Optimal Policy with Limited-Time Commitment

PRELIMINARY AND INCOMPLETE

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Abstract

We consider models where the Full-Commitment Ramsey-optimal policy is time-inconsistent and define a new notion of optimal policy, Limited-Time Commitment. In our setup, successive one-period lived governments can commit to future plans for a limited time only. We discuss a key condition on the mapping from finite sequences of policy instruments to competitive equilibrium allocations that can be checked model by model. If this condition is satisfied, Limited-Time Commitment is sufficient to sustain Full-Commitment outcomes. We show that this condition is verified in several models of optimal fiscal policy studied in the literature, allowing policies derived assuming infinite periods of commitment to be supported with only a finite number of periods of commitment (often a single period). We also study optimal monetary policy in a model where this equivalence result fails, and compute Limited-Time Commitment outcomes numerically. We provide an example where, at the zero lower bound, a single quarter of commitment obtains most of the welfare gains of Full Commitment relative to No Commitment.

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1 Introduction

Governments in advanced economies typically formulate their macroeconomic policies as plans for a finite future horizon. Fiscal policy is generally decided before or at the beginning of the fiscal year and remains fixed for the duration of the year.[1] Fiscal reforms such as VAT rate changes and fiscal consolidation plans are typically announced before their implementation and contain details of short-to-medium run policy plans.[2] Furthermore, the political process may make it hard to change contemporaneous policies, with the result that reforms are often implemented with a delay. Central bankers also expend significant effort communicating their short-to-medium run objectives for monetary policy in order to affect the private sector's inflation expectations.

In contrast, a large part of the macroeconomic literature on optimal policy typically assumes that either a single government at the beginning of time has Full Commitment (FC) into the infinite future, or that in each period there is a government with No Commitment (NC) at all, only able to choose contemporaneous policies. On the face of it, both of these extreme assumptions appear hard to reconcile with the fact that policymakers act and communicate as if they possessed a limited degree of commitment over a finite future horizon. On the one hand, policymakers are only in power for a limited period of time, which makes commitment into the infinite future impossible. On the other hand, political delays and institutional features may place limits on a policymaker's ability to change contemporaneous policy instruments, while allowing for the possibility of planning policy changes for a near future horizon.

Motivated by this apparent distance between observation and theory, in this paper we study optimal policy when successive governments inherit the plans of their predecessors and formulate plans for a finite future horizon. In this formulation, which we call Limited-Time Commitment (LTC), governments cannot commit into the infinite future, but instead only possess the ability to commit for a finite number of periods. LTC thus lies between FC and NC, and we ask the natural question of whether governments with only a finite number of periods of commitment behave more

[1] For example, in the UK the budget is announced on "Budget Day" and typically passed shortly afterwards.

[2] Examples of pre-announced VAT reforms include Japan (announced in 1996, implemented in 1997) and Germany (announced in 2005, implemented in 2007). Alesina et al. (2015) document many examples of multi-year fiscal consolidations.

like governments with full, or no, commitment.

By applying the same Markov-Perfect equilibrium concept introduced by the literature on time-consistent policies to our setup, we show that in several models of optimal policy that have been studied in the literature, a limited number of periods (often a single period) of future commitment is sufficient to sustain the same allocations that arise when a single government has FC and can commit to the infinite future.

The fact that one or few periods of commitment are often sufficient to sustain FC policies and allocations has important consequences both for policy design and for the academic literature. First, it implies that if we want to design commitment technologies that can improve welfare in real-world policy environment, it may be sufficient to ensure that policy-makers are committed and accountable for a short period of time. Second, it implies that even if assuming FC into the infinite future in our models may seem extremely "unrealistic", it actually often leads to the same results that arise in a more empirically plausible setup where there is a succession of governments that make decisions for the near future.

The time-inconsistency problems we study arise from governments formulating optimal policies subject to competitive-equilibrium constraints which contain future values of certain choice variables. We formulate a general framework in which we establish the conditions under which LTC can sustain the FC policy. The reason for the surprising result that this is sometimes possible lies in a key property of the mapping from sequences of policy instruments to sequences of allocations and prices. If it is the case that a finite sequence of future policy instruments is sufficient to uniquely pin down a finite sequence of contemporaneous and future allocations, then the time-inconsistency problem can be resolved if the government can commit to a finite sequence of policies of a certain length.

To build intuition, consider a government elected at time t that chooses taxes in order to maximize welfare subject to competitive equilibrium conditions, including an Euler equation for asset holdings, which contains consumption at $t+1$. The time-inconsistency problem here is that the time-$t+1$ government, who is not subject to the time-t Euler equation, could choose policies leading to a different consumption allocation from the time-t promise. However, if time-$t+1$ consumption is fully pinned down by the time-$t+1$ policy instruments, then the time-t government can prevent the time-$t+1$ government from deviating from her promise if she can commit to taxes one period ahead.

The time-$t+2$ government will also respect the promises of the time-$t+1$ government, creating a chain of commitment which is able to sustain the FC policy.

We discuss the key assumption that needs to be satisfied in order for equivalence of FC and LTC to arise, which can be checked in any model of optimal policy. We show that it is indeed satisfied in many models of optimal fiscal policy that have been analyzed in the literature. In models without capital, where governments choose the timing of labor taxes and government debt, we show that the FC policy can be supported with commitment of the length of the longest maturity bond issued. In models with capital, and governments who choose capital and labor taxes subject to a balanced budget constraint, we show that FC can be supported with commitment equal to the length of time over which the government budget must be balanced. In models with capital but without balanced budgets, we show that the equivalence of FC and LTC arises in the special case of risk neutrality, even though stark differences between FC and NC still exist in this case.

One subtlety in the equivalence of FC and LTC is the role of initial conditions. For example, in the LTC game where governments have one period of commitment, who chooses the time-0 policy? We prove that if the time-0 policy is restricted to equal the time-0 policy the FC government would have chosen, then LTC supports FC for all time periods.[3] This leaves open the question of what happens in the LTC game if we start from the wrong initial condition for time-0 policy. We prove that in two of our fiscal policy examples, starting from the wrong initial condition simply leads to convergence to a different FC allocation, thus maintaining much of the spirit of the equivalence of FC and LTC.

We also study optimal monetary policy in the New Keynesian model, which has well known time-inconsistency issues because of the forward-looking nature of inflation. In this model our key assumption is not satisfied, and LTC policies will differ from the FC policy. We study the optimal policy response to demand shocks that drive the economy to the zero lower bound, as well as cost-push shocks, and compute LTC outcomes numerically. This represents a contribution of its own, as the recent debate on forward guidance and the zero-lower bound hinges crucially on the degree to which central banks can commit to future policies. We show that at the zero lower bound a single pe-

[3]Alternatively, we also prove that if the time-0 government is allowed to choose the time-0 policy in addition to the other policies normally in her control then LTC supports the FC allocation.

riod of commitment goes a long way in recovering welfare losses associated with lack of commitment.

Related Literature. The time-inconsistency of optimal policy has been a central issue in the macroeconomic literature since the 1970s. Kydland and Prescott (1977) highlighted that optimal policy in a dynamic model involves ex-ante promises that appear suboptimal ex-post if the government is allowed to reoptimize at a later date. For this reason, the presence of future (expected) variables in the constraint set of Ramsey-optimal plans rules out the use of standard optimal control techniques.

Despite its importance both in the academic and in the policy debate, this key insight has not lead to a uniform reaction in the literature. A part of the optimal policy research agenda has worked on modifying standard recursive methods to deal with this class of problems under the assumption that the government is endowed with a Full Commitment technology into the infinite future, while another strand of the literature has considered time-inconsistency a central, unavoidable problem and has focused on equilibria with No Commitment instead.

As leading examples of the first approach, consider the Recursive Contracts method formalized in Marcet and Marimon (2011) which adds the Lagrange multipliers on forward-looking constraints as state variables, allowing the reformulation of optimal policy problems as recursive saddle-point problems, or the Promised Utility approach proposed by Abreu et al. (1990), based on the result that past histories can be summarized by promised utility. Relatedly, Kydland and Prescott (1980) proposed a similar recursive method based on the addition of marginal utilities as state variables in the optimal policy problem.

The second part of the literature has focused instead on formulating equilibrium concepts without a commitment technology, starting with the seminal paper on Sustainable Plans by Chari and Kehoe (1990), and the application of Markov NC equilibria in optimal policy games, as for instance in Klein and Ríos-Rull (2003), Krusell et al. (2004) and Klein et al. (2008). In these papers, there is a succession of governments that can only choose contemporaneous policy instruments. Hence, any ability to formulate credible promises about future allocations is ruled out by assumption.

In our paper, we explore an intermediate assumption on the commitment technology, namely that a succession of governments can announce policies for a finite future horizon. We allow for some commitment to future outcomes as in the first strand, but we apply the Markov equilibrium concept typical of the NC literature. We show that in several models that have been studied in the optimal fiscal policy literature (e.g. the seminal paper by Lucas and Stokey, 1983) a few periods of commitment, often only one, are sufficient to sustain the FC outcome as the unique equilibrium of our game. In this sense, one can see our paper as suggesting that while assuming FC into the infinite future may seem extremely "unrealistic", in several models it leads to the same results that arise in a world where finite-lived governments formulate their policies as plans for the near future.

The papers closest to our work are the Quasi Commitment approach of Schaumburg and Tambalotti (2007), and the Loose Commitment approach formulated by Debortoli and Nunes (2010, 2013). These papers assume that a government can formulate a plan into the infinite future, but with some probability in every future period a new government is elected and allowed to change the plan. Like ours, this game represents an intermediate point in the FC vs. NC debate. Differently from these papers, however, LTC gives the government full commitment within a limited time horizon, instead of probabilistic commitment over an infinite horizon. While this may seem a technical difference, it turns out to be important. We show that in some models where LTC with one period of commitment leads to allocations equivalent to FC, Loose Commitment with *on average* one period of commitment leads to allocations closer to NC. In this sense a result of this paper is that once you limit the commitment technology of the government, the details of how you do so matter for the results.[4]

Another example of the importance of how you deviate from FC or NC is a comparison of our results for optimal capital and labor taxation under balanced budgets to those of Klein and Ríos-Rull (2003). They study a game where the time-t government chooses the time-t labor tax and the time-$t+1$ capital tax, thus allowing one period of commitment for capital, but not labor, taxes. In the solution to this game capital taxes are on average high, compared to the FC solution which has average capital taxes near zero. In Section 4.2 we solve a deterministic version of the same model under LTC, and show that if instead the time-t government chooses (and is able to commit to) both

[4]Our result that FC allocations can sometimes be supported with only a few periods of commitment in LTC echoes Schaumburg and Tambalotti's (2007) result that half to three quarters of the welfare gains from commitment can be achieved with Quasi Commitment lasting on average two and five years respectively in a calibrated New Keynesian model.

the time-$t+1$ capital *and* labor taxes then we recover the FC policy and allocation.

A related strand of literature studies the so-called "timeless perspective" in optimal policy (e.g., Woodford, 2011). In this setup, the policy-maker is constrained to choose current policies as if they were part of an optimal FC plan chosen in an infinitely far past. While there is a similarity between this approach and ours – policy-makers are constrained by previous decisions and cannot modify current policies –, there is also a relevant difference. With LTC, under the "right" initial conditions, we recover exactly the FC plan, whereas the optimal "timeless" policy would lead to a different outcome, because it is designed to remove the discrepancy in the FC plan between time-0 policies and the rest of the plan. To make a concrete example, in this paper we consider a deterministic version of a model of the optimal timing of taxes (Lucas and Stokey, 1983). The FC plan calls for a constant tax rate from period 1 onwards, and a different tax rate in period 0. With one period of commitment, LTC sustains the same outcome. The timeless perspective would call for a constant tax rate from period 0, and, importantly, this tax rate is different from both the time-0 and the time-1 tax rate arising under FC.[5]

Domeij and Klein (2005) study tax reform in presence of implementation lags. This approach leads to a similar timing assumption to ours, but with an important distinction: under our main assumption, every government is inheriting the "right" policy from its predecessor, hence FC policies can be sustained in equilibrium, whereas in Domeij and Klein (2005) the focus is on a fiscal reform starting from a suboptimal policy.

Our work on optimal monetary policy at the zero lower bound relates to Adam and Billi (2006, 2007) who show that the zero lower bound increases significantly the welfare losses from lack of commitment in the New-Keynesian model. We provide an example where LTC allows to get close to the welfare with FC, even with a single quarter of commitment. Relatedly, Bodenstein et al. (2012)

[5]A different approach to find an intermediate outcome between FC and NC has been explored in the literature on reputational equilibria, starting with the seminal contribution of Barro and Gordon (1983). Furthermore, several other papers explore the extent to which FC outcomes can be supported in absence of a commitment technology, e.g. Alvarez et al. (2004) and Conesa and Dominguez (2012) explore the role of the maturity structure of debt in imposing restrictions on future governments and allowing to sustain FC plans. Laczó and Rossi (2015) argue that adding more policy instruments brings NC outcomes closer to FC: specifically they allow for consumption taxes in a model of labor and capital taxation and show that this leads to time-consistent policies that are remarkably similar to FC ones.

study optimal monetary at the zero lower bound with Loose Commitment.

The rest of the paper is organized as follows. Section 2 introduces the concept of LTC in a general model and discusses informally its equivalence with FC in a simple example. Section 3 proves formally the main equivalence result. Section 4 applies the result to models of optimal fiscal policy commonly used in the literature and Section 5 discusses the role of initial conditions in two of these examples. Section 6 studies optimal monetary policy in the New-Keynesian model, where our equivalence result fails, and provides numerical results on LTC at the zero lower bound. Section 7 concludes.

2 Full Commitment and Limited-Time Commitment

In this section we describe a general model of optimal policy and we define two notions of optimal policy: the Full-Commitment (FC) Ramsey equilibrium and the Limited-Time Commitment (LTC) equilibrium. We argue that LTC nests both the NC equilibrium, as a special case without any degree of commitment, and the FC equilibrium, in the limit with infinite commitment, under the right initial conditions. In each of these equilibrium concepts, we discuss a specific assumption on the mapping from infinite sequences of taxes to infinite sequences of allocations and prices that needs to be satisfied in order for the government(s) to be able to pin down a single competitive equilibrium by choosing the policy instruments.

We then describe a simple example and use it to provide intuition for our main result: under a stronger assumption on the mapping from *finite* sequences of policies to *finite* sequences of allocations that is satisfied in several models of optimal policy, LTC with a sufficiently long, but finite, commitment, and FC give rise to equivalent outcomes. In other words, commitment to a finite sequence of policies is sufficient to sustain the FC outcome as the unique equilibrium. This result is shown formally in Section 3.

2.1 Environment and competitive equilibrium

Time is discrete and indexed by $t = 0, 1, 2, ...$ The economy is populated by a continuum of households, a continuum of firms and a government or a sequence of governments. A vector of exogenous variables $g_t \in G$ follows a deterministic sequence about which all agents have perfect foresight, with the Markov property that g_t is sufficient information to predict g_{t+1}. We call $b_t \in B$ the endogenous state variables (b_0 being an initial condition), $c_t \in C$ the remaining variables constituting allocations (for instance consumption and hours worked), $p_t \in P \subset \mathbb{R}^{n_P}$ the prices and $\tau_t \in T$ the policy instruments chosen by the government(s).

Agents' preferences are represented by the following utility function:

$$\sum_{t=0}^{\infty} \beta^t r(c_t, b_t, g_t, \tau_t). \tag{1}$$

where $r : C \times B \times G \times T \mapsto \mathbb{R}$ and $\beta \in (0, 1)$. Households and firms take sequences of prices $\{p_t\}_{t=0}^{\infty}$ and policy instruments $\{\tau_t\}_{t=0}^{\infty}$ as given. Households maximize utility subject to their budget constraints (and potentially other constraints such as borrowing constraints). Firms maximize profits subject to their production technologies. All markets clear.

Following the general formulation in Marcet and Marimon (2011), we can summarize these equilibrium conditions with three sequences of constraints: a transition equation for the endogenous states, a set of constraints involving only contemporaneous allocations and a set of constraints involving future variables.

$$b_{t+1} = l(b_t, g_t, c_t, p_t, \tau_t) \tag{2}$$

$$k(b_t, g_t, c_t, p_t, \tau_t) \leq 0 \tag{3}$$

$$h(b_t, g_t, c_t, p_t, \tau_t, ..., b_{t+N}, g_{t+N}, c_{t+N}, p_{t+N}, \tau_{t+N}) = 0. \tag{4}$$

for $N \geq 1$.

