist of agents in $\mathcal{A} \cup B_1$. Second, in period 1, a coalition can propose a new contingent matching that (a) can alter the matchings planned for period 1, (b) cannot alter the matchings planned for $t = 2, m_1 \in M_1$. Part (b) is based on backwards induction: a coalition $A' \cup B'$ takes as given the matchings that will occur for each possible outcome in period 1, that is, it holds $\{\mu(m_1)\}_{m_1\in M_1}$ fixed. However, $A' \cup B'$ may alter the matching at period 1, $\mu(\emptyset)$, which determines which m_1 realizes in the last period, and hence which $\mu(m_1)$.

Definition 4.3. Fix $\mu \in \mathcal{M}$. $\langle A', B', \mu' \rangle$, is a *block of* μ *at* $t = 1$ if:

1. $A' \cup B' \subseteq \mathcal{A} \cup B_1$ can enforce $\mu'(\emptyset)$ over $\mu(\emptyset)$,
2. $(\forall m_1 \in M_1)\mu'(m_1) = \mu(m_1)$,
3. $(\forall a \in A')U(a, m_{\mu'}) > U(a, m_\mu)$, and
4. $(\forall b \in B')V_1(b, m_{\mu'}) > V_1(b, m_\mu)$.

Definition 4.4. A contingent matching μ is *dynamically stable* if it has no blocks at $t = 1$, and $(\forall m_1 \in M_1)$ it has no blocks at m_1. $D(\mathcal{E})$ denotes the set of dynamically stable matchings.

Remark 4.1. In independent and contemporaneous work, Kadam and Kotowski [22] define "dynamic stability" for a two-period matching model in which the set of agents is constant over time but pairings can be revised as time passes. Not only the setting of both models differs, but their definition of dynamic stability applies only to on-path matchings (i.e. to elements of M in my model), while the concept studied here applies to contingent matchings (i.e. to elements of $\mathcal{M}$).

In static allocation of objects with priorities, the stability notion used is *elimination of justified envy*: if $a \in \mathcal{A}$ prefers $b \in \mathcal{B}$ to her match, then b is assigned to an agent with higher priority for b than a. Balinski and Sönmez [7] show that any matching that eliminates justified envy corresponds to a stable matching in a two-sided economy where objects' priorities are regarded as preferences. In the same spirit, I extend dynamic stability to dynamic elimination of justified envy by using objects' priorities, and the derived order over M, discussed in Section 3.1, in Definitions 4.2-4.3. In order to do so, I first modify the definition of what matchings a blocking coalition can enforce: in particular, *b*uildings do not "break up" with *a*pplicants to remain single.

Definition 4.5. Fix a (static) matching $m : A \cup B \mapsto A \cup B$. Coalition $\emptyset \neq A' \cup B' \subseteq A \cup B$ can *enforce* m' over m if the conditions of Definition 4.1 hold, and:

3*. $b \in B' \Rightarrow m'(b) \neq b$.

Definition 4.6. Fix $\mu \in \mathcal{M}$. μ *dynamically eliminates justified envy* if it has no blocks in period 1, and $(\forall m_1 \in M_1)$, it has no blocks at m_1, where blocks are defined using Definition 4.5. Denote by $D_{NE}(\mathcal{E}_{\mathcal{B},\triangleright})$ the set of contingent matchings that eliminate justified envy.

Finally, I extend dynamic stability to the one-sided economy. Using the preferences defined in Section 3.1, *dynamic stability* can be defined as in Definitions 4.2-4.4, using enforcement as defined in Definition 4.5. All the results in Section 5 go through with this definition. However, because objects are indifferent between all matchings in which they are unmatched, this definition does not rule out some counterintuitive matchings. The stability notion presented below, *dynamic Pareto efficiency*, strengthens *dynamic stability* by allowing b in a blocking coalition to be indifferent between the new matching and the one being blocked as long as its matching partners in the blocked matching participate of the blocking coalition. This amounts to changing items 3 in Definition 4.2 and 4 in Definition 4.3 to items 3^*. and 4^*. in the definitions below:

Definition 4.7. Fix $\mu \in \mathcal{M}$ and $m_1 \in M_1$. $\langle A', B', m' \rangle$ is a *block of μ at m_1* if the following hold:

1. $A' \cup B' \subseteq \mathcal{A}(m_1) \cup \mathcal{B}(m_1)$ can enforce m' over $\mu(m_1)$ as in Definition 4.5,
2. $(\forall a \in A') u(a, m'(a)) > u(a, \mu(m_1)(a))$, and

3^*. $(\forall b \in B') \mu(m_1)(b) \notin \mathcal{A}(m_1) \backslash A'$.

Definition 4.8. Fix $\mu \in \mathcal{M}$. $\langle A', B', \mu' \rangle$ is a *block of μ in period* 1 if:

1. $A' \cup B' \subset \mathcal{A} \cup B_1$ can enforce $\mu'(\emptyset)$ over $\mu(\emptyset)$ as in Definition 4.5,
2. $(\forall m_1 \in M_1) \mu'(m_1) = \mu(m_1)$,
3. $(\forall a \in A') U(a, m_{\mu'}) > U(a, m_\mu)$, and

4^*. $(\forall b \in B') \mu(\mu(\emptyset))(b) \notin \mathcal{A} \backslash A'$.

Definition 4.9. A contingent matching $\mu \in \mathcal{M}$ is *dynamically Pareto efficient* if it has no blocks in period 1, and $(\forall m_1 \in M_1)$ it has no blocks at m_1. Denote by $D_P(\mathcal{E}_\mathcal{B})$ the set of dynamically Pareto efficient contingent matchings.

4.2 Core

For two-sided economies, I extend the definition of the core for a static matching market to a dynamic one.

Definition 4.10. Fix an economy with deterministic arrivals. $\mu \in \mathcal{M}$ is in the *core* if there is no agent who would rather remain single than to match according to μ, and $(\nexists a \in \mathcal{A})(\nexists b \in B_1 \cup B_2) : \delta_a^{\mathbb{1}[b \in B_2]} u(a, b) > U(a, m_\mu)$, and $v(a, b) > V_t(b, m_\mu)$.

There are two differences between the core and dynamic stability. First, in the core, even if $b \in B_2$, $\{a, b\}$ can form a blocking coalition. Second, blocking coalitions in the core compare the payoffs they obtain by blocking with the payoffs from μ. In Definition

4.3, a coalition which blocks by waiting in period 1 compares its payoffs at m_μ with the payoffs they obtain at the continuation originated by the block.

In the one-sided economy, the core criterion is *Pareto efficiency*, as defined below. Pareto efficient matchings are key in the existence results in Section 5.2.

Definition 4.11. An on-path matching $m \in M$ is *Pareto efficient* if there is no other on-path matching $m' \in M$ such that $(\forall a \in \mathcal{A}) U(a, m') \geq U(a, m)$, and $(\exists a' \in \mathcal{A})$: $U(a', m') > U(a, m)$.

5 Dynamic Stability: Existence

5.1 *Two-sided economies and the allocation of objects with priorities*

This section presents the main non-existence result: dynamically stable matchings may fail to exist in two-sided economies. Since the instance of an economy used to show this satisfies that no agent on side B blocks by waiting, Proposition 5.1 implies that D_{NE} may be empty as well.

Proposition 5.1. *There exist economies with deterministic arrivals, $\mathcal{E}$, such that $D(\mathcal{E}) = \emptyset$.*

The result is based on the following instance of an economy:

Example 5.1. There are three $\alpha-$agents $\mathcal{A} = \{\alpha_A, \alpha_B, \alpha_C\}$, and four $\beta-$agents that arrive in pairs: $B_1 = \{\beta_{1A}, \beta_{1B}\}$, $B_2 = \{\beta_{2C}, \beta_{2D}\}$. Below, if α ranks (β, t) above (β', t'), then $\delta_\alpha^t u(\alpha, \beta) > \delta_\alpha^{t'} u(\alpha, \beta')$. Moreover, all agents prefer to match over remaining unmatched.

α_A :	$(\beta_{2C}, 0)$	$(\beta_{2C}, 1)$	$(\beta_{1A}, 0)$	$(\beta_{1A}, 1)$	β_{1A} :	α_A		
α_B :	$(\beta_{2C}, 1)$	$(\beta_{2D}, 0)$	$(\beta_{1B}, 0)$	$(\beta_{2D}, 1)$	β_{1B} :	α_B	α_C	
α_C :	$(\beta_{1B}, 0)$	$(\beta_{1B}, 1)$	$(\beta_{2C}, 0)$	$(\beta_{2C}, 1)$	β_{2C} :	α_C	α_A	α_B
					β_{2D} :	α_B		

Table 2: Preferences in Example 5.1

Assume that $(\forall \beta \in B_1 \cup B_2) v(\alpha, \beta) > v(\alpha', \beta) \Rightarrow \delta_\beta v(\alpha, b) > v(\alpha', \beta)$.

I provide a sketch of the non-existence proof in what follows; details can be found in Appendix A.1. First, if $\mu \in D(\mathcal{E})$, then α_C has to be matched: otherwise, she would be unmatched at $t = 2$, and could block with β_{2C}. Thus, any $\mu \in D(\mathcal{E})$ has to match α_C either with β_{2C} at $t = 2$, or β_{1B} at $t = 1$. Second, if α_C matches with β_{2C}, and $\mu \in D(\mathcal{E})$, it has to be that $m_{1\mu}(\alpha_B) = \beta_{1B}$, and $m_{1\mu}(\alpha_A) = \beta_{1A}$: (i) α_B can't be unmatched because she blocks with β_{2D} at $t = 2$, and (ii) α_B blocks matching with β_{2D} with β_{1B} - note that she doesn't like waiting for β_{2D}. However, the unique stable matching in $t = 2$ when only α_A matches with β_{1A} in $t = 1$, matches α_B with β_{2C}, whom she prefers to β_{1B}. Hence, α_B has a block against any such μ by waiting. Similar reasoning rules out that there exists $\mu \in D(\mathcal{E})$, and $m_{1,\mu}(\alpha_C) = \beta_{1B}$.

The market unravels, but in a different way than previously emphasized in the literature. Unraveling usually refers to the case in which matches are made too early, like in National Resident Matching Program (NRMP), as documented by Roth and Xing [43], while here the market unravels because agents want to wait to be matched. This is not merely a difference in timing. In the NRMP, a procedure which produced unstable matchings was used to determine the matching between residents and hospitals. This implied that there existed gains to be made from blocking, and these were realized by matching before the market opened. On the contrary, agents in my model expect that a stable matching will arise in $t = 2$ when they wait. Agents who match at $t = 1$ exit the market, and, hence, free up matching opportunities in $t = 2$ for the remaining agents. These are the gains that the agents who block by waiting want to realize. Moreover, these gains *can* be realized because $t = 2$ matchings are stable. In Example 5.1, when α_A matches with β_A in $t = 1$, she implicitly gives up her priority with β_{2C} above α_B; then, α_B can be matched to β_{2C} in a stable matching in $t = 2$. Thus, the unraveling result is a consequence of the positive externality agents who match early exert on the remaining agents in the market.[7]

5.2 *One-sided economies*

In one-sided economies, dynamically stable matchings always exist:

Proposition 5.2. *Fix a one-sided economy, $\mathcal{E}_{\mathcal{B}}$. Then, for any Pareto efficient matching m, there exists $\mu \in D_P(\mathcal{E}_{\mathcal{B}}) : m_\mu = m$.*

The proof of Proposition 5.2 is in Appendix B. There are two key insights. First, if a matching is Pareto efficient, then there can be no gains from trade amongst agents on side A. In particular, any agent who matches in period 1 has exhausted the value of the option to wait for a better match. This means one can always find period 2 Pareto efficient matchings such that no agent can improve by waiting. However, as illustrated in Example 2.2, not all period 2 Pareto efficient matchings prevent agents from waiting to be matched. The second key insight is the following: a Pareto efficient matching de-

[7]It is also different from the one in Du and Livne [17], who consider a two-period model in which α and $\beta-$agents arrive over time, and in which a stable matching forms in $t = 2$, and unraveling as in Roth and Xing occurs. There are many differences between Du and Livne's model and the one presented here, but the different forms of unraveling are not a consequence of the authors considering a setup in which $\alpha-$agents also arrive in $t = 2$. It is easy to create examples in which there is entry of $\alpha-$agents in $t = 2$ and the market still unravels forward.

termines an "ownership structure" over the objects. If continuation matchings respect said "ownership structure", then no agent has an incentive to wait for a better match.

I now expand on the second insight. When arrivals are deterministic, just as in static environments, one can show that any Pareto efficient matching has the following property. There exists an agent who, at the Pareto efficient matching, obtains her most preferred object in $B_1 \cup B_2$. Remove this agent and this object; then, there exists an agent who, in the Pareto efficient matching, gets her most preferred object out of the remaining ones. Repeat this until no agents or objects remain. This gives an ordering of the agents, where the first agent is the one who obtains her most preferred object, the second agent obtains her most preferred object out of the remaining ones, etc. Hence, any Pareto efficient allocation can be obtained by a *serial dictatorship*. By prescribing that for all m_1, $\mu(m_1)$ is calculated by respecting the order in the serial dictatorship amongst the remaining unmatched agents, one guarantees that Pareto efficient matchings are dynamically stable. Recall Example 2.2. Right of first refusal makes the patient whose donor donated a kidney to someone in the deceased-donor organ waiting list, Alex, the dictator for the ABO compatible deceased-donor kidneys, and hence she obtains her most preferred ABO compatible kidney. However, the matching suggested in the example made Ariel the dictator, since she was obtaining her most preferred kidney.

The observation that any Pareto efficient on-path-matching can be obtained as part of a dynamically Pareto efficient one implies that, in order to achieve efficiency on-path, agents do not need to be promised inefficient matchings "off-path", i.e. after unexpected refusals to receive an object. There are two ways to interpret the result. On the one hand, it rationalizes why UNOS does not punish agents after they reject an offer to match. On the other hand, if one wishes to remain agnostic about the reasons behind the mechanism being used to allocated deceased-donor organs, it says that UNOS has the potential to implement any Pareto efficient matching without the need of resorting to inefficient punishments "off-path". The results in this section indicate that, by carefully designing the waiting lists so as to respect the implicit property rights, in fact, there is no need for such enforcement power. [8]

Remark 5.1. [Pareto efficiency in the allocation of objects with priorities] As discussed in Section 4.1, matchings that eliminate justified envy are constructed by looking at the stable matchings in a two-sided economy where objects' priorities are regarded as preferences. In the two-sided economy, stable matchings are always Pareto efficient. However, they need not be Pareto efficient for each side of the market (see Roth [37]).

[8]Of course, this environment is stylized. In particular, there is no asymmetric information about the quality of the *b*ody part an *a*iling patient is offered. In an ongoing paper, I extend the setup considered here to the case in which such asymmetric information exists, and I find that the optimal mechanism always selects continuations in the Pareto frontier, i.e. it is renegotiation-proof.

Thus, matchings which eliminate justified envy need not be Pareto efficient in the allocation of objects with priorities, where Pareto efficiency is defined as in Definition 4.11. The economy of Example 5.1 is an instance of this. In particular, the matching in which $m_1(\alpha_A) = \beta_{1A}, m_1(\alpha_B) = \beta_{1B}$ and $m_2(\alpha_C) = \beta_{2C}$ cannot be improved upon by any pair (α, β) matching together, and thus eliminates (static) justified envy. However, it is not Pareto efficient: α_B, α_C can improve by exchanging their assigned objects. Moreover, by waiting, α_B can trigger this improvement without causing justified envy since α_A exits the market in period 1. This tension between efficiency and stability in the allocation of objects with priorities is important in the non-existence result. Indeed, the sufficient conditions presented in Sections 6.3-10.3 are such that, under these conditions, there exists an (on-path) matching which cannot be improved upon by agents matching with contemporary objects to which they have priority *and* is Pareto efficient. The discussion in this section implies that said on-path matching can be made part of a contingent matching in D_{NE}.

6 Sufficient conditions for existence

Section 6 studies sufficient conditions on preferences under which dynamically stable matchings exist in two-sided economies, and in the allocation of objects with priorities. In particular, under these conditions, there are core matchings which are dynamically stable. Section 6.1 establishes existence of the core. Sections 6.2 presents the existence results for two-sided economies, and Section 6.3 for the allocation of objects with priorities.

6.1 Benchmark: Core

When arrivals are deterministic, core contingent matchings always exist:

Proposition 6.1 (Gale and Shapley [21])**.** *Let $\mathcal{E}$ be an economy with deterministic arrivals. Then, $C(\mathcal{E}) \neq \emptyset$.*

Let (B_1, B_2) denote the arrivals in $\mathcal{E}$. The result follows immediately from applying Gale and Shapley's algorithm with agents in $\mathcal{A}$ proposing, agents in $B_1 \cup B_2$ on the receiving side, and agent a proposing to $b \in B_t$ before proposing to $b' \in B_{t'}$ if, and only if, $\delta_a^{t-1} u(a, b) > \delta_a^{t'-1} u(a, b')$.

In what follows, a subset of $C(\mathcal{E})$ is particularly important: the set of core matchings that, for each m_1, selects a stable matching for $\langle \mathcal{A}(m_1), \mathcal{B}(m_1) \rangle$. I denote it by $C^*(\mathcal{E})$. Notice that if $\mu \in C(\mathcal{E})$, there exists $\mu' \in C^*(\mathcal{E})$ such that $m_{\mu'} = m_\mu$, since blocking coalitions do not take into account the "off-path" matchings specified by μ when deciding to block (see Section 4.2).

6.2 Two-sided economies: deterministic arrivals

I present conditions on agents' preferences that imply that $D(\mathcal{E}) \neq \emptyset$. Under these conditions, $C^*(\mathcal{E}) \cap D(\mathcal{E}) \neq \emptyset$. In what follows, I introduce two conditions: *almost no discounting* and *no simultaneous cycles*. If one or the other is satisfied, then dynamically stable matchings exist.

Proposition 6.2 below states that if preferences are such that $(\forall a \in \mathcal{A})(\forall b, b' \in \mathcal{B}) u(a,b') > u(a,b) \Rightarrow \delta_a u(a,b') > u(a,b)$, then dynamically stable matchings exist. Fix $\mu \in C^*(\mathcal{E})$. Construct the following $\mu' \in \mathcal{M}$. First, $m_{\mu'} = m_\mu$. Second, for every $t = 2$ continuation generated by a coalition $A' \cup B' \subseteq \mathcal{A} \cup B_1$ blocking by waiting, let $\mu'(m_1) = m_{2,\mu}$. Third, for all other m_1, set $\mu'(m_1) = \mu(m_1)$. The above condition on preferences, which I denote *almost no-discounting*, implies that $\mu' \in C^*(E) \cap D(E)$. In particular, the choice of $\mu'(h^2)$ after blocks by waiting is, indeed, a stable match.[9]

Proposition 6.2. *Consider an economy with deterministic arrivals $\mathcal{E}$ that satisfies almost no discounting. Then,* $(\forall \mu \in C^*(\mathcal{E}))(\exists \mu' \in D(\mathcal{E})) : m_{\mu'} = m_\mu$.

The proof of Proposition 6.2 is not included in the appendix since it follows immediately from the first step in the proof of Proposition 6.3 below. *Almost no discounting* implies that, even though agents who match in $t = 1$ potentially generate incentives for other agents to wait to be matched in $t = 2$, there always exists a stable matching in the continuation in which the agents who wait do not improve on their current matching outcome. Recall Example 5.1: α_B does not want to wait for β_{2D}, but when β_{1B}, β_{2D} are present in the same period, she prefers to match with β_{2D}. This reversal, due to discounting, implies α_B cannot be matched with β_{1B} in $t = 2$.

The following result, Proposition 6.3, shows that whenever $\mu \in C^*(\mathcal{E}) \backslash D(\mathcal{E})$, then there is a *simultaneous preference cycle*, and motivates the next sufficient condition for existence. A simultaneous preference cycle (Definition 6.1) is an alternating sequence of α and $\beta-$agents, where every other position is taken by a $\beta-$agent, such that: (1) each $\alpha-$agent prefers the $\beta-$agent on her right to the $\beta-$agent on her left, and both $\beta-$agents are acceptable to the $\alpha-$agent, and (2) each $\beta-$agent prefers the $\alpha-$agent to his right to the one on his left, and both $\alpha-$agents are acceptable to the $\beta-$agent. The cycles are evaluated using the $t = 1$ preferences of $\alpha-$agents, and the static preferences

[9] Suppose the economy lasts for $T > 2$ periods. There is a continuation in period $T - 1$ such that no one has matched up to $T - 1$. Almost no discounting should hold. Recursively, in period $T - t$, if $u(a,b) > u(a,b')$, then $\delta_a^t u(a,b) > u(a,b')$.

of $\beta-$agents since any core, or dynamically stable matching matches a $\beta-$agent with an $\alpha-$agent when he arrives. $\beta-$agents' intertemporal preferences are important to determine whether a $\beta-$agent wants to wait to be matched to an $\alpha-$agent in $t = 2$. In what follows, I denote a $\beta-$agent of characteristic b that arrives in period t as b_t.

