Selling to Advised Buyers*

Andrey Malenko† Anton Tsoy‡

October 2016

Abstract

In many cases, agents that make purchase decisions are uninformed and rely on the advice of biased experts. For example, the board of the bidder relies on the advice of managers when bidding for a target in a takeover contest. We study how to sell assets to such "advised buyers" if the goal is to maximize revenues or efficiency. In static mechanisms, such as first- and second-price auctions, advisors communicate a coarsening of information, and the revenue equivalence theorem holds. By contrast, in dynamic mechanisms, advisors can often fully communicate their information to buyers, which leads to more efficient allocations. Whether this leads to higher revenues depends on the bias. When advisors are biased for overbidding, an ascending-price auction dominates static formats in both efficiency and expected revenues. When advisors are biased for underbidding, a descending-price auction dominates static mechanisms in efficiency but often results in lower expected revenues.

*We thank Ilona Babenko, Alessandro Bonatti, Will Cong, Peter DeMarzo, Doug Diamond, Glenn Ellison, Bob Gibbons, Zhiguo He, Navin Kartik, Tingjun Liu, Nadya Malenko, Jonathan Parker, Vasiliki Skreta, Konstantin Sorokin, Jean Tirole, Juuso Toikka, and participants at EIEF, MIT Theory Lunch, Queen's University, AMMA 2015, 2015 Stony Brook International Conference on Game Theory, Finance Theory Group (UT Austin), Mechanism Design Workshop at NES, CSEF, SFS Cavalcade (Toronto), FIRS (Lisbon), EFA (Oslo) for helpful comments.

†MIT Sloan School of Management. Email: amalenko@mit.edu.

‡Einaudi Institute for Economics and Finance. Email: tsoianton.ru@gmail.com.

1 Introduction

In many applications, agents that make purchase decisions have limited information about their valuations of the assets for sale. As a consequence, they rely on the advice of informed experts, who however often have misaligned preferences. Consider the following examples:

1. *A firm competing for a target in a takeover contest.* While the board of directors often has formal authority over submitting bids for the target, the firm's managers are more informed about the valuation of the target. They, however, could be prone to overbidding because of career concerns and empire building preferences. A similar conflict of interest occurs if the decision-maker consults an investment banker.

2. *Bidding in spectrum auctions.* Telecommunication companies bidding in spectrum auctions have research teams in charge of preparing for the auction and advising the top management and the board on bidding. They may have different incentives as winning the auction could give a positive signal and help the research team attract future business.

3. *Suppliers competing in procurement.* When a construction company competes on cost for a project in a procurement auction, managers that will work on the project are privately informed about its cost to the firm, while the top management has authority over bidding. The informed managers may have a bias for overstating the cost.

4. *Realtors in real estate transactions.* A buyer of a house gets advice from a realtor about what offer to make. The realtor has private information about the value of the house but may be biased for overpaying.

We call such players "advised buyers" and ask the following question in the paper: Does the presence of advising relationships between buyers and their advisors affect how the seller should design the sale process? We analyze this question both from the position of maximizing expected revenues, which is likely the goal if the designer is the seller, and from the position of allocative efficiency, which could be a more important goal if the designer is the government.

We study a canonical setting where the seller has an asset to auction among a number of potential buyers with independent private values. We depart from it in one aspect: Each potential buyer is a bidder-advisor pair, where the bidder controls bidding decisions (e.g., the board of the firm) but has no information about her valuation, while the advisor (e.g., the firm's manager) knows the valuation but has a conflict of interest. Our initial focus is on the case in which advisors have a bias for overbidding, that is, given value v to the bidder, the advisor's maximum willingness to pay is $v + b$ with $b > 0$. We next consider the case of the bias for underbidding.

Prior to the bidder submitting an offer, the advisor communicates with the bidder via a game of cheap talk. If the sale process consists of a single round of bidding, there is only one round of communication. In contrast, if the sale process consists of multiple rounds, the advisor communicates with the bidder in each round of the auction. In this environment, communication and the design of the sale process interact. On one hand, information transmission from advisors affects bids submitted in the auction and therefore the efficiency and revenues of each auction format. On the other hand, the auction format affects how advisors communicate information to bidders.

We analyze equilibria of the model under the NITS ("no incentive to separate") condition adapted from Chen et al. (2008).[1] We first study static auctions. As one could expect from the classic game of cheap talk (Crawford and Sobel (1982)), communication takes a partition form: All types of the advisor are partitioned into intervals and types in each interval induce the same bid. Even though our game is not a special case of the classic game of cheap talk, as payoffs are endogenous, the logic of Crawford and Sobel (1982) and Chen et al. (2008) still applies. Communication strategies have a partition structure and the equilibrium with the highest number of partitions satisfies the NITS condition. We prove a version of the revenue equivalence theorem for static auctions. We focus on a large class of standard auctions with continuous payments introduced in Che and Gale (2006), including first-price, second-price, and all-pay auctions, and show that all static auctions in this class bring the same expected revenue and feature the same communication between bidders and advisors.

This conclusion changes drastically if the asset is sold via dynamic mechanisms. Consider the ascending-price (English) auction, where the price continuously increases until only one bidder remains. From the position of a bidder and her advisor, the ascending-price auction is a stopping time problem: At what price level to drop out. Because of this, the advisor sends repeated recommendations to his bidder about whether to quit the auction now or not. We show that any equilibrium satisfying the dynamic counterpart of NITS condition has the following structure. The advisor recommends to stay in the auction until the price

[1] Intuitively, the NITS condition says that the weakest type (i.e., the lowest valuation if the advisor has an overbidding bias) has the option to credibly reveal herself. We impose it in each round of the auction, which puts a lower bound on the advisor's expected payoff in each round. Even if the NITS condition does not select the unique equilibrium in the communication game, our main results do not require any further refinement.

reaches the advisor's maximum willingness to pay. In turn, the bidder follows the advisor's recommendation until the price reaches a high enough threshold, at which she drops out irrespectively of what the advisor says then. Thus, the advisors' types perfectly separate at the bottom of the distribution and pool at the top. Moreover, when the value is in the range of perfect information transmission, the bidder overbids: She always exits the auction later than she would had she known her value at the start of the game. When the distribution satisfies natural restrictions, the NITS equilibrium is unique.

The intuition for this equilibrium lies in the irreversibility of the running price in the auction: While a bidder can always bid until a price level higher than the current running price, she cannot go back in time and exit at a price lower than the current running price. Informally, she can improve her offer but cannot renege on her past offers. If the advisor is biased for overbidding, he recommends the bidder to continue bidding and sends the recommendation to quit only when the price reaches the advisor's indifference point, i.e., when the price exceeds the buyer's value by the amount of the bias. When the bidder gets such a recommendation, she backs out her valuation, realizes that it is below the current running price and, hence, quits the auction immediately. When the bidder gets the recommendation to continue bidding, she trades off the value of the advisor's residual private information against the cost of possibly overpaying. If the distribution of valuations has decreasing mean residual lifetime, the solution is to act on the advisor's recommendation unless the price becomes high enough. If the distribution of valuations has increasing mean residual lifetime and the bias is not too high, the advisor's communication strategy is perfectly separating and the bidder always waits for the advisor's recommendation to quit the auction.

Our main result is that the ascending-price auction outperforms static auctions in both efficiency and expected revenues. The first result shows that the NITS equilibrium of the ascending-price auction, when it is unique, is more efficient than any equilibrium of static auctions. While in static auctions communication has a partition structure, in the ascending-price auction advisors fully transmit their information up to a cut-off. This cut-off is higher than the cut-off in the highest partition in the static auction. Intuitively, in the static auction, the cut-off type is indifferent between pooling with higher types (and facing the risk of the bidder having to pay above the advisor's maximum willingness to pay) and with types in the second-highest partition (and facing the risk of losing when the strongest rival is also in the second-highest partition). This indifference implies that the bidder's best guess of its valuation when she learns that it is in the highest partition is strictly higher than the maximum willingness to pay of the cut-off type of the advisor. As a consequence, the advisor with type just above this cut-off could induce the bidder wait until his most preferred price level in the ascending-price auction, implying that the cut-off there is higher. Thus, the ascending-price auction is more efficient.

The second result shows that imposing an additional assumption that the distribution of valuations is such that the virtual surplus of the advisor is increasing in the valuation, the expected revenues in the ascending-price auction are higher than in any NITS equilibrium of the second-price auction. This result may seem surprising, since the equilibrium exit price in the ascending-price auction may be higher for some or lower for other types than the equilibrium bid in the second-price auction. To understand it, think about the seller's optimal auction design problem as selling to advisors directly, where communication between advisors and bidders puts restrictions on what the selling mechanism can be. According to Myerson (1981), the expected revenues equal the expected virtual valuation of the winning advisor less the expected payoff of the advisors with the lowest value. Since the ascending-price auction is more efficient, its expected virtual valuation of the winning advisor is higher. In addition, in the ascending-price auction, the lowest type of the advisor never wins, so his payoff is zero. At the same time, the NITS condition implies that his payoff in the second-price auction cannot be negative, since otherwise he would be better off credibly revealing it. Thus, the ascending-price auction generates higher revenues both because it is more efficient and because it leaves weakly less rents to the lowest type.

Next, we consider the case in which advisors have a bias for underbidding, that is, given value v to the bidder, the advisor's maximum willingness to pay is $v + b$ with $b < 0$. In this case, the ascending-price auction loses its advantage over static formats. If the advisor follows the same strategy of recommending to quit when the running price reaches his maximum willingness to pay, the bidder has no incentive to follow this recommendation. Staying in the auction further is always an option, so the bidder would wait until the price reaches v. Knowing this, the advisor no longer follows the strategy of recommending to quit at price $v + b$. Thus, the argument from the overbidding case does not apply.

One may conjecture that the decreasing-price (Dutch) auction, in which the running price continuously decreases until one bidder accepts it, dominates static auctions in this case. We show that the answer is "yes" for efficiency, but "no" for expected revenues. In this case, the descending-price auction has an equilibrium, which is conceptually similar to the equilibrium in the ascending-price auction with the overbidding bias: Each advisor recommends to stay in the auction (not accept the current price) until the price reaches her optimal bid in the auction, while the bidder follows the advisor's recommendation up to a certain price cut-off. Thus, there is full separation of high types and pooling of low types - the opposite of what happens in the ascending-price auction with overbidding bias. For the same reason, the descending-price auction is more efficient than static formats. However, this equilibrium in the decreasing-price auction can bring lower expected revenue to the seller. The reason is that the separation of advisor's types at the top in the descending-price auction comes at the cost of it occurring at lower prices, since advisors reveal their valuations with delay. Thus, the descending-bid auction often yields lower expected revenues, in contrast to the ascending-price auction always yielding higher expected revenues when the bias is for overbidding.

This paper is related to three strands of the literature. First, it is related to the literature on communication of non-verifiable information (cheap talk), pioneered by Crawford and Sobel (1982). Because our main results rely on the NITS selection condition, our paper is related to Chen et al. (2008), who introduce it.[2] Cheap talk models usually focus on exogenous payoffs of players and exogenous timing of the game (typically, one round of communication). In contrast, the payoffs and the game itself are endogenous in our paper. In particular, by converting the mechanism from a single-round game to a stopping time game for bidders, the seller can make communication between bidders and advisors more efficient, which sometimes (but not always) leads to higher expected revenues. Thus, our paper is related to Grenadier et al. (2016), who study an optimal stopping time problem in the presence of the conflict of interest between the sender and the decision-maker and show that it leads to different equilibria than the analogous static communication game. The main contribution of our paper is to endogenize the design of the game by the seller. Other novel parts are multiple sender-receiver (bidder-advisor) pairs and the ability of the seller to achieve more efficient communication in the dynamic game for any sign of the bias by appropriately designing the auction. The sign of the bias matters because the seller strictly benefits from more efficient communication if the bias is for overbidding, but not necessarily if the bias is for underbidding. A number of papers study cheap talk models with other, less related to ours, dynamic aspects of communication.[3]

Second, the paper is related to the literature on the comparison of auction formats. The central result in this literature is the celebrated revenue equivalence theorem (Myerson (1981), Riley and Samuelson (1981)) and generalized to arbitrary type distributions by Che

[2]It is also related to Kartik (2009) and Chen (2011) who study perturbed versions of the classic cheap talk game with lying costs and behavioral players, respectively, since both variations can be used to motivate the NITS condition.

[3]See Sobel (1985), Morris (2001), Golosov et al. (2014), Ottaviani and Sørensen (2006a,b), Krishna and Morgan (2004), Aumann and Hart (2003).

and Gale (2006). As we show, it continues to hold when bidders are "advised buyers" and sale mechanisms are static, but breaks down when they are dynamic. The revenue equivalence theorem can fail for other reasons, such as affiliation of values (Milgrom and Weber (1982)), bidder asymmetries (Maskin and Riley (2000)), or budget constraints (Che and Gale (1998, 2006), Pai and Vohra (2014)). To our knowledge, we are the first to study the problem of the design of sale procedures when potential buyers are advised by informed experts. Several papers study different auction mechanisms in the presence of information acquisition by bidders.[4] In particular, Compte and Jehiel (2007) show that the ascending-price auction brings higher revenues than static auctions. While this result is similar to our result about revenues when the bias is for overbidding, it follows from a very different argument and relies on the asymmetry of bidders in information endowments and their knowledge of the number of remaining bidders in the auction. Burkett (2015) studies a principal-agent relationship in the auction context when the principal optimally constrains a biased agent with a budget and shows revenues equivalence of first- and second-price auctions in the independent private values setting, which is related to our revenue equivalence result for static auctions.

Finally, several papers study other effects of cheap talk communication in mechanism design and/or trading environments. Matthews and Postlewaite (1989) study pre-play communication in a two-person double auction. Ye (2007) and Quint and Hendricks (2013) study two-stage auctions, where the actual bidding is preceded by the indicative stage, which is a form of cheap talk between bidders and the seller. Kim and Kircher (2015) study how auctioneers with private reservation values compete for potential bidders by announcing cheap-talk messages. Several papers also study the role of cheap-talk communication in non-auction trading environments.[5]

The structure of the paper is as follows. Section 2 introduces the model. Section 3 illustrates our main findings in a canonical uniform example. Section 4 examines static auctions. Section 5 characterizes NITS equilibria of the English auction when advisors have a bias for overbidding. Section 6 analyzes the case of advisors' bias for underbidding. Section 7 gives a quantitative example. Section 8 concludes. All proofs are relegated to Appendix.

[4]Persico (2000), Bergemann and Välimäki (2002), Compte and Jehiel (2007), Bergemann, Välimäki, and Shi (2009), Crémer, Spiegel, and Zheng (2009), and Shi (2012).

[5]See Farrell and Gibbons (1989), Chakraborty and Harbaugh (2010), Levit (2014), Koessler and Skreta (2014), Inderst and Ottaviani (2013).

2 Model

Consider the standard setting with independent private values. There is a single indivisible asset for sale. The value of the asset to the seller is normalized to zero. There are N ex-ante identical potential buyers (bidders). The valuation of bidder i, v_i, is an i.i.d. draw from distribution with c.d.f. F and p.d.f. f. The distribution F has full support on $[\underline{v}, \overline{v}]$ with $0 \leq \underline{v} < \overline{v} \leq \infty$ and satisfies $\int_{\underline{v}}^{\overline{v}} v dF(v) < \infty$. In the analysis, we will frequently refer to the distribution of valuation of the strongest opponent of a bidder. We denote by $\hat{v}$ the maximum of $N-1$ i.i.d. random variables distributed according to F and its c.d.f. by G: $G(\hat{v}) = F(\hat{v})^{N-1}$. We also use a short-hand notation $F(a,b) = F(b) - F(a)$ to denote that a random variable distributed according to F falls in the interval $[a,b]$. Similarly, $G(a,b) = G(b) - G(a)$.

The novelty of our setup is that each bidder i does not know his valuation v_i, but consults an advisor who does. Let advisor i denote the advisor to bidder i. Advisor i knows v_i, but has no information about valuations of other bidders except for their distribution F, which is common knowledge. While advisor i knows v_i, she is biased relative to the bidder. Specifically, the payoffs from acquiring the asset by bidder i are

$$\text{Bidder } i \quad : \quad v_i - p, \tag{1}$$

$$\text{Advisor } i \quad : \quad v_i + b - p, \tag{2}$$

where b is the bias of the advisor. The value that all players get from not acquiring the asset is zero. Bias b is commonly known.[6] Our primary focus is on the preference of advisors for overbidding, $b > 0$, as it is most prominent in applications. In Section 6, we also consider the case of $b < 0$, which shares several similarities with the case of $b > 0$, but also differs from it in a number of important aspects.

Our formulation (1) – (2) captures the empire building motives described in the introduction. For example, consider a publicly traded firm bidding for a target. The board of the firm has formal authority over the bidding process, maximizes firm value, but does not know valuation v_i. Suppose that the CEO of the firm knows v_i, but is biased. Specifically, if the CEO owns fraction α of the stock of the company and gets a private benefit of B from acquiring the target and managing a larger company, her payoff is $\alpha(v_i - p) + B$. More generally, α captures how the CEO's pay is tied to the firm's performance. Normalizing this

[6]For many of our results it is sufficient to assume that b is commonly known by bidders and advisors, while the seller knows only the sign of the bias.

payoff by α and denoting $b = \frac{B}{\alpha}$, we obtain $(1) - (2)$.

In this paper, we compare how different selling mechanisms affect the seller's expected revenue and the allocative efficiency. Several formats are commonly used in practice and studied in the academic literature:

1. **Second-price auction**. Bidders simultaneously submit sealed bids, and the bidder with the highest bid wins the auction and pays the second-highest bid.

2. **First-price auction**. Bidders simultaneously submit sealed bids, and the bidder with the highest bid wins the auction and pay her bid.

3. **Ascending-price (English) auction**. The seller continuously increases the price p, which we refer to as the *running price,* starting from zero. At each price, each bidder decides whether to continue participating or to *quit* the auction. Once a bidder quits, she cannot re-enter the auction. Once only one bidder remains, she wins and pays the price at which the last of her opponents quit the auction.

4. **Descending-price (Dutch) auction**. The seller continuously decreases the price p, which we refer to as the *running price,* starting from a high enough level. At each price, each bidder decides whether to *stop* the auction. The first bidder who stops the auction wins and pays the price at which she stopped the auction.

In all of these auction formats, if a tie occurs, the winner is drawn randomly from the set of tied bidders. We study a rich class of static auctions formally described in Section 4, but restrict attention to the ascending-price and descending-price auctions among dynamic mechanisms.

Communication between bidders and their advisors is modeled as a game of cheap talk. If the auction format is static (i.e., it consists of a single round of bidding), the timing of the game is as follows:

1. Advisor i sends a private message $\tilde{m}_i \in M$ to bidder i where M is some infinite set of messages.

2. Having observed message $\tilde{m}_i$, bidder i chooses her action, i.e., what bid $\beta_i \in \mathbb{R}_+$ to submit.

3. Given all submitted bids $\beta_1, ..., \beta_N$, the asset is allocated and payments are made according to the rule specified by the auction.

We consider Perfect Bayesian Equilibria (PBE) of static auctions. Since all bidders are symmetric, we focus on symmetric PBEs in which all advisors adopt the same communication strategy $m : [\underline{v}, \overline{v}] \to M$ and all bidders follow the same bidding strategy $\beta : M \to \mathbb{R}_+$.[7]

There is in general a multiplicity of equilibria in the game of cheap-talk. We impose the "no incentive to separate" (NITS) condition, adapted from Chen et al. (2008), to select among equilibria in the communication game between bidders and advisors. When $b > 0$, call type $v_w \equiv \underline{v}$ the *weakest type* of advisor.[8] According to the NITS condition, the weakest type has an option to credibly reveal herself if she wants. Thus, an equilibrium violates the NITS condition if the payoff of the weakest type is less than what she would get from revealing herself to the bidder (and having the bidder best-respond to that information). Intuitively, when an advisor is biased for overbidding, every type of the advisor wants to convince the bidder to bid more than the bidder would optimally bid if she knew her value. Thus, it is natural to assume that the recommendation to bid little would be perceived as credible by the bidder. Chen et al. (2008) show that NITS can be justified by perturbations of the cheap-talk game with non-strategic players or costs of lying.[9]

We refer to an equilibrium as *babbling* if regardless of the message received, each bidder plays the same strategy. We refer to an equilibrium of the static auction as *the most informative* if it induces the largest number of actions. As we show later, the most informative equilibrium in the static auctions always satisfies NITS. However, NITS need not select the unique equilibrium, and for the comparison of auction formats, we do not need a selection beyond NITS.