Definition 1. *Given an initial condition b_0, an exogenous sequence $\{g_t\}_{t=0}^{\infty}$ and a policy sequence $\{\tau_t\}_{t=0}^{\infty}$, a* ***competitive equilibrium*** *is a sequence $\{c_t, p_t, b_{t+1}\}_{t=0}^{\infty}$ that satisfies (2), (3) and (4) for $t = 0, 1, ...$*

As is well known in the literature, the presence of the future variables in the constraint set defining competitive equilibria is the reason for the time-inconsistency of FC policies. In Definition 2 we explicitly label the future variables which enter into the constraints as problematic.

Definition 2. *Split* c_t *into its elements* $(c_t^1, ..., c_t^{n_c})$*. For every* $1 \leq s \leq N$*, we call* ***problematic*** *from the perspective of time* t *the elements of* c_{t+s} *which appear in the constraint (4). The same definition applies to elements of* b_{t+s} *(with* $s > 1$*),* p_{t+s} *and* τ_{t+s}*.*

We make the following (standard) regularity assumption.

Assumption 1. *Given an initial condition* b_0 *and an exogenous sequence* $\{g_t\}_{t=0}^{\infty}$*, for all sequences* $\{(\tau_t, c_t, b_{t+1}, p_t)\}_{t=0}^{\infty}$ *satisfying (2), (3), (4),* $\lim_{n\to\infty} \sum_{t=0}^{n} \beta^t r(c_t, b_t, g_t, \tau_t)$ *exists, although it may be plus or minus infinity.*

2.2 Full-Commitment Ramsey equilibrium

In a FC equilibrium, a single benevolent infinitely-lived government endowed with the ability to credibly commit into the infinite future announces a plan at $t = 0$ and then implements it. In order for the government to be able to pin down a unique competitive equilibrium, the following assumption on the mapping from infinite sequences of taxes to allocations is required.

Assumption 2. *Given an initial condition* b_0*, an exogenous sequence* $\{g_t\}_{t=0}^{\infty}$ *and a policy sequence* $\{\tau_t\}_{t=0}^{\infty}$*, there exists a unique sequence* $\{(c_t, b_{t+1}, p_t)\}_{t=0}^{\infty}$ *that satisfies equations (2), (3) and (4) for* $t = 0, 1, ...$

Assumption 2 allows us to map an infinite sequence of policy instruments to a single competitive equilibrium. The FC government solves the following problem:

$$\max_{\{(\tau_t, c_t, b_{t+1}, p_t)\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t r(c_t, b_t, g_t, \tau_t) \tag{5}$$

subject to (2), (3) and (4).

Definition 3. *Let* $\left\{\tau_t^{FC}\right\}_{t=0}^{\infty}$ *be a policy sequence that solves (5). A* ***Full-Commitment (FC) Ramsey equilibrium*** *is the competitive equilibrium associated with* $\left\{\tau_t^{FC}\right\}_{t=0}^{\infty}$*.*

As has been extensively discussed in the literature (e.g. Kydland-Prescott, 1977), the presence of problematic variables in equation (4) induces time-inconsistency in the FC policy. The FC government commits to a certain sequence which is optimal from a $t = 0$ viewpoint, but would no longer be optimal in subsequent periods, if a reoptimization were allowed.

2.3 Limited-Time Commitment equilibrium

The setup can be described as a game, where successive one-period-lived governments indexed by t choose only a finite sequence of policy instruments, taking as given the strategies of the following governments. Each government is benevolent and maximizes (1) subject to the competitive equilibrium conditions (2), (3) and (4). Such a game may often have multiple equilibria. In our exposition, we follow the literature (for instance Klein et al., 2008) in restricting attention to symmetric-Markov equilibria, where each government chooses a common best-response function mapping a small set of "natural" state variables into the chosen sequence of policy instruments.

Let $L = 0, 1, 2, ..$ index the duration of commitment. Let $S^0 \equiv \{1\}$ and $S^L \equiv \{1\} \cup T^L$ for $L > 0$ and let $\tau_t^L \in S^L$ be a vector of policies inherited from governments in power before t. In particular, $\tau_t^L \equiv 1$ for $L = 0$ and $\tau_t^L \equiv (1, \tau_t, \tau_{t+1}, ..., \tau_{t+L-1})$ for $L > 0$. In the case $L = 0$, no policy decision is inherited and LTC nests the NC equilibrium studied for instance by Klein et al. (2008). Hence, we let the corresponding (redundant) state variable be a constant number, which is equivalent to not having a policy-related state variable. Note that this should not be confused with the value of a policy instrument being equal to 1.

The government dated t takes as given the inherited policies τ_t^L and chooses the policy for period τ_{t+L}. In the NC game with $L = 0$, the government is free from policy constraints and chooses the value of the current policy instrument. In the simplest case of Limited-Time Commitment, with $L = 1$ (which we label One-Period Commitment, or OPC), each government takes the current policy as given and chooses the next period policy. Relative to the NC case, a positive level of commitment $L > 0$ implies that the state vector describing the choice problem of a government has to be enriched to include all the policy instruments that have been chosen by previous governments and cannot be changed. Hence, the "natural" state variables of this problem are $\left(b_t, g_t, \tau_t^L\right)$ and accordingly we have to treat τ_0^L as an initial condition (as well as b_0).

Consider the government in power in period t and let $\tilde{V}^{LTC}(b_{t+1}, g_{t+1}, \tau_{t+1}^L)$ be the agent's tail of discounted utility starting in $t+1$ if the next government starts with state variables $\left(b_{t+1}, g_{t+1}, \tau_{t+1}^L\right)$ and all governments from $t+1$ onwards are expected to play a common policy function $\tau_{t+j+L} = \tilde{\tau}^{NC}(b_{t+j}, g_{t+j}, \tau_{t+j}^L)$ for $j = 1, 2, ...$ In order to define $\tilde{V}^{LTC}(b_{t+1}, g_{t+1}, \tau_{t+1}^L)$ more formally, we make the following assumption.

Assumption 2-LTC. *Given a state* $(b_t, g_t, \tau_t^L) \in B \times G \times S^L$*, a function* $\tilde{\tau}^{LTC} : B \times G \times S^L \mapsto T :$ *such that* $\tau_{t+j+L} = \tilde{\tau}^{LTC}(b_{t+j}, g_{t+j}, \tau_{t+j}^L)$ *for* $j = 1, 2, ...$ *and a time* $t + L$ *policy* $\tau_{t+L} \in T$*, the competitive equilibrium system given by (2), (3) and (4) admits a unique time-invariant solution for the vector* (c_t, p_t, b_{t+1}) *given by* $(c_t, p_t, b_{t+1}) = \phi^{LTC}(b_t, g_t, \tau_t^L; \tilde{\tau}^{LTC})$.

When all governments play the policy function $\tilde{\tau}^{LTC}$, the function $\tilde{V}^{LTC}$ satisfies the following functional equation.

$$\tilde{V}^{LTC}(b_t, g_t, \tau_t^L) = r(\phi_1^{LTC}(b_t, g_t, \tau_t^L; \tilde{\tau}^{LTC}), b_t, g_t, \tau_t)$$

$$+\beta \tilde{V}^{LTC}(\phi_3^{LTC}(b_t, g_t, \tau_t^L; \tilde{\tau}^{LTC}), g_{t+1}, \tau_{t+1}^L) \tag{6}$$

where $\tau_{t+L} = \tilde{\tau}^{LTC}(b_t, g_t, \tau_t^L)$ and ϕ_i^{LTC} is the i-th entry of the vector (c_t, p_t, b_{t+1}) implied by the competitive equilibrium mapping ϕ^{LTC}.

The government dated t anticipates that all future governments will play the policy function $\tilde{\tau}^{LTC}$ and solves the following maximization problem

$$\max_{\tau_{t+L}, c_t, p_t, b_{t+1}} r(c_t, b_t, g_t, \tau_t) + \beta \tilde{V}^{LTC}(b_{t+1}, g_{t+1}, \tau_{t+1}^L) \tag{7}$$

subject to $(c_t, p_t, b_{t+1}) = \phi^{LTC}(b_t, g_t, \tau_t^L; \tilde{\tau}^{LTC})$, where future variables appearing in constraint (4), such as c_{t+1}, are given by $\phi_1(b_{t+1}, g_{t+1}, \tau_{t+1}^L; \tilde{\tau}^{LTC})$, and so on.

In a symmetric Markov equilibrium, the solution for the optimal policy instrument is given by $\tau_{t+L} = \tilde{\tau}^{LTC}(b_t, g_t, \tau_t^L)$ and the maximum value is $\tilde{V}^{LTC}(b_t, g_t, \tau_t^L)$. In words, the policy function $\tilde{\tau}^{LTC}$ is associated with a fixed point of the operator defined in (7).

Definition 4. *A **symmetric Markov Limited-Time-Commitment (LTC) equilibrium** is a competitive equilibrium associated with the policy sequence* $\left(\left\{\tilde{\tau}^{LTC}(b_t, g_t, \tau_t^L)\right\}_{t=0}^{\infty}\right)$ *if* $L = 0$ *and with the sequence* $\left(\tau_0, ..., \tau_{L-1}, \left\{\tilde{\tau}^{LTC}(b_t, g_t, \tau_t^L)\right\}_{t=0}^{\infty}\right)$ *if* $L > 0$.

Notice that for $L \to \infty$, if we take as "initial conditions" the FC policy sequence this equilibrium trivially coincides with the FC equilibrium, as in the limit the entire path of taxes coincides with the initial conditions. In this case, LTC nests both NC ($L = 0$) and FC ($L \to \infty$).

2.4 Example

We now discuss the relationship between the equilibrium notions of FC and LTC in the context of a simple deterministic version of the optimal fiscal policy model of Lucas and Stokey (1983). In this model, we informally argue that a single period of commitment ($L = 1$) is sufficient to sustain FC outcomes. To avoid notational clashes with the general framework presented above, here and wherever we refer to a specific model we use upright text to denote variables.

A representative agent has preferences defined over sequences of private consumption $\{\mathrm{c}_t\}_{t=0}^{\infty}$ and labor effort $\{\mathrm{l}_t\}_{t=0}^{\infty}$:

$$\sum_{t=0}^{\infty} \beta^t u(\mathrm{c}_t, \mathrm{l}_t) \tag{8}$$

with standard assumptions $u_{\mathrm{c}} > 0$, $u_{\mathrm{cc}} < 0$, $u_{\mathrm{l}} < 0$, $u_{\mathrm{ll}} < 0$. Their budget constraint is given by

$$\mathrm{c}_t + \mathrm{q}_t \mathrm{b}_{t+1} = \mathrm{w}_t \mathrm{l}_t \left(1 - \tau_t^l\right) + \mathrm{b}_t \tag{9}$$

where q_t is the price of a one-period discount bond issued at t that repays one unit of consumption at $t+1$. The resource constraint reads

$$\mathrm{c}_t + \mathrm{g} = \mathrm{l}_t \tag{10}$$

where g is a constant level of government expenditure that needs to be financed through proportional labor income taxes at rate τ_t^l. Output is produced using a linear technology in labor: $y_t = l_t$, hence firms' profit maximization implies a unit wage: $\mathrm{w}_t = 1$. The government's budget is implied by the agent's budget constraint and the resource constraint:

$$\mathrm{b}_t + \mathrm{g} = \tau_t^l \mathrm{l}_t + \mathrm{q}_t \mathrm{b}_{t+1}. \tag{11}$$

The agent's first order conditions with respect to consumption, labor effort and bonds, together with the resource constraint, can be summarized by an intratemporal optimality condition and an Euler equation.

$$-\frac{u_{\mathrm{l}}(\mathrm{c}_t, \mathrm{c}_t + \mathrm{g})}{u_{\mathrm{c}}(\mathrm{c}_t, \mathrm{c}_t + \mathrm{g})} = 1 - \tau_t^l \tag{12}$$

$$\mathrm{q}_t u_{\mathrm{c}}(\mathrm{c}_t, \mathrm{c}_t + \mathrm{g}) = \beta u_{\mathrm{c}}(\mathrm{c}_{t+1}, \mathrm{c}_{t+1} + \mathrm{g}) \tag{13}$$

In terms of the general notation introduced above, we have $b_t = \mathrm{b}_t$, no exogenous state (g is constant), $c_t = (\mathrm{c}_t, \mathrm{l}_t)$, $p_t = \mathrm{q}_t$, and $\tau_t = \tau_t^l$. Note that we have implicitly solved out for the real wage.

Equation (9) is an example of a constraint like (2), while (10) and (12) are examples of constraints like (3). Equation (13) represents a constraint like (4), involving c_{t+1} as the only "problematic" variable.

We refer to Lucas and Stokey (1983) for a treatment of the FC optimal policy in this setup. However, it is worth stressing that the presence of c_{t+1} in the Euler equation for bonds is the source of time-inconsistency of the FC policy in this model. When $t = 0$, the FC government has an incentive to use the initial allocation to twist the interest rate and decrease the value of outstanding initial debt b_0, hence reducing the distortions required to finance expenditure. In particular, if the FC government starts with a positive (negative) stock of debt, $\mathrm{b}_{-1} < (>)0$, she will choose a lower (higher) tax rate in the first period than in the later periods. Consistently with this initial incentive, any FC promises about allocations c_t with $t > 0$ would be reneged if the government was allowed to reoptimize at t: in order to decrease the value of outstanding debt (assets), b_t, the government would like to offer another tax cut (hike), although she promised not to do this initially.

Consider now the LTC game. When $L = 0$ (NC equilibrium), outstanding debt b_t is the only "natural" state variable and each government chooses the contemporaneous tax rate τ_t^l as a function of b_t. The government in power in each period now has an incentive to twist the interest rate, leading to a deviation from the FC outcomes.[6]

Assume, however, that there is a positive, albeit finite, degree of commitment. In particular, consider the case $L = 1$. Each government takes the contemporaneous tax rate as given and chooses the policy for the following period. The "natural" state variables are (b_t, τ_t^l). Notice that now the government dated t is unable to affect the allocation $(\mathrm{c}_t, \mathrm{l}_t)$, which is entirely pinned down by (10) and (12), given the inherited tax rate τ_t^l. This implies that the government cannot twist the interest rate and affect the value of outstanding debt. Furthermore, by announcing (and committing to) a future tax rate τ_{t+1}^l, the government is effectively choosing (q_t, b_{t+1}) and, importantly, the "problematic" variable c_{t+1}, as well as l_{t+1} from the resource constraint at $t + 1$. Hence, in this model, commitment to a finite sequence of policies, specifically a single future tax rate, is sufficient to uniquely pin down the future variables generating the time-inconsistency, thereby eliminating the

[6] The solution to the NC game in this and similar models is investigated by Krusell et al. (2004) and Debortoli and Nunes (2013).

ability of the future government to act in a way that is inconsistent with the FC plan. By doing so, starting from an initial condition consistent with the FC plan, a chain of successive governments with LTC sustain the whole FC plan as the unique equilibrium.

In the general framework presented above, whether LTC is sufficient to sustain FC outcomes depends on whether finite sequences of policy instruments can uniquely pin down allocations. Our main result, which we prove in the following section, is that an equivalence result between FC and LTC outcomes (with sufficiently large but finite L) holds in a general class of models for which an inherited τ_t^L, together with a commitment to τ_{t+L}, uniquely pin down all the problematic variables appearing in the constraints at t. This implies that FC outcomes can be sustained with a significantly lighter requirement on the policy-maker's ability to control future policies than the one assumed in the standard Ramsey-optimal policy literature.

3 Equivalence result

In this section we first introduce a new notation that will be useful to move towards a recursive formulation of the LTC problem. We then state our key assumption on the mapping from sequences of policies to sequences of allocations and finally prove the equivalence result between the FC and LTC equilibria under this assumption.

3.1 Competitive equilibrium

It is convenient at this point to summarize the variables of the model using $y_t \equiv (b_t, g_t, c_t, p_t, \tau_t) \in Y \subset B \times G \times C \times P \times T$. A sequence $\mathbf{y} \equiv \{y_t\}_{t=0}^{\infty} \in Y^{\infty}$ is denoted a plan. The definition of competitive equilibrium (1) can be straightforwardly restated as a plan satisfying (2), (3), and (4).