Definition 6.1. Fix an economy with deterministic arrivals $\mathcal{E}$. A *simultaneous preference cycle* of length $N + 1$ consists of distinct $b_{0,t_0}a_0, b_{1,t_1}..., b_{N,t_N}, a_N$ such that:

1. $(\forall n = 0, ..., N)\delta_{a_i}^{t_{i+1}-1}u(a_i, b_{i+1,t_{i+1}}) > \delta_{a_i}^{t_i-1}u(a_i, b_{i,t_i}) > u(a_i, a_i)$, and
2. $(\forall n = 0, ..., N)v(a_{i+1}, b_{i,t_i}) > v(a_i, b_{i,t_i}) > v(b_{i,t_i}, b_{i,t_i})$.

where the indices are taken modulo $N + 1$.[10]

Notice that the economy in Example 2.1 has a simultaneous cycle with $\alpha_C\beta_{2C}\alpha_B\beta_{2B}$, and the set of dynamically stable allocations is non-empty and encompasses the core. So, a priori, there is nothing "evil" about preference cycles. However, whenever $\mu \in C^*(\mathcal{E})\backslash D(\mathcal{E})$ there has to be a simultaneous preference cycle.

Proposition 6.3. *Fix an economy with deterministic arrivals, $\mathcal{E}$. If $\mu^* \in C^*(\mathcal{E})\backslash D(\mathcal{E})$, there exists at least one simultaneous cycle in $\mathcal{E}$.*

The proof is in Appendix C.1. The result is similar in spirit to the Decomposition Lemma (Roth and Sotomayor [42]). It follows from comparing the welfare of the two-sides of the market between m_{μ^*} and $m_{\mu'}$ for some μ' obtained from μ^* by a block by waiting. However, this has to be done with care: since different agents match in $t = 2$ under μ^* and μ', reversals due to discounting have to be considered when comparing welfare, while in the Decomposition Lemma preferences are constant across the matchings being compared.

To make the result concrete, recall Example 5.1; there is a simultaneous preference cycle $(\alpha_B\beta_{2C}\alpha_C\beta_{1B})$. It can be shown that $m = (m_1, m_2)$ such that $m_1(\alpha_A) = \beta_{1A}, m_1(\alpha_B) = \beta_{1B}, m_1(\alpha_C) = \alpha_C$, and $m_2(\alpha_C) = \beta_{2C}$ is an on-path matching for a core matching. Hence, there exists $\mu \in C^*(\mathcal{E})$ such that $m_\mu = m$. Recall from the discussion in Section 5.1 that α_B blocked any such μ by waiting: the unique continuation when α_A, β_{1A} match in $t = 1$ matches α_B with β_{2C}, and α_C with β_{1B} in $t = 2$. Since α_B prefers waiting for β_{2C} over matching with β_{1B}, μ has a block by waiting. Since m is on-path for the core, it has to be that β_{2C} prefers α_C over α_B. Moreover, since the $t = 2$ continuation when α_A and β_{1A} match in $t = 1$ is stable, then α_C has to prefer β_{1B} over β_{2C}. Finally,

[10] Romero-Medina and Triossi [36] use the no simultaneous cycles property as a sufficient condition for the core of a many-to-one matching problem with responsive preferences to be unique, and Echenique and Yenmez [19] use a no preference cycle condition in many-to-one matching with preference over colleagues.

since m is on-path for the core, β_{1B} has to prefer α_B over α_C, thus completing the cycle.

Remark 6.1. The no simultaneous cycle property is implied by the *top-coalition property* (Banerjee et al. [9]), and *aligned preferences* (Nierdele and Yariv [32], Clark [11]).[11] Note that since I am not assuming almost no-discounting, these properties have to be checked using the ex-ante preferences of the agents.[12]

In Example 5.1, almost no discounting *and* no simultaneous cycles fail to hold. Indeed, α_B prefers β_{2D} over β_{1B}; however, α_B prefers to match immediately with β_{1B} to wait for β_{2D}. The discussion in the previous paragraph shows there is a simultaneous cycle. Since preferences fail almost no discounting, α_B cannot be matched to β_{1B} when α_A matches in period 1, and she waits to be matched in period 2. However, this does not mean α_B can realize the gain from α_A giving up her priority over β_{2C}: this follows from the simultaneous cycle, which allows α_B to "trade" with α_C her priority at β_{1B} in exchange of her priority at β_{2C}.

The absence of simultaneous cycles implies something stronger than just existence of dynamically stable matchings; it implies that $C^*(\mathcal{E}) = D(\mathcal{E})$. This is recorded in Proposition 6.4, and proved in Appendix C.2:

Proposition 6.4. *Fix $\mathcal{E}$ an economy with deterministic arrivals. If $\mathcal{E}$ has no simultaneous preference cycles, then $C^*(\mathcal{E}) = D(\mathcal{E})$.*

I finish this section by making two remarks:

Remark 6.2 (Singleton blocks by waiting)**.** One could relax Definition 4.3 to allow only for singleton blocks by waiting.[13] In that case, *no simultaneous cycles* can be refined; in particular, it is enough that there are no simultaneous cycles involving $\beta-$agents from B_1 and B_2.

Remark 6.3 (One-sided efficiency vs. stability)**.** Recall the discussion in Remark 5.1: the matching $\mu \in C^*(\mathcal{E})$ such that $m_{1,\mu}(\alpha_A) = \beta_{1A}$, $m_{1,\mu}(\alpha_B) = \beta_{1B}$, and $m_{2,\mu}(\alpha_C) = \beta_{2C}$ is not Pareto efficient. Moreover, by waiting to be matched in period 2, α_B triggers a Pareto improvement for agents in $\mathcal{A}$. This observation is more general (see Lemma C.2): when a coalition of $\alpha-$agents blocks by waiting a core matching, it generates a Pareto improvement for $\alpha-$agents who match in period 2 (same holds for $B' \subseteq B_1$ blocking a core matching by waiting). However, contrary to the allocation of objects with priorities, the tension between stability and one-sided efficiency is not the only force at play. One can construct examples in which no individual agent blocks by

[11] I thank Leeat Yariv for bringing up hers and Clark's paper to my attention.

[12] As with almost no discounting, if the economy lasts for $T > 2$ periods, no simultaneous cycles should hold at each continuation.

[13] All the non-existence examples presented in the paper only involve blocks by waiting by one agent. However, one can construct instances in which no single agent blocks by waiting, but a group does.

waiting, but a pair (a, b) blocks by waiting, not generating Pareto improvements for either side. See Appendix S.4.

6.3 *Allocation of objects with priorities*

I now present sufficient conditions for existence of dynamically stable matchings in the allocation of objects with priorities. As discussed in Remark 5.1, under these conditions, there exists an (on-path) matching which simultaneously eliminates justified envy *and* is Pareto efficient. Hence, under these conditions, $D_{NE} \neq \emptyset$.

Ergin [20] shows that, when priorities satisfy *Ergin-acyclicity*, then the matching obtained by agents on side A making offers to objects on side B is Pareto efficient and strategy-proof. In what follows, I define Ergin-acyclicity, and then state the main existence result for the allocation of objects with priorities when arrivals are deterministic:

Definition 6.2. Fix $b, b' \in \mathcal{B}$. $\{b, b'\}$ form a two-cycle if $(\exists a, a', a'')$ such that $a \rhd_b a' \rhd_b a'' \rhd_{b'} a$. $\mathcal{E}_{\mathcal{B},\rhd}$ satisfies *Ergin-acyclicity* if $(\forall b, b' \in \mathcal{B}) b, b'$ do not form two-cycles.

Proposition 6.5. *Fix an economy with deterministic arrivals. If* $\{\rhd_b\}_{b\in\mathcal{B}}$ *satisfies* Ergin-acyclicity, *then* $D_{NE} \neq \emptyset$.

Proposition 6.5 follows from Theorem 1 in Ergin [20], and is proved in Appendix D.1 using the tools developed in the proof of Proposition 6.3. When arrivals are deterministic, as commented in Section 6.1, there is a matching $\mu \in C^*(\mathcal{E})$ obtained by agents in $\mathcal{A}$ proposing to agents in $B_1 \cup B_2$, where a proposes to b_t before proposing to $b'_{t'}$ if, and only if, $\delta_a^{t-1} u(a, b_t) > \delta_a^{t'-1} u(a, b'_{t'})$. Proposition 6.5 shows that this matching is in $D_{NE}(\mathcal{E})$ when $\{\rhd\}_{b\in\mathcal{B}}$ is Ergin-acyclic. Hence, when there exist pairwise stable matchings which are Pareto efficient, existence of dynamically stable matchings is guaranteed.

Remark 6.4. In two-sided economies, Ergin-acyclicity of $\{v(\cdot, b)\}$ only guarantees that $\alpha-$agents do not block by waiting the matching obtained by side A proposing.[14] However, it may be blocked by agents $b \in B_1$ waiting (see Proposition 11.2).

7 Stochastic arrivals: definitions

I extend the definitions and results from the previous sections to the case when arrivals are stochastic. Section 7.1 defines matchings, and preferences. Section 8 defines the solution concepts. Section 9 presents the existence result for the one-sided economy.

[14] Moreover, if Ergin-acyclicity holds for side A, i.e. $(\nexists a, a' \in \mathcal{A})$ and $(\nexists b_t, b'_{t'}, b''_{t''} \in B_1 \cup B_2)$ such that $\delta_a^{t-1} u(a, b_t) > \delta_a^{t'-1} u(a, b'_{t'}) > \delta_a^{t''-1} u(a, b''_{t''})$, and $\delta_{a'}^{t''-1} u(a', b''_{t''}) > \delta_{a'}^{t-1} u(a', b_t)$, then agents in B_1 cannot block the matching obtained by side B proposing.

Section 10 presents sufficient conditions for existence in two-sided economies, and the allocation of objects with priorities.

7.1 *Stochastic arrivals: Matchings and preferences*

This section mirrors Section 3.1 by extending the definitions presented there to the case of stochastic arrivals. The main difference is that everything is now indexed by the arrival realization (B_1, B_2) and the period 1 matching outcome, and not just by the latter. In what follows, an arrival realization through period t, $(B_1, ..., B_t)$, is denoted by B^t. The set of β-agents who arrived at $\tau \leq t$, according to B^t, is B^t_τ.

Definition 7.1. A $(t, B^t)-$matching is an injective map $m_t(B^t) : \mathcal{A} \cup \bigcup_{\tau=1}^{t} B^t_\tau \mapsto \mathcal{A} \cup \bigcup_{\tau=1}^{t} B^t_\tau$ such that $(\forall a \in \mathcal{A}) m_t(B^t)(a) \in \{a\} \cup \bigcup_{\tau=1}^{t} B^t_\tau$ and $(\forall b \in \bigcup_{\tau=1}^{t} B^t_\tau) m_t(B^t)(b) \in \mathcal{A} \cup \{b\}$. Let $M_t(B^t)$ denote the set of all $(t, B^t)-$matchings.

Definition 7.2. Fix an arrival B^2. A pair $(m_1(B^2_1), m_2(B^2)) \in M_1(B^2_1) \times M_2(B^2)$ is *feasible* if $(\forall a \in \mathcal{A}) m_1(B^2_1)(a) \neq a \Rightarrow m_2(B^2)(a) = m_1(B^2_1)(a)$. Let $M(B^2)$ denote the set of all feasible matchings when arrivals are B^2, $M(B_1)$ the set of all feasible matchings at B_1, and M the set of all feasible matchings.

To define a contingent matching, I introduce a final piece of notation. Fix $B_1 \subseteq \mathcal{B}$. A $t = 2$ history has to account for both the arrival history, and the $t = 1$ outcome. That is, the set of $t = 2$ histories consistent with B_1 is given by $H^2(B_1) = \{(B_1, m_1, B_2) : m_1 \in M_1(B_1), B_2 \subseteq \mathcal{B} \backslash B_1\}$. Let $H^2 \equiv \cup_{B_1 \subseteq \mathcal{B}} H^2(B_1)$.

Definition 7.3. A *contingent matching* μ is a map:

$$\mu : 2^{\mathcal{B}} \cup H^2 \mapsto \bigcup_{B_1 \subseteq \mathcal{B}} \left\{ M_1(B_1) \cup \bigcup_{B_2 \cap B_1 = \emptyset} M_2((B_1, B_2)) \right\},$$
$$\text{s.t.} \begin{cases} (\forall B_1) \mu(B_1) \in M_1(B_1), \text{ and} \\ (\forall h^2 = (B_1, m_1, B_2) \in H^2(B_1)) \mu(h^2) \in M_2(B_1, B_2) \wedge (m_1, \mu(h^2)) \text{ is feasible.} \end{cases}$$

$\mathcal{M}$ denotes the set of contingent matchings; $\mathcal{M}(B_1)$ denotes the set of contingent matchings at B_1.

As before, a contingent matching determines how agents match "on-path":

Definition 7.4. Given $\mu \in \mathcal{M}$, the *on-path matching for* μ is $m_\mu \in \times_{B_1 \subseteq \mathcal{B}} \{M_1(B_1) \times \times_{B_2 \cap B_1 = \emptyset} M_2((B_1, B_2))\}$ defined as follows:

$$\begin{aligned} (\forall B_1 \subseteq \mathcal{B}) m_{1,\mu}(B_1) &= \mu(B_1), \\ (\forall (B_1, B_2)) m_{2,\mu}((B_1, B_2)) &= \mu(B_1, \mu(B_1), B_2). \end{aligned}$$

In the two-sided economy, given an on-path matching $m \in M$, a's utility from matching m is given by:

$$U(a,m) = \mathbb{E}_{B_1}\left[\mathbb{E}_{B_2}\left[\delta_a^{\mathbb{1}[m_1(B_1)(a)=a]} u(a, m_2(B_1,B_2)(a)) \| B_1\right]\right]$$
$$= \mathbb{E}_{B_1}\left[U(a,m;B_1)\right].$$

Fix $B_1 \subseteq \mathcal{B}$. Given $m \in M(B_1)$ and $b \in B_1$, b's utility from m is:

$$V_1(b,m;B_1) = \mathbb{E}_{B_2}\left[\delta_b^{\mathbb{1}[m_1(B_1)(b)=b]} v(m_2(B_1,B_2)(b), b) \| B_1\right],$$

and, for $b \in B_2$, his utility from matching m is given by:

$$V_2(b,m;(B_1,B_2)) = v(m_2(B_1,B_2)(b), b).$$

In the allocation of objects with priorities, fix $B_1 \subseteq \mathcal{B}$. I extend $\rhd_b$ to an (incomplete) order over $M(B_1)$ similarly to Section 3.1. Recall that $a \trianglerighteq_b a'$ if either $a = a'$ or $a \rhd_b a'$. Fix $b \in B_1$, $m, m' \in M(B_1)$ such that $m \neq m'$. If $m_1(B_1)(b) \neq b$, say that b prefers m to m' if $(\forall (B_1,B_2)) m_1(B_1)(b) \trianglerighteq_b m_2'(B_1,B_2)(b)$. In particular, b prefers to match with $m_1(B_1)(b)$ if there is no agent ranked higher than a who matches with b at some (B_1,B_2) under m'. [15]

8 Stochastic arrivals: stability

I proceed to extend the definitions of dynamic stability, and the core to the case of stochastic arrivals. The definition of dynamic stability has to be amended to take into account the arrivals; except for that, the definition coincides with the one in Section 4.1. On the contrary, the definition of the core is not a simple fix from the one presented in Section 4.2. This is explained in Section 8.2.

[15] This extension of $\rhd$ to an order over M is motivated by the following observation. When for all $b \in \mathcal{B}$, $\rhd_b = \rhd$, i.e. all *b*uildings share the same ranking over *a*pplicants, the most natural Pareto efficient allocation in that environment is the one in which the highest ranked agent gets her most preferred choice, the second highest ranked agent gets her most preferred choice out of the remaining feasible matchings, and so on. By extending priorities this way, this Pareto efficient allocation is in the core of the economy in which agents on side B have priorities (see Doval [15] for a definition of the core in the allocation of objects with priorities). Section 12 discusses other ways of extending $\rhd$.

8.1 Dynamic Stability

This section mirrors Section 4.1: I define dynamic stability for two-sided economies, and, then, for the allocation of objects with priorities. Since it should be clear from the definitions below how to extend Definition 4.9 to the case of stochastic arrivals, I omit the definition of dynamic stability and dynamic Pareto efficiency for the one-sided economy.

Fix $\mu \in \mathcal{M}$, an arrival realization B^2, and a period 1 matching $m_1 \in M_1(B_1^2)$. This defines a period 2 history $h^2 = (B_1^2, m_1, B_2^2)$. Let

$$\begin{aligned} \mathcal{A}(h^2) &= \{a \in \mathcal{A} : m_1(a) = a\}, \\ \mathcal{B}(h^2) &= \{a \in \mathcal{A} : m_1(b) = b\} \cup B_2, \end{aligned}$$

denote the unmatched agents at the beginning of period 2, when agents at $(1, B_1^2)$ matched according to m_1, and arrivals in period 2 are B_2^2.

Definition 8.1. Fix $\mu \in \mathcal{M}$, and a history $h^2 \in H^2$. $\langle A', B', m' \rangle$ is a *block of* μ *at* h^2 if:

1. $A' \cup B' \subset \mathcal{A}(h^2) \cup \mathcal{B}(h^2)$ can enforce m' over $\mu(h^2)$,
2. $(\forall a \in A') u(a, m'(a)) > u(a, \mu(h^2)(a))$, and
3. $(\forall b \in B') v(m'(b), b) > v(\mu(h^2)(b), b)$.

Definition 8.2. Fix $\mu \in \mathcal{M}$, and an arrival B_1. $\langle A', B', \mu' \rangle$, is a *block of* μ *at* B_1 if:

1. $A' \cup B' \subseteq \mathcal{A} \cup B_1$ can enforce $\mu'(B_1)$ over $\mu(B_1)$,
2. $(\forall h^2 \in H^2(B_1)) \mu'(h^2) = \mu(h^2)$,
3. $(\forall a \in A') U(a, m_{\mu'}; B_1) > U(a, m_\mu; B_1)$, and
4. $(\forall b \in B') V_1(b, m_{\mu'}; B_1) > V_1(b, m_\mu; B_1)$.

Definition 8.3. A contingent matching μ is *dynamically stable* if $(\forall B_1 \subseteq \mathcal{B})$, it has no blocks at B_1, and $(\forall B_1 \subseteq \mathcal{B})(\forall h^2 \in H_2(B_1))$, it has no blocks at h^2. $D(\mathcal{E})$ denotes the set of dynamically stable matchings.

For the allocation of objects with priorities, given $\mu \in \mathcal{M}$, an arrival B_1, and a history $h^2 \in H^2(B_1)$, blocks at h^2 are defined as in Definition 8.1 using Definition 4.5 instead of Definition 4.1. Below, I define a block of a matching μ at B_1:

Definition 8.4. Fix a contingent matching μ, and a history B_1. $\langle A', B', \mu' \rangle$ is a *block* of μ at B_1 if:

1. $A' \cup B'$ can enforce $\mu'(B_1)$ over $\mu(B_1)$ as in Definition 4.5,

2. $(\forall h^2 \in H^2(B_1))\mu'(h^2) = \mu(h^2)$,

3. $(\forall a \in A')U(a, m_{\mu'}; B_1) > U(a, m_\mu; B_1)$, and

4. $(\forall b \in B')(\forall h^2 = (B_1, \mu(B_1), B_2)) \quad \mu'(B_1)(b) \trianglerighteq_b \mu(h^2)(b)$.

Item 4 in Definition 8.4 incorporates how $b \in B_1$ compares intertemporal allocations; in particular, b blocks with $a \in \mathcal{A}$ only if a is at least as good as all of b's matching partners according to m_μ at B_1.

Definition 8.5. Fix $\mu \in \mathcal{M}$. μ eliminates justified envy if $(\forall B_1 \subseteq \mathcal{B})$ it has no blocks at B_1, and $(\forall B_1 \subseteq \mathcal{B})(\forall h^2 \in H^2(B_1))$, it has no blocks at h^2. Denote by $D_{NE}(\mathcal{E}_{\mathcal{B},\triangleright})$ the set of contingent matchings that eliminate justified envy.