If the auction format is dynamic (i.e., it consists of multiple rounds of bidding), the advisor sends a message to the bidder before each round of bidding. In ascending-price and descending-price auctions, we index rounds by corresponding running prices p.

A *(private) history* of bidder i at the beginning of round p consists of all bidders' actions and all messages sent by advisor i in the previous rounds. We assume that bidders and advisors only observe the running price p, but not the actions of other bidders. In the ascending-price or descending price auctions the history of the bidder i includes the current running price p and messages sent by advisor i up to round p.

A strategy of advisor i is a measurable mapping from the advisor's private information

[7] We use m to denote communication strategies and $\tilde{m}$ for messages in M.

[8] Similarly, when $b < 0$, $v_w \equiv \overline{v}$ is the weakest type of advisor.

[9] Notice that one way of imposing the NITS condition is to specify that there exists an off-equilibrium-path message, such that the bidder believes that it comes from the weakest type of advisor v_w. Since the preferences of players satisfy the single crossing condition, it is not restrictive that not only the weakest but any type of the advisor has the option to signal that the value is v_w.

about the valuation v and a history into a message sent to bidder i after that history. A strategy of bidder i is a measurable mapping from a history and a current message into the action chosen by the bidder. A bidder's posterior belief process is a measurable mapping from a history into the distribution over $[\underline{v}, \overline{v}]$.

We will restrict attention to Perfect Bayesian equilibria in symmetric Markov strategies (PBEM) where the state consists of the auction round p and a bidder's posterior belief about her valuation v. The Markov communication strategy $m_M(v, p, \tilde{\mu})$ gives the message sent in round p when bidder's posteriors are $\tilde{\mu}$ and the advisor's type is v. We focus on equilibria in which communication strategies are pure. The Markov bidding strategy $a_M(\tilde{m}, p, \tilde{\mu})$ gives the bidder's decision in round p to quits/stops the auction ($a_M = 1$) or continue ($a_M = 0$), when her beliefs are $\tilde{\mu}$ and the advisor's last message is $\tilde{m}$.

For dynamic selling mechanisms, we require that the NITS condition holds in every round of the game. Specifically, let[10]

$$v_w(h) = \inf\{v | v \in supp(\mu(h))\}. \tag{3}$$

be the weakest remaining type of the advisor after history h. Similarly to Chen et al. (2008), an equilibrium violates the dynamics version of NITS condition if after history h, the advisor of type $v_w(h)$ is better off claiming that she is the weakest remaining type than playing her equilibrium strategy. To capture this condition, we require that any unexpected message is interpreted as a signal of the weakest type (then the advisor's sequential rationality implies that after any history, the equilibrium strategy is weakly preferred to signaling that you are the weakest type). Formally, the dynamic version of NITS that we impose is stated as follows:

Definition 1. *A PBEM (m_M, a_M, μ) satisfies the NITS condition if the following holds. Consider any p-round history h in which the advisor deviates in round p' for the first time and sends $\tilde{m} \notin \bigcup_{v \in supp(\mu(h'))} m_M(v, p', \mu(h'))$ where h' is a truncation at round p' of history h. Then $\mu(h)$ assigns probability one to $v_w(h')$.*[11]

A couple observations are in order. First, Definition 1 states that after the first unexpected message, the bidder assigns probability one to the weakest type in the round when the deviation happened and never updates her beliefs since then. Second, unlike in the static

[10] For $b < 0$, $v_w(h) = \sup\{v | v \in supp(\mu(h))\}$.

[11] We implicitly assume that the set of messages is rich enough so that there is always an "unused" message in any PBEM.

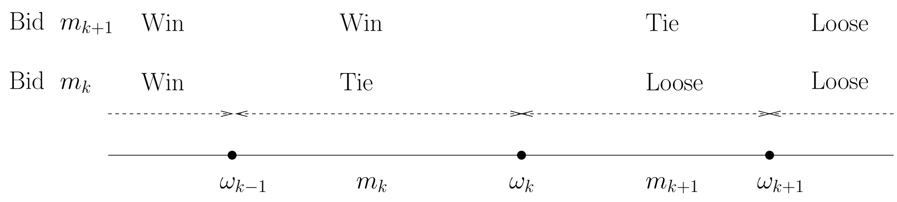


Figure 1: Thresholds in the partition equilibrium of the second-price auctions. Type ω_k of advisor is indifferent between pooling with types in $[\omega_{k-1}, \omega_k)$ by sending message m_k and types in $[\omega_k, \omega_{k+1})$ by sending m_{k+1}. The difference between messages m_k and m_{k+1} is that m_{k+1} wins for sure against types in $[\omega_{k-1}, \omega_k)$ and ties against types in $[\omega_{k-1}, \omega_k)$, while m_k ties with types in $[\omega_{k-1}, \omega_k)$ and looses against types in $[\omega_{k-1}, \omega_k)$.

auction, in the dynamic auctions the weakest type may (and will) change as the auction progresses. Third, the condition in Definition 1 requires that any unsent message is perceived as a signal of the weakest type which is slightly stronger than assuming that the lowest type of advisor does not want to reveal itself in equilibrium. However, given that payoffs satisfy the single crossing condition, these off-equilibium beliefs seem the least restrictive off-the-path beliefs.[12]

3 An Example: Uniform Distribution

We start the analysis by working out a simple example that illustrates the results of the paper: How multiple-round auctions differ from single-round auctions, when bidders rely on informed advisors, and why the direction of the conflict of interest between bidders and advisors is crucial for the optimal design of the sale process. In this example, there are two bidders ($N = 2$), each valuation is an i.i.d. draw from the uniform distribution over $[0, 10]$, and the advisors' bias is $b = 1$ (when the bias is for overbidding) or $b = -1$ (when the bias is for underbidding).

Overbidding bias ($b = 1$). First, consider the second-price auction. Because of the bias, the advisor cannot credibly communicate the valuation to the bidder, and the equilibrium

[12]There are known technical difficulties in defining games in continuous time (see Simon and Stinchcombe (1989)). However, this problem of the outcome indeterminancy in continuous time does not arise in PBEM. This is so, because only the advisor can affect the evolution of posterior beliefs on which both sides condition their strategies. If the advisor deviates to a message that is not expected by the bidder, then in the future the bidder assigns probability one to the weakest type in the round of the deviation and so, the future messages of the advisor are irrelevant and the outcome is uniquely pinned down. If the advisor deviates to a message send by a different type, then such a deviation is not detected and the strategies again uniquely pin down the outcome.

must have a partition structure. Consider the conditions that characterize an equilibrium with K partitions, $[\omega_0, \omega_1]$, ..., $[\omega_{K-1}, \omega_K]$, with $\omega_0 = 0$ and $\omega_K = 10$. Given the advisor's message that conveys that the valuation is in the k^{th} partition, the best response of the bidder is to bid the updated expected valuation, $m_k = (\omega_{k-1} + \omega_k)/2$. This bid is the winning bid with probability one, if the valuation of the rival bidder is below ω_{k-1}, with probability 50%, if the valuation of the rival bidder is between ω_{k-1} and ω_k, and with probability zero, if it is above ω_k (see Figure 1). By inducing the bidder to bid $(\omega_k + \omega_{k+1})/2$ instead of $(\omega_{k-1} + \omega_k)/2$, the advisor increases the probability of winning against types $[\omega_{k-1}, \omega_k]$ from 50% to one and against types $[\omega_k, \omega_{k+1}]$ from zero to 50%. This implies that for the cut-off type of the advisor ω_k, the additional payoff from a higher probability of winning against types $[\omega_{k-1}, \omega_k]$ is equal to the cost from overpaying for the asset when the bidder wins against types $[\omega_k, \omega_{k+1}]$:

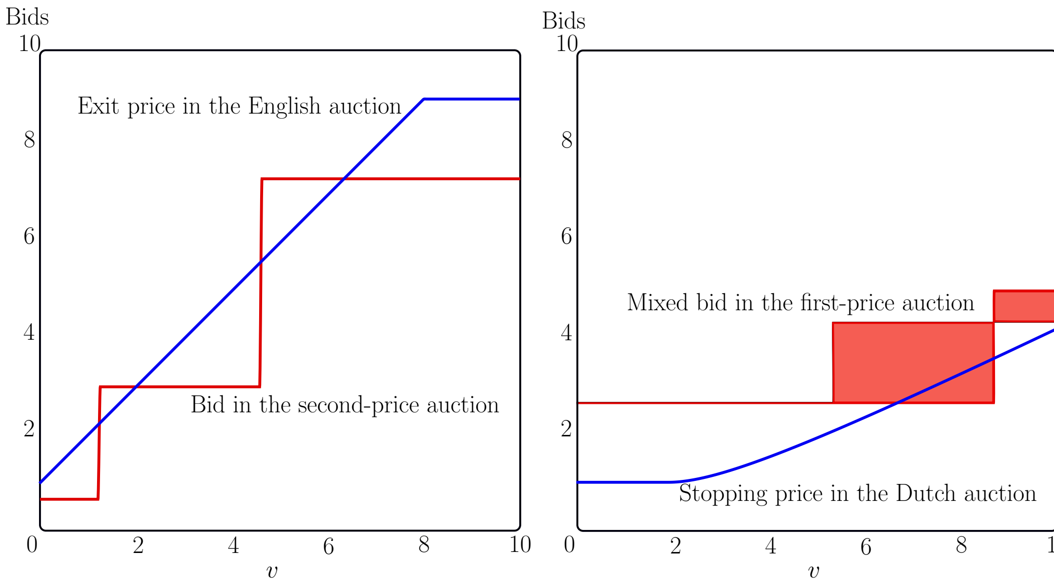
$$\frac{\omega_k - \omega_{k-1}}{10}\left(\omega_k + b - \frac{\omega_{k-1} + \omega_k}{2}\right) = \frac{\omega_{k+1} - \omega_k}{10}\left(\frac{\omega_k + \omega_{k+1}}{2} - \omega_k - b\right),\ k = 1, ..., N-1.$$

This indifference condition simplifies to

$$\omega_{k+1} = 2\omega_k - \omega_{k-1} + 2b,\ k = 1, ..., N-1.$$

When $b = 1$, the most informative equilibrium has three partitions, $\left[0, 1\frac{1}{3}\right]$, $\left[1\frac{1}{3}, 4\frac{2}{3}\right]$, and $\left[4\frac{2}{3}, 10\right]$. The corresponding bids are $\frac{2}{3}$, 3, and $7\frac{1}{3}$ (see Figure 2a). Since the lowest bid is below $b = 1$, this equilibrium satisfies the NITS condition: The weakest type of the advisor ($v = 0$) is better off communicating that the valuation is in $\left[0, 1\frac{1}{3}\right]$ than credibly revealing that $v = 0$. There exist two other equilibria: one with two partitions ($[0, 4]$ and $[4, 10]$) and the babbling equilibrium. Since the lowest bid (2 in the former case; 5 in the latter) exceeds $b = 1$, these equilibria do not satisfy the NITS condition. Indeed, the weakest type of the advisor ($v = 0$) is better off credibly revealing that $v = 0$ and ensuring that she never wins.

Next, consider the ascending-price auction. Now a bidder faces a stopping time problem: At each price p, she decides whether to quit the auction or stay for a little longer. When $b = 1$, there exists the following equilibrium in this auction. Suppose that an advisor with type v plays the threshold strategy of recommending to stay in the auction, if $p < v + 1$, and to quit once p hits $v + 1$ (see Figure 2a). Given this strategy, what is the best response of the bidder? If a bidder gets the recommendation to quit at price $p \in [1, 11]$, she infers that the valuation is equal to $v = p - 1$. Since the current price p is already past the valuation of the



(a) $b > 0$. *English and second-price auction.* (b) $b < 0$. *Dutch and first-price auction.*

Figure 2: Equilibrium bids in static and dynamic auctions. On the horizontal axis is value v. The red region in the second figure depicts the support of the bidder's mixed bidding strategy in the first-price auction.

bidder $p-1$, the best response of the bidder is to quit the auction immediately. If a bidder has received a sequence of recommendations to continue bidding, she trades off the value of waiting for more information against the possibility of overpaying for the asset. As the running price of the auction p increases, the support of bidder's posterior beliefs, $[p-1, 10]$, shrinks, meaning that the residual value of the advisor's private information declines with price p. Therefore, the best response of the bidder to the advisor recommending to stay in the auction is to stay in the auction, as long as $p \leq \hat{p}$, given by

$$0 = \mathbb{E}\left[v | v \geq \hat{p} - 1\right] - \hat{p},$$

which implies $\hat{p} = 9$. Intuitively, $\hat{p} = 9$ is exactly the price at which the bidder is indifferent between winning the auction and getting the valuation of 9 on average (when the auction reaches this price, the bidder's posterior is that $v \in [8, 10]$) and quitting it.

It turns out that this is the unique equilibrium satisfying the NITS criterion. To see this, consider why the equilibrium analogous to the equilibrium with three partitions in the second-price auction violates the NITS condition in the ascending-price auction. In this equilibrium, after the price passes $p = \frac{2}{3}$, the lowest type that remains in the auction is

$v = 1\frac{1}{3}$. According to the equilibrium, no bidder drops out after $p = \frac{2}{3}$ until p reaches 3. This implies that the advisor with type $v = 1\frac{1}{3}$ gets a negative expected payoff, since he wins with probability 50% at price $p = 3$, if the rival bidder's type is in $\left[1\frac{1}{3}, 4\frac{2}{3}\right]$. If the lowest type of the advisor $v = 1\frac{1}{3}$ could credibly reveal itself before p reaches 3, he would do it, since it would lead to the bidder quitting the auction immediately. Hence, the equilibrium with three partitions does not satisfy the NITS condition. By the same logic, any equilibrium that satisfies the NITS condition has the property of separation up to a cut-off. The equilibrium with cut-off p=9 is the unique such equilibrium in this example.

As we have seen, the equilibrium in the ascending-price auction is very different from the equilibrium in the second-price auction. In the former, a bidder bids up to $\min\{v+1, 9\}$, when its valuation is v. In the latter, a bidder bids $\frac{2}{3}$ if $v \in \left[0, 1\frac{1}{3}\right]$, 3 if $v \in \left[1\frac{1}{3}, 4\frac{2}{3}\right]$, and $7\frac{1}{3}$ if $v \in \left[4\frac{2}{3}, 10\right]$. What does this imply for the comparison of revenues and efficiency? It is easy to see that the ascending-price auction is more efficient: Not only there is a separation of types up to $v = 8$, but the pooling interval $[8, 10]$ is contained in the pooling interval in the top partition in the second-price auction $\left[4\frac{2}{3}, 10\right]$. Indeed, the expected valuation of the winning bidder is $6\frac{49}{75}$ in the ascending-price auction and $6\frac{47}{135}$ in the second-price auction. Not only the ascending-price auction is more efficient, but it also generates higher expected revenues than the second-price auction: $4\frac{23}{75}$ versus $3\frac{88}{135}$. The comparison of revenues is not obvious at first glance, since one distribution of bids does not dominate the other (see Figure 2a). Nevertheless, higher expected revenues in the ascending-price auction is a general result.

Underbidding bias ($b = -1$). Consider the case of an under-bidding bias of advisors, $b = -1$. In this case, the most informative equilibrium in the second-price auction has partitions $\left[0, 5\frac{1}{3}\right]$, $\left[5\frac{1}{3}, 8\frac{2}{3}\right]$, and $\left[8\frac{2}{3}, 10\right]$. This is the unique equilibrium satisfying the NITS criterion.[13]

Unlike with $b = 1$, the ascending-price auction does not have the equilibrium in which advisors separate themselves up to a cut-of. To see this, suppose that an advisor with type v plays the threshold strategy of recommending to stay in the auction, if $p < v - 1$, and to quit, otherwise. If the bidder gets the recommendation to quit at price $\tilde{p} \in (0, 9]$, she updates her valuation to $\tilde{p} + 1$. Her best response is thus to stay in the auction until the price hits $\tilde{p} + 1$. Expecting that the bidder will not follow his recommendation, the advisor

[13] The equilibrium with two partitions ($[0, 6]$ and $[6, 10]$) violates the NITS criterion, because the weakest type (now the highest type, $v = 10$) is better off credibly revealing himself. This is because the advisor with type $v = 10$ prefers to win when she faces a rival that bids 8. For the same reason, the equilibrium with one partition $[0, 10]$ violates the NITS criterion.

is better off deviating from recommending the bidder to quit at $p = v - 1$. In other words, truthful communication (even up to a cut-off) is inconsistent with equilibrium for the same reason, it is inconsistent with equilibrium in the second-price auction. This asymmetry arises because the bidder can only adjust bids in one direction: She can improve past bids (wait until a higher price) and thus correct the advisor's underpayment bias, but she cannot do the reverse, that is, go back in time and quit at the lower price, and thus correct the advisor's overpayment bias.

However, the descending-price (Dutch) auction with $b = -1$ has some similarities to the ascending-price auction with $b = 1$. In the descending-price auction, a bidder faces a stopping time problem: At each price p, she chooses whether to accept it and buy the asset at price p or to wait until a marginally lower price and risk losing the auction. Let us construct an equilibrium, which is similar to the equilibrium in the ascending-price auction for the case of $b = 1$. Suppose that the bidder does not stop the auction, unless the advisor recommends to do it or unless the price hits some lower threshold $\underline{p}$. Given this strategy, the optimal price at which the advisor with type v sends a recommendation to stop, $\sigma(v)$, satisfies:

$$\sigma(v) = \arg\max_{p \geq \underline{p}} (v - 1 - p)\, \sigma^{-1}(p), \tag{4}$$

which represents the familiar trade-off between buying the asset for a lower price and having a lower probability of winning. Let v^* denote the lowest type that recommends to stop the auction before $\underline{p}$, and pick v^* and $\underline{p}$ so that the bidder is indifferent between buying the asset or not when the price hits $\underline{p}$, given her belief at that point:

$$\mathbb{E}\left[v | v \leq v^*\right] = \underline{p} = v^* - 1. \tag{5}$$

Combining with (4), we obtain $v^* = 2$, $\underline{p} = 1$, and $\sigma(v) = \frac{v^2 - 2v + 4}{2v}$ (see Figure 2*b*). If a bidder receives a recommendation from her advisor to stop the auction at price $\tilde{p} \in (1, 4.2)$, she infers that the valuation is $\sigma^{-1}(\tilde{p})$. Since $\tilde{p}$ is already below the bidder's optimal stopping point, her best response is to stop the auction immediately.[14] If a bidder has received a sequence of recommendations to continue staying in the auction, she trades off the value of waiting for more information against the possibility of losing the auction. As the running price p goes down, the bidder's posterior belief about the valuation, $[0, \sigma^{-1}(p)]$ shrinks, and her best response is to wait for the recommendation of the advisor until p gets too low, which

[14]The fact that the advisor's optimal stopping price is below the bidder's optimal price follows from the single-crossing property of payoff function (4).

happens to be $\underline{p} = 1$ in this example.

As in the bias for overbidding, the dynamic aspect of communication is crucial for the better information transmission. In particular, we show that the first-price auction, which without the conflict of interest is strategically equivalent to the Dutch auction, is equivalent in terms of information transmission and revenue to the second-price auction. More precisely, in the first price auction the advisor communicates the partition $\left[0, 5\frac{1}{3}\right]$, $\left[5\frac{1}{3}, 8\frac{2}{3}\right]$,$\left[8\frac{2}{3}, 10\right]$, and the bidder plays a mixed bidding strategy (see Figure 2*b*).