Definition 5. *The constraints (2), (3), and (4) define a time-invariant correspondence* $\Gamma^* : Y \mapsto Y^N$*. Let* $Y \subset B \times G \times C \times P \times T$ *be such that* $\Gamma^*(y)$ *is non-empty for all* $y \in Y$*.*

A plan $\mathbf{y}$ thus satisfies competitive equilibrium if and only if $(y_{t+1}, ..., y_{t+N}) \in \Gamma^*(y_t)$ for all $t = 0, 1, ...$ The presence of any terms besides b_{t+1} in determining the feasibility of y_t derives from the problematic elements in (4). We have defined Y as a set such that for any $y_t \in Y$, there exists a continuation sequence starting from $t+1$ which satisfies competitive equilibrium. Note that given this definition, non-emptiness of the correspondence Γ^* simply requires the existence of at least one competitive equilibrium plan.

At $t = 0$ the only predetermined variables are $(b_0, g_0) \in B \times G$. The set of competitive equilibrium plans from a given initial state (b_0, g_0) is denoted by $\Pi^*(b_0, g_0)$, which is constructed as follows. For any $(b_0, g_0) \in B \times G$:

$$\Pi^*(b_0, g_0) = \{\{y_t\}_{t=0}^{\infty} : c_0 \in C, p_0 \in P, \tau_0 \in T, y_0 = (b_0, g_0, c_0, p_0, \tau_0), (y_{t+1}, ..., y_{t+N}) \in \Gamma^*(y_t) \;\; \forall t = 0, 1, ...\} \tag{14}$$

Define $B^* \in B \times G$ as the set of initial conditions for which at least one competitive equilibrium plan exists. $\Pi^*(b_0, g_0)$ is thus non-empty for all $(b_0, g_0) \in B^*$. From now on we take as a maintained assumption that B^* is non-empty, and restrict ourselves to initial conditions in B^*. It is also worth noting at this point that since we can always construct a competitive equilibrium plan by truncating an existing plan at time t, any time-t pair (b_t, g_t) is on a competitive equilibrium plan if and only if it is in B^*. The period utility function is redefined as a function q of y_t, such that $q : Y \mapsto \mathbb{R}$, i.e. $q(y_t) = r(c_t, b_t, g_t, \tau_t)$. The representative agent's utility from $t = 0$ for a given plan, $\mathbf{y}$, is given by:

$$u^*(\mathbf{y}) = \sum_{t=0}^{\infty} \beta^t q(y_t) \tag{15}$$

where it is understood that $\sum_{t=0}^{\infty} \equiv \lim_{n\to\infty} \sum_{t=0}^{n}$.

3.2 Full Commitment

The FC government chooses and commits to an entire path $\{\tau_t\}_{t=0}^{\infty} \in T^{\infty}$ at $t = 0$, taking the initial state $(b_0, g_0) \in B^*$ as given. It is now convenient to restate our Assumptions 1 and 2 with our new notation.

Assumption 1*. *Given any $(b_0, g_0) \in B^*$, for all $\mathbf{y} \in \Pi^*(b_0, g_0)$, $\lim_{n\to\infty} \sum_{t=0}^{n} \beta^t q(y_t)$ exists, although it may be plus or minus infinity.*

Assumption 2*. *Given any $(b_0, g_0) \in B^*$, any path for government policy, $\{\tau_t\}_{t=0}^{\infty} \in T^{\infty}$ which lies on a competitive equilibrium plan $\mathbf{y} \in \Pi^*(b_0, g_0)$ lies on no other plans contained in $\Pi^*(b_0, g_0)$.*

The first assumption ensures that discounted utility converges to a limit, and is hence defined. The second is an invertibility assumption, ensuring that the government is able to pin down a unique path for competitive equilibrium given a path for tax rates. Under this assumption we can equivalently define the government's problem as choosing a path for taxes, or simply choosing the associated plan, $\mathbf{y}$. This allows us to state the FC government's problem, for any $(b_0, g_0) \in B^*$, as:

$$V^*(b_0, g_0) = \sup_{\mathbf{y} \in \Pi^*(b_0, g_0)} u^*(\mathbf{y}) \tag{16}$$

Where $V^* : B^* \mapsto \bar{\mathbb{R}}$ is the supremum, giving the value on an optimal plan. The first two assumptions ensure that this exists and is well defined for any $(b_0, g_0) \in B^*$, since there is always at least one competitive equilibrium plan, and all plans lead to well-defined discounted utilities. For any $(b_0, g_0) \in B^*$, denote the set of plans which achieve the supremum by $\mathbf{y}^{FC}(b_0, g_0)$.

3.3 Limited-Time Commitment

The aim of this subsection is to identify sufficient conditions such that the solutions to the FC and LTC problems coincide. For clarity, we thus avoid restating the LTC problem until these sufficient conditions have been established. The following is our key requirement for equivalence:

Assumption 3*. *There exists an $L \in [N, \infty)$ such that, for any $(b_t, g_t) \in B^*$, any sequence $\{\tau_s\}_{s=t}^{t+L}$ on a competitive equilibrium plan*

1. *implies a unique value for every element of y_t, and hence a unique b_{t+1} from (2), and*
2. *given the implied values of $(b_{t+1}, g_{t+1}, \tau_{t+1}, ..., \tau_{t+L})$, any competitive equilibrium plan $\mathbf{y} \in \Pi^*(b_{t+1}, g_{t+1})$ which contains these variables implies a unique value for any problematic elements of $\{b_{t+s}, c_{t+s}, p_{t+s}\}_{s=1}^{N}$.*

The first half of the assumption allows us to compute today's utility, $q(y_t)$, if we fix $(b_t, g_t, \tau_t, ..., \tau_{t+L})$. It also ensures that the environment is such that we are able to back out a unique equilibrium from a shorter sequence of tax rates. In particular, it allows us to uniquely back out the next endogenous state, b_{t+1}, and hence move the economy forward one period.

The second half of **Assumption 3*** is the more substantial. Note that the problematic elements of $\{b_{t+s}, c_{t+s}, p_{t+s}\}_{s=1}^{N}$ are those which cause time-inconsistency problems. The second half of the assumption states that the we are able to fully control these variables today, since any competitive equilibrium plans starting tomorrow all contain the same values of these problematic elements.

Assumption 3* defines a mapping from $(b_t, g_t, \tau_t, ..., \tau_{t+L})$ to y_t, b_{t+1}, and the problematic elements of $\{b_{t+s}, c_{t+s}, p_{t+s}\}_{s=1}^{N}$. The recursive structure of the definition of competitive equilibrium ensures that this mapping must be time invariant: the relationship between $(b_t, g_t, \tau_t, ..., \tau_{t+L})$ and the variables it pins down does not depend on t.

L is a important number in our setup, as it will turn out to be the number of periods of commitment sufficient to sustain the FC solution as an equilibrium of the LTC game. This allows us to define the state vector for the LTC game when the governments have L periods of commitment: $x_t \equiv (b_t, g_t, \tau_t, ..., \tau_{t+L-1}) \in B \times G \times T^L$. This is the state that the time t government inherits: the natural states, b_t and g_t, and the pre-committed taxes, τ_t to τ_{t+L-1}.[7] Notice that any sequence of $\mathbf{y}$ defines a sequence of $\mathbf{x}$ using the definition of x_t: the elements in $(y_t, ..., y_{t+L-1})$ give us x_t for $t = 0, 1, ...$ Now define the restricted set:

$$X = \left\{x \in B \times G \times T^L : x \text{ lies on at least one path } \mathbf{y} \in \Pi^*(b_0, g_0) \text{ for some } (b_0, g_0) \in B^*\right\} \quad (17)$$

This is the set of values for the state x_t which are compatible with competitive equilibrium. The time-t government in the LTC game then chooses τ_{t+L}. Note that by the second half of **Assumption 3*** and given x_t, choosing τ_{t+L} pins down a unique x_{t+1} (and vice versa), and hence we can equivalently state the government's problem as one of choosing x_{t+1} given x_t. The time-$t+1$ government then inherits the state x_{t+1} and chooses x_{t+2}, and so on. We thus need to establish the time-t government's choices for x_{t+1} which belong to a competitive equilibrium path given the state x_t, which we do in the following lemma. Since x_t and x_{t+1} jointly define a unique value for $(b_t, g_t, \tau_t, ..., \tau_{t+L})$, they also pin down unique values for the variables discussed in **Assumption 3***.

Lemma 1. *There exists a time-invariant transition function $\Gamma : X \mapsto X$ defined s.t. x_{t+1} is on a competitive equilibrium path given x_t iff $x_{t+1} \in \Gamma(x_t)$. For all $x_t \in X$, $\Gamma(x_t)$ is non-empty.*

Proof. $\Gamma : X \mapsto X$ is defined s.t. $x_{t+1} \in \Gamma(x_t)$ iff there exists a $(y_{t+1}, ..., y_{t+N}) \in Y^N$ such that 1) $(y_{t+1}, ..., y_{t+N})$ contains the problematic elements $\{b_{t+s}, c_{t+s}, p_{t+s}\}_{s=1}^{N}$ and b_{t+1} uniquely pinned down by (x_t, x_{t+1}), and 2) $(y_{t+1}, ..., y_{t+N}) \in \Gamma^*(y_t)$. Since Γ^* and the relationship between (x_t, x_{t+1}) and the variables it pins down are both time invariant, defining Γ without time dependence is

[7] We are now omitting the (redundant) constant state 1 that allowed us to nest the NC equilibrium in Section 2. This is because we only focus on non-zero degree of commitment in this section.

possible. $\Gamma(x_t)$ is non-empty for all $x_t \in X$ because being in X implies that x_t lies on at least one competitive equilibrium plan, $\mathbf{y}$. The values $(y_{t+1}, ..., y_{t+L})$ from this plan define a value for x_{t+1} on a competitive equilibrium plan. $\square$

Time inconsistency issues arise from the feasibility of time-t choices depending on the choices of future governments. **Lemma 1** is thus important in proving that LTC overcomes these issues because it establishes that the feasibility of the time-t choice of x_{t+1} does not depend on any future choices, and only depends on the time-t state, x_t.

Finally, let $P = \{(x, y) \in X \times X : y \in \Gamma(x)\}$ be the graph of $\Gamma(x)$, and redefine the utility function as $F : P \mapsto \mathbb{R}$ such that $F(x_t, x_{t+1}) = q(y_t)$. Again, note that **Assumption 3*** allows us to back out a unique y_t given (x_t, x_{t+1}), which is what allows this reformulation. We make the following boundedness assumption on the utility function:

Assumption 4*. *The environment is such that $F(x, y)$ is bounded for all $x \in X$ and $y \in \Gamma(x)$.*

Assumption 4* can be considered a technical assumption, however with interesting implications if it fails. It will turn out to be important for guaranteeing that the FC solution is the unique equilibrium of the LTC game. We discuss why, and how to relax this assumption, in the appendix. Note that this assumption does allow unbounded utility functions, such as CRRA over consumption, as long as competitive equilibrium places a bound on the feasible levels of utility.

Under these assumptions, the LTC game with L periods of commitment can be re-expressed as:

$$v(x_t) = \sup_{x_{t+1} \in \Gamma(x_t)} \left\{ F(x_t, x_{t+1}) + \beta v(x_{t+1}) \right\}, \quad \forall x_t \in X \tag{LTC}$$

This definition of the LTC game is equivalent to the definition in Section 2.3, but the statement is simplified by the above results. In particular, we are able to express both the transition, Γ, and the utility, F, in terms only of today's state, x_t, and today's choice, x_{t+1}. This means that, stated this way, no future choice variables enter the constraints, removing the time inconsistency problem and allowing a standard recursive formulation.

3.4 Equivalence of LTC and FC

Having restated the FC and LTC games in terms of our notation, we now turn to demonstrating the equivalence between the two equilibria. Proofs for several lemmas are relegated to the appendix.

The main proposition is stated below:

Proposition 1. *Consider an L such that* **Assumptions 1* to 4*** *hold, and fix a $(b_0, g_0) \in B^*$. If, in the LTC game, either*

1. *$\{\tau_t\}_{t=0}^{L-1}$ is restricted to be optimal values from the FC game, or*

2. *the time-0 government, in addition to choosing τ_L, is also allowed to choose $\{\tau_t\}_{t=0}^{L-1}$*

then all equilibria of the LTC game generate paths $\mathbf{y} \in \mathbf{y}^{FC}(b_0, g_0)$, and achieve maximum time-0 utility $V^*(b_0, g_0)$.

The bulk of the proof rests on establishing the equivalence between the recursive LTC game and the FC problem, which is done in two steps. We first re-express the FC problem as a "Modified Problem" (MP) where the government chooses paths for x_t instead of y_t. We then show that MP has a recursive formulation equivalent to the LTC game.

To set up the Modified Problem, we first need to define plans in terms of our new state variable: $\mathbf{x} \equiv \{x_t\}_{t=0}^{\infty} \in X^{\infty}$. This allows us to define the set of competitive equilibrium plans, $\mathbf{x}$, starting from a given $x_0 \in X$:

$$\Pi(x_0) = \{\{x_t\}_{t=0}^{\infty} \in X^{\infty} : x_{t+1} \in \Gamma(x_t), \;\; t = 0, 1, ...\}$$

This allows us to redefine the path utilities using $u : \Pi(x_0) \mapsto \bar{\mathbb{R}}$ by $u(\mathbf{x}) = \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1})$. We can then define MP as:

$$V(x_0) = \sup_{\mathbf{x} \in \Pi(x_0)} u(\mathbf{x}) \tag{MP}$$

This problem is to maximize utility given an initial state x_0, by choosing a plan $\mathbf{x}$. Notice that there are thus two differences from the original FC problem. Firstly, the state for the FC problem is just (b_0, g_0), but the state here is $x_0 = (b_0, g_0, \tau_0, ..., \tau_{L-1})$, so the MP problem maximizes utility subject to the initial taxes being taken as given. Secondly, the MP chooses plans for $\mathbf{x}$, whereas the FC problem chooses plans for $\mathbf{y}$. However, the following lemma demonstrates that under our assumptions, choosing plans for $\mathbf{x}$ or $\mathbf{y}$ is equivalent:

Lemma 2. *For all $(b_0, g_0) \in B^*$, each $\mathbf{y} \in \Pi^*(b_0, g_0)$ implies a unique $\mathbf{x} \in \Pi(x_0)$ for some $x_0 \in X$. Conversely, for all $x_0 \in X$, each $\mathbf{x} \in \Pi(x_0)$ implies a unique $\mathbf{y} \in \Pi^*(b_0, g_0)$ for some $(b_0, g_0) \in B^*$.*

Proof in Appendix A. Thus the FC and MP problems are both equivalent, with the time-0 government choosing complete paths for policies, except that the MP problem is restricted in that it cannot choose the initial policies $(\tau_0, ..., \tau_{L-1})$.

Lemma 3. *For any $x_0 \in X$:*

1. *The set of competitive equilibrium paths, $\Pi(x_0)$, is non empty*
2. *For all $\mathbf{x} \in \Pi(x_0)$, $\lim_{n\to\infty} \sum_{t=0}^{n} \beta^t F(x_t, x_{t+1})$ exists, although it may be plus or minus infinity.*

Proof in Appendix A. This lemma ensures that the supremum in the MP is well defined. We can now finish illustrating the tight link between FC and MP. For any $(b_0, g_0) \in B^*$, define the set $Q(b_0, g_0)$ as follows:

$$Q(b_0, g_0) \equiv \left\{(\tau_0, ..., \tau_{L-1}) \in T^L : (b_0, g_0, \tau_0, ..., \tau_{L-1}) \in X\right\}$$

This is the set of L initial government choices that lie on competitive equilibrium paths for a given initial state. The following lemma establishes that as long as these initial choices are chosen correctly, the same paths solve MP and FC, and lead to the same maximum value.

Lemma 4. *For any $(b_0, g_0) \in B^*$, $\sup_{(\tau_0,...,\tau_{L-1})\in Q(b_0,g_0)} V((b_0, g_0, \tau_0, ..., \tau_{L-1})) = V^*(b_0, g_0)$, and the implied plans which achieve the supremum all lie in $\mathbf{y}^{FC}(b_0, g_0)$.*

Proof. **Lemma 2** established the equivalence of $\mathbf{x}$ and $\mathbf{y}$ paths. Suppose that for some $(b_0, g_0) \in B^*$ we had $V^*(b_0, g_0) > \sup_{(\tau_0,...,\tau_{L-1})\in Q(b_0,g_0)} V((b_0, g_0, \tau_0, ..., \tau_{L-1}))$, where the left hand side supremum is achieved with the path $\mathbf{y}$ and the right hand side by the path $\mathbf{x}'$. The path $\mathbf{x}$ defined by $\mathbf{y}$ is a competitive equilibrium and delivers higher utility, contradicting that the right hand side is the supremum. The equivalent argument applies in the opposite direction, leaving equality as the only possibility. □

Note that this lemma also trivially implies that $V((b_0, g_0, \tau_0, ..., \tau_{L-1})) = V^*(b_0, g_0)$ for any $(\tau_0, ..., \tau_{L-1})$ from an optimal FC plan, and that the generated plans all lie in the subset of $\mathbf{y}^{FC}(b_0, g_0)$ which contains plans containing $(\tau_0, ..., \tau_{L-1})$. Having established the link between the FC and MP problems, all that remains is to establish the link between the MP problem and the LTC game. This is done in the final lemma:

Lemma 5.