8.2 Core

To define the core when arrivals are stochastic, one has to accommodate for the fact that, given B_1, a may be matched with different $\beta-$agents depending on the realization of B_2, as long as she doesn't match at $(1, B_1)$, while $b \in B_1$ can match with different $\alpha-$agents depending on the realization of B_2, as long as he hasn't matched at $(1, B_1)$. Hence, a may want to improve on her matching outcome at one particular arrival (B_1, B_2) without changing her matching at other arrivals $(B_1, \tilde{B}_2), \tilde{B}_2 \neq B_2$. Definition 8.6 below allows for these improvements as long as the rest of the $\beta-$agents who match with a are not hurt by her block. Given $B_1 \subseteq \mathcal{B}$, matchings $m, m' \in M$, and an agent $k \in \mathcal{A} \cup B_1$, let $\mathbb{M}(m, m', k; B_1) = \{\tilde{m} \in M(B_1) : (\forall t \in \{1, 2\})(\forall B_1^t = B_1)\tilde{m}_t(B^t)(k) \in \{m_t(B^t)(k), m'_t(B^t)(k)\}\}$ denote the set of all matchings such that agent k is matched to agents to whom k is matched to under m or m'.

Definition 8.6. A contingent matching $\mu \in \mathcal{M}$ is in the *core of economy* $\mathcal{E}$ if $(\forall B_1)(\nexists \langle \mathcal{C}, \mu' \rangle)$ where $\mathcal{C} = \{A', B'_1\} \cup \{B'_2(B^2) : B_1^2 = B_1\}$, $A' \subseteq \mathcal{A}, B'_1 \subseteq B_1$, and $(\forall B^2)B'_2(B^2) \subseteq B_2^2$, such that:

1. $A' \cup B' \cup \bigcup_{B_2 \cap B_1 = \emptyset} B'_2(B^2) \neq \emptyset$,

2. $m_{1,\mu'}(B^t)(A' \cup B'_1) = A' \cup B'_1$, and $(\forall B^2 : B_1^2 = B_1)\, m_{2,\mu'}(B^2)(A' \cup B'_1 \cup B'_2(B^2)) = A' \cup B'_1 \cup B'_2(B^2)$,

3. $(\forall a \in A')U(a, m_{\mu'}; B_1) \geq \max_{\tilde{m} \in \mathbb{M}(m_\mu, m_{\mu'}, a; B_1)} U(a, \tilde{m}, B_1)$,

4. $(\forall b \in B'_1)V_1(b, m_{\mu'}; B_1) \geq \max_{\tilde{m} \in \mathbb{M}(m_\mu, m_{\mu'}, b; B_1)} V_1(b, m_{\mu'}; B_1)$, and $(\forall B^2)\,(\forall b \in B'_2(B^2))$ $v(m_{2,\mu'}(B_1, B_2)(b), b) \geq v(m_{2,\mu}(B_1, B_2)(b), b)$, and

5. $(\exists k \in A' \cup B')$ such that the above inequality is strict.

Denote by $C(\mathcal{E})$ the set of core matchings.

When arrivals are deterministic, Definitions 4.10 and 8.6 are equivalent. Conditions 2. and 3. impose a form of "dynamic consistency" on the blocks[16]: no blocking a (resp., b) can do better by being able to keep one of her (resp., his) former matching partners, possibly dropping another β(resp., α)-agent who she (he) is matched to under μ'.[17]

In one-sided economies with stochastic arrivals, I extend Definition 4.11 to *interim* Pareto efficiency: there should be no Pareto improvements at any B_1.

Definition 8.7. An on-path matching $m \in M$ is *Pareto efficient* if there is no B_1, and no other on-path matching $m' \in M(B_1)$ such that $(\forall a \in \mathcal{A}) \quad U(a, m'; B_1) \geq U(a, m; B_1)$ and $(\exists a' \in \mathcal{A}) : U(a', m'; B_1) > U(a', m; B_1)$.

9 Stochastic arrivals: Existence in one-sided economies

The result in Section 5.2 continues to hold when arrivals are stochastic:

Proposition 9.1. *Fix a one-sided economy, $\mathcal{E}_{\mathcal{B}}$. Then, for any Pareto efficient matching m, there exists $\mu \in D_P(\mathcal{E}_{\mathcal{B}}) : m_\mu = m$.*

The proof of Proposition 9.1 is in Appendix B. When arrivals are stochastic, following the same reasoning as in Section 5.2, any serial dictatorship leads to a Pareto efficient matching: each agent chooses, for each arrival realization, to which object - out of the ones which haven't been claimed so far - she wants to match with. Hence, when arrivals are stochastic, serial dictatorship looks like a multiple-waiting list mechanism, where the order in the waiting list is that of the serial dictatorship.

However, as an example in Appendix B shows, when arrivals are stochastic not all Pareto efficient matchings can be obtained by a serial dictatorship. There are Pareto efficient matchings that satisfy the following. In applying the procedure defined in Section 5.2, one reaches an agent whose favorite matching out of the remaining feasible ones involves objects which, under the Pareto efficient matching, are assigned to some of the remaining agents. In a sense, the Pareto efficient matching is giving priority to these other agents over the objects. This observation is particularly relevant in deceased-donor organ allocation because some organs maybe saved for patients with rare blood types, or tissue compatibilities. In this case, I show there is a way of using the YRMH-IGYT mechanism of Abdulkadiroğlu and Sönmez [1] so that the property rights are respected at each continuation history.

[16] The conditions are vacuous when arrivals are deterministic.

[17] Echenique and Oviedo [18] refer to this notion as the Blair core (see Blair [10]); see also Roth [38].

Example 5.1 implies that dynamically stable matchings may fail to exist in two-sided economies, and the allocation of objects with priorities when arrivals are stochastic. Section 10 studies sufficient conditions on preferences under which dynamically stable matchings exist. As in Section 6, under these conditions, there are core matchings which are dynamically stable. Thus, in Section 10.1, I establish some properties of the core when arrivals are stochastic.

10.1 Benchmark: Core

When arrivals are stochastic, core matchings may fail to exist:

Proposition 10.1 (Theorem 6.2, Doval [15])**.** *When agents discount the future, and arrivals are stochastic, there exist $\mathcal{E}$ such that $C(\mathcal{E}) = \emptyset$.*

The proof is in Appendix S.5. As discussed in Section 8.2, when arrivals are stochastic, an agent $a \in \mathcal{A}$ may be matched to different $\beta-$agents depending on the realization of B_2. Hence, when comparing two matchings m, and m', a is implicitly comparing sets of potential partners. If a discounts the future, this may introduce complementarities: how valuable matching with a $\beta-$agent at one particular realization is depends on who else is available to match at other realizations. To fix ideas, consider the following example. In $t = 1$, $B_1 = \{b_1\}$, and, in $t = 2$, with probability p, $B_2 = \{\overline{b}_2\}$, and with probability $1 - p$, $B_2 = \{\underline{b}_2\}$. Assume $u(a, \overline{b}_2), u(a, \underline{b}_2) > u(a, b_1)$, and $\delta_a[pu(a, \overline{b}_2)+(1-p)u(a, \underline{b}_2)] > u(a, b) > \delta_a[pu(a, \overline{b}_2)+(1-p)u(a, b)]$ -notice that $\delta_a < 1$ for the inequalities to hold-. Then, a's willingness to match with $\overline{b}_2$ depends on whether $\underline{b}_2$ is available to match or not. It is well-known in static many-to-(one)many matching that complementarities preclude existence; however, the observation that these complementarities can be brought about by discounting coupled with the stochasticity of the arrivals is proper to the dynamic setting considered in this paper.

If $(\forall k \in \mathcal{A} \cup \mathcal{B})\delta_k = 1$, these complementarities do not arise, and hence, the core is non-empty.[18] This is recorded for future use in the following proposition:

Proposition 10.2 (Theorem 6.3, Doval [15])**.** *Fix $\mathcal{E}$ such that $(\forall k \in \mathcal{A} \cup \mathcal{B})\delta_k = 1$. Then $C(\mathcal{E}) \neq \emptyset$.*

[18]The condition is stronger than necessary. However, any other condition would impose restrictions both on δ and G (the distribution of arrivals). See the discussion in Doval [15].

As in Section 6.2, this section introduces two kinds of restrictions on preferences: restrictions concerning how patient agents are, and restrictions concerning how much agents agree on what constitutes a good matching.

From Propositions 10.2, and 6.2, it follows that when agents don't discount the future, dynamically stable contingent matchings exist.

Proposition 10.3. *Suppose that* $(\forall k \in \mathcal{A} \cup \mathcal{B})\delta_k = 1$. *Then,* $D(\mathcal{E}) \cap C^*(\mathcal{E}) \neq \emptyset$.

Contrary to Proposition 6.2, it is no longer the case that all core on-path matchings are part of a dynamically stable matching. As an example, suppose $\mathcal{A} = \{a\}$, $B_1 = b_1$, and in $t = 2$ with probability $p < 1$, b_2 arrives. Suppose $u(a, b_2) > u(a, b_1)$. There are two on-path matchings in the core: (i) a matches in $t = 2$ with the best available b, (ii) a matches in $t = 1$ with b. However, only the first is dynamically stable.[19]

Remark 10.1. When arrivals are deterministic, almost no-discounting is only needed to restrict $\alpha-$agents' preferences, while when arrivals are stochastic Proposition 10.3 makes an assumption on the time preferences of agents on both sides. This is because when arrivals are stochastic pairwise stability amongst agents who match in period 1 no longer implies that all $b \in B_1$ who are matched have to match in period 1.

When arrivals are deterministic, the presence of a simultaneous cycle in agents' preferences (Definition 6.1) implies agents do not agree on which agent on the other side constitutes a good match. That is, agents' preferences over matchings are not, in a sense, *aligned*. However, when arrivals are stochastic, no simultaneous cycles does not necessary represent alignment as illustrated below:

Remark 10.2. Suppose a_1, a_2, b arrive in $t = 1$, and with probability p, b^* arrives in $t = 2$. Furthermore, assume $\delta_{a_1}(pu(a_1, b^*) + (1 - p)u(a_1, b)) > u(a_1, b) > \delta_{a_1}pu(a_1, b^*)$; i.e., a_1's favorite match is to match in period 2 with b^*, in case he arrives, and with b, in case b^* doesn't. But she prefers to match in period 1 with b rather than to wait to only match with b^*, if he arrives. If b prefers to match immediately with a_1 to match with a_2, but prefers to match with a_2 to matching in period 2 with a_2 when b^* arrives and

[19] The first matching in the core is denoted a *weakly setwise stable* matching in Doval [15]. The weakly setwise stable set is always a subset of the core, and is non-empty when agents don't discount the future. Under this assumption on time preferences, any weakly setwise stable matching is also dynamically stable. See Doval [15] for the definition of the weakly setwise stable set. Proposition 6.1 in that paper shows that weakly setwise stable matchings select, for each B^2, a matching $m = (m_1, m_2) \in M_1(B_1^2) \times M_2(B_2^2)$ such that m is on-path for the core of an economy in which arrivals are deterministic and given by B^2. It follows from this observation that, when agents don't discount the future, any (on-path) weakly setwise stable matching is part of a dynamically stable matching.

with a_1 when he doesn't, then a_1 matches with him in $t = 1$.[20] Hence, a_1 is negatively affected by b's reticence to wait with her. However, if b^* prefers a_1 to a_2, by waiting, a_1 can always make sure she gets her most preferred matching. In this case, a_1 has a block by waiting, because b is not willing to be a "backup" for a_1. (This example should be reminiscent of job market instances in which a smaller business, b, makes an exploding offer to an attractive applicant a_1 to avoid being used as a backup in case a better business b^* does not make an offer.) Appendix A.2 presents an economy along these lines in which the core is non-empty, but no dynamically stable matching exists.

In the economy described in Remark 10.2, and detailed in Appendix A.2, agents on both sides agree over the ranking of the agents on the other side, and hence no simultaneous cycle exists as in Definition 6.1; however, they do not agree on how to rank the feasible matchings. The condition below, *weak top-coalition* (Banerjee et al. [9]), represents alignment in this environment.[21] Fix $B_1 \subseteq \mathcal{B}$, and a collection $\mathcal{C} = \{A_C, B_{1,C}\} \cup \{B_{2,C}(B^2) : B_{2,C}(B^2) \subseteq B_2^2 \wedge B_1^2 = B_1\}$ where $A_C \cup B_{1,C} \subseteq \mathcal{A} \cup B_1$. Given $k \in A_C \cup B_{1,C}$, let $\mathbb{M}(\mathcal{C}, k; B_1)$ denote the set of matchings in which k matches only with agents in $\mathcal{C}$.

Definition 10.1. Fix a realization $B_1 \subseteq \mathcal{B}$. $\mathcal{C}$ has a *top-coalition* at B_1 if there exists $\mathcal{T}$ such that $A_T \cup B_{1,T} \subseteq A_C \cup B_{1,C}$, and $(\forall B^2) B_{2,T}(B^2) \subseteq B_{2,C}(B^2)$, and there exists $m^* \in M(B_1)$ such that the following holds:

1. $m_1^*(B_1)(A_T \cup B_{1,T}) = A_T \cup B_{1,T}$,
2. $(\forall B^2 : B_1^2 = B_1) m_2^*(B^2)(A_T \cup B_{1,T} \cup B_{2,T}(B^2)) = A_T \cup B_{1,T} \cup B_{2,T}(B^2)$;
3. $(\forall a \in A_T) U(a, m^*; B_1) \geq \max_{m \in \mathbb{M}(\mathcal{C}, a; B_1)} U(a, m; B_1)$,
4. $(\forall b \in B_{1,T} \cap B_1) V(b, m^*; B_1) \geq \max_{m \in \mathbb{M}(\mathcal{C}, b; B_1)} V(b, m; B_1)$, and
5. $(\forall B^2)(\forall b \in B_{2,T}(B^2)) v(m^*(B^2)(b), b) \geq \max\{\max_{a \in A_C} v(a, b), 0\}$.

Definition 10.2. $\mathcal{E}$ satisfies the *weak top-coalition property* at B_1 if there exists $\{\mathcal{C}_i\}_{i=1}^N$, where $\mathcal{C}_i = \{A_i, B_{1,i}\} \cup \{B_{2,i}(B^2) : B_{2,i}(B^2) \subseteq B_2^2 \wedge B_1^2 = B_1\}$, such that:

1. $\{A_i\}_{i=1}^n$ is a partition of $\mathcal{A}$ and $\{B_{1,i}\}_{i=1}^n$ is a partition of B_1,
2. $(\forall B^2)\{B_{2,i}(B^2)\}_{i=1}^n$ is a partition of B_2^2, and
3. $(\forall i \in \{1, ..., n\})$ $\mathcal{C}_i$ is a *top-coalition* of $\mathcal{C}^i = \{\cup_{j=i}^n A_j, \cup_{j=i}^n B_{1,j}\} \cup \{\cup_{j=i}^n B_{2,j}(B^2) : B_1^2 = B_1\}$.

[20] That is, $v(a_1, b_1) > v(a_2, b_1) > \delta_b[pv(a_2, b_1) + (1-p)v(a_1, b_1)]$. Notice that $\delta_b < 1$.

[21] When arrivals deterministic, no simultaneous cycles implies the weak top-coalition property.

$\mathcal{E}$ satisfies the weak top-coalition property if, for all B_1 such that $\sum_{B_2} G(B_1, B_2) > 0$, $\mathcal{E}$ satisfies the weak top-coalition property at B_1.

Definition 10.1 implicitly states that $a \in \mathcal{A}$ who want to be matched in $t = 2$ only match with $\beta-$agents who arrive in period 2, i.e., they need no "backups" as in Remark 10.2. The following result follows from the definition of weak top-coalition:

Proposition 10.4. *If $\mathcal{E}$ satisfies the weak top-coalition property, then $C^*(\mathcal{E}), D(\mathcal{E}) \neq \emptyset$.*

10.3 Allocation of objects with priorities

Ergin-acyclity (Section 6.3) guaranteed that, when arrivals are deterministic, $D_{NE}(\mathcal{E}) \neq \emptyset$. However, when arrivals are stochastic, this is no longer the case[22]:

Proposition 10.5. *There exist economies with stochastic arrivals such that $\{\rhd_b\}$ satisfies Ergin-acyclicity, and $D_{NE} = \emptyset$.*

See Appendix D.2 for details. The failure of existence follows from the effect discussed in Remark 10.2: there are agents who want to use buildings in $t = 1$ as backups in $t = 2$ in case nothing better comes along; a conflict may arise when the building used as backup ranks another agent higher.

The failure of existence of dynamically stable matchings when arrivals are stochastic and priorities are Ergin-acyclic should not be surprising. Kojima [27] shows that, when agents have multi-unit demands, acyclicity no longer guarantees Pareto efficiency of the matching obtained by deferred acceptance. When arrivals are stochastic, the environment considered here resembles a multi-unit demand setting: $a \in \mathcal{A}$ may match with multiple objects (potentially one for each arrival realization). Kojima shows that, when agents have multi-unit demand, another condition, *essentially homogeneous priorities*, guarantees Pareto efficiency of the matching obtained by deferred acceptance. In the setting considered here, it reduces to requiring that $\rhd_b = \rhd$ for all $b \in \mathcal{B}$. It follows that:

Proposition 10.6. *Let $\mathcal{E}_{\mathcal{B},\rhd}$ be such that $(\forall b \in \mathcal{B}) \rhd_b = \rhd$. Then, $D_{NE}(\mathcal{E}_{\mathcal{B},\rhd}) \neq \emptyset$.*

The result follows from the proof of Proposition 9.1, since $\rhd$ can be used to define a serial dictatorship. Moreover, notice that the assumption in Proposition 10.6 implies that the weak top-coalition property holds.

Remark 10.3. Proposition 10.6 is also useful for the allocation of deceased-donor organs, in particular, the allocation of deceased-donor kidneys. The literature has tra-

[22]Nonetheless, *Ergin-acyclicity* is sufficient, in the allocation of objects with priorities, for the non-emptiness of the core when arrivals are stochastic. See Theorem 6.4 in Doval [15].

ditionally modeled kidney exchange as 0-1 preferences on both sides of the market (1=compatible, 0=incompatible) (see Roth et al. [41], Ünver [48], Akbarpour et al. [3], Anderson et al [5]), or as a one-sided market (Roth et al. [40]) as done here. However, the argument can be made that kidneys have priorities over patients just as buildings have priorities over applicants to public housing. The clearing house may wish to prioritize blood type O patients, over types A and B, and the latter over type AB, or sensitized patients over non-sensitized ones. In the case of priority according to blood types, the following priority ranking can be constructed. First, blood type O patients come before $\{A, B, AB\}$, blood type A patients come before $\{B, AB\}$, and blood type B patients come before AB. Second, within each blood type category, complete the ranking by selecting a strict ordering of the agents -for instance, order them according to urgency, or according to benefit of being transplanted-. Then, endow all kidneys with that strict ordering. Proposition 10.6 implies that a dynamically stable matching exists. Moreover, the on-path matching prioritizes agents according to their blood types, in the sense that, until matched, blood type O patients keep the possibility of choosing first, blood type A patients choose second, and so on and so forth.

11 Observability of matching outcomes

I have emphasized so far the role of preferences in generating non-existence of dynamically stable contingent matchings. However, agents' decision to wait or not to be matched in period 2 depends, in part, on who exits the market in period 1. This motivates the analysis in this section. Section 11.1 introduces a weakening of dynamic stability, *correlated dynamic stability*, which allows for randomizing over matchings. This introduces uncertainty over who matches in period 1, and thus, over the benefit of waiting to be matched in period 2. Section 11.2 shows that this uncertainty restores existence in Example 5.1; however, correlated dynamically stable matchings fail to exist (Proposition 11.2). This highlights that lack of preference alignment and discounting may be more important than observability as forces driving non-existence.

11.1 Correlated Dynamic Stability

I begin by defining what is meant by a random matching:

Definition 11.1. A *random contingent matching* μ is a map:

$$\begin{aligned} \mu \quad &: \quad 2^{\mathcal{B}} \cup H^2 \mapsto \bigcup_{B_1 \subseteq \mathcal{B}} \left\{ \Delta(M_1(B_1)) \cup \bigcup_{B_2 \cap B_1 = \emptyset} \Delta(M_2(B_1, B_2)) \right\}, \\ \text{s.t.} \quad & (\forall B_1)\mu(B_1) \in \Delta(M_1(B_1)) \text{ and } (\forall h^2 \in H^2(B_1))\mu(h^2) \in \Delta(M_2^c(h^2)). \end{aligned}$$

I now illustrate how matchings are formed. Agents know μ. Fix $B_1 \subseteq \mathcal{B}$. Nature selects $m_1 \in M_1(B_1)$ according to $\mu(B_1)$. Each agent $k \in \mathcal{A} \cup B_1$ observes $m_1(B_1)(k)$, but not $m_1(B_1)$. At the beginning of $t = 2$, the remaining unmatched agents are informed of $h^2 = (B_1, m_1, B_2)$. Nature selects $m_2(B_1, B_2) \in M_2(B_1, B_2)$ according to $\mu(h^2)$. Agents in $\mathcal{A}(h^2) \cup \mathcal{B}(h^2)$ observe $m_2(B_1, B_2)(k)$, and not $m_2(B_1, B_2)$.