Can we conclude that the descending-price auction is more efficient and generates higher revenues than static auctions when advisors are biased for underbidding, like we did with the ascending-price auction in the case of an overbidding bias? The answer to the first question is a "yes", but to the second one is a "no." The descending-bid auction is indeed more efficient than static auctions: Not only there is a separation of types in $[2, 10]$, but the pooling interval $[0, 2]$ is smaller than the pooling interval in the bottom partition in the first-price auction, $\left[0, 5\frac{1}{3}\right]$. Thus, the descending-bid auction is efficient if the highest valuation is above 2, and results in a lower probability of misallocation, if it is below 2. Indeed, the expected valuation of the winning bidder is $7\frac{2}{25}$ in the descending-price auction and $6\frac{47}{135}$ in the first- and second-price auctions. However, the descending-price auction generates lower revenues than the first-price auction: approximately 2.7 versus $3\frac{88}{135}$. Thus, the first-price auction dominates if the goal of the designer is expected revenues, but the descending-price auction dominates if the goal is efficiency.

The opposite implications for expected revenues occur for the following reason. When advisors are biased for overbidding, the seller's goal of higher expected revenues is aligned with the bias of advisors. In contrast, when advisors are biased for underbidding, the bias goes in the opposite direction from the seller's goal of higher expected revenues.

4 Static Auctions

This section shows that the revenue equivalence theorem extends to the setting when the interests of bidders and advisors are not aligned ($b \neq 0$), if the auction is static in the sense that it admits only a single round of communication between bidders and their advisors. For a rich class of static auctions, we characterize equilibrium communication and show that there is necessarily an efficiency loss due to imperfect communication.

4.1 Revenue Equivalence

After a bidder gets a message $\tilde{m}$ from her advisor, she updates her belief about the value of the asset and decides on the bid. By risk-neutrality, the bidder cares only about her posterior expected value of the asset, which we refer to as her *type* $\theta \equiv \mathbb{E}[v|\tilde{m}] \in [\underline{v}, \overline{v}]$. Let F_θ denote the distribution of a bidder's types, implied by equilibrium at the communication stage (by the symmetry assumption, F_θ is the same for all bidders).

We start by defining a class of standard auctions for which the revenue equivalence theorem holds for arbitrary distributions of values, as shown by Che and Gale (2006):

Definition 2. *[Che and Gale, 2006] Call the static auction a* standard auction with continuous payments *if it satisfies the following conditions:*

1. *the highest bid wins and ties are broken randomly;*

2. *the payment depends only on the bidder's own bid and the highest competing bid, i.e. bidder i pays $\tau_w(\beta_i, \beta_{m(i)})$, if she wins, and $\tau_l(\beta_i, \beta_{m(i)})$, if she loses, where $\beta_{m(i)} = \max_{j \neq i} \beta_j$;*

3. *$\tau_w(0,0) = \tau_l(0, \cdot) = 0$ and $\tau_k(\cdot, \beta_{m(i)})$ is continuous for $k = w, l$, in the relevant domain.*

This is a large class of auctions that includes many popular auctions, such as first-price, second-price, and all-pay auctions. The next theorem establishes revenue equivalence for auctions in this class in our model, when bidders rely on the advice of biased advisors.

Theorem 1. *Suppose that $b \neq 0$ and there is a single round of communication. For any symmetric PBE in a standard auction with continuous payments there exists a symmetric PBE of the second-price auction that generates the same allocation, expected revenue, and distribution of bidders' expected values, F_θ, after the communication stage.*

Our main question is whether the choice of the auction format affects information transmission and through it expected revenues and efficiency. Theorem 1 tells us that it does not if one restricts attention to static auctions. For example, one does not get a better information transmission or higher revenues by switching between first- and second-price auctions.

The proof of Theorem 1 is based on two observations. First, Che and Gale (2006) establish a payoff equivalence for arbitrary distributions of bidders' values: for a fixed distribution of values F_θ, for any bidder's type θ, the expected probability of winning and expected payments are the same across standard auctions with continuous payments. Second, the advisor's incentives to send one message or the other depend on how the information conveyed through messages affects the probability of winning and expected transfer. Since they are the same, the advisor's problem of choosing what message to send is also the same. Thus, if communication strategy m is an equilibrium in some standard auction with continuous payments, it is also an equilibrium in the second-price auction.

4.2 Characterization

Because of payoff equivalence, established in Theorem 1, it is sufficient to study the second-price auction, which has a simple bidding equilibrium: each bidder bids her updated expected valuation of the asset. Given this, it is convenient to refer to messages as bid recommendations and denote the equilibrium messages by conditional expected values $\mathbb{E}[v|\tilde{m}]$. The next theorem characterizes the set of symmetric equilibria of the communication game:

Theorem 2. *Suppose $\overline{v} < \infty$ and $b \neq 0$. Any equilibrium at the communication stage has a partition form, in which types $v \in [\omega_{k-1}, \omega_k)$ send the same message and induce the same bid $m_k = \mathbb{E}[v|v \in [\omega_{k-1}, \omega_k)]$. In an equilibrium with K partitions, thresholds $(\omega_k)_{k=0}^K$ satisfy $\omega_0 = \underline{v}$, $\omega_K = \overline{v}$, and*

$$G(\omega_{k-1}, \omega_k)(1 - \Lambda_k)(\omega_k + b - m_k) = -G(\omega_k, \omega_{k+1})\Lambda_{k+1}(\omega_k + b - m_{k+1}). \tag{6}$$

where

$$\Lambda_k = \frac{1}{G(\omega_{k-1}, \omega_k)} \left(\sum_{n=1}^{N-1} \binom{N-1}{n} \frac{F(\omega_{k-1}, \omega_k)^n F(\omega_{k-1})^{N-1-n}}{n+1} \right) \tag{7}$$

is the probability of winning conditional on a tie at bid m_k.

Theorem 2 implies that in static auctions the conflict of interest results in a coarsening of information transmitted from the advisor to his bidder. As a consequence, the asset is sometimes allocated inefficiently. Theorem 2 is a counter-part of Theorem 1 in Crawford and Sobel (1982) and relies on the same argument, but does not follow from it directly. The difference is that in our case, the payoffs from the communication game are endogenous: the action of a receiver is a bid, and its attractiveness depends on information transmission between other sender-receiver pairs. Equation (6) is the condition that advisor with valuation ω_k is indifferent between sending messages that induce the bidder to bid m_k and m_{k+1}. The left-hand side of (6) is the advisor's benefit from having the bidder bid m_{k+1} instead of m_k:

it increases the probability of winning a tie from Λ_k to 1. The right-hand side of (6) is the cost associated with a higher bid: the bidder can overpay the advisor's maximum willingness to pay, if one of the rival bidder also submits bid m_{k+1}.

Chen et al. (2008) show that in the standard cheap-talk model, there always exist equilibria satisfying the NITS condition, and it selects equilibria that are sufficiently informative. This result also holds in our model:

Proposition 1. *Suppose $b \neq 0$. The equilibrium with the highest number of partitions satisfies the NITS condition.*

5 Ascending-Price Auction

This section solves for equilibria of the ascending-price auction when the bias is for overbidding ($b > 0$) and shows that it dominates static auctions from Section 4 in terms of allocative efficiency and expected revenues.

It turns out that it is sufficient to look for equilibria in which the advisor gives a real-time recommendation of the action ("quit" or "stay") to the bidder, both advisors' and bidders' strategies are of the threshold form, and bidders follow the recommendations of their advisors on equilibrium path. We refer to these equilibria as equilibria in online threshold strategies:

Definition 3. *A PBEM in an ascending-price auction is* in online threshold strategies *if the strategies of each advisor and bidder satisfy:*

$$m(v,p,\mu) = \begin{cases} 1, & \text{if } p \geq \hat{p}(v,\mu), \\ 0, & \text{if } p < \hat{p}(v,\mu), \end{cases} \tag{8}$$

$$a(p,\tilde{\mu}) = \begin{cases} 1, & \text{if } p \geq \bar{p}(\tilde{\mu}), \\ 0, & \text{if } p < \bar{p}(\tilde{\mu}), \end{cases} \tag{9}$$

for some $\hat{p}(\cdot)$ and $\bar{p}(\cdot)$, where $\tilde{\mu}$ denotes the posterior belief of the bidder at price p, having observed her advisor's message in this round. Functions $\hat{p}(\cdot)$ and $\bar{p}(\cdot)$ are such that on equilibrium path the bidder exits the auction the first time her advisor sends message $m = 1$.

Intuitively, at any price p, the advisor sends a binary message to her bidder recommending to quit the auction immediately or stay in it, and on equilibrium path, the bidder follows the advisor's recommendation.

The following lemma shows that the restriction to equilibria in online threshold strategies is without loss of generality:

Lemma 1. *For any pure-strategy PBEM there is a PBEM in online threshold strategies that results in the same bidding behavior on equilibrium path. For any pure-strategy PBEM that satisfies NITS there is a PBEM in online threshold strategies that satisfies NITS and results in the same bidding behavior on equilibrium path.*

The first statement that any equilibrium with a general communication strategy has an equivalent in online threshold strategies. The proof is the manifestation of the sure-thing principle (Savage, 1954), stating that if an action is optimal for a decision-maker in every state, then it must be optimal if she does not know the state. Intuitively, since the advisor's information is only relevant for determining the price level at which the bidder quits the auction, any equilibrium quitting strategy can be achieved by the advisor delaying communication as much as possible, which occurs when she sends a recommendation to quit immediately when the price hits the level at which the bidder is supposed to quit. The second statement implies that the NITS condition is not stronger in equilibria in online threshold strategies than in general: If an equilibrium with a general communication strategy satisfies NITS, then an equivalent in online threshold strategies also satisfies NITS.

5.1 Characterization

This subsection shows that when the bias is for overbidding ($b > 0$), all equilibria of the English auction satisfying the NITS condition are in capped delegation strategies, defined as:

Definition 4. *Online threshold strategies in the English auction are* capped delegation strategies *if for some v^*:*

- $\hat{p}(v, \mu) = \min\{v, v^*\} + b$, *i.e., the advisor with type $v \leq v^*$ starts recommending to quit the auction when the running price reaches her most preferred exit price;*
- *the bidder quits the auction if either the running price increases to $v^* + b$ or she receives message "quit" from the advisor, whichever happens earlier.*

When players follow capped delegation strategies, advisor's types below v^* fully separate over time, while types above v^* pool with each other, since the bidder stops following recommendations when the running price reaches $v^* + b$. If the advisor were submitting the bids herself, she would stay in the auction until price $v + b$. Thus, even though the bidder makes bidding decisions himself, he essentially delegates it to the advisor with the restriction that she cannot stay in the auction beyond price $v^* + b$ ("cap"). The next theorem shows that *all* equilibria in the English auction satisfying NITS are in capped delegation strategies.

Theorem 3. *Suppose that $b > 0$. Then, any PBEM in the English auction that satisfies the NITS condition is in capped delegation strategies with cutoff v^* satisfying:*

$$\begin{aligned} &\textit{if } v^* \in (\underline{v}, \overline{v}), \textit{ then } b = \mathbb{E}[v|v \geq v^*] - v^*; \\ &\textit{if } v^* = \underline{v}, \textit{ then } b \geq \mathbb{E}[v|v \geq v^*] - v^*; \\ &\textit{if } v^* = \overline{v}, \textit{ then } \overline{v} = \infty \textit{ and } b \leq \lim_{s \to \infty} \mathbb{E}[v|v \geq s] - s. \end{aligned} \tag{10}$$

As we see from Theorems 2 and 3, equilibria in the English auction are very different from equilibria in the second-price auction. The difference arises because communication occurs over time. When the advisor recommends the bidder to quit the auction at the current price p, the bidder learns that his valuation is $p - b$. Since it is below the current price, the bidder exits the auction immediately. As the bidder gets a sequence of recommendations to stay in the auction, he updates his belief that his valuation is not too low. His decision whether to continue bidding trades off the benefit of waiting for more information about his valuation against the cost of possibly overpaying. When the current running price is below $v^* + b$, the former dominates, so the bidder follows the advisor's recommendation to stay in the auction. However, when the price reaches a high enough level, $v^* + b$, the bidder learns that the valuation is in a narrow enough partition $[v^*, \overline{v}]$, so that the option value of staying in the auction and waiting for more information does not exceed the cost of possibly overpaying. At this price, the bidder quits the auction regardless of what the advisor recommends.[15] These equilibria are not possible in the second-price auction because of the commitment problem. Indeed, if the advisor with type $v < v^*$ followed the strategy of recommending the bidder to bid $v + b$, the bidder would only bid v. The English auction makes this possible by giving the advisor an opportunity to delay this recommendation to the point when the bidder bid above his (unknown) willingness to pay.

The English auction also has equilibria that are not in capped delegation strategies. In particular, it has equilibria that are outcome-equivalent to equilibria of the second-price auction from Theorem 2. To construct them, we can simply specify that types in $[\omega_{k-1}, \omega_k)$,

[15] Clearly, the communication strategy is optimal for the advisor: It implements the advisor's unconstrained optimal bidding strategy of bidding up to $v + b$, if her type is low enough, and it is impossible to induce the bidder into bidding above $v^* + b$.

which send message m_k in the second-price auction, recommend that the bidder stays in the English auction until price m_k and quit after that. Theorem 3 implies that the NITS condition rules out these equilibria. Intuitively, at the start of the auction, the threshold type ω_k of the advisor is indifferent between the bidder exiting at prices m_k and m_{k+1}: m_{k+1} implies certain winning against types in $[\omega_{k-1}, \omega_k)$ but entails the risk of winning at a higher price m_{k+1} and paying above $\omega_k + b$ then. However, as the running price in the auction exceeds m_k, the bidder learns that the stongest rival bidder would bid at least m_{k+1}, i.e., above the maximum willingness to pay of the advisor. Thus, at this stage type she is better off inducing the bidder to quit immediately, which violates the NITS condition.

For a very large class distributions, the equilibrium satisfying the NITS condition is unique. To see when this is the case, consider the option value to the bidder of following the advisor's recommendation up to price $v^* + b$. Consider a bidder in round $p \in (\underline{v} + b, v^* + b)$ who has not received a recommendation to exit the auction yet. From the fact that the auction reached this stage, the bidder infers that his valuation is in $[p - b, \bar{v}]$, and that there is at least one rival whose valuation is also in $[p - b, \bar{v}]$. Denoting the bidder's posterior probability that n rival bidders have valuations in $[p - b, \bar{v}]$ by $q_n(p)$ and the CDF of the maximum of n i.i.d. random variables distributed according to F by $G_n(\cdot)$, the bidder's option value of following the advisor's recommendation up to price $v^* + b$ can be written as

$$V(p) = \int_{p-b}^{v^*} \frac{1 - F(s)}{1 - F(p - b)} \left(\mathbb{E}[v|v \geq s] - s - b\right) \left(\sum_{n=1}^{N-1} q_n(p)\, dG_n(s|s \geq p - b)\right). \tag{11}$$

Intuitively, if the bidder wins when the strongest rival's valuation is $s < v^*$, he pays $s + b$ and gets, on average, $\mathbb{E}[v|v \geq s]$. The probability of this event is the probability that the strongest rival's valuation is s times the probability that the bidder's valuation is above s, corresponding to the last and the first terms of the expression, respectively.[16] From (11), we can see why v^* must satisfy $\mathbb{E}[v|v \geq v^*] = v^* + b$ when $v^* \in (\underline{v}, \bar{v})$. If $\mathbb{E}[v|v \geq v^*] < v^* + b$, then the bidder would prefer to exit the auction before price $v^* + b$, as the option value of waiting is negative in the neighborhood below $v^* + b$. Similarly, if $\mathbb{E}[v|v \geq v^*] > v^* + b$, then the bidder gets a positive payoff when he wins at a tie at price $v^* + b$. Therefore, the bidder, whose advisor has not recommended to exit before price $v^* + b$, would prefer to wait a little beyond price $v^* + b$, since this would lead to a jump in the probability of winning against a rivel with valuation in $[v^*, \bar{v}]$ to one.

[16] Equation (11) could also include the term, corresponding to the case of winning at a tie at price $v^* + b$. Since it equals zero by Theorem 3, we omit it.

To characterize v^*, we introduce the *mean residual lifetime* function $MRL(s) = \mathbb{E}[v|v \geq s] - s$, a well-studied function in industrial engineering and economics (Bagnoli and Bergstrom (2005)). It turns out that when either of the following conditions holds, the equilibrium is unique:

Assumption A. *$MRL(s)$ is strictly decreasing in s.*

Assumption B. *$MRL(s) > b$ for any $s \in [\underline{v}, \infty)$.*[17]

Decreasing $MRL(s)$ is a natural and intuitive property. In industrial engineering, where $MRL(s)$ captures the expected time before a machine of age s breaks down, decreasing $MRL(s)$ simply means that the machine gets less durable as it ages. In our context, it means that winning at a higher price is worse news for the bidder than winning at a lower price. It holds for many distributions, such as Uniform, Normal, Logistic, Extreme Value, and many others. The next proposition shows that if the MRL is decreasing on a finite support or on an infinite support with a low enough limit, then the equilibrium is unique, and the pooling region is non-empty. This generalizes the example of Section 3 to a large class of distributions.

Proposition 2. *Suppose that $b > 0$, Assumption A holds, and either $\bar{v} < \infty$ or $\bar{v} = \infty$ and $\lim_{v \to \infty} MRL(v) < b$. Then, the unique equilibrium cut-off v^* satisfies $v^* < \bar{v}$. Moreover, the equilibrium is babbling if and only if $MRL(\underline{v}) \leq b$.*

Monotonicity of $MRL(\cdot)$ implies that equation $MRL(v^*) = b$ has at most one solution. Furthermore, a strictly decreasing $MRL(\cdot)$ implies single-crossing: If cut-off type v^* satisfies $MRL(v^*) = b$, then the bidder's value of the option to wait for advisor's recommendation is strictly positive at any price prior to reaching this cut-off, i.e., for any $p < v^* + b$. This implies a unique equilibrium with non-empty separating and pooling intervals, if the bias is low. In contrast, if the bias is high, then the bidder's option value of waiting for the advisor's recommendation is never positive, so babbling occurs in this case.

The next proposition shows that if the distribution satisfies Assumption B, then the equilibrium satisfying NITS is also unique, but it features full separation:

Proposition 3. *Suppose that $b > 0$ and Assumption B holds. Then, the unique equilibrium cut-off v^* satisfies $v^* = \infty$. That is, the bidder always waits for the advisor's recommendation to quit the auction, which type v sends at price $v + b$.*

In this case, although the bidder has formal authority, he effectively fully delegates bidding to the advisor. Intuitively, Assumption B implies that no matter what the current price

[17] This condition can only hold if the support is infinite, $\bar{v} = \infty$.

is, the bidder is always quite optimistic in the sense of holding a posterior that its expected valuation exceeds the current price. For example, if valuations are distributed according to Exponential distribution with parameter λ and $b < \frac{1}{\lambda}$, the unique equilibrium satisfying NITS will feature the bidder always following the advisor's recommendation.[18]

It is worth noting that the equilibrium in the English auction can be informative (and even stronger, fully separating) even if only babbling equilibrium exists in the second-price auction. As an example, consider $N = 2$ and Pareto distribution of valuations, $F(v) = 1 - \left(\frac{1}{v}\right)^2$ on $v \in [1, \infty)$. If $b \in \left(1, \frac{4}{3}\right)$, only the babbling equilibrium exists in the second-price auction, but the English auction has a fully separating equilibrium. Intuitively, since $MRL\,(v) < b$ for low valuations v, winning is bad news for the bidder at the beginning of the auction. However, as the auction continues, the bidder eventually starts getting positive utility from winning. Since the latter is incorporated in the bidder's option value of following the advisor's recommendation, waiting becomes optimal early in the auction too.

5.2 Auction Comparison

We next compare the ascending-bid and static formats in their efficiency and revenues. We will say that an equilibrium in one auction format is *(strictly) more efficient* than an equilibrium in another auction if the former results in a (strictly) higher expected valuation of the winning bidder than the latter. As the next theorem shows, the ascending-bid auction is more efficient than the second-price auction (and by Theorem 1, any standard static auction with continuous payments):

Theorem 4. *Suppose that $b > 0$, and either Assumption A or B holds. Then, the pooling region (if it is non-empty) in the unique equilibrium satisfying NITS in the ascending-bid auction is finer than the top partition in any equilibrium of the second-price auction: $v^* \geq \omega_{K-1}$, with a strict inequality if $v^* > \underline{v}$, i.e., if there is no babbling in the ascending-bid auction. Therefore, the ascending-bid auction is more efficient.*

Higher efficiency of the ascending-bid auction stems from superior information transmission. This result is clear when the equilibrium features full separation, since ascending-bid auction is efficient in this case. However, it is more nuanced when the ascending-bid auction is inefficient. It can be seen from the indifference condition (6) that determines partitions in the second-price auction. For advisor with type ω_{K-1} to be indifferent, the highest bid must exceed the maximum willingness to pay of the advisor with type ω_{K-1}:

[18] Exponential distribution has a constant $MRL\,(s) = \frac{1}{\lambda}$ - the famous memoryless property.