1. *The function V satisfies (LTC).*

2. *Let $\mathbf{x}^* \in \Pi(x_0)$ be a competitive equilibrium plan that attains the supremum in (MP) for initial state x_0. Then*
$$V(x_t^*) = F(x_t^*, x_{t+1}^*) + \beta V(x_{t+1}^*), \quad t = 0, 1, 2, ... \tag{P}$$

3. *If v is a solution to (LTC) and satisfies*
$$\lim_{t\to\infty} \beta^t v(x_t) = 0, \quad \forall \mathbf{x} \in \Pi(x_0), \forall x_0 \in X \tag{BC}$$
then $v = V$.

4. *Let $\mathbf{x}^* \in \Pi(x_0)$ be a competitive equilibrium plan for initial state x_0 satisfying (P) and with*
$$\lim_{t\to\infty} \beta^t V(x_t^*) \leq 0$$
Then $\mathbf{x}^*$ attains the supremum in (MP) for initial state x_0.

Proof. All statements follow from Theorems 4.2-4.5 in Stokey and Lucas (1989). We require that their Assumptions 4.1 and 4.2 hold, which we proved in **Lemma 1** and **Lemma 3**. □

This lemma is the standard statement of recursivity. Note that the original FC problem is not recursive, but the modified problem, where we instead consider picking sequences of x_t, is. We are then able to apply this to proving an equivalence with the LTC game, because the state x_t was chosen to be precisely the state in that game. The first two items of the lemma prove that MP solves the LTC problem: the MP value function $V(x)$ satisfies the LTC recursion, as do the generated optimal plans.

The second two items, combined with **Assumption 4*** can then be used to show that these are the unique function and plans which solve the LTC recursion. **Assumption 4*** states that the return function is bounded, which implies that there is a finite $\bar{F} < \infty$ for which $|F(x, y)| < \bar{F}$ for any $x, y \in X$. This implies that $|v(x)| \leq \bar{F}/(1-\beta)$ for any candidate solution to LTC, including the MP solution, V. This bound, combined with $\beta < 1$, implies that 1) the boundedness condition in point 3 of **Lemma 5** is satisfied for any candidate solution to LTC, meaning that all solutions must be V, and 2) the boundedness condition in point 4 of **Lemma 5** is satisfied for any plan, meaning that any optimal plans in the LTC game must be an optimal plan in MP.

This completes the proof of **Proposition 1**. We have thus proved that the solution to the FC problem can be supported as the unique equilibrium of the LTC game, as long as the initial policies, $\{\tau_t\}_{t=0}^{L-1}$, are either arbitrary forced to be the optimal FC choices, or if the time-0 government is also allowed to optimally choose these policies.

4 Optimal fiscal policy

In this section we show how our equivalence result applies to several models of optimal fiscal policies that have been studied in the literature. We start from the optimal timing of labor taxes bond-only economies such as Lucas and Stokey (1983) and Faraglia et al. (2014) and then move on to study optimal capital taxation as in Klein and Ríos-Rull (2003), Klein et al. (2008) and Debortoli and Nunes (2009). In all these models, we illustrate that **Assumption 3*** holds and hence FC outcomes can be supported as symmetric Markov equilibria in the LTC game. We restrict the set of exogenous states for simplicity, and it is worth noting that all our results go through if extra exogenous states satisfying the Markov property are added. To avoid notational clashes, in the following sections we use upright text to denote variables in the specific models.

4.1 Economies without capital

We start our analysis by considering a deterministic economy without capital and generalize the example presented in Section 2, by allowing for time-varying government spending and long-maturity bonds. An exogenous stream of expenditure $\{\mathrm{g}_t\}_{t=0}^{\infty}$, which for simplicity we assume to be purely wasteful, needs to be financed with labor income taxes $\{\tau_t^l\}_{t=0}^{\infty}$ and debt $\{\mathrm{b}_{t+N}\}_{t=0}^{\infty}$ with generic maturity $N \in [1, \infty)$. This model encompasses a deterministic version of the model studied by Lucas and Stokey (1983) when $N = 1$ as well as models with long-maturity bonds as in Faraglia et al (2014) when $N > 1$.[8] The choice of the label N for maturity is not accidental, as this will indeed turn out

[8]The analysis can easily be extended to multiple maturities, but we restrict ourselves to one bond for expositional simplicity.

to coincide with our definition of N from our general formulation: variables N periods ahead appear in the constraints.

A representative agent has preferences defined over sequences of private consumption $\{\mathrm{c}_t\}_{t=0}^{\infty}$ and labor effort $\{\mathrm{l}_t\}_{t=0}^{\infty}$:

$$\sum_{t=0}^{\infty} \beta^t u(\mathrm{c}_t, \mathrm{l}_t) \tag{18}$$

with standard assumptions $u_{\mathrm{c}} > 0$, $u_{\mathrm{cc}} < 0$, $u_{\mathrm{l}} < 0$, $u_{\mathrm{ll}} < 0$. Their budget constraint is given by

$$\mathrm{c}_t + \mathrm{q}_t \mathrm{b}_{t+N} = \mathrm{w}_t \mathrm{l}_t \left(1 - \tau_t^l\right) + \mathrm{b}_t \tag{19}$$

where q_t is the price of a bond issued at t that repays one unit of consumption at $t + N$.[9] Output equals labor effort, hence the resource constraint reads

$$\mathrm{c}_t + \mathrm{g}_t = \mathrm{l}_t \tag{20}$$

and firms' profit maximization implies a unit wage: $\mathrm{w}_t = 1$. The government's budget is implicitly defined by the agent's budget constraint and the resource constraint.

The agent's first order conditions with respect to consumption, labor effort and bonds, together with the resource constraint, can be summarized by an intratemporal optimality condition and a Euler equation:

$$-\frac{u_{\mathrm{l}}(\mathrm{c}_t, \mathrm{c}_t + \mathrm{g}_t)}{u_{\mathrm{c}}(\mathrm{c}_t, \mathrm{c}_t + \mathrm{g}_t)} = 1 - \tau_t^l \tag{21}$$

$$\mathrm{q}_t u_{\mathrm{c}}(\mathrm{c}_t, \mathrm{c}_t + \mathrm{g}_t) = \beta u_{\mathrm{c}}(\mathrm{c}_{t+N}, \mathrm{c}_{t+N} + \mathrm{g}_{t+N}) \tag{22}$$

This completes the description of the model, allowing us to map it into the general framework of Section 3. In the general notation we have $b_t = (\mathrm{b}_t, ..., \mathrm{b}_{t+N-1})$, $g_t = \mathrm{g}_t$, $c_t = (\mathrm{c}_t, \mathrm{l}_t)$, $p_t = \mathrm{q}_t$, and $\tau_t = \tau_t^l$. Note that we have implicitly solved out for, and ignored, the real wage. The transition Γ^* is defined by the equations (19), (20), (21), and (22). Note that, according to our definition, c_{t+N} is thus the only problematic variable in the time-t constraints, since it appears in (22). We refer to the previous literature for a derivation of FC optimal policy in this model. Here we limit ourselves to a brief discussion of the difference between FC and NC equilibria in this context.

[9] In the case of $N > 1$ we are considering a long-bond economy with "no buy-back": governments cannot repurchase bonds before maturity. This is for expositional purposes, and our results also apply in the case of buy-back.

Equation (22) highlights the source of time-inconsistency of the FC policy in this model. When $t = 0$, the FC government has an incentive to use the initial allocation to decrease the value of outstanding initial debt b_0, and hence reduce the distortions required to finance expenditure. If the government starts with a stock of debt ($\mathrm{b}_{-1} < 0$), the government will enact a tax cut in period 0 to achieve this, relative to later periods. In the special case where government spending is constant, taxes and debt are then constant from period 1 onwards at a level which depends on the initial level of debt. Consistently with the initial incentive to cut taxes, any FC promises about allocations c_t with $t > 0$ would be reneged if the government was allowed to reoptimize at t: in order to decrease the value of outstanding debt, b_t, the government would like to offer another tax cut, although she promised not to do this initially.

The properties of the solution under NC depend crucially on the assumptions about government spending. For $N = 1$, and if government spending is assumed exogenous and constant, Krusell et al. (2004) prove the existence of a "step function" equilibrium which supports multiple steady states for government debt.[10] Debortoli and Nunes (2013) study a version of this economy with $N = 1$ which also features endogenous government spending valued in utility. As discussed above, the FC equilibria features a long run level of debt which depends on the initial level of debt: initial conditions matter. The NC government, on the other hand, has a debt policy which converges to a steady state level b^* regardless of initial conditions. Overall, there are thus large differences in debt-dynamics between the solutions to these models under FC and NC.

In the LTC game with an arbitrary L periods of commitment the government inherits the following states at time t. Firstly there are the natural states, g_t and $(\mathrm{b}_t, ..., \mathrm{b}_{t+N-1})$. Then there are the pre-committed taxes, $(\tau_t^l, ..., \tau_{t+L-1}^l)$. The government then chooses τ_{t+L}^l. Note the similarity to the NC problem, where the government only inherits the natural states, and chooses τ_t^l.

We now show that the FC equilibrium can be supported in the LTC game with $L = N$ periods of commitment. In the Lucas and Stokey economy, which has $N = 1$, this means that just one period of commitment is sufficient to recover FC. To prove this, we need to show that our key assumption on the mapping between sequences of taxes and sequences of allocations, **Assumption 3***, holds

[10] In principle, it is also possible for a smooth equilibrium to exist, although it is challenging to find numerically, making its characterisation hard.

in this model. In other words, we need to show that given the natural states $(\mathrm{g}_t, \mathrm{b}_t, ..., \mathrm{b}_{t+N-1})$, if we fix $(\tau_t^l, ..., \tau_{t+N}^l)$, then (i) we pin down all of $y_t = (\mathrm{b}_t, ..., \mathrm{b}_{t+N-1}, \mathrm{g}_t, \mathrm{c}_t, \mathrm{l}_t, \mathrm{q}_t, \tau_t^l)$, of which we only need to check $(\mathrm{c}_t, \mathrm{l}_t, \mathrm{q}_t)$, and b_{t+N}, and (ii) the problematic variable c_{t+N} is fixed given $(\mathrm{g}_{t+1}, \mathrm{b}_{t+1}, ..., \mathrm{b}_{t+N}, \tau_{t+1}^l, ..., \tau_{t+N}^l)$. To see that this is the case, consider equation (21). Given a tax rate τ_t^l (and an exogenous g_t), this is one equation pinning down one unknown, namely c_t.[11] Hours l_t can then be easily recovered from the resource constraint. Hence a sequence $\left\{\tau_t^l, ..., \tau_{t+N}^l\right\}$ pins down a sequence of consumption and hours $\{(\mathrm{c}_{t+j}, \mathrm{l}_{t+j})\}_{j=0}^{t+N}$. The bond price, q_t can then be recovered from (22) and bonds b_{t+N} from the budget constraint (19), given an outstanding level of debt b_t.

The boundedness restriction in **Assumption 4*** is also satisfied as long as there are no competitive equilibria leading to (negative) infinite value, which is a relatively weak restriction. Depending on the utility function, this can typically be guaranteed by assuming that the government must remain to the left of the peak of the Laffer curve. Having shown that these two assumptions hold, we have proved that the FC solution can be supported by the LTC game.

An important result of this analysis is that the degree of commitment necessary to achieve FC outcomes depends crucially on the the maturity of debt. The longer this maturity, the higher the number of periods of commitment required. Hence for a given planning horizon for fiscal policy, economies with longer debt maturity appear to be more prone to the welfare costs of imperfect commitment.

Notice that in a one-period bond economy, $L = 1$ is sufficient to recover FC. It is simple to prove that this is also the case in a model with endogenous government spending, allowing comparison with the Loose-Commitment results in Debortoli and Nunes (2013). The differences are surprising. Debortoli and Nunes (2013) show that under Loose Commitment, debt will always optimally converge to a steady state value, even if commitment only lasts one average for one period. In contrast, under LTC steady state debt depends on initial debt as it does in FC. This highlights that, once we depart from FC or NC, how we do so can have surprising implications for our models. In this case, one year of LTC, or Loose Commitment which lasts on average one year do not lead to similar equilibria.

[11] Since this equation is nonlinear, an additional, weak regularity assumption is required in order to ensure that the solution is unique, so that a given tax rate pins down a unique allocation. This amounts to ensuring that the left hand side of (21) is either strictly increasing or decreasing in c_t for $\mathrm{c}_t \geq 0$, a condition that is satisfied for standard utility functions, such as separable isoelastic utility in c_t and l_t.

4.2 Capital and labor taxes

We now consider optimal fiscal policy in economies with capital. We study a general model which nests the economies analyzed by Klein and Ríos-Rull (2003), Klein et al. (2008) and Debortoli and Nunes (2010). Specifically, we allow for government consumption to be valued by households and chosen by the government, making the policy instruments labor taxes, τ_t^l, capital taxes, τ_t^k and government spending, g_t. We also let the capital utilization rate, v_t, be endogenous. We begin by discussing a version of the model where the government must balance its budget every period, and then consider a model where the government can borrow and lend from the household.

4.2.1 Balanced budget

The model described in this section is that of Debortoli and Nunes (2010). Klein and Ríos-Rull (2003) and Klein et al. (2008)'s models can be recovered by removing endogenous capital utilization. Household preferences are represented by the utility function:

$$\sum_{t=0}^{\infty} \beta^t u(\mathrm{c}_t, \mathrm{g}_t, \mathrm{l}_t) \tag{23}$$

with $u_{\mathrm{c}} > 0$, $u_{\mathrm{cc}} < 0$, $u_{\mathrm{g}} > 0$, $u_{\mathrm{gg}} < 0$, $u_{\mathrm{l}} < 0$, $u_{\mathrm{ll}} < 0$. Output is produced using a Cobb-Douglas technology that combines capital (with an endogenous utilization rate v_t) and labor:

$$\mathrm{y}_t = (\mathrm{v}_t \mathrm{k}_t)^{\alpha}\, \mathrm{l}^{1-\alpha}. \tag{24}$$

Following Greenwood et al. (2000), capital depreciates at rate $\delta(\mathrm{v}_t)$ with $\delta' > 0$ and $\delta'' > 0$. Depreciation is increasing in the rate of utilization, giving a well defined trade-off which determines the optimal level of utilization. Endogenous utilization makes the capital taxation problem more tractable, because even at time 0 the government faces the cost that higher capital taxes will lower utilization. The resource constraint of the economy reads:

$$\mathrm{c}_t + \mathrm{k}_{t+1} - (1 - \delta(\mathrm{v}_t))\, \mathrm{k}_t + \mathrm{g}_t = (\mathrm{v}_t \mathrm{k}_t)^{\alpha}\, \mathrm{l}_t^{1-\alpha}. \tag{25}$$

Households consume, supply labor, and invest in capital, renting it to firms. Combining the households' and firms' optimality conditions leads to the following conditions:

$$-\frac{u_{\mathrm{l}}(\mathrm{c}_t, \mathrm{g}_t, \mathrm{l}_t)}{u_{\mathrm{c}}(\mathrm{c}_t, \mathrm{g}_t, \mathrm{l}_t)} = \left(1 - \tau_t^l\right)(1 - \alpha)\, (\mathrm{v}_t \mathrm{k}_t)^{\alpha}\, \mathrm{l}_t^{-\alpha} \tag{26}$$

$$\left(1-\tau_t^k\right)\alpha \mathrm{v}_t^{\alpha}\mathrm{k}_t^{\alpha-1}\mathrm{l}_t^{1-\alpha}=\delta'(\mathrm{v}_t) \tag{27}$$

$$u_{\mathrm{c}}(\mathrm{c}_t,\mathrm{g}_t,\mathrm{l}_t)=\beta u_{\mathrm{c}}(\mathrm{c}_{t+1},\mathrm{g}_{t+1},\mathrm{l}_{t+1})\left[1+\alpha\left(\mathrm{v}_{t+1}\mathrm{k}_{t+1}\right)^{\alpha-1}\mathrm{l}_{t+1}^{1-\alpha}\left(1-\tau_{t+1}^k\right)-\delta(\mathrm{v}_{t+1})\right] \tag{28}$$

As in Klein and Ríos-Rull (2003), Klein et al. (2008) and Debortoli and Nunes (2010), we assume that the government budget constraint has to be balanced in every period. This imposes the restriction that $\tau_t^l \mathrm{w}_t \mathrm{l}_t + \tau_t^k \mathrm{r}_t \mathrm{v}_t \mathrm{k}_t = \mathrm{g}_t$, where r_t is the rental rate of capital. Combined with the firm's first order conditions for capital and labor, this gives the condition:

$$\left[\alpha\tau_t^k+(1-\alpha)\,\tau_t^l\right](\mathrm{v}_t\mathrm{k}_t)^{\alpha}\,\mathrm{l}_t^{1-\alpha}=\mathrm{g}_t. \tag{29}$$

That the government must balance its budget every period turns out to be an important assumption in order to obtain equivalence of LTC and FC in this economy, as will be made clear. This completes the statement of the model. In the general notation of Section 3 we have $b_t = \mathrm{k}_t$, $c_t = (\mathrm{c}_t, \mathrm{l}_t, \mathrm{v}_t)$, and $\tau_t = (\tau_t^k, \tau_t^l, \mathrm{g}_t)$. There are no exogenous states (g_t) and we have solved out for all prices (p_t). The transition correspondence, Γ^*, is defined by (25), (26), (27), (28), and (29).