Some subtleties are introduced by the randomization over outcomes, which must be taken into account when defining blocks of μ. First, when $k \in \mathcal{A} \cup B_1$ observes $m_1(B_1)(k)$, she/he can use this, and μ to update her/his information about the realization of $\mu(B_1)$. In particular, if $m_1(B_1)(k) \neq k$, k can use the updated information to calculate the profitability of waiting to be matched in $t = 2$. Second, if $(\hat{k}, k)$ is a candidate blocking pair, where $\hat{k} \neq m_1(B_1)(k)$, then $\hat{k}$'s willingness to participate in the block reveals information about $\hat{k}$'s matching outcome. When deciding whether to match with $\hat{k}$ instead of $m_1(B_1)(k)$, k should use the information revealed by $m_1(B_1)(k)$, and $\hat{k}$'s willingness to match with k.

I now formalize the statements in the previous paragraph. Fix B_1, and suppose agent k observes $m_1(B_1)(k)$. First, when calculating her/his payoffs, k uses $\mu(B_1)$ conditional on $E(m_1(k)) = \{\tilde{m}_1 \in \text{supp } \mu(B_1) : \tilde{m}_1(B_1)(k) = m_1(k)\}$. In a slight abuse of notation, let $U(a, \mu|_{E(m_1(a))}; B_1)$ (resp.,$V(b, \mu|_{E(m_1(b))}; B_1)$) denote a's (resp., b's) utility from μ, conditional on the arrival realization being B_1, and having observed $m_1(a)(resp., m_1(b))$. In particular, if $m_1(B_1)(k) \neq k$, to calculate her/his payoffs from waiting to be matched in $t = 2$, k conditions μ on $C(m_1(k)) = \{h^2 \in H^2(B_1) : h^2 = (B_1, \tilde{m}_1, B_2), \tilde{m}_1(k) = k, \tilde{m}_{1,(k,m_1(k))} \in E(m_1(k))\}$, where $\tilde{m}_{1,(k,m_1(k))} \in M_1$ is the matching that coincides with $\tilde{m}_1$ except that $\tilde{m}_1(k) = m_1(k)$. Second, if $k = a$, and $\hat{k} = b \neq m_1(B_1)(a)$ constitute a candidate blocking pair, a conditions $\mu(B_1)$ on $E(m_1(a), b) = \{\tilde{m}_1 \in E(m_1(a)) : v(a, b) > V(b, \mu|_{E(\tilde{m}_1(b))}, B_1)\}$ to decide whether she should match with b instead of $m_1(B_1)(a)$.

I introduce now an assumption by which the updating brought upon by an offer to match is only relevant for $k \in \mathcal{A} \cup B_1$ such that k is unmatched under $m_1(B_1)$. Consider what k can do after $(k, \hat{k})$ is proposed as a candidate blocking pair, and updating her/his information: (i) decide to go through with her/his match, (ii) match with $\hat{k}$, and (iii) if matched, k could recalculate the profitability of waiting -i.e. replace $E(m_1(k))$ with $E(m_1(k), \hat{k})$ in $C(m_1(k))$-. I rule (iii) out; it is restrictive but relaxing this does not change the non-existence result below. Hence, when a matched k considers to match with $\hat{k}$, k matches with $\hat{k}$ if, and only if, $\hat{k}$ is better than $m_1(B_1)(k)$. Updating is important, then, for those k that are unmatched, and use the information to refine their expectations about their $t = 2$ matching.

Definitions 11.2-11.3 below state formally what is meant by a block when μ is a random contingent match. Definition 11.2 coincides with the previous definition of a block in $t = 2$. However, Definition 11.3 allows only for individual, or pairwise blocks, and hence, weakens the notion of dynamic stability both in the possibility of randomizing over allocations, and the size of the blocking coalitions.

Definition 11.2. Fix a random matching μ, and $h^2 \in H^2(B_1), B_1 \subseteq \mathcal{B}$. $\mu(h^2)$ has a block at h^2 if $\exists m_2 \in \text{supp}\mu(h^2)$ such that one of the following hold:

1. $(\exists a \in \mathcal{A}(h^2)) : u(a, a) > u(a, m_2(a))$, or $(\exists b \in \mathcal{B}(h^2)) : v(b, b) > v(m_2(b), b)$,
2. $(\exists (a, b) \in \mathcal{A}(h^2) \times \mathcal{B}(h^2)) u(a, b) > u(a, m_2(a))$, and $v(a, b) > v(m_2(b), b)$.

Definition 11.3. Fix a random matching μ and $B_1 \subseteq \mathcal{B}$. μ has a block at B_1 if either of the following hold:

1. $(\exists m_1 \in \text{supp } \mu(B_1))(\exists (a, b) \in \mathcal{A} \times \mathcal{B}(B_1))$ such that:

$$u(a, b) > U(a, \mu|_{E(m_1(a),b)}, B_1), \text{and } v(a, b) > V(b, \mu|_{E(m_1(b),a)}, B_1),$$

2. $(\exists m_1 \in \text{supp } \mu(B_1))(\exists a \in \mathcal{A}) : m_1(a) \neq a$, and

$$\delta_a \mathbb{E}_{\mu,B_2}[u(a, m_2(\cdot)(a))||B_1, C(m_1(a))] > u(a, m_1(a)),$$

or $(\exists m_1 \in \text{supp}\mu(B_1))(\exists b \in \mathcal{B}(B_1)) : m_1(b) \neq b$, and

$$\delta_b \mathbb{E}_{\mu,B_2}[v(m_2(\cdot)(b), b)||B_1, C(m_1(b))] > v(m_1(b), b).$$

Definition 11.4. A random contingent matching μ is *correlated dynamically stable* if $(\forall B_1 \in H^1)$ there are no blocks at B_1, and there are no blocks at $h^2 \in H^2(B_1)$. $D_\Delta(\mathcal{E})$ denotes the set of correlated dynamically stable contingent matchings.

11.2 Results

Section 5.1 shows that dynamically stable matchings may fail to exist since agents who match in $t = 1$ exert a positive externality on the remaining agents by freeing up matching opportunities in $t = 2$. For instance, in Example 5.1, when α_A matches at $t = 1$, α_B can match with β_{2C} by waiting; likewise, when α_C matches in $t = 1$, α_A can match with

β_{2C} by waiting. Now, being able to realize the gains by waiting to be matched requires observing who matches in $t = 1$: when α_A matches in $t = 2$, α_B actually prefers to match in $t = 1$, and the same holds for α_A if α_C matches in $t = 2$. Hence, one may prevent the market from unraveling by creating uncertainty over the realization of other agents' $t = 1$ matching outcomes. This is recorded in Proposition 11.1.

Proposition 11.1. *Let $\mathcal{E}$ be as in Example 5.1. Then, $D_\Delta(\mathcal{E}) \neq \emptyset$.*

The details of the proof are in Appendix E.1. However, it is useful to understand how the construction works. Since for each possible outcome in $t = 1$ there is a unique stable matching in $t = 2$ among the remaining agents, the contingent matching only randomizes over the $t = 1$ matching, and this generates uncertainty about the one in $t = 2$. Let μ denote the random contingent matching. The following are the on-path matchings that occur with positive probability (the double bars separate the $t = 1$ and $t = 2$ matchings):

$$m^1 = \left(\begin{array}{cc||cc} \alpha_A & \alpha_B & \alpha_C & \emptyset \\ \beta_{1A} & \beta_{1B} & \beta_{2C} & \beta_{2D} \end{array} \right) m^2 = \left(\begin{array}{cc||cc} \alpha_A & \alpha_C & \alpha_B & \emptyset \\ \beta_{1A} & \beta_{1B} & \beta_{2C} & \beta_{2D} \end{array} \right)$$

$$m^3 = \left(\begin{array}{c||ccc} \alpha_B & \alpha_A & \alpha_C & \emptyset \\ \beta_{1B} & \beta_{1A} & \beta_{2C} & \beta_{2D} \end{array} \right) m^4 = \left(\begin{array}{c||ccc} \alpha_C & \alpha_A & \alpha_B & \emptyset \\ \beta_{1B} & \beta_{2C} & \beta_{2D} & \beta_{1A} \end{array} \right)$$

Focus first on m^1, m^3. When α_B is told that she matches with β_{1B} she does not know whether α_A matches in $t = 1$ or $t = 2$, and hence does not know whether she can be matched with β_{2C} if she waits to be matched in $t = 2$. I show there exist probabilities over m^1 and m^3 such that she does not want to wait. However, note that for this to be possible α_A has to match with β_{1A} at $t = 2$ (m^3). To avoid that α_A blocks μ with β_{1A}, μ places positive probability on m^4, in which α_A matches with β_{2C}. Again, I show that probabilities can be chosen so that conditional on $m_1(\alpha_A) = \alpha_A$, α_A does not want to block with β_{1A}. Now, note that in m^4, α_B wants to block with β_{1B}. This is why μ places positive probability on m^2, in which α_B matches with β_{2C}. Finally, note that whenever β_{1B} wants to block with α_B, β_{1B} can't reveal to α_B whether she will be matched with β_{2C} or β_{2D}, and hence α_B does not want to block, and whenever α_C wants to match with β_{1B}, the latter is matched with α_B.

Restoring existence in the above economy comes at the price of generating an inefficiency: with positive probability (α_A, β_{1A}) match at $t = 2$ when they are both available to match at $t = 1$ (matching m^3). This inefficiency is essential to avoid α_B from deviating in the initial period. Moreover, if μ does not place positive probability on allocations that match α_B at $t = 1$, it is α_A that will find it optimal to wait until the last period to match.

Unfortunately, the set of correlated dynamically stable matchings may be empty as well, as the next proposition shows.

Proposition 11.2. *There exist economies $\mathcal{E}$ such that $D_\Delta(\mathcal{E}) = \emptyset$.*

The proof is in Appendix E.2, and it is based on the following economy:

Example 11.1. Let $\mathcal{A} = \{\alpha_A, \alpha_B, \alpha_C\}, B_1 = \{\beta_{1A}, \beta_{1B}\}, B_2 = \{\beta_{2C}\}$, preferences are written below following the same convention as in Example 5.1.

$\alpha_A :$	$(\beta_{2C}, 0)$	$(\beta_{1B}, 0)$	$(\beta_{1A}, 0)$	$(\beta_{2C}, 1)$	$(\beta_{1B}, 1)$	$(\beta_{1A}, 1)$	$\beta_{1A} :$	α_C	α_A	α_B
$\alpha_B :$	$(\beta_{2C}, 0)$	$(\beta_{2C}, 1)$	$(\beta_{1B}, 0)$	$(\beta_{1B}, 1)$	$(\beta_{1A}, 0)$	$(\beta_{1A}, 1)$	$\beta_{1B} :$	α_C	α_B	α_A
$\alpha_C :$	$(\beta_{2C}, 0)$	$(\beta_{2C}, 1)$	$(\beta_{1B}, 0)$	$(\beta_{1B}, 1)$	$(\beta_{1A}, 0)$	$(\beta_{1A}, 1)$	$\beta_{2C} :$	α_A	α_C	α_B

Table 3: Preferences in Example 11.1

Assume that if $v(\alpha, \beta) > v(\alpha', \beta) \Rightarrow \delta_\beta v(\alpha, \beta) > v(\alpha', \beta)$.

In this economy, all $\alpha-$agents agree that β_{2C} is better than β_{1B}, and β_{1B} is better than β_{1A}. However, α_A is impatient, and prefers to match with β_{1A} at $t = 1$ than to match with β_{2C} at $t = 2$. In this economy, $D(\mathcal{E}) = \emptyset$. If there existed $\mu \in D(\mathcal{E})$, then α_A has to be matched, since otherwise she would block with β_{2C}. However, it can't be that $m_{2,\mu}(\alpha_A) = \beta_{2C}$, and $\mu \in D(\mathcal{E})$: if that were the case, $m_{1,\mu}(\alpha_C) = \beta_{1B}$, and $m_{1\mu}(\alpha_B) = \beta_{1A}$ would hold. This is blocked by (α_A, β_{1A}). Now, if α_A matches with β_{1A}, one can show that α_C has to match in $t = 2$ with β_{2C}, which implies α_B, and β_{1B} match in $t = 1$. The construction in Appendix E.2 shows that when α_B, β_{1B} match in $t = 1$, the unique stable matching in the continuation matches α_C with β_{1A}. Hence, β_{1A} has a block by waiting. Moreover, when β_{1A} waits, this frees α_C for β_{1B}, and β_{1B} wants to wait.

It would seem that by hiding from β_{1A} and β_{1B} whether the other matches or not at $t = 1$ would suffice, as before, to obtain a correlated dynamically stable matching. However, it turns out that to do so, sometimes α_C has to match with β_{1A}, and α_C may want to undo this by matching with β_{1B}. This implies the probability distribution over matchings has to satisfy more constraints. The proof of Proposition 11.2 shows that there is no random contingent matching that satisfies all constraints.

12.1 Discussion

Implementation. Any dynamically stable matching can be sustained as the subgame perfect Nash equilibrium of a two-stage game based on the one introduced by Kalai [23]. In the second stage, agents $a \in \mathcal{A}$ who have not matched in the first stage, and the remaining agents and new entrants on side B simultaneously announce an agent from the other side -or themselves-, coinciding announcements turn into matchings. In the first stage, fixed a realization of the arrivals, B_1, agents in $\mathcal{A} \cup B_1$ play the same game as in stage two, but amongst themselves. When arrivals are deterministic, the coalition-proof Nash equilibria of a second game, in which agents in $\mathcal{A} \cup B_1 \cup B_2$ announce simultaneously agents on the other side coincides with the core (see Kalai [23], Alcalde [4], Kara and Sönmez [24]). The different timings and equilibrium concepts capture the assumptions embodied in the core, and dynamically stable matchings in terms of the agreements agents can form between themselves. These implementation results are presented in the Supplementary Material, Section S.1 .

Lattice When arrivals are stochastic, the "many-to-many" structure of the core (Section 4.2) implies it no longer has the lattice property. This is in contrast with the case of deterministic arrivals. These and other properties which are lost due to the "many-to-many" structure of the core in the case of stochastic arrivals are explored in depth in Doval [15]. However, even when arrivals are deterministic, the set of dynamically stable matchings does not have the lattice structure (see Appendix S.2).

Different intertemporal orders derived from $\triangleright_b$: I only consider objects which are, in a sense, extremely patient: between matching in $t = 1$ with a, and matching at some $(2, B^2)$ with $a' \triangleright a$, the second matching is chosen. A different definition is one in which the objects' ranking is used to decide between different $t = 1$ matching partners, or different matching partners at $(2, B^2)$ for some B^2, but matching in $t = 1$ is prioritized over matching in $t = 2$. What the correct assumption is depends on the setting. The first assumption may be suitable in cases in which one is concerned with market thickness, so that saving the objects to match later is useful - see for instance Akbarpour et al [3]. The second assumption is better suited in settings in which matching right away, instead of who matches with whom is more relevant. Note that if the objects prioritize matching in $t = 1$, it is as if they discount the future. Hence, the conditions for existence will resemble more those of the two-sided market.

Non-storable objects The assumption that objects are storable plays no role in Section 5.2; all results would go through if objects are non-storable. This is good since the allocation of deceased-donor organs falls in the non-storable objects case. However, the result, in Section 5.1, that core matchings fail to be dynamically stable follows from matching opportunities not expiring.[23] Given the applications considered in those sections (adoption markets, job markets, public housing), the assumption is reasonable. But, there is an interesting extension to consider in light of Remark 10.2. In the example presented there, b prefers to match with a less preferred agent a_2 over matching in $t = 2$ with a more preferred agent, a_1, only when a_1's most preferred matching partner, b^*, does not arrive. I interpret this as b tending a_1 an exploding offer to match in $t = 1$.[24] Matching with contracts allows one to model this explicitly: a_1 and b can sign one of two contracts, one in which they either match in $t = 1$, and which expires in $t = 1$ (i.e. the exploding offer), and one in which a gets to pick when to match with b (i.e. an option contract). Of course, the existence of the first contract implies that b can promise not to match in $t = 2$ with a_1, if they both remain unmatched. By allowing for contracts with different terms and expiration dates, matching with contracts allows for a richer set of binding agreements in this setting. However, suppose there is a third contract between a_1 and b, which b can offer to a_1 in $t = 2$. Even if the exploding offer is an option, the question still remains of whether or not, in $t = 2$, b, after a_1 rejects the exploding offer, will offer this contract to a_1. a_1's decision to reject the exploding offer depends on her expectations about b's offer in $t = 2$. Thus, even in the richer setting, all the issues raised in this paper are still relevant. Studying this extension in depth is left for future research.

12.2 Related Literature

The paper relates to several strands of literature. I focus here on the relation with the literature on dynamic matching markets; other related papers have been referenced throughout the main text. To the best of my knowledge, Kurino [29] is the first to introduce into a dynamic matching problem the notion that a matching should be a complete contingent plan; he analyzes a repeated matching market in which men and women match every period, and does not ask for credible continuations (see Example 1 in the paper).[25] In independent and contemporaneous work, Kadam and Kotowski [22] analyze a two-period repeated matching market as in Kurino [29], introducing a stability notion which they also denote dynamic stability, but applies to on-path match-

[23]Proposition S.4 in the Supplementary Material shows that when the core is non-empty, and matching opportunities expire, then the set of dynamically stable matchings is non-empty.

[24]Recall that b's reticence to wait "forces" a_1 to match with him instead of obtaining her most preferred outcome.

[25]Damiano and Lam [13], and Bando [8] analyze similar settings

ings.[26] Pais [33], and Niederle and Yariv [32] study conditions under which a decentralized two-sided market sustains static stable outcomes. Kurino [30], Kennes [26] and Pereyra [34] provide stability notions for allocating objects to overlapping generations of agents, where the main concern is to respect the property rights over the objects induced by the allocation in the previous periods. In a similar vein, Compte and Jehiel [12] analyze a two period model where matching happens in the last period, but the exogenously given allocation in the first period determines the outside option in the second period. Ünver [48] considers the problem of kidney exchange where agents and their paired donors arrive to an exchange pool over time, and characterizes the optimal mechanism for two-way and multi-way exchanges. Leshno [31] considers the problem of allocating objects which arrive over time to a group of agents, who arrive over time, with unobservable preferences for the objects, and assumes objects have to be assigned upon arrival. Akbarpour et al. [3] consider a networked market where agents are indifferent between their feasible matches, but the planner may want the agents to wait to be matched to facilitate others' matchings. Schummer [45] analyzes the issues with influencing waiting lists when agents are served on a first-come-first-serve basis. Thakral [47] considers algorithms for allocating objects with and without priorities, which arrive stochastically over time and have to be allocated upon arrival. None of the existing papers touch upon the issues that are relevant in the present work, namely, the interaction between the value of waiting for a better match, and the assumptions on the binding agreements agents can form, and how this relates to the appropriate notion of stability in each setting.

Some of the ideas presented in the paper have appeared in more recent work. Baccara et al. [6] consider a two-sided market in which agents face a trade-off between matching immediately with a low type and waiting for a high type agent of the opposite side. Kotowski [28] incorporates the idea of contingent stability to a repeated matching market.

13 Conclusions

Dynamic matching markets such as adoption markets, the allocation of public housing, and of deceased-donor organs share three key features: (i) matching opportunities arrive over time, (ii) matching is one-to-one, and (iii) matching is irreversible. In these markets, agents face a dynamic decision problem: an agent can match today, thus, exiting the pool of the unmatched agents, or she may remain unmatched this period, and keep the option of matching with a (potentially better) matching partner tomorrow.

[26] See Remark 4.1

Despite their similarities, the three markets mentioned above deal differently with the option of waiting for a better match: some adoption agencies limit adoptive parents' option to wait to be matched, the PHA dismisses agents from the waiting list after rejecting compatible housing offers, and UNOS does not penalize patients who refuse the offer to receive a transplant. The framework developed in this paper rationalizes the rules implemented by the different clearinghouses.