$\mathbb{E}[v|v \geq \omega_{K-1}] > \omega_{K-1} + b$, or, equivalently, $MRL(\omega_{K-1}) > b$. This implies that the bidder's option value of waiting for the advisor's recommendation is positive at price $\omega_{K-1} + b$. Consequently, types just above ω_{K-1} would recommend the bidder to stay in the auction at this price, and the bidder would follow the recommendation, implying a finer pooling region and higher efficiency of the ascending-bid auction.

We next turn to the comparison of expected revenues. Let $\varphi(v) \equiv v + b - \frac{1-F(v)}{f(v)}$ denote the virtual valuation of advisor with type v. The next theorem shows that if the virtual valuation is increasing, then the ascending-bid auction generates higher expected revenues than the second-price auction:

Theorem 5. *Suppose that $b > 0$, $\varphi(v)$ is strictly increasing, and either Assumption A or B holds. Then the unique equilibrium satifying NITS in the ascending-price auction brings higher expected revenues than any equilibrium satisfying NITS in the second-price auction. It brings strictly higher revenues if $v^* > \underline{v}$, i.e., if there is no babbling in the ascending-bid auction.*

The result of Theorem 2 may seem surprising: It is a priori not clear if the ascending-bid auction should bring higher expected revenue. In the example in Section 3, the bids in the second-price and the ascending-bid auction are not clearly ordered (see Figure 2a). Moreover, Bergemann and Pesendorfer (2007) study the seller's problem of joint mechanism design (how to sell) and static information design (how much information about her valuation should each bidder learn), and they show that the optimal information structure is represented by partitions. One crucial difference of our model is that a switch from the second-price auction to the ascending auction results not only in higher efficiency but also in biased bidding: The bidder with any valuation $v < v^*$ stays in the auction even after the price passes its maximum willingness to pay v, because she does not know it yet.[19]

The key idea of Theorem 5 is to view the seller's problem as the problem of selling directly to informed advisors, where communication between advisors and bidders puts a restriction on the set of mechanisms that can be implementable. By the envelope formula in Myerson (1981), we can write the seller's expected revenues as the expected virtual valuation of the winning advsior less the payoff of the lowest type:

$$\mathbb{E}\left[\sum_{i=1}^{N} \varphi(v_i)\, p_i(\boldsymbol{v})\right] - NU_A(\underline{v}), \tag{12}$$

[19]Other differences are that the particular information structure arising via communication is not guaranteed to be optimal for the seller with full flexibility to design information that bidders get and that the second-price auction is not optimal with discrete types (see Che and Gale (2006)).

where $p_i(\boldsymbol{v})$ is the probability that bidder i wins the auction if the types are $\boldsymbol{v} = (v_1, \ldots, v_N)$ and $U_A(\underline{v})$ is the expected payoff of type $\underline{v}$ of the advisor from the auction. In (12), the auction format determines $p_i(\cdot)$ and $U_A(\underline{v})$. Higher efficiency of the ascending-price auction together with increasing virtual valuation implies the first term in (12) is higher in the ascending-price auction than in the second-price auction. The NITS criterion guarantees that the expected payoff of the lowest type is non-negative in the second-price auction, while it is zero in the ascending-bid auction. Together, these two effects imply that the ascending-bid auction generates higher expected revenues.

While the seller's problem of maximizing over all possible mechanisms goes beyond the scope of the paper, we can say that the ascending-bid auction with an appropriate reserve price is the globally optimal mechanism when it features full separation (i.e., when Assumption B holds). The argument is as follows. We know from Myerson (1981) that if the seller were to sell directly to informed advisers, the ascending-bid auction with a reserve price $r = \varphi^{-1}(0) + b$ would achieve the highest possible expected revenues. Since the seller's problem of selling to bidders relying on the advice of informed advisors is a constrained problem of selling to advisors directly, the optimal mechanism in the former cannot generate higher expected revenues than the optimal mechanism in the latter. Finally, when there is full separation, the ascending-bid auction in which the seller sells to bidders relying on the advice of advisors is identical to selling to advisors directly, since in equilibrium bidders behave as if they fully delegate bidding to advisors.[20] We summarize this result in the following corollary:

Corollary 1. *Suppose that $b > 0$, $\varphi(v)$ is strictly increasing, and Assumption B holds. Then, the ascending-price auction with a reserve price $r = \varphi^{-1}(0) + b$ is optimal.*

5.3 Comparative Statics

The characterization of equilibria in the English auction yields interesting comparative statics.

Proposition 4. *Suppose MRL is strictly decreasing. Then in PBEM of the English auction:*

1. *The auction efficiency, $\mathbb{E}[\max_i\{v_i\}]$, is decreasing in b.*
2. *The revenue is increasing in b in the neighborhood of $b = 0$ and decreasing in b in the neighborhood of $\hat{b}$ where $\hat{b} \equiv \mathbb{E}[v] - \underline{v}$.*

[20] Proposition 3 can be easily modified to allow for a reservation price by simply assuming that the seller starts increasing price from the reservation price.

3. *For any bias $b > 0$, there exists $N(b)$ and $\varepsilon(b) > 0$ such that for all $N > N(b)$, the seller prefers bias $b - \varepsilon(b)$ to bias b.*

4. *Equilibrium strategies do not depend on N.*

Naturally, as the conflict of interest decreases, the pooling region decreases and the equilibrium becomes more efficient. Interestingly, the revenue is generally non-monotone and the seller prefers moderate values of the advisor's bias. In particular, for low values of the bias, the seller prefers to increase the bias, while for sufficiently high values of the bias the seller prefers to lower the bias. Intuitively, there are two contrary forces that affect the revenue as b increases. On the one hand, as the bias increases, the distribution of advisors' values increases in the sense of first-order stochastic dominance, and hence, the revenue of the seller increases. On the other hand, the bias reduces the informativeness of the communication, as it lowers v^* (see equation (10)). Initially the seller benefits from the increase in the bias, as it shift the distribution of advisors' values to the right. However, as the bias increases, the cost of reduced communication eventually outweighs the benefit from a first-order stochastic shift of advisors' values. In the uniform example discussed above, the revenue of the seller equals $\mathbb{E}[\min\{v_1, v_2, v^*\}] + b = \frac{1}{100}v^*(100 - 10v^* + \frac{1}{3}v^{*2}) + b$ and equation (10) implies $v^* = \max\{0, 10 - 2b\}$. Therefore, the revenue has an inverse-U-shaped and attains maximum for $b \approx 3.54$.

By the third statement of Proposition 4, as the environment becomes more competitive, the seller prefers a lower bias of advisors. The intuition is that as N increases, it becomes more likely that the second order statistic is above v^*. However, pooling above v^* does not allow the seller to fully benefit from such a shift in the distribution of $\hat{v}$. Therefore, with the increase in N, the seller prefers lower biases as they allow for a finer discrimination of higher types.

By the last statement of Proposition 4, when MRL is decreasing, the number of bidders does not affect the communication. This is quite striking, as the following argument seems correct at first sight. When N goes to infinity, the highest value is close to the second highest value. Therefore, the bidder almost certainly overpays b and so, should get a negative expected utility from following the recommendation of the advisor. This reasoning however does not account for the pooling at the top, and in particular, the realizations of types when the second highest value is below v^*, but the highest value is above v^*.[21]

Proposition 4 sheds light on the communication in static auctions. In static auctions, the dependence of the communication on N is more convoluted than in the English auction.

[21] This can be clearly seen in the uniform example in subsection 3 with N bidders. The option value of

The number of bidders N enters recursion (6) in a complicated way and from it, it is not clear how N affects the communication partition. In particular, it is not a priori clear whether we approach perfect communication as $N \to \infty$. However, from Theorem 4, the communication partition in the English auction is finer than the partition generated by any static auction. This implies that the communication in static auctions does not become perfect as we increase the competitiveness of the auction.

5.4 Discussion

This section discusses the generality of the characterization and the role of commitment.

Generality The characterization in Theorem 3 can be extended in several directions. First, Theorem 3 can be immediately generalized with obvious changes of notations to the case of bidders with different biases b_i and value distributions F_i. The argument of Theorem 3 does not rely on the symmetry of the PBEM and it is still valid even if the distribution of competing bids is exogenous or is generated by an equilibrium play of bidders with different biases or distributions of values. In particular, when MRL functions are strictly decreasing for all bidders, all thresholds v_i^* are characterized by (10) where the expectation now is taken with respect to F_i and b is replaced by b_i.

Second, the analysis of the English auction does not change with the introduction of the reserve price which essentially boils down to the truncation of the distribution of values. It is a standard result in the auction literature that the reserve prices can increase the auction revenue (Myerson (1981)). In fact, in the English auction the optimal reserve price can be easily computed: it is simply the minimum of the optimal reserve price $\varphi^{-1}(0)$ and $v^* + b$.

Third, the characterization does not require the seller to know the magnitude of b. It is sufficient that it is common knowledge that there is a conflict of interest on the bidder's side and $b > 0$.

---

the bidder at stage 0 is given by

$$\int_0^{v^*} \left(\frac{v}{10}\right)^{N-1} (v - b - \mathbb{E}[s|s < v])dv + \int_{v^*}^{10} \left(\frac{v^*}{10}\right)^{N-1} (v - b - \tfrac{1}{2}v^*)dv,$$

where the first term accounts for the realizations of the bidder's type below v^*, while the second term for realizations of type above v^*. The argument above accounts for the first, but not the second term. The first term is indeed negative and converges to zero at speed $\left(\frac{v^*}{10}\right)^{N-1} \frac{1}{N}$, while the second term is positive and converges to zero at a slower speed of $\left(\frac{v^*}{10}\right)^{N-1}$. Thus, even when $N \to 0$, the bidder's value does not become negative, although it converges to zero, as the distribution of the second-highest type gets more and more concentrated at $\overline{v}$.

The generality of Theorem 3 is in contrast with the characterization of equilibria in static auctions in Theorem 2 which becomes more complicated for asymmetric bidders and reserve prices. For example, in asymmetric auctions threshold types are generally different for different bidders and pinned down by a complex system of recursive equations. Similarly, while one can easily compute an optimal reserve price in the English auction, this is not the case in static auctions. Indeed, if the seller restricts bids in the second price auction to be above some r, then this affects the equilibrium communication. After the introduction of the reserve price, the distribution of values is $F(\cdot|v \in [r, \overline{v}])$ and generally the partition of types in equilibrium changes, which in turn changes which types tie with each other in equilibrium. Hence, determining the optimal revenue price is more difficult in static auctions. We find the simplicity and robustness of the English auction to be an appealing property, which is in stark contrast with the complexity of static auctions.

Let us mention that the dynamic auction formats have advantages over static formats in the presence of the advisors' bias only when there are several bidders, but does not hold in negotiations where there is only one buyer. First, when there is only one buyer, it is efficient to allocate to this buyer by posting a price $\underline{v}$, as $\underline{v} \geq 0$. Second, under the increasing virtual valuation φ and $\mathbb{E}[v|v > \varphi^{-1}(0)] \geq \varphi^{-1}(0)$, in the negotiation, it is optimal to post price $\varphi^{-1}(0)$. Indeed, such a price is optimal when the seller sells directly to the advisor. Moreover, if the advisor simply tells the bidder whether to buy or not at price $\varphi^{-1}(0)$, under $\mathbb{E}[v|v > \varphi^{-1}(0)] \geq \varphi^{-1}(0)$, it is optimal for the bidder to follow the advisor's recommendation. Intuitively, when there is only one buyer, a coarse information is sufficient to implement both efficient and optimal allocations. Thus, there is no advantage of using dynamic auction formats. When there are several buyers, the seller needs to extract finer information about values to implement efficient and optimal outcomes. Therefore, there is a benefit in using the dynamic auction formats, as they enable a better information transmission.

Commitment We assume that there is no formal contract that allows the bidder to commit to a certain bidding strategy. Interestingly, in the English auction, the bidder essentially delegates bidding to the advisor with the cap on the bid level. We next show that even if the bidder could commit to a bidding strategy, it would not be different from the delegation strategy implemented without such a commitment.

Consider the following *auction with contracts.* Suppose that each bidder can commit to a contract that specifies a mapping from advisor's type announcements into bids in the auction.[22] In the beginning, bidders commit to such contracts. Then advisors communicate their private information, and bidders bid in the auction abiding to their contracts. The following proposition shows that offering a delegation contract, in which bidding is unrestricted up to a cutoff $v^* + b$, is an equilibrium of this game.

Proposition 5. *Suppose MRL is decreasing, f is differentiable, and* $\sup_{v \in [\underline{v}, \overline{v}]} (\ln f(v))' \leq \frac{1}{b}$*. Strategies described in Theorem* 3 *also constitute a PBE of the English or second-price auction with contracts.*

The proof follows from a general analysis of the delegation problem in Amador and Bagwell (2013).[23] Proposition 5 states that if all bidders in the English auction follow the advisor's recommendation up to threshold v^*, then the bidder would not gain from the ability to commit to a contract. In particular, committing to a coarse information transmission is not optimal, and the bidder prefers to use a finer information of the advisor even though it comes at cost of implementing an optimal bid of the advisor. Moreover, the same outcome can be also implemented in the second-price auction, which contrasts with the drastic difference between equilibrium outcomes without commitment.

There is normally a drastic difference between the outcomes with and without commitment. The second-price auction is a typical example. Without commitment only coarse communication is possible, while with contracts, there is a PBE in bidders commit to a delegation up to a cut-off. This difference is familiar from the difference between the outcomes of the cheap-talk games (no commitment) and the optimal delegation (commitment). Surprisingly, such a difference disappears once we consider dynamic auctions: the no-commitment outcome in the English auction is also a PBEM outcome in the English auction when bidders can commit to contracts. Our analysis suggests what type of commitment the bidder needs in order to achieve an optimal outcome. In the English auction, although the bidder does not have a full commitment, he has a partial commitment not to lower bids after the information is revealed which is sufficient to implement an optimal contract.

6 Bias toward Underbidding

Motivated by the empire-building and career concerns, we focused so far on the bias toward overbidding. This section considers the case $b < 0$ when advisors are biased toward under-

[22] By the revelation principle, restricting attention to direct mechanisms is without loss of generality.

[23] Their analysis does not apply directly, as their concavity assumptions on payoff functions are not satisfied in the English auction. In the proof, we perturb the contracting problem so that we can apply their results, and then relate the solution of the perturbed problem to the solution of our contracting problem.

bidding that could be more relevant in procurement auctions.[24] As in the case of overbidding bias, dynamic auction formats can attain higher efficiency than static auctions, but unlike the case of overbidding bias, can lead to a lower revenue.

Theorems 1 and 2 hold for both $b > 0$ and $b < 0$ and so, the analysis of static auctions is similar to the case of bias toward overbidding: the communication strategy in static auctions has a partition structure, and we can focus on the second-price auction in the comparison of static and dynamic auction.

Let us now turn to dynamic auctions. When $b < 0$, if the bidder knew the value, then she would submit a higher bid than the advisor. Then the English auction does not have an advantage over static auctions, as it only restricts the bidder to submit bids lower than the running price. However, now the Dutch auction can allow for a better information transmission, as it restricts the bidder from submitting bids above the running price. A relevant statistics of the distribution is the *mean-advantage-over-inferiors, MAI*, defined as $MAI(s) \equiv s - \mathbb{E}[v|v \leq s]$. Most of the commonly used distributions have strictly increasing MAI.[25] The next theorem constructs a partially revealing PBEM of the Dutch auction.

Theorem 6. *Suppose $\mathbb{E}[v] - \overline{v} < b < 0$ and MAI is strictly increasing. Let v^* be the solution to*

$$\mathbb{E}[v|v \leq v^*] = v^* + b. \tag{13}$$

There exists a PBEM of the Dutch auction characterized by $\{\sigma(\cdot), v^*\}$ as follows. The advisor of type $v > v^*$ sends message "stop" when the running price p reaches $\sigma(v) \equiv \mathbb{E}[\max\{\hat{v}, v^*\} + b|\hat{v} \leq v]$ and the advisor of type $v < v^*$ sends "stop" when the running price p reaches $\sigma(v^*)$. The bidder immediately stops the auction after she receives the message "stop" or when the running price p reaches $\sigma(v^*)$.

As with the overbidding bias, the PBEM communication strategies constructed in Theorem 6 are fully separating above v^* and drastically differ from the partition equilibrium in static auctions. The partially perfect separation is possible because the Dutch auction restricts bidders not to increase the bids. Because of the underbidding bias of the advisor, the optimal price of stopping the auction for the bidder is higher than for the advisor. Thus, it is optimal for her to stop immediately after she gets a recommendation from the advisor.

[24] The bias toward underbidding is also relevant in takeover contests when the management has the "quiet life" preference (Bertrand and Mullainathan (2003)): incorporating additional business requires additional effort from managers and managers prefer not to increase the size of the firm.

[25] In particular, all distributions with a strictly concave log of c.d.f. have increasing MAI (see Bagnoli and Bergstrom (2005) for related results and a list of distributions with log-concave c.d.f.).

Moreover, increasing MAI implies that before the price reaches $v^* + b$, the option value of waiting for a recommendation exceeds the utility of winning at a current running price.

Function $\sigma(v)$ in Theorem 6 is the equilibrium bidding strategy in the Dutch auction if bids were submitted directly by advisors, and it is implemented for $v > v^*$. However, unlike the case $b > 0$, the pooling happens at the bottom of the distribution, not at the top, and it always occurs in the PBEM constructed. The reason for this is that at a certain stage, the uncertainty of the bidder about her value is sufficiently reduced. Then the bidder prefers stopping the auction immediately to guarantee the victory, rather than trying to win at a lower price, but facing the risk of loosing the auction.

We next turn to the comparison of auction formats.

Theorem 7. *Suppose $b < 0$. The PBEM of the Dutch auction in Theorem* 6 *is more efficient than any PBE of the second-price auction.*

As in the case of overbidding bias, because of the better separation in the communication strategy, the Dutch auction is more efficient than any static auction. This is true despite that there is a pooling at the bottom in the Dutch auction. We show that generally the pooling region below v^* is always smaller than any first element of the partition equilibrium in any static auction, i.e. $v^* \leq \omega_1$.

Unlike the case $b > 0$, the revenue comparison is ambiguous when the advisor is biased toward underbidding. We can proceed as in Theorem 5 to break down the expected revenue into two parts as in (12): the part increasing with the auction efficiency and the part decreasing with the rent of the lowest type. Because of the higher efficiency the first term in (12) is higher in the Dutch auction. However, the rent of the lowest type is lower in the second-price auction. Indeed, because the first element of the partition in the second-price auction is larger than the pooling region in the Dutch auction ($\omega_1 > v^*$), type $\underline{v}$ wins with a lower probability in the Dutch auction and he also pays a lower price conditional on winning.[26] Therefore, the second-price auction may bring higher revenue than the Dutch auction, if the second term in (12) dominates the first term. This was indeed the case in the uniform example in Section 3.

Finally, notice that the NITS condition does not play as important role in the revenue comparison as in the case of $b > 0$. When $b < 0$, the weakest type of advisor is the highest type remaining in the game in round p. NITS puts restrictions on the utility of the highest type, while for the revenue comparison, the utility of the lowest type matters.

[26] The advisor of type $\underline{v}$ gets negative profit in both auctions, and pays $m_1 = \mathbb{E}[v|v < \omega_1]$ in the second-price auction, and $v^* + b = \mathbb{E}[v|v \leq v^*]$ in the Dutch auction. Since $\omega_1 > v^*$, the latter price is smaller.