The source of time-inconsistency in this model is the incentive of the government to promise low capital taxes in order to foster capital accumulation and then to tax capital ex post once it has been installed. More formally, the government is constrained by the Euler equation, (28), which contains the problematic variables c_{t+1}, v_{t+1} and l_{t+1}, and next period's capital tax, τ_{t+1}^k. In the notation of our general formulation we have $N = 1$, with variables one period ahead appearing in the constraints.

Once again we refer to these previous papers for the derivation of the optimal policy under FC and NC. We limit ourselves to proving that FC outcomes can be supported as equilibrium of the LTC game. It will turn out that the government can sustain the FC solution with $L = 1$ periods of commitment in this model. In the LTC game with $L = 1$ periods of commitment the government inherits the following states at time t. Firstly there is the natural state, k_t. Then there are the pre-committed policies, $(\tau_t^k, \tau_t^l, \mathrm{g}_t)$. The government then chooses $(\tau_{t+1}^k, \tau_{t+1}^l, \mathrm{g}_{t+1})$.

We need to show that **Assumption 3***, holds in this model for $L = 1$. In other words, we need to show that given the natural state k_t, if we fix $(\tau_t^k, \tau_t^l, \tau_{t+1}^k, \tau_{t+1}^l, \mathrm{g}_t, \mathrm{g}_{t+1})$, then (i) we pin down all of $y_t = (\mathrm{k}_t, \mathrm{c}_t, \mathrm{l}_t, \mathrm{v}_t, \tau_t^k, \tau_t^l, \mathrm{g}_t)$, of which we only need to check $(\mathrm{c}_t, \mathrm{l}_t, \mathrm{v}_t)$, and k_{t+1}, and (ii) the problematic variables $(\mathrm{c}_{t+1}, \mathrm{l}_{t+1}, \mathrm{v}_{t+1})$ are fixed given $(\mathrm{k}_{t+1}, \tau_{t+1}^k, \tau_{t+1}^l, \mathrm{g}_{t+1})$.

To see that this is the case, notice that given the government's state, $\left(\mathrm{k}_t, \tau_t^k, \tau_t^l, \mathrm{g}_t\right)$, equations (25), (26), (27) and (29) form a system of four (non-linear) equations in four unknowns, namely (c_t, l_t, v_t, k_{t+1}), uniquely pinning them down and satisfying the first two requirements.[12] By the same logic, by choosing $\left(\tau_{t+1}^k, \tau_{t+1}^l, \mathrm{g}_{t+1}\right)$, the government thus pins down $(\mathrm{c}_{t+1}, \mathrm{l}_{t+1}, \mathrm{v}_{t+1})$ which are the problematic variables in equation (28).

The boundedness restriction in **Assumption 4*** is satisfied under similar, weak conditions as in the last section. Since these two assumptions hold, we have shown the equivalence of the LTC and FC solutions in this model.

It is worth noting the importance of the balanced budget assumption for this result. The balanced budget equation, (29), is one of the four equations we used to pin down equilibrium with only a finite number of periods of taxes. Intuitively, to sustain commitment the time-t government would like to force the government at $t+1$ to pick certain values of c_{t+1} and the other problematic variables. Committing to $\left(\tau_{t+1}^k, \tau_{t+1}^l, \mathrm{g}_{t+1}\right)$ is not enough to guarantee this, since future governments can always influence c_{t+1} by changing future taxes, and hence investment.

The balanced budget assumption stops future governments from being able to do this, since this would influence government revenue, potentially unbalancing the budget. It turns out that the government at $t+1$ must set the value of c_{t+1} chosen at time t in order to balance her budget, which constrains her feasible choices of $\left(\tau_{t+2}^k, \tau_{t+2}^l, \mathrm{g}_{t+2}\right)$. Thus without assuming a balanced budget our theorem would not hold, and LTC would not support the FC solution.

Debortoli and Nunes (2010) solve the balanced-budget model with Loose Commitment, and find differences from the FC solution. In particular, they find that under Loose Commitment the capital tax rate does not converge to zero, as it does under FC. Since LTC supports the FC solution, LTC and Loose Commitment again deliver different results in this framework.

Finally, in the appendix we provide an extension where the budget must be balanced across a fixed number of periods. In particular, we suppose that the government can issue one-period bonds, denoted b_t, but that every M periods the government must set $b_{t+1} = 0$ and not issue any bonds. This captures the idea of medium run fiscal constraints placed upon the government, such as yearly

[12] The solution is unique under the same regularity assumption described in footnote 11. To see this, take the ratio of (27) and (29) to form a single equation in v_t. Since $\delta(\mathrm{v}_t)$ is strictly increasing, this pins down a unique v_t. The remaining variables can be solved recursively: (29) gives a unique l_t, (26) a unique c_t and (25) a unique k_{t+1}.

balanced-budget restrictions in a quarterly model. In this case LTC can support FC with $L = M$ periods of commitment.

This extension is interesting for two reasons. Firstly, it illustrates the key role that the length of time over which the budget must be balanced plays in achieving FC in this model. This leads to potentially important policy implications, especially for countries that are implementing multi-annual budget plans, such as several European countries. Secondly, it provides an example where $L > N$, and the number of periods of commitment required to support FC exceeds the number of periods ahead that choice variables appear in the constraints. In the previous two examples we had $L = N$ in both cases.

4.2.2 Unbalanced budget

In this section we consider an extension of the above model where the government is able to borrow and lend from the household using a one-period bond. In this case we introduce the variable b_t denoting government borrowing, and replace the balanced budget equation, (29), with:

$$\mathrm{q}_t \mathrm{b}_{t+1} + \left[\alpha\tau_t^k + (1-\alpha)\,\tau_t^l\right](\mathrm{v}_t\mathrm{k}_t)^\alpha\,\mathrm{l}_t^{1-\alpha} = \mathrm{g}_t + \mathrm{b}_t \tag{30}$$

$$\mathrm{q}_t = \beta\frac{u_\mathrm{c}(\mathrm{c}_{t+1},\mathrm{g}_{t+1},\mathrm{l}_{t+1})}{u_\mathrm{c}(\mathrm{c}_t,\mathrm{g}_t,\mathrm{l}_t)} \tag{31}$$

Where q_t is the price of the bond, and (31) is the household's Euler equation pricing the bond. In the general notation of Section 3 we have $b_t = (\mathrm{k}_t, \mathrm{b}_t)$, $c_t = (\mathrm{c}_t, \mathrm{l}_t, \mathrm{v}_t)$, $p_t = \mathrm{q}_t$, and $\tau_t = (\tau_t^k, \tau_t^l, \mathrm{g}_t)$. There are no exogenous states (g_t). The transition correspondence, Γ^*, is defined by (25), (26), (27), (28), (30), and (31).

In general, our theorem does not hold in this model. This is because, as discussed in the previous section, it is not possible to pin down the problematic variables with a finite sequence of policy instruments. For example, investment, and hence consumption, will depend on the entire infinite sequence of future taxes, making it impossible to pin down the problematic variable c_{t+1} with only a finite sequence of policies.

However, it is possible to prove that our theorem holds in the special case of linear utility from consumption. In particular, consider a version of the model with constant government spending, g, which is purely wasteful, a utility function $u(\mathrm{c}_t, \mathrm{g}_t, \mathrm{l}_t) = \mathrm{c}_t - v(\mathrm{l}_t)$, and no capital utilisation margin, giving $\mathrm{v}_t = 1$. The function $v(\mathrm{l}_t)$ is assumed to satisfy the usual conditions $v'(\mathrm{l}_t) > 0$ and $v''(\mathrm{l}_t) > 0$. The equations of the model are now:

$$\mathrm{c}_t + \mathrm{k}_{t+1} - (1-\delta)\,\mathrm{k}_t + \mathrm{g}_t = \mathrm{k}_t^{\alpha}\mathrm{l}_t^{1-\alpha}. \tag{32}$$

$$v'(\mathrm{l}_t) = \left(1-\tau_t^l\right)(1-\alpha)\mathrm{k}_t^{\alpha}\mathrm{l}_t^{-\alpha} \tag{33}$$

$$\beta\mathrm{b}_{t+1} + \left[\alpha\tau_t^k + (1-\alpha)\,\tau_t^l\right]\mathrm{k}_t^{\alpha}\mathrm{l}_t^{1-\alpha} = \mathrm{g} + \mathrm{b}_t \tag{34}$$

$$1 = \beta\left[\alpha\mathrm{k}_{t+1}^{\alpha-1}\mathrm{l}_{t+1}^{1-\alpha}\left(1-\tau_{t+1}^k\right) + 1 - \delta\right] \tag{35}$$

Since there is no utilization margin we also add the constraint that capital taxes cannot exceed an upper bound $\bar{\tau}^k$. Even with linear utility from consumption, there is still a meaningful distinction between the FC and NC solutions to this model. A government with FC will choose to have the period-0 capital tax at the maximum level, and then set capital taxes from period 1 onwards to zero.[13] Labor taxes will be constant from period 1 onwards. A government with NC, on the other hand, will have the temptation to tax capital once it is installed, and is going to set capital taxes to the maximum level for a long time, potentially forever.

To see that LTC can support FC in this special case, note that the only problematic variable is now l_{t+1}. In the notation of our general formulation we have $N = 1$, with variables one period ahead appearing in the constraints. We need to show that **Assumption 3***, holds in this model for $L = 1$. In other words, we need to show that given the natural state $(\mathrm{k}_t, \mathrm{b}_t)$, if we fix $(\tau_t^k, \tau_t^l, \tau_{t+1}^k, \tau_{t+1}^l)$, then 1) we pin down all of $y_t = (\mathrm{k}_t, \mathrm{c}_t, \mathrm{l}_t, \tau_t^k, \tau_t^l)$, of which we only need to check $(\mathrm{c}_t, \mathrm{l}_t)$, and k_{t+1}, and 2) the problematic variable l_{t+1} is fixed given $(\mathrm{k}_{t+1}, \tau_{t+1}^k, \tau_{t+1}^l)$.

To see that this is the case note that, given k_t and τ_t^l, (33) pins down a unique l_t, defining a function $\mathrm{l}_t = l(\mathrm{k}_t, \tau_t^l)$. Combined with (35) and the time-t government's choice of $(\tau_{t+1}^k, \tau_{t+1}^l)$ this uniquely pins down $(\mathrm{k}_{t+1}, \mathrm{l}_{t+1})$. Finally, the resource constraint, (32) uniquely pins down c_t.

[13]Straub and Werning (2015) show that in the presence of an upper bound on capital taxation, it is possible that the Chamley-Judd result that capital taxes converge to zero does not hold for sufficiently high initial government debt, and the upper bound optimally binds asymptotically. We assume that initial government debt is low enough to avoid this case.

5 The role of initial conditions

Our equivalence result proved in Section 3 relies on initial policy instruments being consistent with the FC plan. In this section we explore the consequences of letting governments with LTC inherit arbitrary policies for the properties of optimal policy with LTC. In two specific models, we argue that the economy converges in a single period to a FC equilibrium consistent with different initial conditions for the endogenous state variables. In this sense, LTC sustains an allocation that shares the same qualitative properties as the FC solution.

5.1 Arbitrary initial conditions in the labor tax model

In this subsection, we study the effects of starting from an arbitrary initial condition for policy in the deterministic version of Lucas and Stokey (1983) with LTC. We have established that starting from initial conditions given by the FC policy sequence, LTC sustains FC outcomes with a length of commitment given by the longest outstanding debt maturity. We now ask the question of what happens if a government with LTC game inherits an arbitrary initial policy, potentially different from the one implied by the FC policy path. We show that the economy converges in one period to another FC equilibrium, consistent with a different level of initial debt. We also provide simple formulas to evaluate the welfare cost of starting from a "wrong" initial policy.

For simplicity of exposition, consider the version of the model with one-period debt and assume that government expenditure is constant.[14] Using the intratemporal optimality condition at time 0, we can obtain hours as an implicit function h of the tax rate from

$$\frac{v'(h(\tau_0^l))}{u'(h(\tau_0^l) - \mathrm{g})} = 1 - \tau_0^l \tag{36}$$

Using this function, the government budget constraint in period 0 can then be expressed as

$$u'(h(\tau_0) - \mathrm{g})\left(\mathrm{b}_0 + \mathrm{g} - \tau_0 h(\tau_0)\right) = \beta u'(\mathrm{c}_1)\mathrm{b}_1 \tag{37}$$

Let $a_0 \equiv u'(h(\tau_0^l) - \mathrm{g})\left(\mathrm{b}_0 + \mathrm{g} - \tau_0 h(\tau_0^l)\right)$ and note that this variable is a function only of the initial debt and initial tax: $a_0 = a(\mathrm{b}_0, \tau_0^l)$. The economic interpretation of this variable is the (marginal utility) value of the resources that the time-0 government needs to raise on the bond market.

[14] This assumption can be easily relaxed without affecting the main insight. However, perfect tax smoothing arises only under a stronger assumption on preferences, such as CRRA in both consumption and labor.

Let $s(\tau_t^l) \equiv u'(h(\tau_t^l) - \mathrm{g})\left(\mathrm{g} - \tau_1 h(\tau_t^l)\right)$. By adding and subtracting $s(\tau_1^l)$ on the right-hand side of (37), we get

$$a_0 = \beta\left(a_1 + s(\tau_1^l)\right). \tag{38}$$

Note that that the problem of the government at $t = 0$ is affected by (b_0, τ_0^l) only through their effect on a_0. This is because this government cannot affect hours worked at $t = 0$. Hence the government's optimization problem can be formulated in the following recursive form.[15]

$$W(a) = \max_{(a',\tau')} \beta\left[u(h(\tau') - \mathrm{g}) - v(h(\tau')) + W(a')\right] \tag{39}$$

subject to the transition $a' = \beta^{-1}a - s(\tau')$. Note that this recursive form ignores contemporaneous utility, which is an any case fixed from the government's point of view.

The FC policy for this model is fully characterized by two functions $\tau_0^{FC}(\mathrm{b}_0)$ and $\tau_1^{FC}(\mathrm{b}_0)$ as $\tau_t^l = \tau_1^{FC}(\mathrm{b}_0)$ for all $t \geq 1$. The government chooses a perfectly smooth tax from $t = 1$ onwards and uses the tax rate at $t = 0$ to affect the utility value of initial debt by affecting the initial allocation in order to decrease the amount of distortions needed to finance expenditure and service the debt. The allocation is also constant from $t = 1$ onwards.

In order for debt not to explode with a constant tax rate and constant hours and consumption, it has to be the case that a_t is also constant. Hence, we can get the optimal tax rate from period 1 onwards from the transition equation for a in steady-state:

$$\tau^*(a_0) = s^{-1}\left(\frac{1-\beta}{\beta}a_0\right) \tag{40}$$

and the value function satisfies

$$W(a_0) = \frac{\beta}{1-\beta}\left[u(h(\tau^*(a_0)) - g) - v(h(\tau^*(a_0))\right] \tag{41}$$

Total welfare starting from arbitrary initial conditions (b_0, τ_0^l) is then given by

$$V(\mathrm{b}_0, \tau_0^l) = u(h(\tau_0^l - \mathrm{g})) - v(h(\tau_0^l)) + W(a_0). \tag{42}$$

[15] This is an alternative formulation relative to the more general recursive formulation used to prove Proposition 1. It holds in this model because the welfare-relevant component of the allocation $(\mathrm{c}_t, \mathrm{l}_t)$ is fixed from the point of view of the government dated t.