In order to understand why markets that look (a priori) similar have in place so different rules, the present paper proceeds in three steps. First, because matching markets with properties (i)-(iii) above have not been studied in the literature, I define a natural stability notion for these markets. I denote it *dynamic stability*. Second, I show that for two-sided economies (and, hence, for the allocation of objects with priorities) dynamically stable matchings may fail to exist. This non-existence problem arises precisely because agents have the option to wait for better matching opportunities. Adoption markets and the allocation of public housing correspond, respectively, to two-sided economies, and the allocation of objects with priorities. In contrast, in markets such as the allocation of deceased-donor organs dynamically stable matchings always exist. It follows from these two observations that in the first two markets a central clearing house which regulates agents actions, in particular, what happens after they reject a current offer in order to wait for a better match, is needed to guarantee the allocation is efficient. Third, in markets such as adoption markets, and public housing, I provide sufficient conditions under which dynamically stable matchings exist. Hence, under these conditions, the clearing house need not restrict agents' actions in order to guarantee efficiency.

A Proofs of Section 5.1

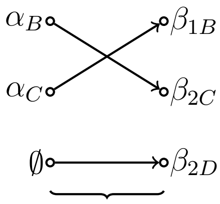
A.1 Proof of Proposition 5.1

I start by specifying the matchings at the different continuation histories; the terms in the curly brackets indicate who matched in $t = 1$ (if any):

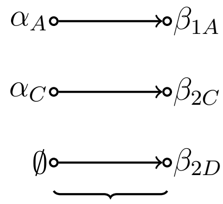
Step 1: There is no dynamically stable matching μ in which α_C is unmatched. Hence, in any $\mu \in D$ she matches with β_{1B}, or β_{2C}. This is because if α_C is unmatched, she blocks with β_{2C}.

Step 2: There is no $\mu \in D$ such that α_C matches with β_{2C}. Said contingent matching would have to match α_B with β_{1B}, and α_A with β_{1A}. However, α_B blocks by waiting to be matched with β_{2C} since that is her outcome in the unique continuation when α_A, β_{1A} match in $t = 1$.

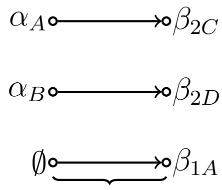
Step 3: There is no dynamically stable matching in which α_C matches with β_{1B}. Said contingent matching would have to match α_B with β_{2C} - otherwise (α_B, β_{1B}) block-, and α_A



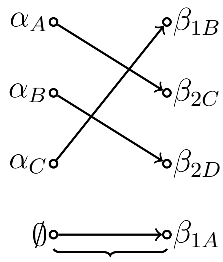
(α_A, β_{1A}) at $t = 1$



(α_B, β_{1B}) at $t = 1$



(α_C, β_{1B}) at $t = 1$



$\emptyset$ at $t = 1$

with β_{1A}. However, this is blocked by α_A waiting to be matched in $t = 2$ since when α_C, β_{1B} matches in $t = 1$, α_A gets matched with β_{2C} in $t = 2$.

Steps 1-3 imply that $D(\mathcal{E}) = \emptyset$.

A.2 Remark 10.2: Non-existence with stochastic arrivals

Example A.1. There exist economies $\mathcal{E}$ with stochastic arrivals such that $D(\mathcal{E}) = \emptyset$.

Proof. Let $\mathcal{A} = \{\alpha_1, \alpha_2\}$, $\mathcal{B} = \{b_1, b_2\}$, and supp $G = \{(b_1, b_2), (b_1, \emptyset)\} = \{\overline{B}^2, \underline{B}^2\}$, where $r_1 = G(\underline{B}^2)$. Preferences are given by:

$$\begin{aligned}
(\forall b \in \mathcal{B}) v(\alpha_2, b) &> v(\alpha_1, b) \\
(\forall \alpha \in \mathcal{A}) u(\alpha, b_2) &> u(\alpha, b_1) \\
v(\alpha_2, b_1) > v(\alpha_1, b_1) &> \delta_{b_1}[r_1 v(\alpha_2, b_1) + (1 - r_1) v(\alpha_1, b_1)] \\
u(\alpha_1, b_1) &> \delta_{\alpha_1}[r_1 u(\alpha_1, b_1) + (1 - r_1) u(\alpha_1, b_2)] \\
\delta_{\alpha_2}[r_1 u(\alpha_2, b_1) + (1 - r_1) u(\alpha_2, b_2)] > u(\alpha_2, b_1) &> \delta_{\alpha_2}(1 - r_1) u(\alpha_2, b_2)
\end{aligned}$$

Both $\beta-$agents agree that α_2 is the best, while both $\alpha-$agents agree that b_2 is the best. However, there is uncertainty about b_2's arrival. α_1 prefers to match immediately with b_1 to wait and match with the best available $\beta-$agent. α_2 holds exactly the opposite preferences. Moreover, α_2 is willing to match with b_1 if the alternative is to match only with b_2 if he arrives. Finally, b_1 prefers to match with α_1 to match contingently with α_1 on $\overline{B}^2$ and with α_2 on $\underline{B}^2$.

Note that if both α_1 and α_2 are present in $t = 1$, any stable match assigns the best available $\beta-$agent to α_2, and the remaining (if any) to α_1, and if only one $\alpha-$agent remains she gets the best remaining $\beta-$agent (if any). Moreover, note that any dynamically stable contingent matching has to satisfy that: (i) no $\beta-$agent remains unmatched, and (ii) b_1 is matched at $t = 1$.

Given the above, consider the contingent matching μ that has the on-path matching: $m_1(B_1)(\alpha_1) = b_1, m_1(B_1)(\alpha_2) = \alpha_2, m_2(\underline{B}^2)(\alpha_2) = \alpha_2, m_2(\overline{B}^2)(\alpha_2) = b_2$, and specifies $\mu(h^2)$ taking into account the above. However, μ is blocked at $t = 1$ by (α_2, b_1), since α_2 prefers to match with b_1, and b_1 gets his most preferred $\alpha-$agent.

The only other possible contingent matching is μ' that has the on-path matching: $m_1'(B_1)(\alpha_1) = \alpha_1, m_1'(B_1)(\alpha_2) = b_1, m_2'(\underline{B}^2)(\alpha_1) = \alpha_1, m_2'(\overline{B}^2)(\alpha_1) = b_2$. However, since $\mu'(h^2)(\alpha_2) = b_2$ for $h^2 = (B_1, \tilde{m}_1, B_2), \tilde{m}_1(k) = k$, and $\mu'(h^2)(\alpha_2) = b_1$ for $h^2 = (B_1, \tilde{m}_1, \emptyset), \tilde{m}_1(k) = k$, α_2 blocks by waiting to be matched at $t = 2$. □

B Proofs of Section 5.2

B.1 *Proof of Proposition 5.2 for economies with deterministic arrivals*

The proof follows from the following definition, and two lemmas.

Let $g : \{1, ..., |\mathcal{A}|\} \mapsto \mathcal{A}$ be an ordering of the $\alpha-$agents. Consider the matching $m^g \in M$ such that $g(1)$ gets her most preferred element in $B_1 \cup B_2$, $b^*(g(1))$; $g(2)$ gets her most preferred element in $B_1 \cup B_2 \backslash \{b^*(g(1))\}$; $g(l)$ gets her most preferred element in $B_1 \cup B_1 \backslash \bigcup_{i=1}^{l-1}\{b^*(g(i))\}$; where most preferred is defined using the preferences at the initial period. For each $h^2 \in H^2$, let $m_2^g(h^2)$ be the allocation for the submarket $\langle \mathcal{A}(h^2), \mathcal{B}(h^2) \rangle$ obtained by the same procedure as before using ordering $g(B_1) = g|_{\mathcal{A}(h^2)}$ and letting $\alpha-$agents pick objects in $\mathcal{B}(h^2)$.

Definition B.1. The sequential dictatorship contingent matching when agents are ordered according to g, μ^g, is the element of $\mathcal{M}$, such that $m_{\mu^g} = m^g$; and for all $h^2 \in H^2$, $\mu(h^2)|_{\mathcal{A}(h^2) \cup \mathcal{B}(h^2)} = m_2^g(h^2)$.

Lemma B.1. Fix an economy $\mathcal{E}$ with deterministic arrivals; and fix $g : \{1, ..., |\mathcal{A}|\} \mapsto \mathcal{A}$ an injective function. Then, μ^g defined in Definition B.1 is dynamically Pareto efficient.

Proof. By construction, it is only necessary to show that there are no blocks at $t = 1$.

Step 1: If (A', μ') is a block of μ^g at $t = 1$ then there exists $a \in A'$ such that $\mu'(B_1)(a) = a$.

Proof. By contradiction. Suppose not. Then,

$$(\forall a \in A')\mu'(B_1)(a) \in \cup_{a' \in A'}\{\mu^g(B_1)(a')\} \cup \{b \in B_1 : (\forall \overline{a} \in \mathcal{A})b \neq \mu^g(B_1)(\overline{a})\}$$

Moreover, since $(\forall l \geq 0)\{b \in B_1 : (\forall \overline{a} \in \mathcal{A})b \neq \mu^g(B_1)(\overline{a})\} \subset B_1 \cup B_2 \backslash \cup_{k=1}^{l} \{b^*(g(k))\}$ (where, for $l = 0$, $\cup_{k=1}^{l}\{b^*(g(k))\} = \emptyset$) the unassigned buildings at $t = 1$ are available to $\alpha-$agent $g(l+1)$ for all $l \geq 0$, and it cannot be the case that $\exists a \in A' : \mu'(B_1)(a) \in \{b \in B_1 : (\forall \overline{a} \in \mathcal{A})b \neq \mu^g(B_1)(\overline{a})\}$.

Thus, $(\forall a \in A')\mu'(B_1)(a) \in \cup_{a' \in A'}\{\mu^g(B_1)(a')\}$. Let $\underline{a} = \arg\min\{g^{-1}(a) : a \in A'\}$. Note that:

$$\mu'(B_1)(\underline{a}) \in \bigcup_{a' \in A'} \{\mu^g(B_1)(a')\} \subseteq B_1 \cup B_1 \backslash \bigcup_{k=1}^{g^{-1}(\underline{a})-1} \{b^*(g(k))\}$$

Hence, $\mu'(B_1)(\underline{a})$ was available for $\underline{a}$ to choose when it was her turn under g, which contradicts that $u(\underline{a}, \mu'(B_1)(\underline{a})) > U(\underline{a}, m^g, B_1)$. □

Step 2: There are no blocks (A', μ') at $t = 1$ such that $(\exists a \in A') : \mu'(B_1)(a) = a$.

Proof. By contradiction. Let (A', μ') be a block of μ^g at B_1. By step 1, it follows that there exists $a \in A' : \mu'(B_1)(a) = a$. Moreover, letting $\underline{a} = \arg\min\{g^{-1}(a) : a \in A'\}$, the proof of Step 1 shows that $\mu'(B_1)(\underline{a}) = \underline{a}$.

Let $h^2_{m_{\mu'}}$ be the continuation history induced by the block, i.e. $h^2_{m_{\mu'}} = (B_1, m_{1,\mu'}, B_2)$, and let

$m_{2,\mu'} = \mu(h^2_{m_{\mu'}})|_{\mathcal{A}(h^2_{m_{\mu'}})\cup\mathcal{B}(h^2_{m_{\mu'}})}$.

Let $b^* = m_{2,\mu'}(\underline{a})$. Note that (i) $b^* \in \bigcup_{k=1}^{g^{-1}(\underline{a})-1}\{b^*(g(k))\}$, and (ii) $b^* \in B_2$. The first follows from the fact that otherwise b^* would have been available for $\underline{a}$ to choose. The second follows from the fact that b^* can't have been assigned in $t = 1$ to another $\alpha-$agent, b^* can't have been assigned at $t = 1$ to $\hat{a} \in A'\backslash\{\underline{a}\}$ by definition of $\underline{a}$, and Step 1, and, finally, b^* can't be an unassigned building available at $t = 1$ which would contradict (i).

Hence, there exists $a^* \in \mathcal{A} : m^g(a^*) = b^*$, and $g^{-1}(a^*) < g^{-1}(\underline{a})$.

Define $a_0 = \arg\min\{g^{-1}(a) : a \in \mathcal{A}(h^2_{m_{\mu'}})\}$. Note that $g^{-1}(a_0) \leq g^{-1}(a^*) < g^{-1}(\underline{a})$. Note that $b^*(a_0)$ continues to be a_0's choice at $h^2_{m_{\mu'}}$. This follows from the fact that the set of buildings available at $h^2_{m_{\mu'}}$, $\mathcal{B}(h^2_{m_{\mu'}})$ is such that:

$$\mathcal{B}(h^2_{m_{\mu'}}) \cap B_1 = \{b \in B_1 : \mu'(B_1)(b) = b\}$$
$$= \{b \in B_1 : \mu(B_1)(b) = b\} \cup \{b \in B_1 : \mu(B_1)(b) \in A' \wedge \mu'(B_1)(b) = b\}$$

Hence, all the buildings available because of the block were available for a_0 to choose from when it was her turn under g. Therefore, she chooses $b^*(a_0)$ again.

Inductively, $(\forall a \in \mathcal{A}(h^2_{m_{\mu'}})) : g^{-1}(a) \in \{g^{-1}(a_0) + 1, ..., g^{-1}(f^*)\}$, $b^*(a)$ continues to be their most preferred choice out of $\mathcal{B}(h^2_{m_{\mu'}})\backslash \bigcup_{k=1}^{g^{-1}(a)-1}\{b^*(g(k))\}$. In particular, a^* chooses b^*. This contradicts that $\underline{a}$ obtains b^* at $h^2_{m_{\mu'}}$, and hence, that there is a block at B_1. □

□

Lemma B.2. Fix an economy with deterministic arrivals. Let $m \in M$ be a Pareto efficient matching. Then, there exists $g : \{1, ..., |\mathcal{A}|\} \mapsto \mathcal{A}$ such that $m^g = m$.

Proof. **Step 1:** In any Pareto efficient allocation $m^* \in M$, there exists $a \in \mathcal{A}$ such that $m^*(a) \in \arg\max_{m\in M} U(a, m)$.

Proof. Suppose not. Then, $(\forall a \in \mathcal{A})\ m^*(a) \notin \arg\max_{m\in M} U(a, m)$. Consider the following graph. Nodes are given by $\mathcal{A} \cup B_1 \cup B_2$. Edges are as follows: each $a \in \mathcal{A}$ points to her most favorite element in $\{a\} \cup B_1 \cup B_2$. Each $b \in B_1 \cup B_2$ points to the $\alpha-$agent to which it was allocated under m^* (or to itself otherwise).

Note that since $(\forall a \in \mathcal{A})m^*(a) \notin \arg\max_{m\in M} U(a, m)$, no $a \in \mathcal{A}$ points to $m^*(a)$. Moreover, since m^* is Pareto efficient, no $a \in \mathcal{A}$ points to an unassigned building.

Claim: If $\exists a \in \mathcal{A} : m^*(a) \neq (a, a)$ then there exists a cycle. Fix $a_0 \in \mathcal{A}$, and follow the directed path starting at a_0, $p = (a_0b_0a_1b_1...a_nb_n)$. Note that each $b_l \in p$ is such that $b_l = m^*(a_{l+1})$, and note that a_{l+1} never points to b_l. Since the graph is finite, the path stops at some point. I claim it forms a cycle. Say the path ends at an unreached $b \in B_1 \cup B_2$, such that $m^*(b) = \hat{a}$. Then, b has to point to $\hat{a}$, a contradiction to the path ending (note that the discussion before implies that the path cannot end at a $b : m^*(b) = b$). Say that the path ends at an unreached $\hat{a} \in \mathcal{A}$. But then $\hat{a}$ points to his most favorite available building. Hence, there is a cycle. Moreover, each agent is better off with the building his pointing at than with his own. Hence, we obtain a Pareto improvement by having the agents in the cycle exchange their own buildings, a contradiction to m^* being Pareto efficient. □

Applying inductively the proof of Step 1, Step 2 below follows. Define:

$$\begin{aligned}
\mathcal{A}^1 &= \{a \in \mathcal{A} : m^*(a) \in \arg\max_{m \in M} U(a, m, B_1)\} \\
M^1 &= \{m' \in M : (\forall a \in \mathcal{A}^1) m'(a) = m^*(a)\} \\
\mathcal{A}^n &= \bigcup_{i=1}^{n-1} \mathcal{A}^i \cup \{a \in \mathcal{A} : m^*(a) \in \arg\max_{m \in M^{n-1}} U(a, m, B_1)\} \\
M^n &= \{m' \in M : (\forall a \in \mathcal{A}^n) m'(a) = m^*(a)\}
\end{aligned}$$

Step 2: $(\forall n \geq 1)(\exists a \in \mathcal{A} \backslash \mathcal{A}^n) : m^*(a) \in \arg\max_{m \in M^n} U(a, m, B_1)$.

Step 3: Define $g : \{1, ..., |\mathcal{A}\} \mapsto \mathcal{A}$ to be such that $(\forall n < n')(\forall a \in \mathcal{A}^n)(\forall a' \in \mathcal{A}^{n'}) g^{-1}(a) < g^{-1}(a')$. Then, $m^g = m^*$.

Proof. This follows from revealed preference, and the definitions in Steps 1 and 2. □

□

B.2 Proof of Proposition 5.2 for economies with stochastic arrivals

The proof of Proposition 5.2 for economies with stochastic arrivals has two parts. First, I show that the analog of μ^g for economies with stochastic arrivals is dynamically Pareto efficient. Then, I show that not all Pareto efficient allocations can be achieved by a serial dictatorship, and, after defining a different contingent matching based on the YRMH-IGYT mechanism of Abdulkadiroğlu and Sönmez [1], I show that the contingent matchings generated in that way are dynamically Pareto efficient.

B.2.1 Serial Dictatorship

Let $g : \{1, ..., |\mathcal{A}|\} \mapsto \mathcal{A}$ be an ordering of the $\alpha-$agents. Considering the allocation $m^g \in M$ constructed as follows. Firm $g(1)$ chooses: $m^*_{g(1)} \in \arg\max_{m \in M} U(g(1), m)$. Define $M(g(1)) = \{m' \in M : m'(g(1)) = m^*_{g(1)}(g(1))\}$. Inductively, define $M(g(1)...g(n)) = \{m' \in M : (\forall i \in \{1, ..., n\}) m'(g(i)) = m^*_{g(i)}(g(i))\}$, and let
$m^*_{g(n+1)} \in \arg\max_{m \in M(g(1),...,g(n))} U(g(n+1), m)$. Let $m^g \in M$ be the allocation constructed in that way. For each h^2, let $m^g(h^2)|_{\mathcal{A}(h^2),\mathcal{B}(h^2)}$ be the allocation obtained by serial dictatorship using $g|_{\mathcal{A}(h^2)}$. Let $\mu^g \in \mathcal{M}$ be the contingent matching such that $m_{\mu^g} = m^g$, and for all h^2, $\mu(h^2)|_{\mathcal{A}(h^2)} = m^g(h^2)|_{\mathcal{A}(h^2)}$.[27] The proof of Lemma B.3 follows from the same steps as the proof of Lemma B.1 (though with slightly more notation) so it is stated without proof:

Lemma B.3. Fix an economy $\mathcal{E}$. Then, μ^g is dynamically Pareto efficient.

B.2.2 YRMH-IGYT mechanism

Example B.1 (Not all Pareto efficient allocations can be obtained by serial dictatorship)**.** Consider the following economy based on the one in Proposition A.1. There are two $\alpha-$agents

[27]Recall that, by definition of a contingent matching, the matching at h^2 specifies that the $\alpha-$agents and buildings that were matched at $t = 1$ continue to be matched together

$\mathcal{A} = \{a_1, a_2\}$, and arrivals are as follows: $\underline{B}^2 = \{b_1, \emptyset\}, \overline{B}^2 = \{b_1, b_2\}$. Let $r_1 = Pr(\underline{B}^2)$. Preferences are given as follows:

$$\begin{aligned} u(a_i, b_2) &> u(a_i, b_1), i = 1, 2 \\ \delta_{a_2}[r_1 u(a_2, b_1) + (1 - r_1) u(a_2, b_2)] &> u(a_2, b_1) > \delta_{a_2}(1 - r_1) u(a_2, b_2) \\ u(a_1, b_1) &> \delta_{a_1}[r_1 u(a_1, b_1) + (1 - r_1) u(a_1, b_2)] \end{aligned}$$

The following matching is Pareto efficient: $m_1(B_1)(a_1) = a_1, m_2(\underline{B}^2)(a_1) = a_1$, $m_2(\overline{B}^2)(a_1) = b_2, m_1(B_1)(a_2) = m_2(\underline{B}^2)(a_2) = m_2(\overline{B}^2)(a_2) = b_1$. Pareto efficiency follows from: (i) any allocation that makes a_2 better off has to match her with b_2 at $\overline{B}^2$, and with b_1 at $\underline{B}^2$, which implies a_1 can only match with b_1 at $\overline{B}^2$, and $u(a_1, b_1) < u(a_1, b_2)$, and (ii) any allocation that makes a_1 better off implies that the highest payoff a_2 can obtain is $\delta_{a_2}(1 - r_1) u(a_2, b_2)$.