7 Quantitative Example: Auctions of Companies

In this section, we assess the quantitative implications of our analysis, applying the model to auctions of companies. The analysis of Section 4 proves that the ascending-price auction dominates static mechanisms in both efficiency and revenues. However, these results do not imply that the difference is meaningful quantitatively. Suppose that each bidder i is a firm, consisting of the board of directors and the manager. The board has formal authority over submitting bids but has no information about firm's valuation of the target v_i, except for the prior distribution. The manager knows v_i, but has a bias $b > 0$ for overpayment.

To get a plausible value of b, we use the following argument. There is a strong empirical evidence that the compensation of CEO and other top executives is increasing in the absolute size of the firm. This dependence leads to their bias for overpaying for the target. On the other hand, overpaying for the target results in the destruction of firm value and ultimately in a poor performance of the acquirer's stock price. Since the wealth of top managers is sensititve to their company's stock price, there is a limit to which they are willing to overpay for the target. Bias b is the point at which the positive effect on compensation of higher firm size is exactly offset by the negative effect on compensation due to firm value destruction. To get the estimate of b, we use CEO compensation regressions from Harford and Li (2007) and the characteristics of the typical deal from Betton et al. (2008). Since the market leverage ratio of the median target is 13.1% and the median ratio of the deal size to the acquirer's assets is 31%, the ratio of the deal size to the acquirer's equity for a typical deal is $31\% \times \frac{1}{0.869} = 35.67\%$. Since the median acquisition premium is 39%, the ratio of the pre-deal target's equity to the acquirer's equity is $35.67\% \times \frac{1}{1.39} = 25.66\%$. Assume that after the acquisition, the sales of the combined company increase by the same amount, i.e., by 25.66% in perpetuity. Using the estimate of Harford and Li (2007), this increase in sales leads to an increase in the acquirer's CEO compensation by $0.435 \times \log(1 + 0.2566) = 4.32\%$ every year. In addition, acquiring the target is associated with an increase in the CEO compensation of 3.7%, irrespectively of the increase in sales, in the year of the acquisition. Thus, the positive effect of acquiring the target on CEO compensation for a typical deal is 8.02% in the year of the deal and 4.32% in every subsequent year. Using the expected tenure of 6 years and discounting at 10%, the present value of the positive effect is 22.52% of the CEO's annual compensation. On the other hand, overbidding by b (normalizing the pre-acquisition value of the target's equity to one), reduces the acquirer's equity value by $b \times 0.2566$. Since the porftolio value of equity incentives is 9.5 times the CEO annual pay (Table II in Harford and Li (2007)), the negative effect on the CEO wealth is $9.5 \times b \times 25.66\%$ of the CEO's annual compensation. The estimate of overpayment bias b is thus $\frac{0.2252}{9.5\times 0.2566} = 9.2\%$.[27] For example, if the value of target under its current ownership is \$1billion and the true value of the target to the acquirer is \$1.4 billion, the maximum willingness to pay for the target by the CEO is \$1.492 billion.

To get a plausible distribution of valuations, we use the estimates from Gorbenko and Malenko (2014). We normalize the value of the target under its current management to one. Using data on bids and assuming lognormal distribution, Gorbenko and Malenko (2014) estimate that the valuations of strategic bidders are distributed with parameters $\mu = 0.167$ and $\sigma = 0.258$. We use this distribution, truncated at one, for the distribution of valuations in our numerical example. We assume that there are $N = 4$ btidders.

The results are presented in Table 1. Firs, consider the ascending-price auction. The unique equilibrium satisfying the NITS criterion is that each bidder increases her bid until her advisor recommends to stop doing so. In other words, the estimated value of b is low enough to imply informative communication. Second, consider static auctions (for concreteness, the second-price auction). The most informative equilibrium in this case consists of three partitions, $[1, 1.12]$, $[1.12, 1.38]$, and $[1.38, \infty]$. The corresponding expected valuations are 1.06, 1.24, and 1.64. In the second-price auction, each advisor communicates that the valuation is in one of the three partitions, and the bidder submits one of the three expected values. The comparison of expected revenues is striking. The expected takeover premium is 49% in the ascending-price auction, which is 23% higher than the expected takeover premium in static auctions (21%). The comparison of efficiency is less striking, but the difference is also sizable: The expected valuation of the winning bidder is 1.65 in the ascending-price auction, but 1.57 in the static auction. As the comparison of expected bidders' payoffs illustrates, an increase in revenues largely occurs because of the more aggressive bidding among bidders. Overall, we conclude that the result that the ascending-price auction is more efficient and brings more revenues than static auctions is quantitatively very large, at least for the application of auctions of companies.

8 Conclusion

This paper studies the interaction between the information transmission and bidding in auctions. We characterize equilibria in different auction formats and show the efficiency/revenue

[27] This value is likely an underestimate, since it ignores non-financial benefits of the acquirer's CEO, such as the preference for power and empire-building, and since the sales of the combined firm may exceed the sum of the individual companies' sales due to synergies.

	Ascending-price	Static	Ratio
Exp. Revenues	1.49	1.21	1.23
Exp. Valuation of Winner	1.65	1.57	1.05
Exp. Payoff of Bidder	0.04	0.16	0.25

Table 1: Expected Revenue and efficiency of ascending-price and static auctions.

comparison. In static auctions, the revenue-equivalence result holds giving, in particular, the equivalence of the first- and second- price auctions. However, dynamic auctions, such as the English and the Dutch auctions, are generally more efficient than static auctions. This happens because in dynamic auctions the set of bids available to the bidder shrinks over time. Therefore, by sending the information later in the game, the advisor can induce the bidder to choose a more favorable action and hence, would provide a more refined information to the bidder. Moreover, the English auction also dominates static auctions in terms of revenue when advisors are biased toward overbidding, the case most relevant empirically.

A Appendix

A.1 Proofs for Section 4

Proof of Theorem 1. Consider a standard static auction $\mathcal{A}$ with continuous payments and a symmetric PBE in it. Let $m_{\mathcal{A}} : [\underline{v}, \overline{v}] \mapsto M$ be the equilibrium communication strategy, $F_{\theta,\mathcal{A}}$ be the distribution of each bidder's types generated by $m_{\mathcal{A}}$, and $\beta_{\mathcal{A}} : \Theta_{\mathcal{A}} \mapsto \mathbb{R}_+$ be the equilibrium bidding strategy, where $\Theta_{\mathcal{A}}$ is the support of $F_{\theta,\mathcal{A}}$. Let $x(\theta)$ and $t(\theta)$ be type θ's equilibrium expected probability of winning and expected payment, respectively.

We first use the results of Che and Gale (2006) to argue that if bidders' types are drawn i.i.d. from $F_{\theta,\mathcal{A}}$, the symmetric PBE $\beta_{\mathcal{S}}$ in the second-price auction $\mathcal{S}$ implies the same expected probabilities of winning and payments $x(\theta)$and $t(\theta)$. Since this result follows directly from Che and Gale (2006), we simply outline the argument. Lemma 2 in Che and Gale (2006) shows that a symmetric equilibrium of a standard auction with continuous payments admits an efficient allocation, i.e., for any realization of bidders' types (which in our case are drawn i.i.d. from $F_{\theta,\mathcal{A}}$), a bidder with the highest type wins the auction. This implies that function $x(\theta)$ is the same across such auctions. Proposition 1 in Che and Gale (2006) shows that for standard auctions with continuous payments their conditions (A1) and (A2) hold. Condition (A1) implies that their inequality (3) holds as equality. This in conjunction with the envelope condition for the bidder's payoff (their equation (4)) and condition (A2) implies that function $t(\theta)$ is the same across standard auctions with continuous payments.

We next show that the communication strategy $m_{\mathcal{A}}$ is also an equilibrium communication strategy in the second-price auction. By contradiction, suppose that is is not. Then, there exists value v, such that the advisor is better off sending message m' instead of $m_{\mathcal{A}}(v)$. Let θ' denote the type generated by message m'. Since $x(\theta)$ and $t(\theta)$ are the same in both auctions, it must be that $(v+b)\,x(\theta') - t(\theta') > (v+b)\,x(\theta) - t(\theta)$, where θ' and θ denote the types generated by messages m' and $m_{\mathcal{A}}(v)$, respectively. However, this implies that the advisor must also be better off sending message m' instead of $m_{\mathcal{A}}(v)$in auction $\mathcal{A}$. Hence, $m_{\mathcal{A}}$ is not an equilibrium communication strategy in auction $\mathcal{A}$, which is a contradiction. Hence, $m_{\mathcal{A}}$ is also an equilibrium communication strategy in the second-price auction.

Thus, we have constructed a symmetric PBE in the second-price auction with the same communication strategy $m_{\mathcal{A}}$ as in $\mathcal{A}$. Moreover, we have shown that given that bidders' types are drawn i.i.d. from $F_{\theta,\mathcal{A}}$, the two auctions exhibit payoff equivalence (functions $x(\theta)$ and $t(\theta)$ are the same) and thus, yield the same expected revenues. Moreover, the two auctions allocate the asset to the bidder with the highest type θ.

□

Derivation of Λ_k. Define Λ_k as the probability of a bidder with bid $m_k = \mathbb{E}[v|v \in [\omega_{k-1}, \omega_k)]$ winning a tie, conditional on the tie taking place at bid $m_k = \mathbb{E}[v|v \in [\omega_{k-1}, \omega_k)]$. Without loss of generality, we refer to this bidder as bidder N and to his rivals as bidders $i \in \{1, 2, ..., N-1\}$. Since ties are broken randomly,

$$\Lambda_k = \mathbb{E}\left[\frac{1}{\tilde{n}_k + 1}|\hat{v} \in [\omega_{k-1}, \omega_k)\right],$$

where $\tilde{n}_k = \sum_{i=1}^{N-1} 1\{v_i \in [\omega_{k-1}, \omega_k)\}$ is a random variable, denoting the number of rival bidders with the same bid m_k. Re-writing,

$$\begin{aligned}
\Lambda_k &= \sum_{n=1}^{N-1} \frac{1}{n+1} \frac{\Pr[\tilde{n}_k = n, \hat{v} \in [\omega_{k-1}, \omega_k)]}{G(\omega_{k-1}, \omega_k)} \\
&= \sum_{n=1}^{N-1} \frac{1}{n+1} \frac{\Pr\left[\tilde{n}_k = n, \sum_{i=1}^{N-1} 1\{v_i < \omega_{k-1}\} = N-1-n\right]}{G(\omega_{k-1}, \omega_k)} \\
&= \sum_{n=1}^{N-1} \frac{1}{n+1} \frac{\begin{pmatrix} N-1 \\ n \end{pmatrix} F(\omega_{k-1}, \omega_k)^n\, F(\omega_{k-1})^{N-1-n}}{G(\omega_{k-1}, \omega_k)},
\end{aligned}$$

which coincides with expression (7) in Theorem 2.

Proof of Theorem 2. Denote by $q(\tilde{\beta})$ and $t(\tilde{\beta})$ the expected probability of winning and the expected payment, respectively, for a bidder submitting bid $\tilde{\beta}$ in the second-price auction given that other players follow equilibrium strategies m and β. Let $Q = \{q(\tilde{\beta}), \tilde{\beta} \in \mathbb{R}_+\}$ and $t(q) = \min_{\tilde{\beta}\in\mathbb{R}_+ : q=q(\tilde{\beta})} t(\tilde{\beta})$. Then the bidder and the advisor play the cheap-talk game with the bidder's action $q \in Q$ and payoffs

$$\text{Bidder} \quad : \quad qv - t(q), \tag{14}$$

$$\text{Advisor} \quad : \quad q(v+b) - t(q). \tag{15}$$

We first show that the equilibrium has an interval partition form. Suppose there is $\tilde{Q} \subseteq Q$ such that $\min \tilde{Q} < \max \tilde{Q}$ and $\tilde{Q}$ is dense in $\overline{Q} = [\min \tilde{Q}, \max \tilde{Q}]$. Then there is an interval of advisor's types $\tilde{V}$ that fully separate and induce all q in $\tilde{Q}$. Let $q', q \in \tilde{Q}$ and $v', v \in \tilde{V}$ be such that in equilibrium, v chooses q, and v' chooses q'. Then $v' \geq \frac{t(q')-t(q)}{q'-q} \geq v$ when $v' > v$ and $v' \leq \frac{t(q)-t(q')}{q-q'} \leq v$ when $v' < v$. By letting v' converge to v, we get that for all $v \in \tilde{V}$, $t'(q(v)) = v$ where $q(v)$ is the equilibrium action chosen by bidder with value v. For $v \in \tilde{V}$, the advisor's marginal utility at $q(v)$ is $v + b - t'(q(v)) > 0$ and so, the advisor prefers to induce a higher action than $q(v)$ which is a contradiction to the sequential rationality of the communication strategy. Therefore, there is no $\tilde{Q} \subseteq Q$ such that $\min \tilde{Q} < \max \tilde{Q}$ and $\tilde{Q}$ is dense in $\overline{Q} = [\min \tilde{Q}, \max \tilde{Q}]$ and so, to characterize equilibria of the second-price auction, we need to determine incentives of threshold types of the advisor ω_k. Consider any such type ω_k. In the second-price auction, a message is simply an expected value of the bidder m_k. Let $\hat{m}$ be the message of the highest bidder among $N-1$ opponents of the bider. From submitting a message m_k, type ω_k gets utility

$$G(\omega_{k-1})\mathbb{E}[\omega_k + b - \hat{m}|\hat{v} < \omega_{k-1}] + G(\omega_{k-1}, \omega_k)\Lambda_k(\omega_k + b - m_k),$$

where the expected utility from bidding m_k when the other bidders submit bids below m_k and when some bidders tie with the bidder is captured by the first and second terms, respectively. Here, Λ_k gives the expected number of other bidders with who tie on m_k and its expression is provided in the statement of the theorem. Analogously, from submitting a message m_{k+1}, type ω_k gets utility

$$G(\omega_{k-1})\mathbb{E}[\omega_k + b - \hat{m}|\hat{v} < \omega_{k-1}] + G(\omega_{k-1}, \omega_k)(\omega_k + b - m_k) + G(\omega_k, \omega_{k+1})\Lambda_{k+1}(\omega_k + b - m_{k+1}).$$

Type ω_k should be indifferent between the two which gives the equation (6). Thus, any PBE communication strategy can be described by a solution $(\omega_k)_{k=0}^{K}$ to the recursion (6) where $\omega_0 = \underline{v}$ and $\omega_{K+1} = \overline{v}$. We next show that K is bounded from above by some $\overline{K} < \infty$.

Claim 1. *If $\omega_{k+1} = \omega_k$, then $k = 0$ when $b > 0$ and $k = K$ when $b < 0$.*

Proof: Suppose by contradiction that $b > 0$ and $\omega_{k+1} = \omega_k$ for some $0 < k \leq K$ (the argument for $b < 0$ and $0 \leq k < K$ is symmetric). This implies that $G(\omega_k, \omega_{k+1}) = 0$ and so, from (6),

$G(\omega_{k-1}, \omega_k)(1-\Lambda_{k-1})(\omega_k + b - m_k) = 0$ which in turn implies that $\omega_k + b = m_k$ or $\omega_{k-1} = \omega_k$. If $\omega_{k-1} < \omega_k$, then $m_k < \omega_k < \omega_k + b$ which is a contradiction. If $\omega_{k-1} = \omega_k$, then choose j so that $\omega_{k-j-1} < \omega_{k-j} = \cdots = \omega_{k-1} = \omega_k = \omega_{k+1}$ and the argument proceeds as in the case $\omega_{k-1} < \omega_k$. *q.e.d.*

Claim 2. *If $b > 0$, then there exists $\varepsilon > 0$ such that for all k, either $\omega_{k+1} - \omega_k > \varepsilon$ and $0 < k \leq K$ or $\omega_{k+1} = \omega_k$ and $k = 0$. If $b < 0$, then there exists $\varepsilon > 0$ such that for all k, either $\omega_{k+1} - \omega_k > \varepsilon$ and $0 \leq k < K$ or $\omega_{k+1} = \omega_k$ and $k = K$.*

Proof: Again we prove the claim for $b > 0$ and the similar argument applies for $b < 0$. Suppose $\omega_{k+1} - \omega_k > 0$ and so, $0 < k \leq K$ by Claim 1. Since $\omega_k + b - m_k > \omega_k + b - m_{k+1}$, it follows from (6) and $\omega_{k+1} - \omega_k > 0$ that

$$\omega_k + b \leq \mathbb{E}[v|v \in [\omega_k, \omega_{k+1})]. \tag{16}$$

If to contradiction for any $\varepsilon > 0$, there were an equilibrium such that $\omega_{k+1} - \omega_k < \varepsilon$, then for such equilibrium $\mathbb{E}[v|v \in [\omega_k, \omega_{k+1})] \leq \omega_k + \varepsilon < \omega_k + b$ which would contradict (16) for $\varepsilon < b$. Thus, there is $\varepsilon > 0$ such that $\omega_{k+1} - \omega_k > \varepsilon$ for $0 < k \leq K$ *q.e.d.*

Claims 1 and 2 imply that there is at most a finite number of partitions in the communication strategy.

□

Proof of Proposition 1. The proof is an adaptation of the proof of Proposition 1 from Chen, Kartik, and Sobel (2008) for our problem. It is useful to introduce the following notations:

$$\Psi(\omega_{k-1}, \omega_k) = G(\omega_{k-1}, \omega_k)(1-\Lambda_k) = \sum_{n=1}^{N-1} \binom{N-1}{n} F(\omega_{k-1}, \omega_k)^n F(\omega_{k-1})^{N-1-n} \frac{n}{n+1},$$
$$\Phi(\omega_k, \omega_{k+1}) = G(\omega_k, \omega_{k+1})\Lambda_{k+1} = \sum_{n=1}^{N-1} \binom{N-1}{n} F(\omega_k, \omega_{k+1})^n F(\omega_k)^{N-1-n} \frac{1}{n+1}.$$

Denote $m(\omega_{k-1}, \omega_k) = \mathbb{E}[v|v \in (\omega_{k-1}, \omega_k)]$ and

$$H(\omega_{k-1}, \omega_k, \omega_{k+1}) \equiv \Psi(\omega_{k-1}, \omega_k)(\omega_k + b - m(\omega_{k-1}, \omega_k)) + \Psi(\omega_k, \omega_{k+1})(\omega_k + b - m(\omega_k, \omega_{k+1})). \tag{17}$$

Note that an equilibrium with K partitions is given by recursion $H(\omega_{k-1}, \omega_k, \omega_{k+1}) = 0$, with $\omega_0 = \underline{v}$ and $\omega_K = \overline{v}$. To prove the statement, we will show that if an equilibrium with K partitions, $\omega = (\omega_0, \omega_1, ..., \omega_K)$, violates the NITS condition, then for all $k = 1, ..., K$, there exists a solution to recursion $H(\omega_{k-1}, \omega_k, \omega_{k+1}) = 0$, with $k+1$ partitions, ω^k, that satisfies $\omega^k_0 = \underline{v}$, $\omega^k_k > \omega_{k-1}$, and $\omega^k_{k+1} = \omega_k$. After this result is established, the statement of the proposition follows from the

Figure 3: Illustration for the proof of Theorem 1.

following argument. By contradiction, suppose that the most informative equilibrium (i.e., one with $\bar{K}$ partitions) violates the NITS condition. Applying the result above for $k = \bar{K}$, there must be a solution to $H(\omega_{k-1}, \omega_k, \omega_{k+1}) = 0$ with $\bar{K}+1$ partitions satisfying boundary conditions $\omega_0^{\bar{K}} = \underline{v}$ and $\omega_{\bar{K}+1}^{\bar{K}} = \omega_{\bar{K}} = \bar{v}$. By Theorem 2, this is an equilibrium, which contradicts the statement that $\bar{K}$ is the highest number of equilibrium partitions.