We now argue that the LTC policy and allocation starting from (b_0, τ_0^l), with $\tau_0^l \neq \tau_0^{FC}(\mathrm{b}_0)$ converges to a another FC policy (and allocation), indexed by a different debt level.

Let $\tilde{\mathrm{b}}_0$ be the solution to the following non-linear equation[16]

$$a(\tilde{\mathrm{b}}_0, \tau^{FC}(\tilde{\mathrm{b}}_0)) = a(\mathrm{b}_0, \tau_0^l). \tag{43}$$

Then, the government at time 0 solves the problem defined in (39) starting from $a_0 = a(\tilde{\mathrm{b}}_0, \tau^{FC}(\tilde{\mathrm{b}}_0))$ and the policy and allocation from $t = 1$ onwards will coincide with the ones implied by the FC equilibrium starting from $\tilde{\mathrm{b}}_0$. In particular, we will have $\tau_t^l = \tau^*(a_0) = \tau_1^{FC}(\tilde{\mathrm{b}}_0)$ for all $t \geq 1$.

In order to assess the welfare cost of starting from any initial tax, it is sufficient to compare the value attained by the FC policy starting from $t = 0$ with the value defined in (42).

5.2 Arbitrary initial conditions in the capital tax model

In the special case of the capital tax model with linear utility from consumption for which our theorem holds, we can also prove that the equilibrium of the LTC game converges to a different FC solution if a generic time-t government inherits "incorrect" policies. Recall that in this model the FC solution for $t > 0$ features zero capital taxes and constant labor taxes.

To prove this, it is convenient to combine the competitive equilibrium constraints into a single implementability constraint:

$$\frac{1}{\beta}\mathrm{k}_t + \mathrm{b}_t = \mathrm{c}_t - v_{l,t}\mathrm{l}_t + \mathrm{k}_{t+1} + \beta \mathrm{b}_{t+1} \tag{44}$$

This constraint combines all the time-t constraints except for the capital Euler equation, and also incorporates the time-$t-1$ Euler equation.[17] We can write the government's problem recursively. Note that we can use $(\mathrm{b}_t, \mathrm{k}_t, \mathrm{l}_t)$ as the only states since we can use the time-t labor condition, (33), and $t-1$ Euler, (35) to infer what time-t taxes they imply, instead of holding the taxes as additional states.

$$W(\mathrm{b}_t, \mathrm{k}_t, \mathrm{l}_t) = \max_{\mathrm{b}_{t+1}, \mathrm{k}_{t+1}, \mathrm{l}_{t+1}} \mathrm{k}_t^{\alpha}\mathrm{l}_t^{1-\alpha} + (1-\delta)\,\mathrm{k}_t - \mathrm{g} - \mathrm{k}_{t+1} - v(\mathrm{l}_t) + \beta W(\mathrm{b}_{t+1}, \mathrm{k}_{t+1}, \mathrm{l}_{t+1}) \tag{45}$$

[16] Under standard regularity conditions a unique solution exists.

[17] Thus the constraint requires that the time-t capital tax is consistent with the optimal choice of k_t made one period before. Hence the following discussion applies to any deviation of $t+1$ policy from the FC plan which is announced at time-t, not surprise deviations. The result that we converge to another FC plan also holds for surprise deviations, but the convergence simply takes one period longer.

where the maximization is subject to (44). Denote by λ_t the multiplier on (44), then the capital, bond and labor first order conditions give respectively:

$$1 - \lambda_t = \beta\left(\alpha \mathrm{k}_t^{\alpha-1}\mathrm{l}_t^{1-\alpha} + 1 - \delta\right) - \lambda_{t+1} \tag{46}$$

$$\lambda_t = \lambda_{t+1} \tag{47}$$

$$(1-\alpha)\mathrm{k}_{t+1}^{\alpha}\mathrm{l}_{t+1}^{-\alpha} - v'(\mathrm{l}_{t+1}) - \lambda_{t+1}(v''(\mathrm{l}_{t+1})\mathrm{l}_{t+1} + v'(\mathrm{l}_{t+1})) = 0 \tag{48}$$

We can combine the capital and bond first order conditions to give:

$$1 = \beta\left(\alpha \mathrm{k}_t^{\alpha-1}\mathrm{l}_t^{1-\alpha} + 1 - \delta\right) \tag{49}$$

This is just the household's capital Euler equation with zero capital taxes. Hence we have shown that regardless of the initial condition at time-t, a government with LTC will always immediately set $\tau_{t+1}^k = 0$. Labor taxes will be constant from period $t+1$ onwards because both the multiplier and capital in (48) are constant, implying constant hours, and hence constant labor taxes. The level these constant labor taxes must be set at can be solved for by iterating the government's budget constant forward and imposing transversality. These constant labor taxes must be the solution to a FC game for a different value of the initial endogenous state variables.

6 Optimal monetary policy

6.1 New Keynesian model

We now turn to investigating the implications of LTC for optimal monetary policy in the New Keynesian model. This framework has well known time-consistency issues because of the forward-looking nature of inflation. Hence there are important differences between the FC and NC solutions as discussed, for instance, by Clarida et al (1999).

Particularly when the zero lower bound on the nominal interest rate is taken into consideration, lack of commitment becomes a relevant issue, as shown by Adam and Billi (2007). The recent debate on forward guidance in monetary policy relies on the central bank being able to credibly commit to a future path for interest rates in order to facilitate the exit of the economy from a liquidity trap.

In the real world, such commitment can only be limited in time. This makes this class of models a particularly interesting environment for our theory.

It turns out that in the New Keynesian model of monetary policy **Assumption 3*** does not hold, with the consequence that the LTC game cannot support the FC equilibrium for any finite L. Thus the model serves as a useful laboratory to investigate the properties of the LTC game in cases more general than those where **Proposition 1** holds and we will use numerical methods to solve for optimal policy with LTC.

For simplicity, we work with the log-linearized version of the model, with policy minimizing a squared loss function. The log-linearized version of the model can be stated as follows:

$$\pi_t = \beta \pi_{t+1} + \kappa \mathrm{y}_t + \mathrm{e}_t \tag{50}$$

$$\mathrm{y}_t = \mathrm{y}_{t+1} - \sigma \left(\mathrm{i}_t - \pi_{t+1} \right) + \mathrm{g}_t \tag{51}$$

Where all variables refer to deviations from a zero-inflation steady state. π_t gives inflation, and y_t the output gap, with the first equation being the New Keynesian Philips Curve (NKPC), and the second the IS equation. We focus on two shocks, a cost-push shock, e_t, and a demand shock, g_t. We investigate each shock individually. For the demand shock experiments we also impose the zero lower bound (ZLB) constraint, $\mathrm{i}_t \geq -r^*$. Welfare is summarised by the loss function:

$$\sum_{t=0}^{\infty} \beta^t \left(-\frac{1}{2}\pi_t^2 - \frac{\lambda}{2}\mathrm{y}_t^2 \right) \tag{52}$$

In this framework, $N = 1$, because π_{t+1} and y_{t+1} appear in the constraints, which is the source of the time-inconsistency problem. However we require the entire future sequence $\{\mathrm{i}_s\}_{s=t}^{\infty}$ to pin down π_t and y_t. Hence we cannot find a finite L with which to support the FC solution as an equilibrium to the LTC game, and the solutions will differ. This allows us to investigate how close the solution to the LTC game can be to the FC or NC solutions in a situation where our equivalence result does not hold. The results will clearly be situation and model specific, so the results of this section should not be considered general to all situations where our theorem does not hold.

For the LTC experiments, we assume that the government has one period of commitment ($L = 1$). The model is calibrated to quarterly frequency, and we choose standard parameter values, as summarized in, for example, Bodenstein, Hebden, and Nunes (2012). They are reported in Table 1, and derivations are relegated to the appendix.

6.2 Zero Lower Bound

We investigate a negative shock to demand at time zero, $\mathrm{g}_0 < 0$, which only lasts for one period. This allows us to abstract from the non-linearities driven by an occasionally binding constraint. We will compare the paths of inflation, output gap and interest rates with FC, NC and LTC and we will compute welfare in these three scenarios given the path of the shock. A full analysis of optimal policy with occasionally binding ZLB with LTC is left for future work.

The linearized NK model does not have any endogenous states, and so the only states for the LTC government are the exogenous states and the pre-committed interest rate, i_t. This contrasts to the NC government, who has no states, and the FC government who solves an entire sequence problem. We assume that g_0 is large enough (in absolute value) to drive the economy to the ZLB at time 0.

Before stating the results, it is helpful to analyse the problem of the LTC government. From time 1 onwards there are no shocks, so the problem of the government is to choose i_{t+1} given her inherited i_t to maximize value, $V(\mathrm{i}_t)$. The linear quadratic structure (plus the assumption of a non-binding ZLB) allows us to guess and verify linear policy functions of the form $\pi_t = a_\pi i_t$ and $y_t = a_y i_t$. Plugging these into the time-t NKPC and IS equations yields:

$$y_t = \alpha_y i_t + \delta_y i_{t+1} \tag{53}$$

$$\pi_t = \alpha_\pi i_t + \delta_\pi i_{t+1} \tag{54}$$

Where $\alpha_y = -\sigma$, $\delta_y = a_y + \sigma a_\pi$, $\alpha_\pi = \alpha_y \kappa$, and $\delta_\pi = a_\pi \beta + \delta_y \kappa$. Given these policy functions, the time-t government chooses i_{t+1} understanding that it will affect π_t and y_t (via π_{t+1} and y_{t+1}) according to the above, and also affect $V(\mathrm{i}_{t+1})$. The time-t problem can be expressed as:

$$V(\mathrm{i}_t) = \max_{\mathrm{i}_{t+1}} -\frac{1}{2}\pi_t^2 - \frac{\lambda}{2}\mathrm{y}_t^2 + \beta V(\mathrm{i}_{t+1}) \tag{55}$$

Subject to (53) and (54). This leads to a linear policy rule for the interest rate: $\mathrm{i}_{t+1} = a_i \mathrm{i}_t$, and a quadratic form for the value function: $V(\mathrm{i}_t) = \frac{1}{2} a_v \mathrm{i}_t^2$ with $a_v < 0$. Two interesting things emerge from this. Firstly, the time-0 government might want to set $\mathrm{i}_1 < 0$ to help stimulate the economy, but this will come at a welfare cost from period 1 onwards, since $V(\mathrm{i}_1) < 0$ if $\mathrm{i}_1 \neq 0$.

Secondly, if she does this, the future governments will "fight" against her decision. This is because $a_i < 0$, so all future governments will set interest-rate deviations of the opposite sign to those they inherit. This is intuitive: In the absence of shocks, all future governments want to set zero interest rates. Hence, if they inherit an interest rate which is non-zero, they will set the next interest rate to the opposite sign to try and offset its effect on the behavior of the private sector, which depends on the whole infinite sequence of future interest rates.

This makes the problem of the time-0 government tricky. We allow her to choose i_0 and i_1. If the shock is severe enough that i_0 hits the ZLB, she might consider setting a positive i_1. This is for the well-known reason that doing so will increase π_1, reducing the time-0 real interest rate and stimulating demand. With only one period of commitment this becomes harder, since future governments will try to offset her decision.

Figure 1 shows the paths of output gap, inflation and nominal interest rate following the demand shock. The red line gives the responses under NC. The lack of commitment and shocks from $t = 1$ onwards means all variables revert to zero from time 1. The time-0 deviations are large, with both output and inflation falling. The black line gives the FC responses. Here the central bank is able to tame the time-0 recession by promising to keep interest rates low in future periods, leading to a boom in period 1. The increase in π_1 reduces the real interest rate at time 0, reducing the output and inflation deviations.

Finally, the blue line gives the responses under LTC. We see that LTC does not do as well as FC, but the responses are closer to FC than NC with only one period of commitment, which is a surprisingly positive result. The previously-mentioned offsetting behavior of future central banks is visible here. The time-0 bank uses her one period of commitment to set a low i_1. This leads the time-1 bank to set a positive i_2, offsetting some of the benefit of setting a low i_1 and setting off a cycle of alternating rates. Note how the time-0 bank, anticipating this behavior, thus sets an even lower i_1 than the FC bank.

In terms of welfare, LTC does surprisingly well. The NC government achieves a welfare loss 153% larger than FC, whereas the loss under OPC is only 27% larger than FC. This implies that one single quarter of commitment allows to avoid most of the welfare losses associated with going from Full to No Commitment in this experiment.

6.3 Cost push shock

In this section we analyze the response to a positive cost-push shock at time 0, which decays according to $\mathrm{e}_{t+1} = \rho_e \mathrm{e}_t$. In contrast to the last section, this allows us to analyze the effectiveness of LTC in response to persistent shocks in a situation where our main proposition does not hold. We choose $\rho_e = 0.9$.

The commitment issues in response to cost-push shocks have been extensively studied. The key insight is that the central bank might want to influence π_{t+1} to offset e_t in the NKPC, and spread the cost of a shock over several periods. This, however, requires commitment, and while the central bank is able to do this with FC, it is not with NC. LTC represents an intermediate case.

The results are demonstrated in Figure 2, which gives the impulse responses of the economy to the cost push shock under FC, NC, and LTC. As before, the choice of the initial policy, here i_0, is clearly important, and we allow the time-0 government to also choose i_0.

The responses under FC and NC are familiar. The NC central bank has no ability to affect the future, and achieves deviations of output and inflation much more severe than the FC bank. The LTC bank is able to partially dampen the deviation in the initial period compared to the NC case, but the equilibrium remains closer to NC than FC. Inflation is always lower than NC, but the output gap may be larger in later periods. The oscillations due to the offsetting behavior of each bank on interest rates are less apparent due to the exogenous dynamics of e_t, although they are still visible for y_t. Despite this, the LTC government still achieves a welfare gain relative to NC, although in this case welfare remains closer to NC than FC.

7 Conclusion

In this paper we have studied optimal policy in economies where successive one-period lived governments formulate plans for a finite horizon. We have emphasized a key condition on the mapping from policy instruments to allocations. If this condition is satisfied in a given model, then this Limited-Time Commitment game with sufficiently long but finite commitment can sustain the same outcomes that would arise if there were a single government at time 0, endowed with Full Commitment into the infinite future.

We have argued that this is indeed the case for a number of economies that have been studied in the fiscal policy literature. In this sense, we have provided a case for assuming Full Commitment in those models: even with a much lighter commitment assumption we can find the same results.

Additionally, we have shown that the length of commitment required to sustain Full Commitment allocations in these models is related to intuitive model features: in a model of labor taxes without capital, we require commitment equal to the length of the longest maturity bond. In a model of capital taxation with balanced budgets, we require commitment equal to the length of time over which the budget must be balanced.

Another result of our analysis is that once we start making assumptions that limit the government's commitment technology, the details of how we do it can be quite important: we obtain equivalence with Full Commitment in cases where the Loose Commitment approach (probabilistic commitment into the infinite future) would lead to outcomes more similar to No Commitment.

We have also presented a case of optimal monetary policy subject to the zero lower bound, where the key condition for equivalence with Full Commitment fails. We compute results numerically and show that even a small amount of commitment may be sufficient to give most of the welfare difference between Full Commitment and No Commitment.

Finally, in this paper we have worked with deterministic economies, and we leave a general characterization of optimal policy with Limited-Time Commitment in stochastic economies for future work. However, we conjecture that our main equivalence result will survive in stochastic models, provided that governments can commit to finite sequences of state-contingent policy instruments.