However, this allocation cannot be obtained by a serial dictatorship: any allocation that makes a_1 the dictator, matches a_1 with b_1 at $t = 1$, and any allocation that makes a_2 the dictator matches a_2 as in (i) above. Still, as the next lemma shows, there is away of constructing the contingent matching that implements any Pareto efficient allocation.

Lemma B.4. Let $m \in M$ be a Pareto efficient matching. Then, there exists $\mu \in \mathcal{M}$ such that $m_\mu = m$, and μ is dynamically Pareto efficient.

Proof. Fix m a Pareto efficient matching. Fix an order $g : \mathcal{A} \mapsto \{1, ..., |\mathcal{A}|\}$. Let $\mu \in \mathcal{M}$ be as follows:

1. $m_\mu = m$. For each initial history, B_1, and for each arrival history $B^2 = (B_1, B_2)$, and for each $h^2 \in H^2(B_1)$, Let $B^2(h^2)$ be the projection of h^2 onto $\{(B_1, B_2) : (B_1, B_2) \in (2^{\mathcal{B}})^2\}$.

2. For all $h^2 \in H^2(B_1)$, let $E(h^2) = \{a \in \mathcal{A}(h^2) : m_2(B^2(h^2)) \in \mathcal{B}(h^2)\}$, and let $\mathcal{B}_O(h^2) = m_2(B^2(h^2))(E(h^2))$. $E(h^2)$ are the existing tenants, $\mathcal{B}_O(h^2)$ are the owned buildings, $\mathcal{A}(h^2) \backslash E(h^2)$ are the new tenants, and $\mathcal{B}(h^2) \backslash \mathcal{B}_O(h^2)$ are the vacant houses. Let $\mu(h^2)$ be the allocation obtained from YRMH-IGYT algorithm applied to $\langle E(h^2), \mathcal{A}(h^2), \mathcal{B}_O(h^2), \mathcal{B}(h^2) \backslash \mathcal{B}_O(h^2) \rangle$ using the order $g|_{\mathcal{A}(h^2)}$. (Note that potentially $E(h^2)$ may be empty, so that $\mu(h^2)$ is a serial dictatorship).

Remark B.1. The following hold: (i) $\mu(h^2)$ is Pareto efficient for all h^2, and (ii) Let h^2 be such that $E(h^2) \neq \emptyset$, then all $a \in E(h^2)$ obtain at least $u(a, m_2(B^2(h^2))(a))$.

The remark implies that only blocks at B_1 need to be ruled out. Towards a contradiction, fix B_1, and let $\langle A', \mu' \rangle, \mu' \in \mathcal{M}(B_1)$ be a block of μ at B_1. Let $H^2_{A'}(B_1) = \{h^2 \in H^2(B_1) : h^2 = (B_1, \mu'(B_1), B_2), B_2 \subseteq \mathcal{B}\}$, and let $m_{\mu'}$ be the on-path matching for μ' at B_1. Note that $(\forall a \in \mathcal{A} \backslash A') : m_1(B_1)(a) \neq a, U(a, m_\mu; B_1) = U(a, m_{\mu'}, B_1)$. Moreover, $(\forall a \in \mathcal{A} \backslash A') : m_1(B_1)(a) = a, (\forall h^2 \in H_{A'}(B_1)) m_2(B^2(h^2))(a) \in \mathcal{B}(h^2)$, whenever $m_2(B^2(h^2))(a) \neq a$. Therefore, for all such a, $U(a, m_{\mu'}, B_1) \geq U(a, m_\mu; B_1)$, since a owns her original assignment at h^2 under the YRMH-IGYT mechanism. Finally, by definition of a block $(\forall a \in A') U(a, m_{\mu'}, B_1) > U(a, m_\mu; B_1)$. This contradicts m being Pareto efficient. Hence, there are no blocks at B_1. □

C Proofs of Section 6.2

C.1 *Proof of Proposition 6.3*

Some lemmas are needed for the proof.

Lemma C.1. Let $\mu^* \in C^*(\mathcal{E})$ Then, μ is dynamically stable if, and only if, there is no $\langle A', B', \mu'\rangle$, $A' \cup B' \subset \mathcal{A} \cup B_1$ such that $(\forall k \in A' \cup B')m_{1,\mu'}(B_1)(k) = k$ and $\langle A', B', \mu'\rangle$ is a block of μ^* at B_1.

Lemma C.1 follows immediately from Definition 4.10.

Definition C.1 (Objection to early matching)**.** Given $\mu \in \mathcal{M}$, a block of μ at B_1 $\langle A', B', \mu'\rangle$ such that $(\forall k \in A' \cup B')m_{1,\mu'}(B_1)(k) = k$ is referred to as an *objection to early matching*.

Lemma C.2. Fix an economy with deterministic arrivals, $\mathcal{E}$, and $\mu^* \in C^*(\mathcal{E})$. Let $\langle A', B', \mu'\rangle$ be a block of μ^* at $t = 1$. Suppose that $A', B' \neq \emptyset$. $(\forall b \in B_1)$ if $m_{1,\mu^*}(b) \in A'$, then $b \notin B'$.

Proof. Towards a contradiction, assume that $(\exists a \in A') : m_{1,\mu^*}(a) \in B'$. By Lemma C.1, $(\forall k \in A' \cup B')m_{1,\mu'}(k) = k$. Consider the $t = 2-$matching market originated by the block; i.e. letting $h^2_{m_{\mu^*}} = (B_1, m_{1,\mu^*}, B_2), h^2_{m_{\mu'}} = (B_1, m_{1,\mu'}, B_2)$, it follows that:

$$\begin{aligned}\mathcal{A}(h^2_{m_{\mu'}}) &= \mathcal{A}(h^2_{m_{\mu^*}}) \cup m_{1,\mu^*}(B') \cup A', \\ \mathcal{B}(h^2_{m_{\mu'}}) &= \mathcal{B}(h^2_{m_{\mu^*}}) \cup m_{1,\mu^*}(A') \cup B'\end{aligned}$$

Let $m_{2,\mu'} = \mu^*(h^2_{m_{\mu'}})$. The proof follows from the following steps:

Step 1 $(\forall a \in A')m_{2,\mu'}(a) \in m_{1,\mu^*}(A') \cup B_2$, and $m_{2,\mu^*}(m_{2,\mu'}(a)) \neq m_{2,\mu'}(a)$.

Proof. First, $m_{2,\mu'}(a) \in m_{1,\mu^*}(A') \cup B_2$. Suppose not, then $m_{2,\mu'}(a) \in B' \cup \{b \in B_1 : m_{1,\mu^*}(b) = b\}$. This contradicts m_{μ^*} being an on-path matching for a core contingent matching. Clearly, if $m_{2,\mu'}(a) \in m_{1,\mu^*}(A')$, then $m_{2,\mu^*}(m_{2,\mu'}(a)) \neq m_{2,\mu'}(a)$. Hence, assume $m_{2,\mu'}(a) \in B_2$. Then, if $m_{2,\mu^*}(m_{2,\mu'}(a)) = m_{2,\mu'}(a)$, $(a, m_{2,\mu'}(a))$ block μ^* according to the core definition because $m_{2,\mu'}$ is individually rational, and $v(a, m_{2,\mu'}(a)) > v(m_{2,\mu'}(a), m_{2,\mu'}(a))$. Hence, $m_{2,\mu^*}(m_{2,\mu'}(a)) \neq m_{2,\mu'}(a)$. □

Now, if $m_{2,\mu'}(A') = m_{1,\mu^*}(A')$, the proof is done because $\mu^* \in C(\mathcal{E})$ implies that $(\forall b \in m_{2,\mu'}(A'))$ $v(m_{2,\mu'}(b), b) < v(m_{2,\mu^*}(b), b)$. However, this implies that $(\forall b \in m_{1,\mu^*}(A'))b \notin B'$. Hence, assume that $m_{2,\mu'}(A') \neq m_{1,\mu^*}(A')$. Hence, by Step 1, $(\exists \hat{a} \in A') : m_{2,\mu'}(\hat{a}) \in B_2$. Let $n = |\{a \in A' : m_{2,\mu'}(a) \in B_2\}|$.

Moreover, note that $(\forall a \in A')$ $\delta_a u(a, m_{2,\mu'}(a)) > u(a, m_{1,\mu^*}(a))$. Therefore, it has to be that $(\forall b \in m_{2,\mu'}(A'))v(m_{2,\mu'}(b), b) < v(m_{2,\mu^*}(b), b)$, otherwise μ^* would not be in the core of $\mathcal{E}$.

Step 2 Let $\mathcal{A}^{\mu' \succ \mu^*}(h^2_{m_{\mu^*}}) = \{a \in \mathcal{A}(m^2_{\mu^*}) : u(a, m_{2,\mu'}(a)) > u(a, m_{2,\mu^*})(a)\}$, and $\mathcal{B}^{c,\mu' \succ \mu^*}(h^2_{m_{\mu^*}}) = \{b \in B_2 : m_{2,\mu'}(b) \in \mathcal{A}^{\mu' \succ \mu^*}(h^2_{m_{\mu^*}})\}$. Then, $\mathcal{A}^{\mu' \succ \mu^*}(h^2_{m_{\mu^*}}) \neq \emptyset$, and $m_{2,\mu^*}(\mathcal{B}^{c,\mu' \succ \mu^*}(h^2_{m_{\mu^*}})) \subsetneq \mathcal{A}^{\mu' \succ \mu^*}(h^2_{m_{\mu^*}})$.

Proof. To see that $\mathcal{A}^{\mu' \succ \mu^*}(h^2_{m_{\mu^*}}) \neq \emptyset$, note that $\hat{a} \in A'$ is such that $m_{2,\mu'}(\hat{a}) \in B_2$, and $m_{2,\mu^*}(m_{2,\mu'}(\hat{a})) \in \mathcal{A}$. Let $\tilde{a} = m_{2,\mu^*}(m_{2,\mu'}(\hat{a}))$. If $u(\tilde{a}, m_{2,\mu'}(\tilde{a})) < u(\tilde{a}, m_{2,\mu^*}(\tilde{a}))$, then $(\tilde{a}, m_{2,\mu'}(\hat{a}))$ block $m_{2,\mu'}$, a contradiction. Hence, $\tilde{a} \in \mathcal{A}^{\mu' \succ \mu^*}(h^2_{m_{\mu^*}})$.

Take $b \in \mathcal{B}^{c,\mu^* \succ \mu'}(h^2_{m_{\mu^*}})$. Clearly, $m_{2,\mu^*}(b) \neq b$, since otherwise $(b, m_{2,\mu'}(b))$ block m_{2,μ^*}, a contradiction. Then, $m_{2,\mu^*}(b) \neq b$. If $m_{2,\mu^*}(b) \notin \mathcal{A}^{\mu' \succ \mu^*}(h^2_{m_{\mu^*}})$, then since $m_{2,\mu'}(m_{2,\mu^*}(b)) \neq b$, it has to be that $u(m_{2,\mu^*}(b), b) < u(m_{2,\mu^*}(b), m_{2,\mu'}(m_{2,\mu^*}(b)))$. If $v(m_{2,\mu^*}(b), b) > v(m_{2,\mu'}(b), b)$, then $(m_{2,\mu^*}(b), b)$ block $m_{2,\mu'}$, a contradiction. Then,
$v(m_{2,\mu^*}(b), b) < v(m_{2,\mu'}(b), b)$, and hence $(m_{2,\mu'}(b), b)$ block m_{2,μ^*}, a contradiction. Therefore, $m_{2,\mu^*}(b) \in \mathcal{A}^{\mu' \succ \mu^*}(h^2_{m_{\mu^*}})$ □

Step 2 implies that $m_{2,\mu^*}(\mathcal{B}^{c,\mu'\succ\mu^*}) \subset \mathcal{A}^{\mu'\succ\mu^*}(h^2_{m_{\mu^*}})$. However, the inclusion is strict because $m_{2,\mu^*}(m_{2,\mu'}(\tilde{a})) \in \mathcal{A}^{\mu'\succ\mu^*}(h^2_{m_{\mu^*}})$, and $m_{2,\mu'}(\tilde{a}) \notin \mathcal{B}^{c,\mu'\succ\mu^*}$. Actually, it follows that $|\mathcal{B}^{c,\mu^*\succ\mu'}| \leq |m_{2,\mu^*}(A')| - n$.

Step 3 $m_{2,\mu'}(\mathcal{A}^{\mu'\succ\mu^*}(h^2_{m_{\mu^*}})) \cap (\{b \in B_1 \cup B_2 : m_{2,\mu^*}(b) = b\} \cup B') = \emptyset$.

Proof. Towards a contradiction, suppose that $(\exists a \in \mathcal{A}^{\mu'\succ\mu^*}(h^2_{m_{\mu^*}})) : m_{2,\mu^*}(m_{2,\mu'}(a)) = m_{2,\mu'}(a)$. Since $m_{2,\mu'}$ is individually rational and a prefers her outcome under $m_{2,\mu'}$ to the one under m_{2,μ^*}, then $(a, m_{2,\mu'}(a))$ block μ^* according to the core definition, a contradiction. Similarly, if $m_{2,\mu'}(a) \in B'$, then $(a, m_{2,\mu'(a)})$ block μ^*, a contradiction. The result follows. □

Note that $|\mathcal{B}^{c,\mu^*\succ\mu'}(h_{m^2_{2,\mu^*}})| \leq |\{b \in B^2 : m_{2,\mu^*}(b) \in \mathcal{A}^{\mu'\succ\mu^*}(h^2_{m_{\mu^*}})\}| - n = |\mathcal{A}^{\mu'\succ\mu^*}(h^2_{m_{\mu^*}})| - n$, and $|\{b \in B_1 : m_{1,\mu^*}(b) \in A' \wedge b \notin B' \wedge m_{2,\mu'}(b) \notin A'\}| \leq n - 1$, where the last inequality follows from assuming that there is at least one $b \in m_{1,\mu^*}(A') : b \in B'$. This is a contradiction because by Step 3 the remaining $\beta-$agents can't be matched to $\alpha-$agents in $\mathcal{A}^{\mu'\succ\mu^*}$, and $(\forall a \in \mathcal{A}^{\mu'\succ\mu^*}(h^2_{m_{\mu^*}}))m_{2,\mu'}(a) \neq a$, and $|m_{2,\mu'}(\mathcal{A}^{\mu'\succ\mu^*}(h^2_{m_{\mu^*}}))| = |\mathcal{A}^{\mu'\succ\mu^*}(h^2_{m_{\mu^*}})|$. Therefore, $(\forall b \in m_{1,\mu^*}(A'))b \notin B'$.

Similarly, it follows that $(\forall a \in m_{1,\mu^*}(B'))a \notin A'$. □

Lemma C.3. Fix an economy with deterministic arrivals $\mathcal{E}$, and fix $\mu^* \in C^*(\mathcal{E})$. Let $\langle A', B', \mu'\rangle$ be a block of μ^* at $t = 1$. Let $h^2_{m_{\mu'}} = (B_1, m_{1,\mu'}, B_2)$ denote the $t = 2$ history generated by the block, and let $m_{2,\mu'}$ be the second period on-path matching for μ', i.e. $m_{2,\mu'} = \mu^*(h^2_{m_{\mu'}})$. Also, let $h^2_{m_{\mu^*}} = (B_1, m_{1,\mu^*}, B_2)$ denote the on-path history for μ^*. Define:

$$\begin{aligned}
\mathcal{A}^{\mu'\succ\mu^*}(h^2_{m_{\mu'}}) &= \{a \in \mathcal{A}(h^2_{m_{\mu'}}) : \delta_a^{\mathbf{1}[m_{2,\mu'}(a)\in B_2]}u(a, m_{2,\mu'}(a)) > \delta_a^{\mathbf{1}[m_{2,\mu^*}(a)\in B_2]}u(a, m_{2,\mu^*}(a))\} \\
\mathcal{A}^{\mu^*\succ\mu'}(h^2_{m_{\mu'}}) &= \{a \in \mathcal{A}(h^2_{m_{\mu'}}) : \delta_a^{\mathbf{1}[m_{2,\mu'}(a)\in B_2]}u(a, m_{2,\mu'}(a)) < \delta_a^{\mathbf{1}[m_{2,\mu^*}(a)\in B_2]}u(a, m_{2,\mu^*}(a))\} \\
\mathcal{B}^{\mu'\succ\mu^*}(h^2_{m_{\mu'}}) &= \{b \in \mathcal{B}(h^2_{m_{\mu'}}) : v(m_{2,\mu'}(b), b) > v(m_{2,\mu^*}(b), b)\} \\
\mathcal{B}^{\mu^*\succ\mu'}(h^2_{m_{\mu'}}) &= \{b \in \mathcal{B}(h^2_{m_{\mu'}}) : v(m_{2,\mu'}(b), b) < v(m_{2,\mu^*}(b), b)\}
\end{aligned}$$

The following hold:

$$\begin{aligned}
\mathcal{A}^{\mu'\succ\mu^*}(h^2_{m_{\mu'}}) &\longleftrightarrow^{\mathrm{m}_{2,\mu'}}_{\mathrm{m}_{2,\mu^*}} \mathcal{B}^{\mu^*\succ\mu'}(h^2_{m_{\mu'}}) \\
\mathcal{A}^{\mu^*\succ\mu'}(h^2_{m_{\mu'}}) &\longleftrightarrow^{\mathrm{m}_{2,\mu'}}_{\mathrm{m}_{2,\mu^*}} \mathcal{B}^{\mu'\succ\mu^*}(h^2_{m_{\mu'}}) \\
\{a \in \mathcal{A}(h^2_{m_{\mu^*}}) : m_{2,\mu^*}(a) = a\} &= \{a \in \mathcal{A}(h^2_{m_{\mu'}}) : m_{2,\mu'}(a) = a\} \\
\{b \in \mathcal{B}(h^2_{m_{\mu^*}}) : m_{2,\mu^*}(b) = b\} &= \{b \in \mathcal{B}(h^2_{m_{\mu'}}) : m_{2,\mu'}(b) = b\}
\end{aligned}$$

Proof. Note that by definition:

$$\begin{aligned}
\mathcal{A}^{\mu'\succ\mu^*}(h^2_{m_{\mu'}}) &= A' \cup \mathcal{A}^{\mu'\succ\mu^*}(h^2_{m_{\mu^*}}), \\
\mathcal{B}^{\mu'\succ\mu^*}(h^2_{m_{\mu'}}) &= B' \cup \mathcal{B}^{\mu'\succ\mu^*}(h^2_{m_{\mu^*}})
\end{aligned}$$

It follows from Lemma A.3 that $m_{2,\mu'}(\mathcal{A}^{\mu'\succ\mu^*}(h^2_{m_{\mu'}})) = m_{1,\mu^*}(A') \cup \{b \in B_2 : m_{2,\mu^*}(b) \in \mathcal{A}^{\mu'\succ\mu^*}(h^2_{m_{\mu^*}})\} = m_{2,\mu^*}(A' \cup \mathcal{A}^{\mu'\succ\mu^*}(h^2_{m_{\mu^*}})) \subseteq \mathcal{B}^{\mu^*\succ\mu'}(h^2_{m_{\mu'}})$. Also, it follows from Lemma A.3 that $m_{2,\mu'}(\mathcal{B}^{\mu'\succ\mu^*}(h^2_{m_{\mu'}})) = m_{1,\mu^*}(B') \cup \{a \in \mathcal{A}(h^2_{m_{\mu^*}}) : m_{2,\mu^*}(a) \in \mathcal{B}^{\mu'\succ\mu^*}(h^2_{m_{\mu^*}})\} =$

$m_{2,\mu^*}(B' \cup \mathcal{B}^{\mu' \succ \mu^*}(h^2_{m_{\mu^*}})) \subseteq \mathcal{A}^{\mu^* \succ \mu'}(h^2_{m_{\mu'}})$.