We show the result by induction on k. For brevity, we consider only the case $b > 0$ here (case $b < 0$ is analogous). As an induction base, consider $k = 1$. If the equilibrium with K partitions $\omega = (\omega_0, \omega_1, ..., \omega_K)$ violates the NITS, it must be that $\underline{v} + b < m(\underline{v}, \omega_1)$, and hence, $H(\underline{v}, \underline{v}, \omega_1) < 0$. At the same time, $H(\underline{v}, \omega_1, \omega_1) > 0$, since $\omega_1 > m(\underline{v}, \omega_1)$ and $b > 0$. By continuity, there exists $x \in (\underline{v}, \omega_1)$ at which $H(\underline{v}, x, \omega_1) = 0$. Hence, the claim holds for $k = 1$: $\omega^1 = \left(\omega_0^1, \omega_1^1, \omega_2^1\right) = (\underline{v}, x, \omega_1)$ solves $H(\omega_{k-1}, \omega_k, \omega_{k+1}) = 0$ with $\omega_0^1 = \underline{v}$, $\omega_1^1 > \omega_0 = \underline{v}$, and $\omega_2^1 = \omega_1$. Consider the difference $H\left(\omega_k^k, \omega_k, \omega_{k+1}\right) - H(\omega_{k-1}, \omega_k, \omega_{k+1})$:

$$\begin{aligned}
&\Psi\left(\omega_k^k, \omega_k\right)\left(\omega_k + b - m\left(\omega_k^k, \omega_k\right)\right) - \Psi(\omega_{k-1}, \omega_k)(\omega_k + b - m(\omega_{k-1}, \omega_k)) \\
=&F(\omega_k)^{N-1}\left(\sum_{n=1}^{N-1}\binom{N-1}{n}\left(\frac{F\left(\omega_k^k, \omega_k\right)}{F(\omega_k)}\right)^n\left(\frac{F\left(\omega_k^k\right)}{F(\omega_k)}\right)^{N-1-n}\frac{n}{n+1}\right)\left(\omega_k + b - m\left(\omega_k^k, \omega_k\right)\right) \\
&- F(\omega_k)^{N-1}\left(\sum_{n=1}^{N-1}\binom{N-1}{n}\left(\frac{F(\omega_{k-1}, \omega_k)}{F(\omega_k)}\right)^n\left(\frac{F(\omega_{k-1})}{F(\omega_k)}\right)^{N-1-n}\frac{n}{n+1}\right)(\omega_k + b - m(\omega_{k-1}, \omega_k)).
\end{aligned}$$

Since $\omega_k^k > \omega_{k-1}$, we have two implications. First, $m\left(\omega_k^k, \omega_k\right) > m(\omega_{k-1}, \omega_k)$, implying $\omega_k + b - m\left(\omega_k^k, \omega_k\right) < \omega_k + b - m(\omega_{k-1}, \omega_k)$. Second, binomial distribution with success probability $\frac{F(\omega_{k-1}, \omega_k)}{F(\omega_k)}$ dominates binomial distribution with success probability $\frac{F\left(\omega_k^k, \omega_k\right)}{F(\omega_k)}$ in the sense of first-order stochastic dominance, implying $\Psi\left(\omega_k^k, \omega_k\right) < \Psi(\omega_{k-1}, \omega_k)$. Therefore, $H\left(\omega_k^k, \omega_k, \omega_{k+1}\right) - H(\omega_{k-1}, \omega_k, \omega_{k+1}) < 0$. Since $H(\omega_{k-1}, \omega_k, \omega_{k+1}) = 0$, we conclude that $H\left(\omega_k^k, \omega_k, \omega_{k+1}\right) < 0$.

On the other hand, since $\omega_k > m\left(\omega_k^k, \omega_k\right)$ and $b > 0$, we have $\omega_k + b > m\left(\omega_k^k, \omega_k\right)$, implying $H\left(\omega_k^k, \omega_k, \omega_k\right) > 0$. This, $H\left(\omega_k^k, \omega_k, \omega_{k+1}\right) < 0$, and continuity imply that there exists $x \in (\omega_k, \omega_{k+1})$ at which $H\left(\omega_k^k, \omega_k, x\right) = 0$. Since $\omega_{k+1}^k = \omega_k$, the same x satisfies $H\left(\omega_k^k, \omega_{k+1}^k, x\right) = 0$. That is, there exists a solution in which the (k+1)st partition ends at $\omega_{k+1}^k = \omega_k$ and the (k+2)nd partition ends at $x < \omega_{k+1}$. By continuity, there exists a solution to the recursion in which the (k+2)nd partition ends at any $\omega \in \left(\omega_{k+1}^k, \omega_{k+1}\right)$. By continuity, for one such ω, denoted ω_{k+1}^{k+1}, the (k+2)nd partition ends exactly at ω_{k+1}, i.e., $\omega_{k+2}^{k+1} = \omega_{k+1}$. Hence, there exists a solution to recursion $H(\omega_{k-1}, \omega_k, \omega_{k+1}) = 0$, with $k+2$ partitions, ω^{k+1}, that satisfies $\omega_0^{k+1} = \underline{v}$, $\omega_{k+1}^{k+1} > \omega_k$, and $\omega_{k+2}^{k+1} = \omega_{k+1}$. This completes the proof of the inductive step.

□

A.2 Proofs for Section 5

Proof of Lemma 1. The proof of the first statement follows the argument of the proof of Lemma IA.2 in Grenadier, Malenko, and Malenko (2016). Specifically, for any pure-strategy PBEM we construct an equilibrium in online threshold strategies that results in the same bidding behavior on equilibrium path.

Consider any pure-strategy equilibrium E with some strategies $\bar{m}(v, p, \mu)$ and $\bar{a}(p, \tilde{\mu})$. It implies an equilibrium exit price $\bar{\tau}(v)$, which is the price at which the bidder exits the auction, if the valuation is v, provided that the bidder and her advisor play the equilibrium strategies $\bar{m}(\cdot)$ and $\bar{a}(\cdot)$. Note that $\bar{\tau}(v)$ must be weakly increasing in v. To see this, suppose by contradiction that $\bar{\tau}(v_1) > \bar{\tau}(v_2)$ for some $v_1 \in [\underline{v}, \bar{v})$ and $v_2 \in (v_1, \bar{v}]$. Since the advisor's payoff from acquiring the asset at any price p is higher for type v_2 than for type v_1 ($v_2 + b - p > v_1 + b - p$), the advisor's continuation value from not exiting the auction at any price p cannot be lower for type v_2 than for type v_1. The payoff from exiting the auction at any current price p does not depend on the type and equals zero. Thus, $\bar{\tau}(v_2) \geq \bar{\tau}(v_1)$. Let $\varrho \equiv \{p : \exists v \in [\underline{v}, \bar{v}] \text{ such that } \bar{\tau}(v) = p\}$ be the set of prices at which the bidder exits the auction for some realization of v. It will be convenient to define $v_l(p) \equiv \inf\{v : \bar{\tau}(v) = p\}$ and $v_h(p) \equiv \sup\{v : \bar{\tau}(v) = p\}$ for any $p \in \varrho$. We extend the definition of $v_l(p)$ for any $p \notin \varrho$ by setting $v_l(p) \equiv \inf\{v : \bar{\tau}(v) \geq p\}$.

Consider an online threshold strategy of the advisor, $m(v, p, \mu)$, with $\hat{p}(v, \mu) = \bar{\tau}(v)$ and the following belief updating rule of the bidder. For any belief μ, price p, and message $\tilde{m}$ such that there is $v \in supp(\mu(h))$ with $m(v, p, \mu) = \tilde{m}$, belief μ is updated via the Bayes rule. Any other message $\tilde{m}$ (i.e., a message for which there is no $v \in supp(\mu(h))$ with $m(v, p, \mu) = \tilde{m}$) is treated as some message $\tilde{m}'$ for which there is some $v \in supp(\mu(h))$ with $m(v, p, \mu) = \tilde{m}'$, and belief μ is updated following message $\tilde{m}$ in the same way as following message $\tilde{m}'$.[28] Given this, the posterior belief of the bidder for any history h is as follows. A sequence of messages $m = 0$ for all prices $p' \leq p$ up to price p implies that the bidder's posterior belief is given by the prior distribution of valuations truncated from below at $v_l(p)$. A sequence of messages $m = 0$ for all prices $p' < p'' \in \varrho$ and message $m = 1$ at price $p'' \in \varrho$ and any history of messages after that results in the bidder's posterior belief given by the prior distribution of valuations truncated at $v_l(p'')$ from below and at $v_h(p'')$ from above. Any history involving off-equilibrium messages leads to the posterior belief equivalent to one of these two posterior beliefs by construction of the updating rule. Given this, consider an online threshold strategy of the bidder, $a(p, \tilde{\mu})$, with $\bar{p}(\tilde{\mu}) = \mathbb{E}[v | v \geq v_l(p)]$ for the posterior belief

[28] Intuitively, according to this updating rule, the bidder effectively ignores unexpected messages. As a consequence, it is sufficient to consider only deviations to on-path (expected) messages. Because no deviation to an off-path message can beneficial, we do not lose any equilibria by focusing on this belief updating rule.

$\tilde{\mu}$ in the history of the first kind (i.e., when the advisor never recommended quitting at one of prices $p \in \varrho$ in the past), and with $\bar{p}(\tilde{\mu}) = \mathbb{E}[v|v \in [v_l(p''), v_h(p'')]]$ for the posterior belief $\tilde{\mu}$ in the history of the second type (i.e., when the advisor recommended to quit the auction at price $p'' \in \varrho$). Let E' denote a combination of these online threshold strategies of the advisor and the bidder and the belief updating rule. Below we show that E' is indeed an equilibrium and that it results in the same equilibrium exit price $\bar{\tau}(v)$ as equilibrium E.

For the collection of strategies and beliefs E' to be an equilibrium, we need to verify the incentive compatibility (IC) conditions of the advisor and the bidder.

1 - IC of the advisor. First, we verify that the advisor is not better off deviating from (8) with $\hat{p}(v,\mu) = \bar{\tau}(v)$. Because of the above definition of the off-path beliefs, it is sufficient to consider only deviations to $m \in \{0,1\}$ at $p \in \varrho$. First, consider a deviation of type v to $m = 1$ at $p \in \varrho$ at which $p < \bar{\tau}(v)$. This deviation is equivalent to mimicking the communication strategy of type $v' : \bar{\tau}(v') = p$. Since mimicking the communication strategy of type v' is not profitable for type v in equilibrium E (otherwise, it would not be an equilibrium), it is also not profitable here. Second, consider a deviation of type v to $m = 0$ at $p = \bar{\tau}(v)$. Depending on her communication strategy at later prices, this deviation will result in exit at price $\bar{\tau}(v')$ for some $v' \geq v_h(\bar{\tau}(v))$. Hence, any such deviation is equivalent to mimicking the communication strategy of type v'. Since it is not profitable for type v in equilibrium E, it is also not profitable here.

2 - IC of the bidder after observing $m = 1$ at $p \in \varrho$ and $m = 0$ before. We argue that $\bar{p}(\tilde{\mu}) \leq p$ in this case, so the bidder's best response is to quit the auction immediately. Given this history, the bidder's posterior belief is that $v \in [v_l(p), v_h(p)]$. Because the bidder expects the advisor to follow (8) with $\hat{p}(v,\mu) = \bar{\tau}(v)$, she expects the advisor to send $m = 1$ at any later price. Since the bidder expects to not learn anything new about v, her optimal exit strategy is given by the expected valuation, i.e., $\mathbb{E}[v|v \in [v_l(p), v_h(p)]]$. It follows that the bidder exists immediately if $p \geq \mathbb{E}[v|v \in [v_l(p), v_h(p)]]$. Next, we show that $\bar{\tau}(p)$ is equilibrium E must satisfy this condition at any $p \in \varrho$. Since exiting at price p is optimal for the bidder for any realization $v \in [v_l(p), v_h(p)]$ of the valuation, it must be that $p \geq \mathbb{E}\left[v|\mathcal{H}_p^E\right]$ for any history $\mathcal{H}_p^E$ induced by equilibrium communication of the advisor with type $v \in [v_l(p), v_h(p)]$. It follows that $p \geq \max_{\mathcal{H}_p^E \in \mathbb{H}_p^E} \mathbb{E}\left[v|\mathcal{H}_p^E\right]$, where $\mathbb{H}_p^E$ denotes the set of such histories. Using the law of iterated expectations and fact that the maximum of a random variable cannot be below its mean,

$$
\begin{aligned}
p &\geq \max_{\mathcal{H}_p^E \in \mathbb{H}_p^E} \mathbb{E}\left[v|\mathcal{H}_p^E\right] \geq \mathbb{E}\left[\mathbb{E}\left[v|\mathcal{H}_p^E\right]|\mathcal{H}_p^E \in \mathbb{H}_p^E\right] \\
&= \mathbb{E}\left[v|\mathcal{H}_p^E \in \mathbb{H}_p^E\right] = \mathbb{E}\left[v|v \in [v_l(p), v_h(p)]\right].
\end{aligned}
$$

Therefore, when the bidder observes message $m = 1$ at $p \in \varrho$ for the first time, she finds it optimal to quit the auction immediately.

3 - IC of the bidder after observing a sequence of messages $m = 0$ up to price $p < \bar{\tau}(\bar{v})$. We argue that $\bar{p}(\tilde{\mu}) > p$ for any such history, i.e., it is optimal for the bidder to wait. Given this history, the bidder's posterior is that $v \in [v_h(p'), \bar{v}]$ for highest $p' \in \varrho$ satisfying $p' < p$. Consider equilibrium E and any history $\tilde{\mathcal{H}}_p^E$ induced by equilibrium communication of the advisor with type $v \in [v_h(p'), \bar{v}]$. Denote the set of such histories by $\tilde{\mathbb{H}}_p^E$ Since the bidder finds it optimal to wait, the payoff from waiting is weakly below the payoff from quitting the auction immediately (i.e., zero) for any such history $\tilde{\mathcal{H}}_p^E$. Since strategy profile E' results in weakly higher expected information revelation after price p, the fact that waiting is optimal for any history $\tilde{\mathcal{H}}_p^E \in \tilde{\mathbb{H}}_p^E$ implies that waiting is also optimal when the bidder expects the advisor to follow (8) with $\hat{p}(v, \mu) = \bar{\tau}(v)$.

Therefore, the collection of strategies and beliefs E' is an equilibrium. Furthermore, on equilibrium path, the advisor with type v recommends to quit the auction when the price reaches $\bar{\tau}(v)$, and the bidder exits the auction immediately. Therefore, E' results in the same bidding behavior as E.

The second statement of the lemma can be proved by contradiction. Consider equilibrium E that satisfies the NITS condition, and suppose that an equilibrium in online threshold strategies with the same bidding behavior violates with NITS. Therefore, there exists price p such that the advisor with type $v_l(p)$ is better off credibly revealing itself at that point than getting the expected (as of information at price p) payoff in equilibrium E'. Hence, the time-0 expected payoff of the advisor of type $v_l(p)$ from sending message $m = 0$ until price p and credibly revealing itself then exceeds the time-0 expected payoff of the advisor of type $v_l(p)$ in equilibrium E'. Now, consider equilibrium E, and the strategy of the advisor of type $v_l(p)$ to send equilibrium message $\bar{m}(v, p', \mu)$ for all $p' < p$ and to credibly reveal itself at price p (by definition of $v_l(p)$, type $v_l(p) = \inf\{v | v \in supp(\mu(h))\}$ for any history induced by this message strategy up to price p). Since bidding behavior of other bidders is the same and since the bidder's reaction to the advisor credibly revealing itself at price p is the same as in equilibrium E', the time-0 expected payoff of the advisor of type $v_l(p)$ from this strategy is the same as the time-0 expected payoff of the advisor of type $v_l(p)$ from sending message $m = 0$ until price p and credibly revealing itself then in equilibrium E', which is strictly higher than the time-0 equilibrium expected payoff of the advisor of type $v_l(p)$. Therefore, equilibrium E also violates the NITS condition, which is a contradiction. Therefore, there exists an equilibrium in online threshold strategies with the same bidding behavior as equilibrium E that satisfies NITS. □

Proof of Theorem 3. By Lemma 1, it is without loss of generality to focus on equilibria in online threshold strategies. In the proof of Lemma 1, we introduced function $\bar{\tau}(v)$, which denotes the equilibrium exit price of the bidder if the advisor's type is v. In an equilibrium in online threshold strategies, $\bar{\tau}(v)$ is also the first price at which the advisor with type v sends message "quit" to the bidder.

Any equilibrium generates partition Π of $[\underline{v}, \bar{v}]$ satisfying $\bar{\tau}(v) = \bar{\tau}(v')$ for any $v, v' \in \pi$ for any element $\pi \in \Pi$. Since $\bar{\tau}(v)$ is weakly increasing, any $\pi \in \Pi$ is an interval (possibly consisting of one element). We say that types in $\pi \in \Pi$ *pool* if $\bar{\tau}(v)$ is constant on $v \in \pi$, i.e., these types start sending message "quit" at the same price. We say that types in $[v', v'']$ *separate*, if $\bar{\tau}(v)$ is strictly increasing on $[v', v'']$, i.e., these types start sending message "quit" at different prices. Let Π^P and $\Pi^S \equiv [\underline{v}, \overline{v}] \backslash \Pi^P$ be the sets of all types that pool with some other type and that separate, respectively. Denote by $\partial \Pi^P$ the boundary of Π^P.

Babbling ($\bar{\tau}(v) = \mathbb{E}[v]\ \forall v$) is an equilibrium of the English auction, and it satisfies NITS if and only if $\mathbb{E}[v] \leq \underline{v} + b$. This proves case $v^* = \underline{v}$ of the theorem. Hence, we can consider the case in which there is a non-trivial information transmission in equilibrium.

Claim 3. *For any $\pi, \pi' \in \Pi^P$, π and π' are not adjacent.*

Proof: By contradiction, suppose that there are two adjacent intervals of types, π and π', such that $\bar{\tau}(v) = p\ \forall v \in \pi$ and $\bar{\tau}(v) = p'\ \forall v \in \pi'$. Without loss of generality, $p' > p$. Consider the advisor with type $\tilde{v}$ on the boundary of π and π'. By continuity, the advisor with type $\tilde{v}$ is indifferent between her bidder quitting the auction at prices p and p'. The benefit of the latter is winning against types in π, while the cost is risking to win against types in π' and paying p'. The indifference of type $\tilde{v}$ implies that $p' > \tilde{v} + b$. Consider running price $\frac{p'+p}{2}$. Type $\tilde{v}$ is the weakest remaining type at this price. Since $p' > \tilde{v} + b$, following her equilibrium strategy of waiting to send recommendation $m = 1$ until price p'generates negative expected payoff to the advisor at this point. In contrast, claiming that she is the weakest remaining type at the current price of $\frac{p'+p}{2}$ will lead to the bidder quitting immediately, yielding the payoff of zero to the advisor. This contradicts the NITS condition. *q.e.d.*

We next show that whenever types within an interval separate, they start recommending to quit the auction at their most preferred time.