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A Proofs

Proof of Lemma 2. For the first statement: For all $(b_0, g_0) \in B^*$, each $\mathbf{y} \in \Pi^*(b_0, g_0)$ defines a unique path $\mathbf{x}$, including its initial element x_0. Since $\mathbf{y}$ is a competitive equilibrium, we know that $(y_{t+1}, ..., y_{t+N}) \in \Gamma^*(y_t)$, and hence, by the definition of Γ, that $x_{t+1} \in \Gamma(x_t)$, for all $t = 0, 1, ...$, and that $x_0 \in X$. Thus $\mathbf{x} \in \Pi(x_0)$. For the converse: For all $x_0 \in X$, each $\mathbf{x} \in \Pi(x_0)$ means that $x_{t+1} \in \Gamma(x_t)$ for all $t = 0, 1, ...$ The definition of Γ means that each pair (x_t, x_{t+1}) along the path defines a unique sequence $(y_t, y_{t+1}, ..., y_{t+N})$ which satisfies $(y_{t+1}, ..., y_{t+N}) \in \Gamma^*(y_t)$. This is thus a competitive equilibrium path $\mathbf{y} \in \Pi^*(b_0, g_0)$, with $(b_0, g_0) \in B^*$ taken from y_0. □

Proof of Lemma 3. Point 1 follows trivially from the non-emptiness of Γ for any $x \in X$. For point 2, note that **Lemma 2** established that any $\mathbf{x} \in \Pi(x_0)$ has a unique associated path $\mathbf{y} \in \Pi^*(b_0, g_0)$ for some $(b_0, g_0) \in B^*$. By **Assumption 1*** we know that for this $\mathbf{y}$, $\lim_{n\to\infty} \sum_{t=0}^{n} \beta^t r(b_t, a_t, g_t)$ exists, although it may be plus or minus infinity. Given that we defined F as $F(x_t, x_{t+1}) = r(a_t, b_t, g_t)$, it must also be that $\lim_{n\to\infty} \sum_{t=0}^{n} \beta^t F(x_t, x_{t+1})$ exists. □

B Additional notes

B.1 Note on role of bounded returns assumption

At this point it is worth briefly outlining the role of bounded returns (**Assumption 4***) in the preceding proof. Without this assumption it is only possible to prove that the FC solution is *an* equilibrium of the LTC game, and not the *unique* equilibrium. This is because without it we cannot guarantee that the boundedness conditions in **Lemma 5** hold, meaning that the LTC game could have other equilibria.

While this may seem like a technical assumption, it is often easy to construct a second equilibrium when it fails. This equilibrium is a "shut down" equilibrium where the government achieves utility of $-\infty$. To see when this is possible, consider a model where there exists a y s.t. $F(x, y) = -\infty$ for all $x \in X$, which we denote $\hat{y}(x)$. We can construct an equilibrium where all governments play $\hat{y}(x)$ as follows. Guess that all future governments play $\hat{y}(x)$, then the continuation value for today's government is $-\infty$, giving the optimisation problem:

$$v(x_t) = \sup_{x_{t+1} \in \Gamma(x_t)} \left\{ F(x_t, x_{t+1}) + \beta(-\infty)) \right\}, \quad \forall x_t \in X \tag{56}$$

Since the government can only achieve utility $-\infty$ regardless of its choice, it might as well also play $\hat{y}(x)$. This equilibrium is essentially one of mutual destruction: if all future governments shut down the economy, I might as well too. While this might seem problematic for our results, it is worth noting that it really must be possible for the government to achieve $-\infty$ utility for this to matter: any arbitrarily low bound on utility is enough to remove this equilibrium.

It is also worth noting that the assumption can be relaxed slightly, if we instead assume that the state variables can not grow "too fast". This follows from Section 4.3 of Stokey and Lucas (1989).

B.2 Multi-period balanced budget

In this section we consider a government who faces the constraint that she must balance her budget every M periods. There are many ways to implement this which lead to LTC supporting FC, and we illustrate one method here. In particular, we suppose that the government can issue one-period bonds, denoted b_t, which are priced according the the agent's Euler equation:

$$\mathrm{q}_t u_{\mathrm{c}_t} = \beta u_{\mathrm{c}_{t+1}} \tag{57}$$

Where $u_{\mathrm{c}_t} \equiv u_{\mathrm{c}}(\mathrm{c}_t, \mathrm{g}_t, \mathrm{l}_t)$. The government's budget is now:

$$\mathrm{q}_t \mathrm{b}_{t+1} + \left[\alpha \tau_t^k + (1-\alpha)\, \tau_t^l\right] (\mathrm{v}_t \mathrm{k}_t)^{\alpha}\, \mathrm{l}_t^{1-\alpha} = \mathrm{g}_t + \mathrm{b}_t \tag{58}$$

Substituting in the bond Euler gives:

$$\beta \frac{u_{\mathrm{c}_{t+1}}}{u_{\mathrm{c}_t}} \mathrm{b}_{t+1} + \left[\alpha \tau_t^k + (1-\alpha)\, \tau_t^l\right] (\mathrm{v}_t \mathrm{k}_t)^{\alpha}\, \mathrm{l}_t^{1-\alpha} = \mathrm{g}_t + \mathrm{b}_t \tag{59}$$

We implement the balanced budget assumption by assuming that the government cannot issue bonds once every M periods. Give each period an index $m_t \in \{1, 2, ..., M\}$ denoting its position in the cycle, with $m_t = M$ denoting the last period of the cycle (where the government can't issue debt) and $m_t = 1$ denoting the first (where there is thus no inherited debt to repay). To fix ideas, consider a model where one period is a quarter, and the government must balance her yearly budget.

This means that $M = 4$, and the government cannot issue any bonds in the fourth quarter of every year.

The rest of the model equations are as in the baseline model where the government must balance the budget every period: (25), (26), (27) and (28), to which we add the government budget, (59), and the restriction that $\mathrm{b}_{t+1} = 0$ if $m_t = M$. We now prove that we can support FC with LTC with $L = M$ periods of commitment.

To prove **Assumption 3***, we need to prove that given $(\mathrm{k}_t, \mathrm{b}_t)$ and $(\tau_t^k, \tau_t^l, \mathrm{g}_t, ..., \tau_{t+M}^k, \tau_{t+M}^l, \mathrm{g}_{t+M})$ we pin down 1) all of $y_t = (\mathrm{k}_t, \mathrm{b}_t, \mathrm{c}_t, \mathrm{l}_t, \mathrm{v}_t, \tau_t^k, \tau_t^l, \mathrm{g}_t)$, of which we only need to check $(\mathrm{c}_t, \mathrm{l}_t, \mathrm{v}_t)$, 2) next period's endogenous state, $(\mathrm{k}_{t+1}, \mathrm{b}_{t+1})$, and 3) the problematic variables $(\mathrm{c}_{t+1}, \mathrm{l}_{t+1}, \mathrm{v}_{t+1})$.

This has do be done separately for each position in the cycle, but the procedure is similar in all cases. First consider a period where $m_t = 1$, at the beginning of the cycle. We can forward (57) from t to $t + M - 1$ to yield:

$$\sum_{s=0}^{M-1} u_{\mathrm{c}_{t+s}} \left[\alpha \tau_{t+s}^k + (1-\alpha)\, \tau_{t+s}^l\right] (\mathrm{v}_{t+s}\mathrm{k}_{t+s})^{\alpha}\, \mathrm{l}_{t+s}^{1-\alpha} = \sum_{s=0}^{M-1} u_{\mathrm{c}_{t+s}} \mathrm{g}_{t+s} \tag{60}$$

Combining this with

- M resource constraints, (25), from t to $t + M - 1$
- M labor FOCs, (26), from t to $t + M - 1$
- M utilization FOCs, (27), from t to $t + M - 1$
- $M - 1$ capital Euler equations, (28), from t to $t + M - 2$

gives $4 \times M$ equations in $4 \times M$ unknowns, $\{\mathrm{c}_{t+s}, \mathrm{l}_{t+s}, \mathrm{v}_{t+s}, \mathrm{k}_{t+s+1}\}_{s=0}^{M-1}$, pinning down everything we need. Note that we have used the fixed government instruments $(\tau_t^k, \tau_t^l, \mathrm{g}_t, ..., \tau_{t+M-1}^k, \tau_{t+M-1}^l, \mathrm{g}_{t+M-1})$ in these equations. b_{t+1} is found from (60).

Now consider a period where $m_t = M$. Since $\mathrm{b}_{t+1} = 0$ in this period, the government budget, (60) gives:

$$\left[\alpha \tau_t^k + (1-\alpha)\, \tau_t^l\right] (\mathrm{v}_t \mathrm{k}_t)^{\alpha}\, \mathrm{l}_t^{1-\alpha} = \mathrm{g}_t + \mathrm{b}_t \tag{61}$$

Combined with (25), (26), (27) and (28) this gives four equations in four unknowns, $(\mathrm{c}_t, \mathrm{l}_t, \mathrm{v}_t, \mathrm{k}_{t+1})$. By the same logic as for the the period with $m_t = 1$, we can pin down $\{\mathrm{c}_{t+s}, \mathrm{l}_{t+s}, \mathrm{v}_{t+s}, \mathrm{k}_{t+s+1}\}_{s=1}^{M}$

using:

$$\sum_{s=1}^{M} u_{\mathrm{c}_{t+s}} \left[\alpha\tau_{t+s}^{k} + (1-\alpha)\,\tau_{t+s}^{l}\right] (\mathrm{v}_{t+s}\mathrm{k}_{t+s})^{\alpha}\, \mathrm{l}_{t+s}^{1-\alpha} = \sum_{s=1}^{M} u_{\mathrm{c}_{t+s}} \mathrm{g}_{t+s} \tag{62}$$

Combined with

- M resource constraints, (25), from $t+1$ to $t+M$
- M labor FOCs, (26), from $t+1$ to $t+M$
- M utilization FOCs, (27), from $t+1$ to $t+M$
- $M-1$ capital Euler equations, (28), from $t+1$ to $t+M-1$

A similar procedure can be used for periods with $m_t \in (2, 3, ..., M-1)$.

C Optimal monetary policy derivations

C.1 Cost push, FC

$$L = \sum_{t=0}^{\infty} \beta^{t} \left(-\frac{1}{2}\pi_t^2 - \frac{\lambda}{2}\mathrm{y}_t^2 + \mu_t\left(\pi_t - \beta\pi_{t+1} - \kappa\mathrm{y}_t - \mathrm{e}_t\right)\right) \tag{63}$$

$$\frac{\partial}{\partial \pi_t} \Rightarrow \pi_t = \mu_t - \mu_{t-1} \tag{64}$$

$$\frac{\partial}{\partial \mathrm{y}_t} \Rightarrow \mathrm{y}_t = -\frac{\kappa}{\lambda}\mu_t \tag{65}$$

Combine the FOCs with the NKPC to solve for policy function of the form $\pi_t = a\mu_{t-1} + b\mathrm{e}_t$. This leads to the following equation:

$$\left(1 - a\beta + \frac{\kappa^2}{\lambda}\right)(a\mu_{t-1} + b\mathrm{e}_t) = \left(a\beta - \frac{\kappa^2}{\lambda}\right)\mu_{t-1} + (1 + \beta b\rho_e)\,\mathrm{e}_t \tag{66}$$

For this equation to hold for any e_t or μ_{t-1} we require:

$$\left(1 - a\beta + \frac{\kappa^2}{\lambda}\right) a - a\beta + \frac{\kappa^2}{\lambda} = 0 \tag{67}$$

$$\left(1 - a\beta + \frac{\kappa^2}{\lambda}\right) b - 1 - ab\rho_e = 0 \tag{68}$$

This allows us to solve for a and b, giving the inflation policy function.

We then find μ_t from:

$$\mu_t = \pi_t + \mu_{t-1} = (1+a)\mu_{t-1} + b\mathrm{e}_t \tag{69}$$

Notice that for this equation to be stable we require that $-1 \leq 1 + a \leq 1$. Solving for a leads to a quadratic equation, and imposing stability actually tells us which of the two solutions to pick. We then get y_t from:

$$\mathrm{y}_t = -\frac{\kappa}{\lambda}\mu_t \tag{70}$$

C.2 Cost push, NC

Now the policy function can only depend on e_t, so we guess $\pi_t = a\mathrm{e}_t$. The NC game can be set up by replacing π_{t+1} in the PC with $a\rho_e\mathrm{e}_t$.

$$V(\mathrm{e}_t) = -\frac{1}{2}\pi_t^2 - \frac{\lambda}{2}\mathrm{y}_t^2 + \beta V(\mathrm{e}_{t+1}) + \mu_t\left(\pi_t - \kappa\mathrm{y}_t - (1 + a\rho_e\beta)\mathrm{e}_t\right) \tag{71}$$

This gives the following FOCs:

$$\frac{\partial}{\partial \pi_t} \Rightarrow \pi_t = \mu_t \tag{72}$$

$$\frac{\partial}{\partial \mathrm{y}_t} \Rightarrow \mathrm{y}_t = -\frac{\kappa}{\lambda}\mu_t \tag{73}$$

Combining these and plugging in to the PC, and then solving for a gives:

$$a = \frac{1}{1 + \frac{\kappa^2}{\lambda} - \beta\rho_e} \tag{74}$$

Then get y_t from:

$$\mathrm{y}_t = -\frac{\kappa}{\lambda}\pi_t \tag{75}$$

C.3 Cost push, LTC

To solve the LTC game with $L = 1$, we guess that the future policy functions take the form $\pi_t = a_\pi\mathrm{i}_t + b_\pi\mathrm{e}_t$ and $\mathrm{y}_t = a_y\mathrm{i}_t + b_y\mathrm{e}_t$. Plugging these into the PC and IS curve gives:

$$\mathrm{y}_t = \alpha_y\mathrm{i}_t + \beta_y\mathrm{e}_t + \delta_y\mathrm{i}_{t+1} \tag{76}$$

$$\pi_t = \alpha_\pi\mathrm{i}_t + \beta_\pi\mathrm{e}_t + \delta_\pi\mathrm{i}_{t+1} \tag{77}$$

Where $\alpha_y = -\sigma$, $\beta_y = b_y\rho_e + b_\pi\rho_e$, $\delta_y = a_y + \sigma a_\pi$, $\alpha_\pi = \alpha_y\kappa$, $\beta_\pi = 1 + \beta_y\kappa + \beta b_\pi\rho_e$, and $\delta_\pi = a_\pi\beta + \delta_y\kappa$. The time-$t$ problem can be expressed as:

$$V(\mathrm{i}_t, \mathrm{e}_t) = \max_{\mathrm{i}_{t+1}} -\frac{1}{2}\pi_t^2 - \frac{\lambda}{2}\mathrm{y}_t^2 + \beta V(\mathrm{i}_{t+1}, \mathrm{e}_{t+1}) \tag{78}$$

Subject to the above two equations. We also guess that $V(\mathrm{i}_t, \mathrm{e}_t) = \frac{1}{2}\left(a_v\mathrm{i}_t^2 + b_v\mathrm{e}_t^2 + 2c_v\mathrm{i}_t\mathrm{e}_t\right)$. Plugging these in:

$$V(\mathrm{i}_t, \mathrm{e}_t) = \max_{\mathrm{i}_{t+1}} -\frac{1}{2}\left(\alpha_\pi\mathrm{i}_t + \beta_\pi\mathrm{e}_t + \delta_\pi\mathrm{i}_{t+1}\right)^2 - \frac{\lambda}{2}\left(\alpha_y\mathrm{i}_t + \beta_y\mathrm{e}_t + \delta_y\mathrm{i}_{t+1}\right)^2 + \frac{\beta}{2}\left(a_v\mathrm{i}_{t+1}^2 + b_v\rho_e^2\mathrm{e}_t^2 + 2c_v\rho_e\mathrm{i}_{t+1}\mathrm{e}_t\right) \tag{79}$$

Taking the i_{t+1} FOC:

$$\frac{\partial}{\partial \mathrm{i}_{t+1}} \Rightarrow \mathrm{i}_{t+1} = a_i\mathrm{i}_t + b_i\mathrm{e}_t \tag{80}$$

Where:

$$a_i = \frac{\alpha_\pi\delta_\pi + \lambda\alpha_y\delta_y}{a_v\beta - \delta_\pi^2 - \lambda\delta_y^2} \tag{81}$$

$$b_i = \frac{\beta_\pi\delta_\pi + \lambda\beta_y\delta_y - \beta c_v\rho_e}{a_v\beta - \delta_\pi^2 - \lambda\delta_y^2} \tag{82}$$

To solve for the values of these parameters we do the following. We have guessed seven parameters: $(a_\pi, b_\pi, a_y, b_y, a_v, b_v, c_v)$, and we need seven equations to solve for them. First, plug (80) and the guessed policy functions for π_t and y_t into (76) and (77):

$$a_y\mathrm{i}_t + b_y\mathrm{e}_t = \left(\alpha_y + a_i\delta_y\right)\mathrm{i}_t + \left(\beta_y + b_i\delta_y\right)\mathrm{e}_t \tag{83}$$

$$a_\pi\mathrm{i}_t + b_\pi\mathrm{e}_t = \left(\alpha_\pi + a_i\delta_\pi\right)\mathrm{i}_t + \left(\beta_\pi + b_i\delta_\pi\right)\mathrm{e}_t \tag{84}$$