It remains to show that the weak inclusions are, in fact, equalities. This will also prove the last two statements in the proposition. Suppose that $m_{2,\mu^*}(A' \cup \mathcal{A}^{\mu' \succ \mu^*}(h^2_{m_{\mu^*}})) \subsetneq \mathcal{B}^{\mu^* \succ \mu'}(h^2_{m_{\mu'}})$, and let $b \in \mathcal{B}^{\mu^* \succ \mu'}(h^2_{m_{\mu'}}) \backslash m_{2,\mu^*}(A' \cup \mathcal{A}^{\mu' \succ \mu^*}(h^2_{m_{\mu^*}}))$. Note that $m_{2,\mu'}(b) = b$. If $m_{2,\mu^*}(b) = b$, then we have a contradiction. Then, assume that $m_{2,\mu^*}(b) \neq b$. Then, it has to be that $m_{2,\mu^*}(b) \in \mathcal{A}^{\mu^* \succ \mu'}(\cdot)$, otherwise $(b, m_{2,\mu^*}(b))$ block $m_{2,\mu'}$. However, this contradicts that $m_{2,\mu'}(\mathcal{A}^{\mu' \succ \mu^*}(h^2_{m_{\mu'}})) = m_{2,\mu^*}(A' \cup \mathcal{A}^{\mu' \succ \mu^*}(h^2_{m_{\mu^*}}))$. Hence, $m_{2,\mu^*}(\mathcal{A}^{\mu' \succ \mu^*}(h^2_{m_{\mu'}})) = m_{2,\mu'}(\mathcal{A}^{\mu' \succ \mu^*}(h^2_{m_{\mu'}})) = \mathcal{B}^{\mu^* \succ \mu'}(h^2_{m_{\mu'}})$. The other statements are shown by similar steps. □

I now use the previous lemmas to prove Proposition 6.3:

Proof. Fix an economy with deterministic arrivals $\mathcal{E}$, and fix $\mu^* \in C^*(\mathcal{E})$. Suppose $\langle A', B', \mu' \rangle$ is a block at B_1. By Lemma C.1, it has to be that $\langle A', B', \mu' \rangle$ is an objection to early matching. Let $m_{\mu'}$ be the on-path allocation for μ', and let $h^2_{m_{\mu'}}$ be the $t = 2$ history generated by the block, and let $h^2_{m_{\mu^*}}$ be the $t = 2$ history that was on-path for μ^*. Let $m_{2,\mu'} = \mu(h^2_{m_{\mu'}})$.

Suppose that $A' \neq \emptyset$. Then, by Lemma C.3,

$$\mathcal{A}^{\mu' \succ \mu^*}(h^2_{m_{\mu'}}) \quad \overset{\mathrm{m}_{2,\mu'}}{\underset{\mathrm{m}_{2,\mu^*}}{\longleftrightarrow}} \quad \{b \in \mathcal{B}(h^2_{m_{\mu'}}) : m_{2,\mu^*}(b) \in \mathcal{A}^{\mu' \succ \mu^*}(h^2_{m_{\mu'}})\}$$

Construct the following graph. The nodes are $\mathcal{A}^{\mu' \succ \mu^*}(h^2_{m_{\mu'}}) \cup \{b \in \mathcal{B}(h^2_{m_{\mu'}}) : m_{2,\mu^*}(b) \in \mathcal{A}^{\mu' \succ \mu^*}(h^2_{m_{\mu'}})\}$, and there is a directed edge from a to b if $m_{2,\mu'}(a) = b$, and there is a directed edge from b to a if $m_{2,\mu^*}(a) = b$. Fix $a \in A'$, and follow the path starting from a. I claim that the path cycles. Since the graph is finite, the path stops at some node. Say that the node at which it stops is an unreached b: this is a contradiction since $b \in \{b \in \mathcal{B}(h^2_{m_{\mu'}}) : m_{2,\mu^*}(b) \in \mathcal{A}^{\mu' \succ \mu^*}(h^2_{m_{\mu'}})\}$ implies that there is a directed edge from b to $m_{2,\mu^*}(b) \neq b$. Likewise, the path cannot end at an unreached $a \in \mathcal{A}^{\mu' \succ \mu^*}(h^2_{m_{\mu'}})$. Hence, the path cycles. Therefore, we have $(a_1 b_1 a_2 b_2 ... a_N b_N)$ where[28]:

$$\begin{aligned}
(\forall i = 1...N) m_{2,\mu'}(a_i) &= b_i \\
(\forall i = 1, ..., N) m_{2,\mu^*}(a_{i+1}) &= b_i \\
(\forall i = 1, ..., N) \delta_{a_i}^{\mathbf{1}[b_i \in B_2]} u(a_i, b_i) &> \delta_{a_i}^{\mathbf{1}[b_{i+1} \in B_2]} u(a_i, b_{i+1})
\end{aligned}$$

and, since μ^* is in the core, it has to be that:

$$(\forall i = 1, ..., N) v(a_{i+1}, b_i) \quad > \quad v(a_i, b_i)$$

Therefore, there is a simultaneous cycle. A similar result holds using $\mathcal{B}^{\mu' \succ \mu^*}(h^2_{m_{\mu'}})$ when $B' \neq \emptyset$. □

C.2 *Proof of Proposition 6.4*

Proof. Suppose towards a contradiction that $(\exists \mu \in D(\mathcal{E})) : \mu \notin C^*(\mathcal{E})$. Then, there exists $a \in \mathcal{A}$, and $b \in B_2$ such that: $\delta_a u(a, b) > u(a, m_{1\mu}(a))$, and $v(a, b) > v(m_{2\mu}(b), b)$. Now, since $\mu \in D(\mathcal{E})$, then a does not have a block by waiting. Hence, letting $\mu' \in \mathcal{M}$ be such that $\mu'(B_1)(k) = \mu(B_1)(k), k \notin \{a, m_{1\mu}(a)\}$, $\mu'(B_1)(k) = k$ otherwise, and $(\forall h^2 \in H^2(B_1)) \mu'(h^2) =$

[28] Indices are modulo N

$\mu(h^2)$, it follows that $\delta_a u(a, m_{2,\mu'}(a)) < u(a, m_{1\mu}(A))$. Hence, $v(m_{2,\mu'}(b), b) > v(a, b)$ by stability of $m_{2\mu'}$ at $h^2_{\mu'} = (B_1, m_{1\mu'}, B_2)$.

Let $h^2_\mu = (B_1, m_{2\mu}, B_2)$. Define $B^{\succ} = \{\tilde{b} \in \mathcal{B}(h^2_\mu) : v(m_{2\mu'}(\tilde{b}), \tilde{b}) > v(m_{2\mu}(\tilde{b}), \tilde{b})\}$, and $F^{\succ,c} = \{\tilde{a} \in \mathcal{A}(h^2_\mu) : m_{2\mu'}(a) \in B^{\succ}\}$.

Note $B^{\succ} \neq \emptyset$ because $b \in B^{\succ}$. Moreover, $|B^{\succ} \backslash \{b\}| \geq 1$. Note first that if $m_{2\mu}(m_{2\mu'}(b)) = m_{2\mu'}(b)$, then $(m_{2\mu'}(b), b)$ block $m_{2\mu}$, a contradiction. Then, let $b' = m_{2\mu}(m_{2\mu'}(b))$. Second, note that if $b' \notin B^{\succ}$, then since $m_{2\mu'}(b') \neq m_{2\mu}(b')$, it has to be that $v(m_{2\mu'}(b'), b') < v(m_{2\mu}(b'), b')$. If $m_{2\mu'}(b)$ prefers b to b', then $(m_{2\mu'}(b), b)$ block $m_{2\mu}$; otherwise, if $m_{2\mu'}(b)$ prefers b' to b, then $(m_{2\mu'}(b), b')$ block $m_{2\mu'}$, both of which are contradictions. Hence, $b' \in B^{\succ}$.

I claim that $m_{2\mu}(A^{\succ,c}) \subseteq A^{\succ}$. Take $\tilde{a} \in \mathcal{A}^{c,\succ}$. First, note that $m_{2\mu}(\tilde{a}) \neq \tilde{a}$, otherwise $(\tilde{a}, m_{2\mu'}(\tilde{a}))$ block $m_{2\mu}$, a contradiction. Suppose then that $\tilde{b} = m_{2\mu}(\tilde{a}) \notin B^{\succ}$. Then, by strict preferences, and $m_{2\mu'}(\tilde{b}) \neq \tilde{a}$, it follows that $v(m_{2\mu'}(\tilde{b}), \tilde{b}) < v(\tilde{a}, \tilde{b})$. An argument similar to the one in the previous paragraph shows that this leads to a contradiction. Hence, $\tilde{b} \in B^{\succ}$, and the claim follows.

Thus, $m_{2\mu'}(B^{\succ,c}) \subseteq B^{\succ}$, and $m_{2\mu}(A^{\succ,c}) \subseteq B^{\succ}$. Note that both $m_{2\mu'}, m_{2\mu}$ are one-to-one and map $A^{\succ,c}$ into $B^{\succ}$. By Lemma 2.3 in Roth and Sotomayor [42], there exists $\emptyset \neq \hat{A} \subset A^{\succ,c}$ such that $m_{2\mu}(\hat{A}) = m_{2\mu'}(\hat{A}) (\subseteq B^{\succ})$.

I claim that $|\hat{A}| \geq 2$. Suppose not. Then, $\hat{A} \neq \emptyset$ implies that $|\hat{A}| = 1$, i.e. $\hat{A} = \{\tilde{a}\}$. Thus, there exists $\tilde{b} \in B^{\succ}$ such that $m_{2\mu}(\tilde{b}) = m_{2\mu'}(\tilde{b})$, a contradiction to the definition of $B^{\succ}$. Hence, $|\hat{A}| \geq 2$.

Finally, consider a (directed) graph $\mathcal{G} = \langle \mathcal{N}, E \rangle$, with nodes $\mathcal{N} = \hat{A} \cup m_{2\mu}(\hat{A})$, and edges $(\tilde{a}, \tilde{b}) \in E$ if $\tilde{b} = m_{2\mu}(\tilde{a})$, and $(\tilde{b}, \tilde{a}) \in E$ if $\tilde{b} = m_{2\mu'}(\tilde{a})$. Pick $\tilde{a} \in \hat{A}$, and follow the path starting from $\tilde{a}$. Since the graph is finite, the path stops. The definition of $\hat{A}$ implies that the patch cycles: it can't stop at an unreached a or b. By the previous step, this implies there exists a preference cycle, a contradiction. Therefore, $\mu \in C^*(\mathcal{E})$. □

D Proofs of Section 6.3

D.1 Proof of Proposition 6.5

The proof follows from three steps. First, Proposition D.1 below states a property of cycles when priorities satisfy acyclicity. Second, fix $\mu \in C^*(\mathcal{E})$ such that m_μ coincides with the matching obtained by side A proposing to agents on side B. The proof of Proposition 6.3 implies that if there is a block by waiting, there has to be a cycle. The first step determines the cycle's length. Third, the workings of deferred acceptance imply that the cycle is a two-cycle, a contradiction.

Definition D.1. Let $\{\rhd_b\}_{b \in \mathcal{B}}$ be a priority structure for buildings. A sequence $(a_1, b_1, ..., a_N, b_N)$, $N \geq 2$ is an *improvement cycle* if $a_1 \rhd_{b_1} a_2 ... \rhd_{b_N} a_N$.

Proposition D.1. *If $\{\rhd_{b \in \mathcal{B}}\}$ satisfies Ergin-acyclicity, and $(a_1, b_1, ..., a_N, b_N)$ is an improvement cycle, then $N = 2$.*

Proof. Let $(a_1, b_1, ..., a_N, b_N)$ be an improvement cycle, and suppose $N > 2$. Then, for all $i \in \{1, ..., N-1\} a_i \rhd a_{i+1}$, and $a_N \rhd_{b_N} a_1$.

I claim that acyclicity implies $(\forall i)(\forall j \neq i, i+1)\ a_j \rhd_{b_i} a_{i+1}$. First, note that acyclicity implies that $(\forall i) a_{i-1} \rhd_{b_i} a_{i+1}$, where $i = 1 \Rightarrow i-1 = N$; otherwise, $a_i \rhd_{b_i} a_{i+1} \rhd_{b_i} a_{i-1} \rhd_{b_{i-1}} a_i$, violating Ergin acyclicity. Hence, $a_i \rhd_{b_i} a_{i+1}$, and $a_{i-1} \rhd_{b_i} a_{i+1}$. Similarly, it must be the case that $a_{i-2} \rhd_{b_i} a_{i+1}$; otherwise $a_{i-1} \rhd_{b_i} a_{i+1} \rhd_{b_i} a_{i-2} \rhd_{b_{i-2}} a_{i-2}$ holds, violating acyclicity. Thus, $a_i \rhd_{b_i} a_{i+1}$, $a_{i-1} \rhd_{b_i} a_{i+1}$, and $a_{i-2} \rhd_{b_i} a_{i+1}$, and we can proceed inductively and complete this for all $j \notin \{i, i+1\}$.

Now, take $j \notin \{i, i+1\}$. If $a_j \triangleright_{b_i} a_i \triangleright_{b_i} a_{i+1} \triangleright_{b_{j-1}} a_j$ - note that $j-1 \neq i$. Then, it has to be that $a_i \triangleright_{b_i} a_j \triangleright_{b_i} a_{i+1} \triangleright_{b_{i-1}} a_i$, a contradiction.[29] Thus, $N = 2$. $\square$

Now, consider $\mu \in C^*(\mathcal{E})$ such that m_μ coincides with the outcome of deferred acceptance with side A proposing to side B. Suppose $\mu \notin D(\mathcal{E})$. Then, there exists $A' \subseteq \mathcal{A}$ such that A' blocks by waiting. It follows from the proof of Proposition 6.3 that $\exists(b_1, a_1, ..., b_N, a_N)$ such that $\delta_{a_i}^{\mathbf{1}[b_{i+1} \in B_2]} u(a_i, b_{i+1}) > \delta_{a_i}^{\mathbf{1}[b_i \in B_2]} u(a_i, b_i)$, and $a_i \triangleright_{b_i} a_{i+1}$. By Proposition D.1, it follows that $N = 2$. Hence, write the cycle as (b_1, a_1, b_2, a_2). Moreover, from the proof of Proposition 6.3 it follows that $b_1 = m_{2,\mu}(a_1), b_2 = m_{2,\mu}(a_2)$.

Let r be the last step of the algorithm in which an agent in $\{a_1, a_2\}$ makes an offer (and, is accepted by) the building to whom she is matched under m_μ. Without loss of generality, say that a_1 proposes to b_1 at step r.

Let $l \in \{1, 2\}$. Since $u(a_l, m_{2,\mu}(a_{l+1})) > u(a_l, m_{2,\mu}(a_l))$, a_l was rejected by $m_{2,\mu}(a_{l+1})$ before step r. Hence, at step $r-1$, $m_{2,\mu}(a_{l+1})$ was matched to some agent, and at $r-1$, b_1 had an upstanding offer from some agent and it is not a_2; otherwise a_2 gets rejected in step r when a_1 makes an offer to b_1. Hence, $(\exists \hat{a} \notin \{a_1, a_2\})$such that $a_1 \triangleright_{b_1} \hat{a} \triangleright_{b_1} a_2$. Since $a_2 \triangleright_{b_2} a_1$, there is a two-cycle, a contradiction. Therefore, $\mu \in D(\mathcal{E})$.

D.2 *Proof of Proposition 10.5*

Consider the following economy. $\mathcal{A} = \{a_1, a_2\}$, $\mathcal{B} = \{b_{1,1}, b_{1,2}, \overline{b}_{2,1}, \underline{b}_{2,1}\}$. Objects' rankings are:

$$\begin{array}{ccc} a_1 & \triangleright_{b_{1,1}} & a_2 \\ a_2 & \triangleright_{b_{1,2}} & a_1 \\ a_2 & \triangleright_{\overline{b}_{2,1}} & a_1 \\ a_1 & \triangleright_{\underline{b}_{2,1}} & a_2 \end{array}$$

Arrivals are as follows. Let $B_1 = \{b_{1,1}, b_{1,2}\}, \overline{B}_2 = \{\overline{b}_{1,2}\}, \underline{B}_2 = \{\underline{b}_{2,1}\}$. Then, $G(B_1, \overline{B}_2) = p$, and $G(B_1, \underline{B}_2) = 1 - p$. Agents' preferences are as follows:

$$\delta_{a_1}[pu(a_1, \overline{b}_{1,2}) + (1-p)u(a_1, \underline{b}_{1,2})] > u(a_1, b_{1,1}) > \delta_{a_1}[pu(a_1, b_{1,1}) + (1-p)u(a_1, \underline{b}_{1,2})]$$
$$\delta_{a_2}[pu(a_2, \overline{b}_{1,2}) + (1-p)u(a_2, b_{1,1})] > u(a_2, b_{1,1}) > u(a_2, b_{1,2}) > \delta_{a_2}[pu(a_2, \overline{b}_{1,1}) + (1-p)u(a_2, b_{1,2})]$$

All objects that do not appear in the above rankings are deemed unacceptable. There is only one matching in the core: $m_{1,\mu}(B_1)(a_2) = b_{1,1}$, $m_{2,\mu}(B_1, \overline{B}_2)(a_1) = \overline{b}_{2,1}, m_{2,\mu}(B_1, \underline{B}_2) = \underline{b}_{2,1}$. That this is the unique matching in the core follows from two observations. First, there is no matching in the core which both agents match in period 1. If a_1 matches with $b_{1,1}$ in $t = 1$, then a_2 matches with $b_{1,2}$ in $t = 1$ as well. Hence, a_1 can block with $\overline{b}_{2,1}, \underline{b}_{2,1}$. Second, there is no matching in which a_2 matches in period 2. In this case, she would match with $b_{1,1}$ at $(B_1, \underline{B}_2)$, but this is blocked by $a_1, b_{1,1}$.

Note, however, that the unique (on-path) matching in the core cannot be part of a dynamically stable contingent matching: a_2 can block by waiting and get her most preferred matching.

[29] Here is where $N > 2$ matters most, worst case scenario $j = i - 1$

E Proofs of Section 11

E.1 *Proof of Proposition 11.1*

From the proof of Proposition 5.1, it follows that for each $h^2 \in H^2$ there is a unique stable matching. Let the random matching μ be as follows. With probability pq (the probabilities are defined below) $(\alpha_B, \beta_{1B}), (\alpha_A, \beta_{1A})$ are matched in $t = 1$, with probability $q(1-p)$, $(\alpha_C, \beta_{1B}), (\alpha_A, \beta_{1A})$ are the ones to be matched in $t = 1$, with probability $(1-q)p$ (α_B, β_{1B}) are the only ones to be matched in $t = 1$, and with probability $(1-q)(1-p)$ (α_C, β_{1B}) are the only ones that match in $t = 1$, that is μ puts positive probability on the following matchings (given the continuations in the proof of Proposition 5.1):

$$\underbrace{\left(\begin{array}{cc||cc} \alpha_A & \alpha_B & \alpha_C & \emptyset \\ \beta_{1A} & \beta_{1B} & \beta_{2C} & \beta_{2D} \end{array}\right)}_{pq} \underbrace{\left(\begin{array}{cc||cc} \alpha_A & \alpha_C & \alpha_B & \emptyset \\ \beta_{1A} & \beta_{1B} & \beta_{2C} & \beta_{2D} \end{array}\right)}_{(1-p)q}$$

$$\underbrace{\left(\begin{array}{c||ccc} \alpha_B & \alpha_A & \alpha_C & \emptyset \\ \beta_{1B} & \beta_{1A} & \beta_{2D} & \beta_{2D} \end{array}\right)}_{p(1-q)} \underbrace{\left(\begin{array}{c||ccc} \alpha_C & \alpha_A & \alpha_B & \emptyset \\ \beta_{1B} & \beta_{2C} & \beta_{2D} & \beta_{1A} \end{array}\right)}_{(1-p)(1-q)}$$

where:

$$\begin{aligned} p &= \frac{\delta_{\alpha_A} u(\alpha_A, \beta_{2C}) - u(\alpha_A, \beta_{1A})}{\delta_{\alpha_A}(u(\alpha_A, \beta_{2C}) - u(\alpha_A, \beta_{1A}))} \\ q &= \frac{u(\alpha_B, \beta_{1B}) - \delta_{\alpha_B} u(\alpha_B, \beta_{2D})}{\delta_{\alpha_B}(u(\alpha_B, \beta_{2C}) - u(\alpha_B, \beta_{2D}))} \end{aligned}$$

I show that the random contingent matching defined this way has no blocks. Recall that the definition of correlated dynamic stability requires that we consider only coalitions of size 2 and 1.

Step 1: No $\alpha-$agent or $\beta-$agent blocks by waiting to be matched.