Claim 4. *If $\bar{\tau}(v)$ is strictly increasing on (v', v''), then $\bar{\tau}(v) = v + b$ for any $v \in (v', v'')$.*

Proof: By contradiction, suppose there is $v \in (v', v'')$ with $\bar{\tau}(v) \neq v + b$. Then, either $\bar{\tau}(v) > v + b$ or $\bar{\tau}(v) < v + b$. First, consider the former case. Since v is interior, there exists a subset of (v', v'') of types $v + \varepsilon > v$ with positive measure with $\bar{\tau}(v) > v + \varepsilon + b$. Since $\bar{\tau}(\cdot)$ is strictly increasing, we have $\bar{\tau}(v + \varepsilon) > \bar{\tau}(v) > v + \varepsilon + b$. Therefore, any such type $v + \varepsilon$ is better off mimicking the communication strategy of type v to ensure exit at price $\bar{\tau}(v)$ instead of $\bar{\tau}(v + \varepsilon)$: by doing this, the advisor ensures that the bidder does not win when the valuation of the strongest rival is in $(v, v + \varepsilon)$, in which case the bidder overpays relative to the advisor's maximum willingness to pay of $v + \varepsilon + b$. Hence, it cannot be that $\bar{\tau}(v) > v + b$. Second, consider the case $\bar{\tau}(v) < v + b$. Now, there exists a subset of (v', v'') of types $v - \varepsilon < v$ with positive measure with $\bar{\tau}(v) < v - \varepsilon + b$. Since $\bar{\tau}(\cdot)$ is strictly increasing, we have $\bar{\tau}(v - \varepsilon) < \bar{\tau}(v) < v - \varepsilon + b$. Therefore, any such type $v - \varepsilon$ is better off mimicking the communication strategy of type v to ensure exit at price $\bar{\tau}(v)$ instead of $\bar{\tau}(v - \varepsilon)$: by doing this, the advisor ensures that the bidder wins when the valuation of the strongest rival is in $(v-\varepsilon, v)$, in which case the advisor gets a positive payoff, since the bidder pays below the advisor's maximum willingness to pay. Therefore, it cannot be that $\bar{\tau}(v) < v+b$. We conclude that $\bar{\tau}(v) = v+b$. *q.e.d.*

Claim 5. *If* $\Pi^P \neq \emptyset$ *and* $\Pi^S \neq \emptyset$*, then* Π^P *contains a single interval* $\pi^P = [v^*, \bar{v}]$*, where* $v^* > \underline{v}$*.*

Proof: By contradiction, suppose that Π^P contains more than one interval or that it contains one interval that it to the left of Π^S. In the former case, Claim (3) implies that the intervals are not adjacent. Therefore, there is an interval $\pi \in \Pi^P$ that lies to the left of an interval in Π^S. Let $v \in \pi$ be the highest type in this partition. Since it must be indifferent between separation and pooling and $\bar{\tau}(v) = v+b$ in the separation region by Claim 2, we have $v+b = \bar{\tau}(w)$ for any $w \in \pi$. Therefore, $\bar{\tau}(w) > w+b$ for any $w \in \pi$, $w \neq v$. In particular, it holds for the lowest type in the partition, $w' = \min_{w\in\pi} w$. However, this violates the NITS condition. Indeed, consider running price $p = \frac{\bar{\tau}(w)+w'+b}{2}$. Type w' is the weakest remaining type of the advisor at this price. Since $p > w'+b$, following her equilibrium strategy of waiting until price $\bar{\tau}(w)$, $w \in \pi$ to send recommendation $m = 1$ generates negative expected payoff to the advisor at the current point. In contrast, claiming that she is the weakest remaining type at the current price will lead to the bidder quitting immediately, yielding the payoff of zero. Therefore, Π^P contains only one interval that lies to the right of Π^S, i.e., the interval is of the form $\pi^P = [v^*, \bar{v}]$ for some $v^* > \underline{v}$. *q.e.d.*

Claim 6. *Cut-off* v^* *satisfies* (10)*.*

Proof: Case $v^* = \underline{v}$ (the babbling equilibrium) was covered before Claim (3). Consider case $v^* = \bar{v}$, i.e., $\Pi^P = \emptyset$. By Claim 4, $\bar{\tau}(v) = v+b$ for any $v \in (\underline{v}, \bar{v})$. If $\bar{v} < \infty$, the upper bound on the bidder's utility in round $p = v+b$ is $\lim_{v\to\bar{v}} (\bar{v} - (v+b)) = -b\ \bar{v} - (v+b) \overset{v\to\bar{v}}{\to} -b$, which contradicts the optimality of the bidder to follow the advisor's recommendation. Hence, it must be that $\bar{v} = \infty$. Next, by contradiction, suppose that $b > \lim_{s\to\infty} \mathbb{E}[v|v \geq s] - s$. By continuity, there is $\overline{s} < \infty$ such that $b > \mathbb{E}[v|v \geq s] - s$ for any $s > \overline{s}$. If the bidder wins in any round $p \geq s+b$, then her expected utility equals $\mathbb{E}[v|v \geq s] - s - b < 0$ and so, the value of following the advisor's recommendations is negative, which is a contradiction. Therefore, it must be that $b \leq \lim_{s\to\infty} \mathbb{E}[v|v \geq s] - s$.

Finally, consider case $v^* \in (\underline{v}, \bar{v})$. By contradiction, suppose that $v^* + b \neq \mathbb{E}[v|v \geq v^*]$. By indifference of type v^*, it must be that $\bar{\tau}(v) = v^* + b$ for any $v \in [v^*, \bar{v}]$. If $v^* + b < \mathbb{E}[v|v \geq v^*]$, then $\bar{\tau}(v)$, $v > v^*$ violates the incentive compatibility condition of the bidder. To see this, consider running price just below $v^* + b$. The equilibrium behavior prescribes the advisor to exit the auction in the next instant, which is below his maximum willingness to pay of $\mathbb{E}[v|v \geq v^*]$. By waiting a little beyond price $\bar{\tau}(v) = v^* + b$, the bidder ensures that he wins the auction with probability one and pays below his estimated valuation of $\mathbb{E}[v|v \geq v^*]$. Since this strategy results in a discontinuous upward jump in the expected utility of the bidder, he is better off deviating. Hence, it cannot be that $v^* + b < \mathbb{E}[v|v \geq v^*]$. If $v^* + b > \mathbb{E}[v|v \geq v^*]$, then $\bar{\tau}(v) = v^* + b$, $v \geq v^*$ violates the incentive compatibility condition of the bidder, because he would prefer to exit the auction slightly earlier.

Consider the running price $p = v^* + b - \varepsilon$ for an infinitesimal positive ε and suppose that the bidder has got a sequence of recommendations $m = 0$. His posterior belief is that the valuation is in the range $(v^* - \varepsilon, \overline{v}]$. Suppose that the bidder follows his equilibrium play. If $v \in (v^* - \varepsilon, v^*)$ and the bidder wins, he pays $v + b$ above his valuation v. If $v \in (v^*, \bar{v}]$ and the bidder wins, he pays $\bar{\tau}(v) = v^* + b$, which is, on average, above his valuation v ($\mathbb{E}[v|v \geq v^*]$). Since the bidder wins with positive probability, his expected payoff from following the equilibrium play is negative. In contrast, immediate exit yields zero expected payoff. Hence, the bidder is better off deviating and exiting the auction immediately. Therefore, $v^* + b = \mathbb{E}[v|v \geq v^*]$. *q.e.d.*

The proof of Theorem 3 follows from Claims 4, 5, and 6. □

Proof of Proposition 2. Since $MRL(\cdot)$ is strictly decreasing, equation $MRL(v) = b$ has at most one solution. First, consider case $MRL(\underline{v}) > b$. If $\bar{v} < \infty$, then $MRL(\bar{v}) = 0$. Since $MRL(\cdot)$ is strictly decreasing (and continuous, since the distribution has full support on $[\underline{v}, \bar{v}]$), there exists a unique solution $v \in (\underline{v}, \bar{v})$ to $MRL(v) = b$. Similarly, if $\bar{v} = \infty$, then, since $MRL(\cdot)$ is strictly decreasing and $\lim_{v \to \infty} MRL(v) < b$, there exists a unique solution $v \in (\underline{v}, \bar{v})$ to $MRL(v) = b$. Denote it by v^*. Since $MRL(\cdot)$ is strictly decreasing and $MRL(v^*) = b$, $s + b - \mathbb{E}[v|v \geq s]$ for any $s < v^*$. Therefore, the bidder's option value of waiting, $V(p)$ in (11), is strictly positive for all $p < v^* + b$. Hence, this cut-off indeed corresponds to an equilibrium. By Theorem 3, it cannot be that $v^* = \underline{v}$, since $MRL(\underline{v}) > b$, and $v^* = \bar{v}$, since either $\bar{v} < \infty$ or $\bar{v} = \infty$ and $b > \lim_{v \to \infty} MRL(v)$. Hence, the equilibrium is unique.

Second, condier case $MRL(\underline{v}) \leq b$. Since $MRL(\cdot)$ is strictly decreasing, $MRL(v) < b$ for any $v > \underline{v}$. Hence, by Theorem 3 the only candidate for equilibrium is $v^* = \underline{v}$, i.e., babbling. Clearly, it is a PBEM. To see that it satisfies the NITS condition, note that the implied expected payoff to type $\underline{v}$ of the advisor is $\frac{1}{N}(b - MRL(\underline{v})) \geq 0$. Therefore, the NITS condition is satisfied. Hence, babbling is indeed the unique equilibrium in this case. □

Proof of Proposition 3. Since $MRL(v) > b$ for all $v \in [\underline{v}, \infty)$, by Theorem 3 the only candidate for equilibrium is $v^* = \infty$. It follows from (11) that whenever $MRL(v) > b$ for all $v \in [\underline{v}, \overline{v}]$, the bidder's option value of waiting for the advisor's recommendation is always positive. Hence, $v^* = \infty$ indeed corresponds to an equilibrium. Moreover, it satisfies the NITS condition, since any type of the advisor gets his unconstrained optimal bidding strategy. □

Proof of Theorem 4. Under Assumption A, the number of partitions K must be finite. Since ω_{K-1} satisfies equation (6), $\omega_{K-1} + b - \mathbb{E}[v|v \geq \omega_{K-1}] < 0$ or $b < MRL(\omega_{K-1})$. On the other hand, $b = MRL(v^*)$. Since $MRL(\cdot)$ is strictly decreasing, $v^* > \omega_{K-1}$. Under Assumption B, the unique equilibrium under NITS of the English auction is fully separating ($v^* = \infty$), while any equilibrium in the second-price auction has a partition structure (by the same logic as in the proof of Theorem 2 for $\overline{v} < \infty$). □

Proof of Theorem 5. See the argument after the theorem in the main text. □

Proof of Corollary 1. See the argument after the theorem in the main text. □

Proof of Proposition 4. The first statement follows directly from equation (10). Let us prove the second statement. The revenue from the English auction is given by

$$b + \int_{\underline{v}}^{v^*} \hat{v} dH(\hat{v}) + (1 - H(v^*))v^*, \tag{18}$$

where H is the distribution of the second order statistic. The derivative of (18) with respect to b equals $1 + (1 - H(v^*))\frac{d}{db}v^*$.We can find $\frac{d}{db}v^*$ by the implicit function theorem from (10),

$$\tfrac{d}{db}v^* = -\left(1 - \tfrac{f(v^*)}{1-F(v^*)}b\right)^{-1}.$$

Since MRL is strictly decreasing, $\frac{d}{db}v^* < 0$ for $b \in [0, \overline{b}]$ and so, $1 - \frac{f(v^*)}{1-F(v^*)}b > 0$. Thus, the derivative of the revenue with respect to b equals

$$\frac{H(v^*) - \frac{f(v^*)}{1-F(v^*)}b}{1 - \frac{f(v^*)}{1-F(v^*)}b}. \tag{19}$$

When b is close to 0, v^* is close to $\overline{v}$. Using (10) and L'Hospital's Rule, we get

$$\begin{aligned} \lim_{v^* \to \overline{v}} \frac{f(v^*)}{1 - F(v^*)} b &= \lim_{v^* \to \overline{v}} \frac{f(v^*) \int_{v^*}^{\overline{v}} (v - v^*) dF(v)}{(1 - F(v^*))^2} \\ &= f(\overline{v}) \lim_{v^* \to \overline{v}} \frac{1}{2f(v^*)} = \frac{1}{2}. \end{aligned}$$

Since $H(\overline{v}) = 1$, expression (19) is positive for small b. When b is approaches $\overline{b}$, v^* approaches $\underline{v}$ and so, $\frac{f(v^*)}{1-F(v^*)}b \to f(\underline{v})\overline{b}$ while $H(v^*) \to 0$. Therefore, expression (19) is negative for sufficiently large b, which completes the proof of the first statement.

To prove the third statement, notice that $\lim_{N\to\infty} H(v^*) = 0$. Therefore, there exists $N(b)$ such that $H(v^*) - \frac{f(v^*)}{1-F(v^*)}b < 0$ for all $N > N(b)$. Since $H(v^*)$ decreases monotonically in N, there is $\varepsilon(b) > 0$ such that a decrease in b to $b - \varepsilon(b)$ is beneficial for the seller for sufficiently large N. The last statement follows from Proposition 2. □

Proof of Proposition 5. We consider the English auction with contracts and show that if all other bidders offer the delegation contracts with caps on the bids at $v^* + b$, then this contract indeed maximizes each bidder's payoff.

When all bidders offer delegation contracts with the cap at $v^* + b$, there is a one-to-one mapping between bids and pairs consisting of the probability of winning q and expected payment $t(q)$ defined as follows. To each bid p corresponds

$$q = \begin{cases} 1 \\ q^* \\ G(p-b) \end{cases} \qquad t(q) = \begin{cases} \mathbb{E}[\hat{v}+b|\hat{v}<v^*]G(v^*) + (v^*+b)(1-G(v^*)) & \text{if } p > v^*+b, \\ \mathbb{E}[\hat{v}+b|\hat{v}<v^*]G(v^*) + q^*(v^*+b) & \text{if } p = v^*+b, \\ \mathbb{E}[\hat{v}+b|\hat{v}\le p-b] & \text{if } p < v^*+b, \end{cases} \tag{20}$$

where q^* denotes the expected number of bidders with value above $\hat{v}$. Thus, we can equivalently formulate the problem of the bidder as the design of the contract which maps each report of type v by the advisor into the probability of winning q and expected payment $t(q)$. The preferences over q's of the bidder and the advisor are

$$\text{Bidder}: \qquad qv - t(q), \tag{21}$$

$$\text{Advisor}: \qquad q(v+b) - t(q). \tag{22}$$

The bidder designs a contract that solves the program A:

$$\begin{aligned} &\max_{\mathbf{q}(\cdot)\in Q} \int_{\underline{v}}^{\overline{v}} (\mathbf{q}(v)v - t(\mathbf{q}(v)))\, dF(v) \\ &s.t.\ v \in \arg\max_{v'\in[\underline{v},\overline{v}]} \{(v+b)\mathbf{q}(v') - t(\mathbf{q}(v'))\} \textit{ for all } v, \end{aligned} \tag{23}$$

where Q is the set of all measurable functions from $[\underline{v},\overline{v}]$ into $[0,\overline{q}) \cup \{q^*, 1\}$ and $\overline{q} = \lim_{p\uparrow v^*+b} G(p-b)$. We want to show that $\mathbf{q}^A = \begin{cases} G(v) & \text{if } v < v^* \\ q^* & \text{if } v \geq v^* \end{cases}$ solves program A. In fact, we will show a stronger statement that $\mathbf{q}^A$ solves program A in which the set Q is replaced by the set of all measurable functions from $[\underline{v},\overline{v}]$ into $[0,\overline{q}] \cup \{q^*, 1\}$ where $t(\overline{q}) = \lim_{q\uparrow\overline{q}} t(q)$. As an auxiliary step let us derive some properties of function $t(\cdot)$.

Claim 7. *$t(\cdot)$ is strictly increasing, strictly convex, twice differentiable on $[0,\overline{q})$, and*

$$\frac{t(1)-t(q^*)}{1-q^*} = \frac{t(1)-t(\overline{q})}{1-\overline{q}} = v^*+b = \lim_{q\uparrow\overline{q}} t'(q). \tag{24}$$

Proof: Function $t(\cdot)$ specified in (20) is derived under the assumption that other bidders delegate bidding to their advisors with a cap on bids at v^*+b. Then for an advisor with type $v<v^*$ who submits bids himself, it is strictly optimal to bid $v+b$ in such an auction, i.e. for all $v, v' < v^*$, $q = G(v)$, $q' = G(v')$, it holds $q(v+b) - t(q) > q'(v+b) - t(q')$. This implies that $v+b > \frac{t(q)-t(q')}{q-q'}$ whenever $q > q'$ and $v+b < \frac{t(q')-t(q)}{q'-q}$ whenever $q < q'$. Since the range of G on $[\underline{v}, v^*)$ is $[0,\overline{q})$, this in turn implies that $t(\cdot)$ is strictly increasing and strictly convex on $[0,\overline{q})$. Differentiability of $t(\cdot)$ follows from $t(q) = \mathbb{E}[\hat{v}+b|\hat{v} \le G^{-1}(q)]$ for $q < \overline{q}$.

We next show (24). By (10), the bidder with the expected value $\mathbb{E}[v|v \geq v^*]$ gets expected profit 0 from winning at price v^*+b. Therefore, she is indifferent between $(\overline{q}, t(\overline{q}))$, $(q^*, t(q^*))$, and $(1, t(1))$ and so, they lie on the same line (her indifference curve) with slope $\mathbb{E}[v|v \geq v^*] = v^* + b$. As was mentioned above, for $q < \overline{q}$ and $v = G^{-1}(q)$, it holds $q = \arg\max_{q' \in [0,\overline{q})} q'(v+b) - t(q')$ and so, by the differentiability of $t(\cdot)$, $G^{-1}(q)+b = t'(q)$, which implies that $\lim_{q \uparrow \overline{q}} t'(q) = \lim_{q \uparrow \overline{q}} G^{-1}(q)+b = v^*+b$. *q.e.d.*

Equation (24) implies that the bidder's profit from $\mathbf{q}^A$ and $\mathbf{q}^B = \begin{cases} G(v) & \text{if } v < v^* \\ \overline{q} & \text{if } v \geq v^* \end{cases}$ is the same. We next show that if $\mathbf{q}$ solves program A and $\mathbf{q}(v) = q^*, \mathbf{q}(v) = 1$ for some v, v', then $\tilde{\mathbf{q}}$ such that $\tilde{\mathbf{q}}(v) = q^*$ for $\{v : \tilde{\mathbf{q}}(v) = 1\}$ and $\tilde{\mathbf{q}}(v) = \mathbf{q}(v)$ for the rest v also solves program A. Indeed, if the image of $\mathbf{q}$ contains both q^* and 1, then for all types of the advisor above v^*, and only for them, $\mathbf{q}(v) = 1$. Then by Claim 7, the expected profit of the bidder conditional on $v > v^*$ is zero. Again by Claim 7, this profit will not change, if all such types choose a report corresponding to the probability of winning q^*. Analogously, we can show that if $\mathbf{q}$ solves program A and $\mathbf{q}(v) = q^*, \mathbf{q}(v) = 1$ for some v, v', then $\tilde{\mathbf{q}}$ such that $\tilde{\mathbf{q}}(v) = 1$ for $\{v : \tilde{\mathbf{q}}(v) = q^*\}$ and $\tilde{\mathbf{q}}(v) = \mathbf{q}(v)$ for the rest v also solves program A. Thus, it sufficies to show that $\mathbf{q}^B$ solves program A in which the set Q is replaced by the set Q' of all measurable functions from $[\underline{v}, \overline{v}]$ into either $[0, \overline{q}] \cup \{q^*\}$ or $[0, \overline{q}] \cup \{1\}$. We will prove the statement for the case of Q' mapping $[\underline{v}, \overline{v}]$ into $[0, \overline{q}] \cup \{q^*\}$ and the other case follows by the analogous argument. Define the program B as follows:

$$\max_{\mathbf{q}(\cdot) \in Q'} \int_{\underline{v}}^{\overline{v}} (\mathbf{q}(v)v - t(\mathbf{q}(v)))\, dF(v)$$
$$\text{s.t. } v \in \arg\max_{v'} \{(v+b)\mathbf{q}(v') - t(\mathbf{q}(v'))\} \text{ for all } v.$$

It follows from Claim 7 that while $t(\cdot)$ is strictly convex below $\overline{q}$, it cannot be extrapolated to a strictly convex function to the whole interval $[0, 1]$, as points $(\overline{q}, t(\overline{q}))$, $(q^*, t(q^*))$, and $(1, t(1))$ lie on the same line. Thus, we cannot directly apply the results of Amador and Bagwell (2013) to program B. Instead, the approach is to perturb the program B so that we can apply their result to this perturbed program and then relate the solution of the perturbed program to the solution of the program B. We next show that $\mathbf{q}^B$ defined above is a solution to program B. We perturb function $t(\cdot)$ on $[0, q^*]$ as follows $t_\varepsilon(q) = t(q) + \varepsilon \max\{0, (q - \overline{q})^3\}$. Consider the auxiliary program B_ε:

$$\max_{\mathbf{q}(\cdot) \in Q''} \int_{\underline{v}}^{\overline{v}} (\mathbf{q}(v)v - t_\varepsilon(\mathbf{q}(v)))\, dF(v)$$
$$\text{s.t. } v \in \arg\max_{v'} \{(v+b)\mathbf{q}(v') - t_\varepsilon(\mathbf{q}(v'))\} \text{ for all } v,$$

where Q'' is the set of all measurable functions from $[\underline{v}, \overline{v}]$ to $[0, q^*]$. Unlike $t(\cdot)$, function $t_\varepsilon(\cdot)$ is strictly convex and twice differentiable and so, we can apply Proposition 1 in Amador and Bagwell

(2013) to get the following lemma (the proof is below):

Lemma 2. $\mathbf{q}^B$ *is a solution to program* B_ε *for any* ε.