In order for these equations to hold for any value of i_t and e_t we require:

$$a_y = \alpha_y + a_i\delta_y \tag{85}$$

$$b_y = \beta_y + b_i\delta_y \tag{86}$$

$$a_\pi = \alpha_\pi + a_i\delta_\pi \tag{87}$$

$$b_\pi = \beta_\pi + b_i\delta_\pi \tag{88}$$

These are four of our seven equations. The final three come from the value function. Plugging the guess for the value function on both sides of the value function definition, along with the guessed policy functions for π_t and y_t and the optimal i_{t+1} choice gives:

$$a_v \mathrm{i}_t^2 + b_v \mathrm{e}_t^2 + 2c_v \mathrm{i}_t \mathrm{e}_t =$$

$$-\left(a_\pi \mathrm{i}_t + b_\pi \mathrm{e}_t\right)^2 - \lambda\left(a_y \mathrm{i}_t + b_y \mathrm{e}_t\right)^2 + \beta\left(a_v (a_i \mathrm{i}_t + b_i \mathrm{e}_t)^2 + b_v \rho_e^2 \mathrm{e}_t^2 + 2c_v \rho_e (a_i \mathrm{i}_t + b_i \mathrm{e}_t)\mathrm{e}_t\right) \tag{89}$$

Collecting terms on the right hand side:

$$a_v \mathrm{i}_t^2 + b_v \mathrm{e}_t^2 + 2c_v \mathrm{i}_t \mathrm{e}_t =$$

$$\left(-a_\pi^2 - \lambda a_y^2 + \beta a_v a_i^2\right)\mathrm{i}_t^2 + \left(-b_\pi^2 - \lambda b_y^2 + \beta a_v b_i^2 + \beta b_v \rho_e^2 + \beta b_i 2c_v \rho_e\right)\mathrm{e}_t^2$$

$$+2\left(-a_\pi b_\pi - \lambda a_y b_y + \beta a_v a_i b_i + \beta a_i c_v \rho_e\right)\mathrm{i}_t \mathrm{e}_t \tag{90}$$

For this to hold for any values of the state we require:

$$a_v = -a_\pi^2 - \lambda a_y^2 + \beta a_v a_i^2 \tag{91}$$

$$b_v = -b_\pi^2 - \lambda b_y^2 + \beta a_v b_i^2 + \beta b_v \rho_e^2 + \beta b_i 2c_v \rho_e \tag{92}$$

$$c_v = -a_\pi b_\pi - \lambda a_y b_y + \beta a_v a_i b_i + \beta a_i c_v \rho_e \tag{93}$$

C.4 ZLB, FC

We maximize subject to: 1) $\mathrm{i}_0 \geq -r^*$, 2) time-0 IS curve only (assume that don't hit ZLB from time 1, hence IS stops being relevant), 3) whole sequence of NKPC. Stated like this, it is a primal problem apart from i_0:

$$L = \sum_{t=0}^{\infty} \beta^t \left(-\frac{1}{2}\pi_t^2 - \frac{\lambda}{2}\mathrm{y}_t^2 + \mu_t\left(\pi_t - \beta\pi_{t+1} - \kappa \mathrm{y}_t\right)\right) + \gamma\left(\mathrm{y}_0 - \mathrm{y}_1 + \sigma \mathrm{i}_0 - \sigma\pi_1 - \mathrm{g}_0\right) + \theta\left(\mathrm{i}_0 + r^*\right) \tag{94}$$

The FOCs for inflation and output from period 2 onwards are the same as the cost push shock model:

$$\frac{\partial}{\partial \pi_t} \Rightarrow \pi_t = \mu_t - \mu_{t-1} \tag{95}$$

$$\frac{\partial}{\partial \mathrm{y}_t} \Rightarrow \mathrm{y}_t = -\frac{\kappa}{\lambda}\mu_t \tag{96}$$

And from $t = 2$ onwards we are only subject to the NKPC, so we can solve for exactly the same policy functions as in the cost push case. Hence from $t = 2$ we have the policy functions $\pi_t = a\mu_{t-1} + b\mathrm{e}_t$ with a and b exactly the same as before, and $\mathrm{e}_t = 0$. We then solve the remaining system, $(\pi_0, \mathrm{y}_0, \pi_1, \mathrm{y}_1, \mu_0, \mu_1, \gamma, \mathrm{i}_0, \theta)$, using the following procedure. Firstly, impose $\mathrm{i}_0 = -r^*$. Then solve for seven unknowns $(\pi_0, \mathrm{y}_0, \pi_1, \mathrm{y}_1, \mu_0, \mu_1, \gamma)$ from the following seven equations:

$t = 1$ FOCs:

$$\frac{\partial}{\partial \pi_1} \Rightarrow \pi_1 = \mu_1 - \mu_0 - \frac{\gamma\sigma}{\beta} \tag{97}$$

$$\frac{\partial}{\partial \mathrm{y}_1} \Rightarrow \mathrm{y}_1 = -\frac{\kappa\mu_1}{\lambda} - \frac{\gamma}{\beta\lambda} \tag{98}$$

$t = 0$ FOCs:

$$\frac{\partial}{\partial \pi_0} \Rightarrow \pi_0 = \mu_0 \tag{99}$$

$$\frac{\partial}{\partial \mathrm{y}_0} \Rightarrow \mathrm{y}_1 = -\frac{\kappa\mu_0}{\lambda} + \frac{\gamma}{\lambda} \tag{100}$$

And the time 0 and 1 NKPCs, and the time 0 IS:

$$\mathrm{y}_0 = \mathrm{y}_1 + \sigma r^* + \sigma\pi_1 + \mathrm{g}_0 \tag{101}$$

$$\pi_0 = \beta\pi_1 + \kappa \mathrm{y}_0 \tag{102}$$

$$\pi_1 = \beta a\mu_1 + \kappa \mathrm{y}_1 \tag{103}$$

Finally, take the i_0 FOC:

$$\frac{\partial}{\partial \mathrm{i}_0} = \sigma\gamma + \theta = 0 \Rightarrow \theta = -\sigma\gamma \tag{104}$$

If $\theta < 0$ then the ZLB doesn't actually bind, and we can just set all variables to zero instead.

C.5 ZLB, NC

Since the shock is only for one period, with NC we have $\pi_t = \mathrm{y}_t = \mathrm{i}_t = 0$ from period 1 onwards. The period 0 NKPC and IS (with binding ZLB) then give:

$$\mathrm{y}_0 = \sigma r^* + \mathrm{g}_0 \tag{105}$$

$$\pi_0 = \kappa \mathrm{y}_0 \tag{106}$$

C.6 ZLB, OPC

OPC from period 1 onwards is just the solution to the cost push shock with $\mathrm{e}_t = 0$. Then in period 0 the government faces the constraints:

$$\mathrm{y}_0 = \mathrm{y}_1 - \sigma \mathrm{i}_0 + \sigma \pi_1 + \mathrm{g}_0 \tag{107}$$

$$\pi_0 = \beta \pi_1 + \kappa \mathrm{y}_0 \tag{108}$$

Where $\pi_1 = a_\pi \mathrm{i}_1$, and $\mathrm{y}_1 = a_y \mathrm{i}_1$. These become:

$$\mathrm{y}_0 = \alpha_y \mathrm{i}_0 + \beta_y \mathrm{g}_0 + \delta_y \mathrm{i}_1 \tag{109}$$

$$\pi_0 = \alpha_\pi \mathrm{i}_0 + \beta_\pi \mathrm{g}_0 + \delta_\pi \mathrm{i}_1 \tag{110}$$

Where $\alpha_y = -\sigma$, $\beta_y = 1$, $\delta_y = a_y + \sigma a_\pi$, $\alpha_\pi = \alpha_y \kappa$, $\beta_\pi = \beta_y \kappa$, and $\delta_\pi = a_\pi \beta + \delta_y \kappa$. The time-0 problem can be expressed as:

$$\max_{\mathrm{i}_0 \geq -r^*, \mathrm{i}_1} -\frac{1}{2} \left(\alpha_\pi \mathrm{i}_0 + \beta_\pi \mathrm{g}_0 + \delta_\pi \mathrm{i}_1\right)^2 - \frac{\lambda}{2} \left(\alpha_y \mathrm{i}_0 + \beta_y \mathrm{g}_0 + \delta_y \mathrm{i}_1\right)^2 + \frac{\beta}{2} a_v i_{t+1}^2 \tag{111}$$

This leads to an optimal choice for i_{t+1}:

$$\frac{\partial}{\partial \mathrm{i}_{t+1}} \Rightarrow \mathrm{i}_1 = a_i \mathrm{i}_0 + b_i \mathrm{g}_0 \tag{112}$$

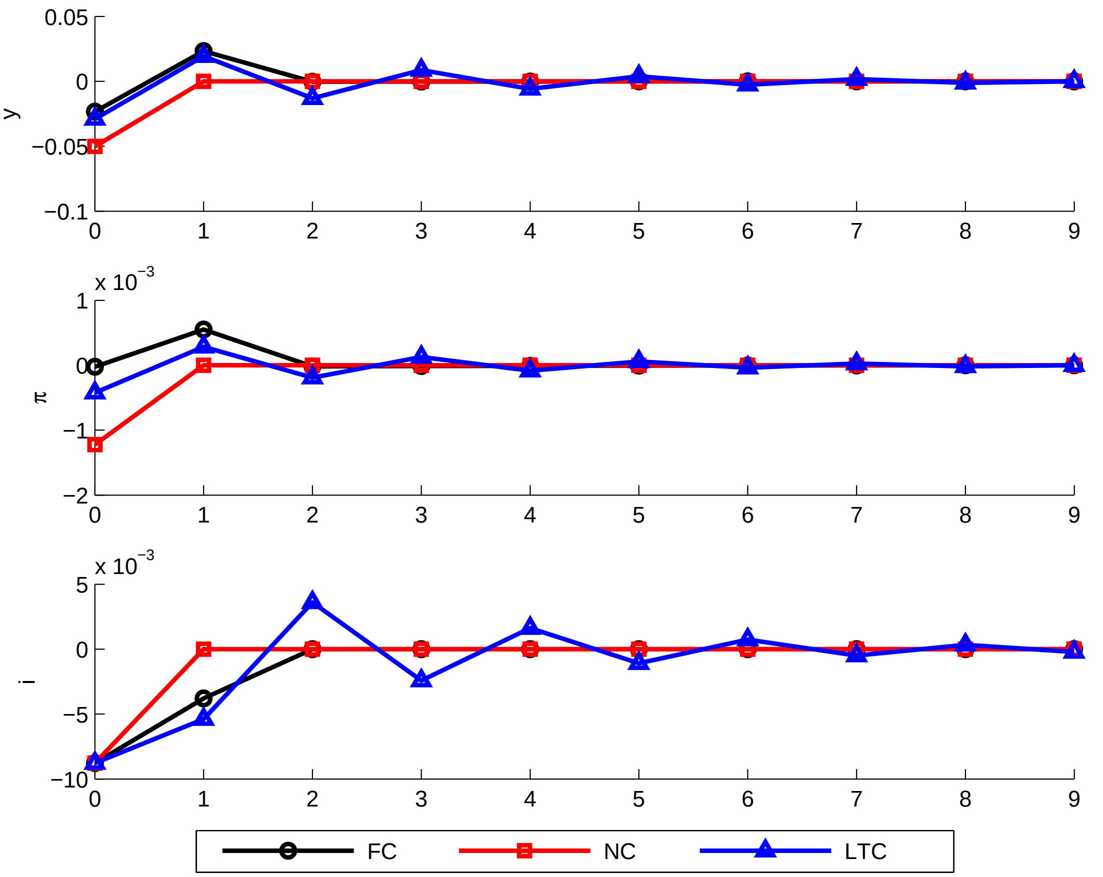
Where:

$$a_i = \frac{\alpha_\pi \delta_\pi + \lambda \alpha_y \delta_y}{a_v \beta - \delta_\pi^2 - \lambda \delta_y^2} \tag{113}$$

$$b_i = \frac{\beta_\pi \delta_\pi + \lambda \beta_y \delta_y}{a_v \beta - \delta_\pi^2 - \lambda \delta_y^2} \tag{114}$$

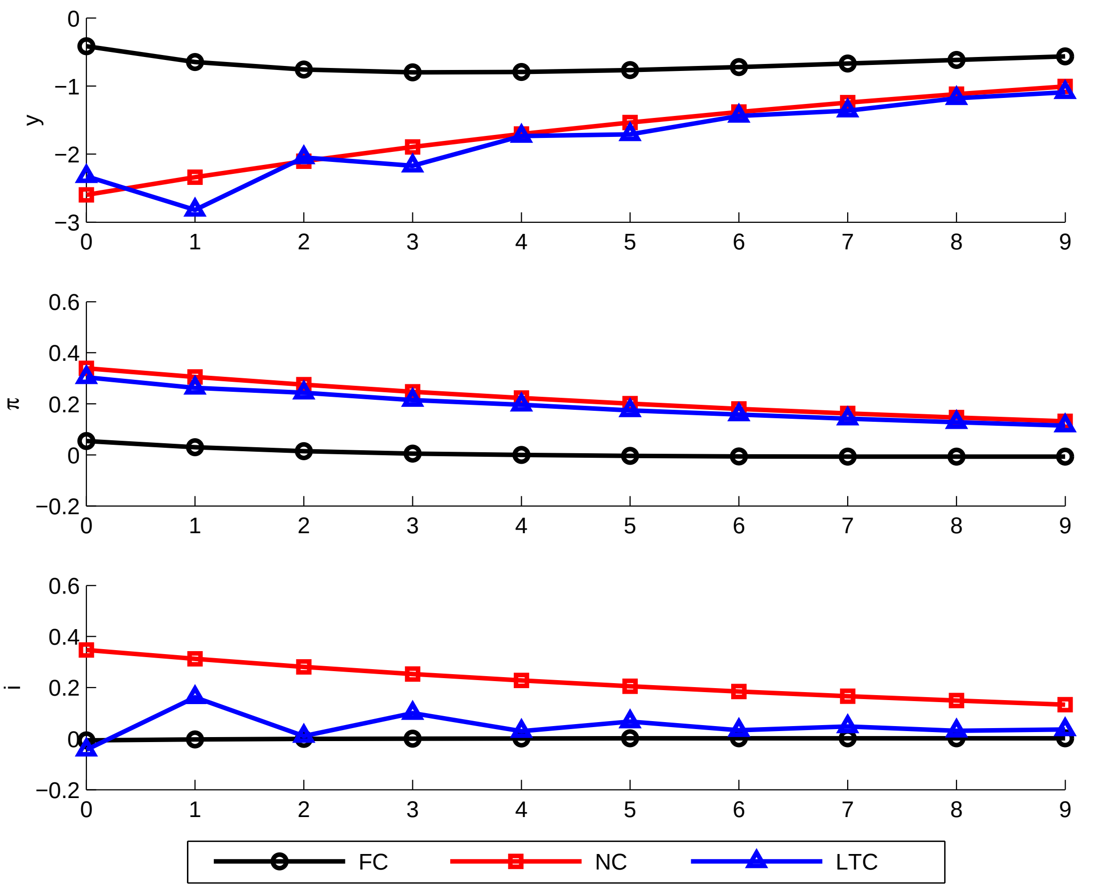
D Figures

Figure 1: Impulse response to demand shock at the ZLB



Response to a one period shock that drives the economy to the ZLB at $t = 0$ *under different policy regimes.* y *refers to the output gap,* π *to inflation, and* i *to the nominal interest rate.*

Figure 2: Impulse response to cost-push shock, $\rho_e = 0.9$



Response to a cost-push shock with $\rho_e = 0.9$ *at* $t = 0$ *under different policy regimes. y refers to the output gap,* π *to inflation, and i to the nominal interest rate.*

E Optimal monetary policy calibration

r^* is set to the steady state interest rate with zero inflation, $1/\beta - 1$. The output-sensitivity of inflation is given by $\kappa = (1-\nu)(1-\beta\nu)\nu^{-1}(\sigma^{-1}+\omega)(1+\omega\theta)^{-1}$, and the weight on output in the loss function by $\lambda = \kappa/\theta$.

Table 1: Calibration

	Interpretation	Value
β	Discount factor	0.9917
σ	Interest sensitivity of consumption	6.25
ν	Probability can't adjust price	0.66
ω	Elasticity of firm marginal cost to output	0.47
θ	Elasticity of substitution between varieties	7.66
ρ_e	Autocorrelation of cost push shock	0.9
r^*	ZLB constraint	0.0084
ρ_e	Autocorrelation of cost push shock	0.9
κ	Inflation sensitivity to output	0.0244
λ	Weight on output in loss function	0.0032