1. When α_B is informed that she is matched with β_{1B}, if she decides to wait, she obtains:

   $$q\delta_{\alpha_B} u(\alpha_B, \beta_{2C}) + (1-q)\delta_{\alpha_B} u(\alpha_B, \beta_{2D}) = u(\alpha_B, \beta_{1B})$$

   where I used that α_A conditions on her being matched with β_{1B}, and the definition of q.

2. When β_{1B} is informed that he is matched with α_B, since α_B is his most preferred $\alpha-$agent, there is no benefit in waiting to be matched.

3. When α_A is informed she is matched with β_{1A}, by waiting to be matched in $t = 2$, she obtains:

   $$p\delta_{\alpha_A} u(\alpha_A, \beta_{1A}) + (1-p)\delta_{\alpha_A} u(\alpha_A, \beta_{2C}) = u(\alpha_A, \beta_{1A})$$

   where I used the definition of p. Hence, α_A does not block by waiting to be matched in $t = 2$.

4. Notice that β_{1A} can never gain by matching later, hence there is no block involving β_{1A}.

5. Finally, when α_C is announced she matches with β_{1B}, since that is her preferred match, she does not deviate.

Step 2: There are no blocks of size 2.

1. Notice that when α_B is told she matches in $t = 2$, her expected payoff is:

$$q\delta_{\alpha_B} u(\alpha_B, \beta_{2C}) + (1 - q)\delta_{\alpha_B} u(\alpha_B, \beta_{2D}) = u(\alpha_B, \beta_{1B})$$

   Hence, she has no incentives to contact β_{1B} to match with him in the initial period.

2. When α_A is told she matches in $t = 2$, her expected payoff is:

$$p\delta_{\alpha_A} u(\alpha_A, \beta_{1A}) + (1 - p)\delta_{\alpha_A} u(\alpha_A, \beta_{2C}) = u(\alpha_A, \beta_{1A})$$

   and, therefore, she does not block with β_{1A}.

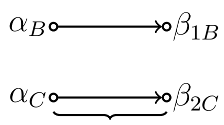
3. When α_C is told she matches in the last period, she knows that β_{1B} matches with α_B, and hence, β_{1B} does not want to block. Likewise, when β_{1A} is told that he does not match in the initial period, α_A does not want to block.

Steps 1 and 2 imply that the contingent matching is correlated dynamically stable.

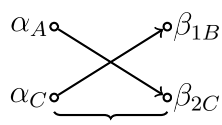
E.2 Proof of Proposition 11.2

First, I show that there is a unique stable matching for each possible outcome in $t = 1$, which implies that all contingent matchings have to specify said matchings as continuations. Second, I show that $D(\mathcal{E}) = \emptyset$. Finally, I show that under certain conditions on payoffs, $D_\Delta(\mathcal{E}) = \emptyset$.

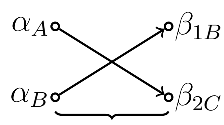
For the first step, the uniqueness of the continuation stable matchings follows from all $\alpha-$agents agreeing statically on the rank of the $\beta-$agents. The following are the unique stable matchings for the possible $t = 1$ outcomes:



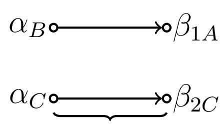
(α_A, β_{1A}) at $t = 1$



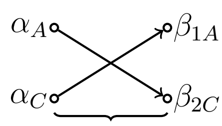
(α_B, β_{1A}) at $t = 1$



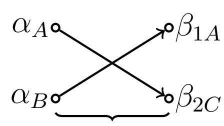
(α_C, β_{1A}) at $t = 1$



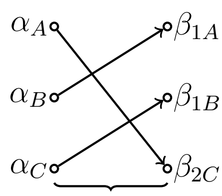
(α_A, β_{1B}) at $t = 1$



(α_B, β_{1B}) at $t = 1$



(α_C, β_{1B}) at $t = 1$



$\emptyset$ at $t = 1$

For the second step, note the following: (i) Any dynamically stable contingent matching has to match two $\alpha-$agents with two $\beta-$agents in $t = 1$, (ii) As argued in the main text, there is no dynamically stable contingent matching that matches $(\alpha_A, \beta_{1A}), (\alpha_B, \beta_{1B})$ in $t = 1$, since this is blocked by β_{1A}, (iii) Any dynamically stable contingent matching that matches α_C in the initial period, has to match α_C with β_{1B}, (iv) There is no dynamically stable contingent matching that matches (α_C, β_{1B}) and has α_A wait to be matched with β_{2C} since α_A blocks it with β_{1A}, (v) There is no dynamically stable contingent matching that matches α_C with β_{1B}, and α_A with β_{1A}, since α_C blocks by waiting to be matched with β_{2C}. Hence, there is no dynamically stable contingent matching.

For the third step, I prove the following lemma:

Lemma E.1. Assume that the following holds:

$$\begin{aligned}
\delta_{\alpha_C}[Bu(\alpha_C, \beta_{2C}) &+ (1-B)u(\alpha_C, \beta_{1B})] > u(\alpha_C, \beta_{1B}) \\
u(\alpha_C, \beta_{1B}) &> \delta_{\alpha_C}[Bu(\alpha_C, \beta_{1B}) + (1-B)(Au(\alpha_C, \beta_{1A}) + (1-A)u(\alpha_C, \beta_{1B}))] \\
A &= \frac{v(\alpha_A, \beta_{1A}) - \delta_{\beta_{1A}} v(\alpha_B, \beta_{1A})}{\delta_{\beta_{1A}}(v(\alpha_C, \beta_{1A}) - v(\alpha_B, \beta_{1A}))} \\
B &= \frac{\delta_{\beta_{1B}} v(\alpha_C, \beta_{1B}) - v(\alpha_B, \beta_{1B})}{\delta_{\beta_{1B}}(v(\alpha_C, \beta_{1B}) - v(\alpha_B, \beta_{1B}))}
\end{aligned}$$

Then, $D_\Delta(\mathcal{E}) = \emptyset$.

Proof. Let μ be a random contingent matching.

Step 1: If $m_1 \in \text{supp } \mu(B_1)$, then $m_1(\alpha_C) \neq \beta_{1A}$.

Proof. Otherwise, (α_C, β_{1B}) block. □

Step 2: If $m_1 \in \text{supp } \mu(B_1)$, then $m_1(\alpha_B) \neq \beta_{1A}$.

Proof. If $m_1 \in \text{supp } \mu(B_1)$ is such that $m_1(\alpha_B) = \beta_{1A}$, then it has to be that $m_1(\alpha_A) = \beta_{1B}$, otherwise, (α_A, β_{1A}) block. However, this means that (α_B, β_{1B}) block. □

Similarly, we have that:

Step 3: If $m_1 \in \text{supp } \mu(B_1)$, then $m_1(\alpha_A) \neq \beta_{1B}$.

Proof. If $m_1 \in \text{supp}\mu(B_1)$ and $m_1(\alpha_A) = \beta_{1B}$, then $m_1(\alpha_B) = \alpha_B$, and there exists $p > 0$ such that:

$$U(\alpha_B, \mu|_{E(m_1(\alpha_B))}, B_1) = \delta_{\alpha_B}[pu(\alpha_B, \beta_{2C}) + (1-p)u(\alpha_B, \beta_{1A})] \geq u(\alpha_B, \beta_{1B}).$$

However, $p > 0$ implies that $\exists \tilde{m}_1 \in E(m_1(\alpha_B))$ such that $m_1(\alpha_C) = \beta_{1A}$ - otherwise, α_B cannot be matched to $\beta-$agent β_{2C} in $t = 2$ -, and by Step 1 this contradicts that $\mu \in D_\Delta(\mathcal{E})$. Thus, if $m_1 \in \text{supp}\mu(B_1)$, then $m_1(\alpha_A) \neq \beta_{1B}$. □

Then, consider the following $t = 1$ matchings:

$$\begin{array}{llll}
m^1: & m_1^1(\alpha_A) = \beta_{1A} & m_1^1(\alpha_C) = \beta_{1B} & m_1^1(\alpha_B) = \alpha_B \\
m^2: & m_1^2(\alpha_A) = \alpha_A & m_1^2(\alpha_C) = \beta_{1B} & m_1^2(\alpha_B) = \alpha_B \\
m^3: & m_1^3(\alpha_A) = \beta_{1A} & m_1^3(\alpha_C) = \alpha_C & m_1^3(\alpha_B) = \beta_{1B} \\
m^4: & m_1^4(\alpha_A) = \beta_{1A} & m_1^4(\alpha_C) = \alpha_C & m_1^4(\alpha_B) = \alpha_B \\
m^5: & m_1^5(\alpha_A) = \alpha_A & m_1^5(\alpha_C) = \alpha_C & m_1^5(\alpha_B) = \beta_{1B} \\
m^6: & m_1^6(\alpha_A) = \alpha_A & m_1^6(\alpha_C) = \alpha_C & m_1^6(\alpha_B) = \alpha_B
\end{array}$$

Let $p_i = \mu(B_1)(m_1^i), i = 1, 2, ..., 6$.

Step 4: If $p_1 + p_2 > 0$, then $\frac{p_1}{p_1+p_2} \leq \frac{(1-\delta_{\alpha_C})u(\alpha_C,\beta_{1B})}{\delta_{\alpha_C}(u(\alpha_C,\beta_{2C})-u(\alpha_C,\beta_{1B}))} \equiv D$.

Proof. Conditional on being informed that she matches with β_{1B} at $t = 1$, α_C should not have incentives to wait to be matched in $t = 2$, that is:

$$u(\alpha_C, \beta_{1B}) \geq \delta_{\alpha_C} \left[\frac{p_1}{p_1 + p_2} u(\alpha_C, \beta_{2C}) + \frac{p_2}{p_1 + p_2} u(\alpha_C, \beta_{1B}) \right].$$

The result follows immediately. □

Corollary E.1. *If $p_2 = 0$, then $p_1 = 0$.*

Step 5: If $p_1 + p_3 + p_4 > 0$, then $\frac{p_3}{p_1+p_3+p_4} \leq \frac{v(\alpha_A,\beta_{1A})-\delta_{\beta_{1A}}v(\alpha_B,\beta_{1A})}{\delta_{\beta_{1A}}(v(\alpha_C,\beta_{1A})-v(\alpha_C,\beta_{1A}))} \equiv A$.

Proof. Conditional on being informed that he matches with α_A at $t = 1$, β_{1A} should not want to wait to be matched in $t = 2$, that is:

$$v(\alpha_A, \beta_{1A}) \geq \delta_{\beta_{1A}} \left[\frac{p_1 + p_4}{p_1 + p_3 + p_4} v(\alpha_B, \beta_{1A}) + \frac{p_3}{p_1 + p_3 + p_4} v(\alpha_C, \beta_{1A}) \right].$$

The result follows from the above equation. □

Step 6: If $p_3 + p_4 + p_5 + p_6 > 0$, then:

$$\begin{aligned} u(\alpha_C, \beta_{1B}) \leq & \\ & \delta_{\alpha_C} \left[\begin{array}{c} \frac{p_3+p_4}{p_3+p_4+p_5+p_6} u(\alpha_C, \beta_{1B}) + \frac{p_5}{p_3+p_4+p_5+p_6} u(\alpha_C, \beta_{1A}) \\ + \frac{p_6}{p_3+p_4+p_5+p_6} u(\alpha_C, \beta_{1B}) \end{array} \right]. \end{aligned} \tag{1}$$

Proof. Conditional on being informed that she does not match in $t = 1$, α_C should not want to block with β_{1B}, that is equation (1) should hold. □

Step 7: If $p_3 + p_5 > 0$, then $\frac{p_3}{p_3+p_5} \geq \frac{\delta_{\beta_{1B}}v(\alpha_C,\beta_{1B})-v(\alpha_B,\beta_{1B})}{\delta_{\beta_{1B}}(v(\alpha_C,\beta_{1B})-v(\alpha_B,\beta_{1B}))}$.

Proof. Conditional on being informed that he matches in $t = 1$, β_{1B} should not want to wait, that is:

$$v(\alpha_B, \beta_{1B}) \geq \delta_{\beta_{1B}} \left[\frac{p_3}{p_3 + p_5} v(a_3, b_3) + \frac{p_5}{p_5 + p_3} v(\alpha_C, \beta_{1B}) \right]$$

□

Step 8: If $p_2 + p_5 + p_6 > 0$, then $\frac{p_5}{p_2+p_5+p_6} \geq A$.

Proof. This follows from β_{1A} not wanting to block with α_A when he is informed that he matches in $t = 1$. □

Step 9: If $p_4 + p_6 > 0$, then $\frac{p_4}{p_4+p_6} \leq B$.

Proof. This follows from the requirement that β_{1B} does not want to block with α_B when he finds out that he matches in $t = 2$. □

Corollary E.2. *If $p_6 = 0$, then $p_4 = 0$.*

Step 10: $p_1 + p_2 < 1$.

Proof. Otherwise, (α_A, β_{1A}) block when they are informed they match in $t = 2$. □

Corollary E.3. $p_3 + p_4 + p_5 + p_6 > 0$

Step 11: There is no random contingent matching that is correlated dynamically stable when $p_2 = 0$.

Proof. If $p_2 = 0$, then, by Corollary E.1, $p_1 = 0$. Hence, the probabilities have to satisfy the following inequalities:

$$\frac{p_3}{p_3 + p_4} \leq A \qquad \frac{p_5}{p_5 + p_6} \geq A$$
$$\frac{p_3}{p_3 + p_5} \geq B \qquad \frac{p_6}{p_6 + p_4} \leq B$$
$$p_3 + p_4 + p_5 + p_6 = 1 \qquad \text{Equation (1)}$$

Moreover, from the no-blocking constraints if follows that $p_3, p_4, p_5, p_6 > 0$. In fact, Steps 7 and 8, together with $p_2 = 0$, imply that $p_3 p_5 > 0$. Moreover, $p_4 = 0$ leads to a contradiction since it implies that when β_{1A} learns $m_1(\beta_{1A}) = \alpha_A$, he knows that $m_1 = m_1^3$, and he blocks by waiting. Finally, assuming $p_6 = 0$ also leads to a contradiction since in that case when β_{1B} learns that $m_1(\beta_{1B}) = \beta_{1B}$, he knows $m_1 = m_1^4$, and he blocks with α_B. The inequalities can be written as follows:

$$p_3 \leq \frac{A}{1-A} p_4 \qquad p_5 \geq \frac{A}{1-A} p_6$$
$$p_3 \geq \frac{B}{1-B} p_5 \qquad p_4 \leq \frac{B}{1-B} p_6$$

From equation (1), it follows that p_3, p_4 have to be as large as possible, hence:

$$p_3 = \frac{A}{1-A} p_4$$
$$p_4 = \frac{B}{1-B} p_6$$

Moreover, it has to be the case that $p_5 > 0$, otherwise the above inequalities imply that $p_i = 0, i \in \{3,4,5,6\}$, a contradiction. Then, $\frac{A}{1-A}p_6 \le p_5 \le \frac{1-B}{B}p_3 = \frac{A}{1-A}p_6$. Hence, we obtain: $p_3 = AB, p_4 = B(1-A), p_5 = A(1-B), p_6 = (1-A)(1-B)$. However, from the assumption in the statement of the Lemma, these probabilities can't satisfy (1).Therefore, it follows that $p_2 > 0$. □

Step 12: There is no random contingent matching that is correlated dynamically stable, and has $p_6 = 0$.

Proof. If $p_6 = 0$, then $p_4 = 0$ by Corollary E.2. Hence, (1) implies that:

$$\frac{p_3}{p_3 + p_5} \ge \frac{u(\alpha_C, \beta_{1B}) - \delta_{\alpha_C} u(\alpha_C, \beta_{1A})}{\delta_{\alpha_C}(u(\alpha_C, \beta_{2C}) - u(\alpha_C, \beta_{1A}))} \equiv C$$

and, it follows that $B < C$ since $u(\alpha_C, \beta_{1A}) < u(\alpha_C, \beta_{1B})$. Therefore, the following inequalities have to hold:

$$p_1 \le \frac{D}{1-D}p_2$$
$$p_3 \le \frac{A}{1-A}p_1 \qquad p_5 \ge \frac{A}{1-A}p_2$$
$$p_3 \ge \frac{C}{1-C}p_5$$

The equations imply that: $p_2\frac{A}{1-A} \le p_5 \le \frac{1-C}{C}p_3 \le \frac{1-C}{C}\frac{A}{1-A}p_1 \le \frac{1-C}{C}\frac{A}{1-A}\frac{D}{1-D}p_2$. Since $p_2 > 0$, it has to be that: $\frac{D(1-C)}{(1-D)C} \ge 1$. However, this is if, and only if,$(1-\delta_{\alpha_C})u(\alpha_C, \beta_{1B}) \ge u(\alpha_C, \beta_{1B}) - \delta_{\alpha_C} u(\alpha_C, \beta_{1A}) \Leftrightarrow u(\alpha_C, \beta_{1B}) \le u(\alpha_C, \beta_{1A})$, which contradicts $u(\alpha_C, \beta_{1A}) < u(\alpha_C, \beta_{1B})$. Therefore, $p_6 > 0$. □

Remark E.1. The no blocking constraints imply that $p_3, p_5, p_2 > 0$, and $p_4 + p_1 > 0$. Step 8 and $p_2 p_6 > 0$ imply that $p_5 > 0$. Hence, Step 7 implies that $p_3 > 0$. Finally, $p_3 > 0$ and Step 5 imply that $p_1 + p_4 > 0$.

Step 13: There is no correlated dynamically stable matching with $p_1 = 0$.

Proof. If $p_1 = 0$, then $p_4 > 0$, and the following inequalities have to hold:

$$p_3 \le \frac{A}{1-A}p_4 \qquad p_5 \ge \frac{A}{1-A}(p_2 + p_6)$$
$$p_3 \ge \frac{B}{1-B}p_5 \qquad p_4 \le \frac{B}{1-B}p_6$$

Equation (1)

Now, the inequalities imply that: $\frac{A}{1-A}p_4 \ge p_3 \ge \frac{B}{1-B}p_5 \ge \frac{B}{1-B}\frac{A}{1-A}(p_2 + p_6) \ge \frac{B}{1-B}\frac{A}{1-A}p_2 + \frac{A}{1-A}p_4$, which implies $p_2 = 0$, a contradiction since $p_2 > 0$. □

Step 14: It cannot be the case that $p_4 = 0$.

Proof. If so, applying Step 14, the probabilities satisfy: $p_2\frac{D}{1-D} \ge p_1 \ge \frac{1-A}{A}p_3 \ge \frac{B}{1-B}\frac{1-A}{A}p_5 \ge \frac{B}{1-B}p_2 + \frac{B}{1-B}p_6$. The assumption in the statement implies that $B > D$, hence, $p_2 = 0$, and $p_6 = 0$, a contradiction. Therefore, it has to be that $p_4 > 0$. □

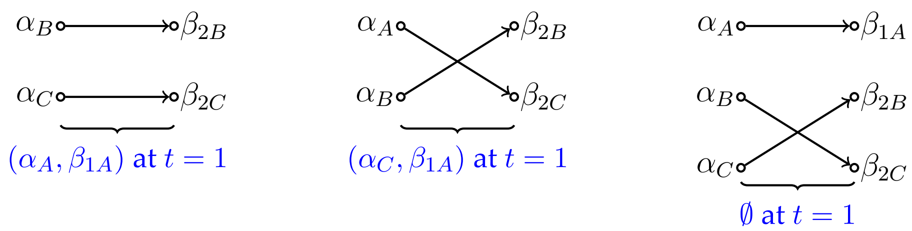
Step 15: It cannot be the case that $p_4 > 0$.

Proof. In this case, the inequalities imply that: $\frac{A}{1-A}(p_1 + p_4) \geq p_3 \geq \frac{B}{1-B}p_5 \geq \frac{B}{1-B}\frac{A}{1-A}(p_2 + p_6)$. Thus, $p_1 + p_4 \geq \frac{B}{1-B}(p_2 + p_6) \geq \frac{B}{1-B}p_2 + p_4$. This implies: $p_1 \geq \frac{B}{1-B}p_2$, and Step 4 implies $p_1 \leq \frac{D}{1-D}p_2$. Since $B > D$, this cannot be. □

Steps 1-15 imply that $D_\Delta(\mathcal{E}) = \emptyset$. □

F EXAMPLE 2.1

I now complete the description of the contingent matchings described in Example 2.1. Let μ denote the matching that has the core allocation on-path, and let μ' be the contingent matching that has the non-core allocation on-path. They only differ in the specification of the continuation matching after (α_A, β_{1A}) match together, so the following constitute the off-path, $t = 2$, matchings for both μ, μ':



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