We are now in the position to show that $\mathbf{q}^B$ solves program B. Suppose to contradiction another $\tilde{\mathbf{q}}^B$ solves program B, but not $\mathbf{q}^B$. First, if the range of $\tilde{\mathbf{q}}^B$ does not contain q^*, then $\tilde{\mathbf{q}}^B \in Q''$ which is a contradiction to Lemma 2. Thus, we suppose that the range of $\tilde{\mathbf{q}}^B$ contains q^*. Let us perturb $\tilde{\mathbf{q}}^B$ as follows. Denote by $\tilde{Q}$ the range of $\tilde{\mathbf{q}}^B$ and let $\tilde{\mathbf{q}}^B_\varepsilon(v) = \max_{q\in\tilde{Q}}\{(v+b)q - t(q)\}$. By construction, $\tilde{\mathbf{q}}^B_\varepsilon$ satisfies constraints of program B_ε and moreover, the measure of types for which $\tilde{\mathbf{q}}^B_\varepsilon$ differs from $\tilde{\mathbf{q}}^B$ converges to zero as $\varepsilon \to 0$. Then for some positive c_0, c_1, c_2,

$$
\begin{aligned}
\int_{\underline{v}}^{\overline{v}} \left(\tilde{\mathbf{q}}^B(v)v - t(\tilde{\mathbf{q}}^B(v))\right) dF(v) &> \int_{\underline{v}}^{\overline{v}} \left(\mathbf{q}^B(v)v - t(\mathbf{q}^B(v))\right) dF(v) + c_0 \\
&> \int_{\underline{v}}^{\overline{v}} \left(\mathbf{q}^B(v)v - t_\varepsilon(\mathbf{q}^B(v))\right) dF(v) + c_0 \\
&\geq \int_{\underline{v}}^{\overline{v}} \left(\tilde{\mathbf{q}}^B_\varepsilon(v)v - t_\varepsilon(\tilde{\mathbf{q}}^B_\varepsilon(v))\right) dF(v) + c_0 \\
&\geq \int_{\underline{v}}^{\overline{v}} \left(\tilde{\mathbf{q}}^B_\varepsilon(v)v - t(\tilde{\mathbf{q}}^B_\varepsilon(v))\right) dF(v) + c_0 - c_1\varepsilon \\
&\geq \int_{\underline{v}}^{\overline{v}} \left(\tilde{\mathbf{q}}^B(v)v - t(\tilde{\mathbf{q}}^B(v))\right) dF(v) + c_0 - c_1\varepsilon - c_2\varepsilon
\end{aligned}
$$

where the first inequality follows from the fact that $\mathbf{q}^B \in Q'$ and satisfies the constraint of program B but does not solve program B, the second inequality follows from $t_\varepsilon(v) \geq t(v)$ for all $v \in [\underline{v}, \overline{v}]$, the third inequality follows from the optimality of $\mathbf{q}^B$ in program B_ε, the forth inequality follows from the construction of function $t_\varepsilon(\cdot)$, the last inequality follows from the construction of $\mathbf{q}^B_\varepsilon$. By taking $\varepsilon \to 0$, we get a contradiction, thus, $\mathbf{q}^B$ solves program B. □

Proof of Lemma 2. In Table 2, we verify that conditions (c1), (c2), (c3') of Proposition 1 in Amador and Bagwell (2013) as well as their concavity and differentiability assumptions are satisfied for program B_ε. Therefore, by Proposition 1 in Amador and Bagwell (2013), q^B_ε is a solution to program B_ε. □

A.3 Proofs for Section 6

Proof of Theorem 6. First, observe that since $\underline{v} - \mathbb{E}[v|v \leq \underline{v}] + b < 0$ and $\overline{v} - \mathbb{E}[v] + b > 0$, there exists a solution v^* to (13), and it is unique, as MAI is strictly increasing.

To prove that conjectured strategies constitute a PBEM, we need to show that the advisor sends the message "stop" at the optimal time given that the bidder follows his recommendations, and that the bidder prefers to follow recommendations of the advisor and not stop earlier.

Amador and Bagwell's notations and assumptions	Our model's counter-parts
$\gamma \in \Gamma = [\underline{\gamma}, \overline{\gamma}]$ distributed according to F	$v \in [\underline{v}, \overline{v}]$ distributed according to F
$\pi \in \Pi = [0, \overline{\pi}]$	$q \in [0, q^*]$
$b(\pi)$	$bq - t_\varepsilon(q)$
Agent's payoff: $\gamma\pi + b(\pi)$	Advisor's payoff: $(v+b)q - t_\varepsilon(q)$
Principal's payoff: $w(\gamma, \pi)$	Bidder's payoff: $vq - t_\varepsilon(q)$
γ_L	$\underline{v}$
$\pi_f(\gamma) \in \arg\max_{\pi\in\Pi} \gamma\pi + b(\pi)$	$q_f(v) \in \arg\max_{q\in[0,q^*]}(v+b)q - t_\varepsilon(q)$ which implies $v + b = t'_\varepsilon(q_f(v))$.
γ_H solves $\int_{\gamma_H}^{\overline{\gamma}}(w_\pi(\gamma, \pi_f(\gamma_H))dF(\gamma) = 0$	v^* solves $\int_{v^*}^{\overline{v}}(v - t'_\varepsilon(v^*))dF(v) = 0$ or equivalently (10)
$\kappa = \inf_{(\gamma,\pi)\in\Gamma\times\Pi}\left\{\frac{w_{\pi\pi}(\gamma,\pi)}{b''(\pi)}\right\}$	$\kappa = \inf_{(v,q)\in[\underline{v},\overline{v}]\times[0,q^*]}\left\{\frac{\frac{\partial^2}{\partial q^2}(vq - t_\varepsilon(q))}{\frac{d^2}{dq^2}(bq - t_\varepsilon(q))}\right\} = 1$
Assumption 1	
w continuous in γ and π	by Claim 7
$w(\gamma, \cdot)$ concave and twice differentiable in π	
b is strictly concave and twice differentiable in π	
π_f is twice differentiable and $\pi'_f(\gamma) > 0$	by $q'_f(v) = \frac{1}{t''_\varepsilon(q_f(v))} > 0$ and f differentiable
w_π is continuous in γ	by Claim 7
Assumptions of Proposition 1 in Amador and Bagwell (2013)	
(c1) $\kappa F(\gamma) - w_\pi(\gamma, \pi_f(\gamma))f(\gamma)$ is nondecreasing.	$F(v) - (v - t'(q_f(v)))f(v) = F(v) - bf(v)$ is nondecreasing whenever $(\ln f(v))' \leq \frac{1}{b}$.
(c2) For all $\gamma' \in [\gamma_H, \overline{\gamma}]$, $(\gamma' - \gamma_H)\kappa - \int_{\gamma'}^{\overline{\gamma}} w_\pi(\tilde{\gamma}, \pi_f(\gamma_H))\frac{f(\tilde{\gamma})}{1-F(\gamma)}d\tilde{\gamma} \geq 0$.	For all $v' \in [v^*, \overline{v}]$, $v' - v^* - \int_{v'}^{\overline{v}}(\tilde{v} - v^* - b)\frac{f(\tilde{v})}{1-F(v')}d\tilde{v} =$ $b + v' - \mathbb{E}[v \mid v \geq v'] \geq 0$ by the decreasing MRL.
(c3') $\omega_\pi(\underline{\gamma}, \pi_f(\underline{\gamma})) \leq 0$.	$\underline{v} - t'(q_f(\underline{v})) = -b < 0$

Table 2: The first column represents assumptions and conditions of Proposition 1 from Amador and Bagwell (2013). The second column represents corresponding variables in our model and verifies that their assumptions hold in our setup.

First, we show that strategy $\sigma(\cdot)$ is optimal for the advisor. The advisor of type v solves the following problem

$$\max_{\sigma}(v+b-\sigma)G(\sigma^{-1}(\sigma)), \tag{25}$$

for which the first-order condition is

$$g(v)(v+b) = (G(v)\sigma(v))' \tag{26}$$

Solving (26) with the initial condition $\sigma(v^*) = v^* + b$, we get

$$\begin{aligned}
\sigma(v) =& \frac{G(v^*)}{G(v)}(v^*+b) + \frac{1}{G(v)}\int_{v^*}^{v} g(\hat{v})(\hat{v}+b)d\hat{v} = \\
& b + \frac{G(v^*)}{G(v)}v^* + \frac{1}{G(v)}\left(\int_{\underline{v}}^{v} \hat{v}dG(\hat{v}) - \int_{\underline{v}}^{v^*} \hat{v}dG(\hat{v})\right) = \\
& b + \mathbb{E}[\hat{v}|\hat{v}<v] + \frac{G(v^*)}{G(v)}\left(v^* - \mathbb{E}[\hat{v}|\hat{v}<v^*]\right) = \\
& b - v + \frac{1}{G(v)}\left(\left(v - \mathbb{E}[\hat{v}|\hat{v}<v]\right)G(v) - \left(v^* - \mathbb{E}[\hat{v}|\hat{v}<v^*]\right)G(v^*)\right) = \\
& b - v + \frac{1}{G(v)}\left(vG(v) - v^*G(v^*) - \int_{v^*}^{v} \hat{v}dG(\hat{v})\right) = \\
& b - v + \frac{1}{G(v)}\int_{v^*}^{v} G(\hat{v})d\hat{v}
\end{aligned} \tag{27}$$

It follows from the first line that $\sigma(v) = \mathbb{E}[\max\{v^*, \hat{v}\} + b|\hat{v} < v]$. Thus, if the bidder follows the recommendation of the advisor, then the strategy to stop when $p = \sigma(v)$ is optimal for the advisor when $v > v^*$. The maximized function in (25) has a single crossing property in (v, σ). Hence, since type v^* prefers to stop in round v^*, all types below v^* prefer to stop in round v^* as well to stopping earlier. Therefore, the advisor's strategy is optimal.

The maximized function in (25) has a single crossing property in (b, σ). Thus, if the bidder knew v, then she would prefer to stop the auction earlier than round $\sigma(v)$. Therefore, it is optimal for her to stop when she gets the message from the advisor with type $v > v^*$. To finish the proof, we show that the bidder does not want to stop the auction earlier. Let $v_p \equiv \sigma^{-1}(p)$ for all $p > p^*$. Recall that $\hat{v}$ denotes the value of the opponent bidder distributed according to $G(\hat{v})$. The expected utility of the bidder in round p from following the recommendation of the advisor is

$$\begin{aligned}
&\mathbb{E}\left[(v-\sigma(v))1\{v>\hat{v}\}|\hat{v},v<v_p\right]= \\
&\mathbb{E}\left[(v-\sigma(v))1\{v>\hat{v}\}|v^*<v<v_p,\hat{v}<v_p\right]\frac{F(v_p)-F(v^*)}{F(v_p)}= \\
&\int_{v^*}^{v_p}\left(-b-\frac{1}{G(v)}\int_{v^*}^{v}G(\hat{v})d\hat{v}\right)\frac{G(v)}{G(v_p)}\frac{dF(v)}{F(v_p)}= \\
&\frac{1}{G(v_p)F(v_p)}\int_{v^*}^{v_p}\left(-bG(v)+\int_{v^*}^{v}G(\hat{v})d\hat{v}\right)dF(v)= \\
&\frac{1}{G(v_p)F(v_p)}\left(-bG(v_p)F(v_p)+F(v_p)\int_{v^*}^{v_p}G(\hat{v})d\hat{v}+bG(v^*)F(v^*)+\int_{v^*}^{v_p}F(v)(bg(v)-G(v))dv\right)= \\
&-b+\frac{1}{G(v_p)}\int_{v^*}^{v_p}G(\hat{v})d\hat{v}+b\frac{G(v^*)F(v^*)}{G(v_p)F(v_p)}+\frac{1}{G(v_p)F(v_p)}\int_{v^*}^{v_p}F(v)(bg(v)-G(v))dv \qquad (28)
\end{aligned}$$

where the first equality is by the fact that $\mathbb{E}\left[(v-\sigma(v))1\{v>\hat{v}\}|v<v^*,\hat{v}<v_p\right]=0$, in the second equality we use (27) to substitute for $\sigma(v)$ and the rest is algebraic manipulations and the integration by parts. On the other hand, if the bidder deviates and stops the auction in round p, then her expected utility is given by

$$\mathbb{E}[v|v<v_p]-\sigma(v_p)=\mathbb{E}[v|v<v_p]-v_p-b+\frac{1}{G(v_p)}\int_{v^*}^{v_p}G(\hat{v})d\hat{v}, \qquad (29)$$

where again we used (27) to substitute for $\sigma(v_p)$. We need to show that (29) is less than (28):

$$bG(v^*)F(v^*)+\int_{v^*}^{v_p}F(v)(bg(v)-G(v))dv-(\mathbb{E}[v|v<v_p]-v_p)G(v_p)F(v_p)\geq 0. \qquad (30)$$

Equivalently, using $G(v)=F^{N-1}(v)$ and $g(v)=(N-1)F^{N-2}(v)f(v)$, we can rewrite the left-hand side of (30) as follows

$$\phi(v_p)=bF^N(v^*)+\int_{v^*}^{v_p}\left(b(N-1)F^{N-1}(v)f(v)-F^N(v)\right)dv-F^{N-1}(v_p)\int_{\underline{v}}^{v_p}vdF(v)+v_pF^N(v_p).$$

Then

$$\begin{aligned}
\frac{d}{dv_p}\phi(v_p)&=b(N-1)F^{N-1}(v_p)f(v_p)+ \\
&(N-1)F^{N-2}(v_p)f(v_p)\int_{\underline{v}}^{v_p}vdF(v)-v_pF^{N-1}(v_p)f(v_p)+v_pNF^{N-1}(v_p)f(v_p) \\
&=(N-1)F^{N-1}(v_p)f(v_p)\left(b-\mathbb{E}[v|v\leq v_p]-v_p\right)
\end{aligned}$$

is positive for $v_p>v^*$, as MAI is strictly increasing and $\mathbb{E}[v|v\leq v^*]=v^*+b$ by (13). Since $\phi(v^*)=0$, $\int_{v^*}^{v_p}\phi(v)dv>0$ and so, the inequality (30) indeed holds proving the optimality of the bidder. □

Proof of Theorem 7. We want to show that there is no partition $(\omega_k)_{k=1}^K$ induced by the equilibrium of the second-price auction such that $\omega_k \in [\underline{v}, v^*]$. Since v^* is the unique solution to (13) and $\underline{v} + b - \mathbb{E}[v|v \leq \underline{v}] = b < 0$, $\omega_k + b - \mathbb{E}[v|v \leq \omega_k] < 0$. Therefore,

$$\omega_k + b - \mathbb{E}[v|v \in [\omega_{k-1}, \omega_k)] \quad \leq \quad \omega_k + b - \mathbb{E}[v|v < \omega_k] < 0,$$

which contradicts the fact that the first term in (6) should be positive. □

References

Amador, M. and Bagwell, K.: 2013, The theory of optimal delegation with an application to tariff caps, *Econometrica* **81**(4), 1541–1599.

Aumann, R. and Hart, S.: 2003, Long cheap talk, *Econometrica* **71**(6), 1619–1660.

Bagnoli, M. and Bergstrom, T.: 2005, Log-concave probability and its applications, *Economic theory* **26**(2), 445–469.

Bertrand, M. and Mullainathan, S.: 2003, Enjoying the quiet life? corporate governance and managerial preferences, *Journal of political Economy* **111**(5), 1043–1075.

Betton, S., Eckbo, B. E. and Thorburn, K. S.: 2008, Corporate takeovers, *Handbook of Empirical Corporate Finance* pp. 291–429.

Burkett, J.: 2015, Endogenous budget constraints in auctions, *Journal of Economic Theory* **158**, 1–20.

Chakraborty, A. and Harbaugh, R.: 2010, Persuasion by cheap talk, *The American Economic Review* **100**(5), 2361–2382.

Che, Y.-K. and Gale, I.: 1998, Standard auctions with financially constrained bidders, *The Review of Economic Studies* **65**(1), 1–21.

Che, Y.-K. and Gale, I.: 2006, Revenue comparisons for auctions when bidders have arbitrary types, *Theoretical Economics* **1**(1), 95–118.

Chen, Y.: 2011, Perturbed communication games with honest senders and naive receivers, *Journal of Economic Theory* **146**(2), 401–424.

Chen, Y., Kartik, N. and Sobel, J.: 2008, Selecting cheap-talk equilibria, *Econometrica* **76**(1), 117–136.

Compte, O. and Jehiel, P.: 2007, Auctions and information acquisition: sealed bid or dynamic formats?, *The Rand Journal of Economics* **38**(2), 355–372.

Crawford, V. P. and Sobel, J.: 1982, Strategic information transmission, *Econometrica* **50**(6), 1431–1451.

Farrell, J. and Gibbons, R.: 1989, Cheap talk can matter in bargaining, *Journal of Economic Theory* **48**(1), 221–237.

Golosov, M., Skreta, V., Tsyvinski, A. and Wilson, A.: 2014, Dynamic strategic information transmission, *Journal of Economic Theory* **151**, 304–341.

Gorbenko, A. S. and Malenko, A.: 2014, Strategic and financial bidders in takeover auctions, *The Journal of Finance* **69**(6), 2513–2555.

Grenadier, S. R., Malenko, A. and Malenko, N.: 2016, Timing decisions in organizations: Communication and authority in a dynamic environment, *American Economic Review, Forthcoming* .

Harford, J. and Li, K.: 2007, Decoupling ceo wealth and firm performance: The case of acquiring ceos, *The Journal of Finance* **62**(2), 917–949.

Inderst, R. and Ottaviani, M.: 2013, Sales talk, cancellation terms and the role of consumer protection, *The Review of Economic Studies* p. rdt005.

Kartik, N.: 2009, Strategic communication with lying costs, *The Review of Economic Studies* **76**(4), 1359–1395.

Kim, K. and Kircher, P.: 2015, Efficient competition through cheap talk: the case of competing auctions, *Econometrica* **83**(5), 1849–1875.

Koessler, F. and Skreta, V.: 2014, Sales talk, *Available at SSRN 2465174* .

Krishna, V. and Morgan, J.: 2004, The art of conversation: eliciting information from experts through multi-stage communication, *Journal of Economic theory* **117**(2), 147–179.

Levit, D.: 2014, Advising shareholders in takeovers, *Available at SSRN 1866379* .

Maskin, E. and Riley, J.: 2000, Asymmetric auctions, *The Review of Economic Studies* **67**(3), 413–438.

Matthews, S. A. and Postlewaite, A.: 1989, Pre-play communication in two-person sealed-bid double auctions, *Journal of Economic theory* **48**(1), 238–263.

Milgrom, P. R. and Weber, R. J.: 1982, A theory of auctions and competitive bidding, *Econometrica: Journal of the Econometric Society* pp. 1089–1122.

Morris, S.: 2001, Political correctness, *Journal of political Economy* **109**(2), 231–265.

Myerson, R. B.: 1981, Optimal auction design, *Mathematics of operations research* **6**(1), 58–73.

Ottaviani, M. and Sørensen, P. N.: 2006a, Professional advice, *Journal of Economic Theory* **126**(1), 120–142.

Ottaviani, M. and Sørensen, P. N.: 2006b, Reputational cheap talk, *The Rand journal of economics* **37**(1), 155–175.

Pai, M. M. and Vohra, R.: 2014, Optimal auctions with financially constrained buyers, *Journal of Economic Theory* **150**, 383–425.

Quint, D. and Hendricks, K.: 2013, Indicative bidding in auctions with costly entry.

Riley, J. G. and Samuelson, W. F.: 1981, Optimal auctions, *The American Economic Review* **71**(3), 381–392.

Simon, L. K. and Stinchcombe, M. B.: 1989, Extensive form games in continuous time: Pure strategies, *Econometrica: Journal of the Econometric Society* pp. 1171–1214.

Sobel, J.: 1985, A theory of credibility, *The Review of Economic Studies* **52**(4), 557–573.

Ye, L.: 2007, Indicative bidding and a theory of two-stage auctions, *Games and Economic Behavior* **58**(1), 181–207.