Market Selection and the Information Content of Prices

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Abstract

We study price formation in a large, common-value auction where buyers choose, based on their private information, between bidding in the auction and an outside option. The distribution of bidders participating in the auction is determined endogenously in equilibrium. We first focus on an outside option whose value is state dependent and positively correlated with the common-value object on auction. If the outside option's expected value is non-negative and its variance is positive, then information is not aggregated in the auction in any equilibrium. We then turn to a model where bidders choose to participate in one of two concurrently operating auction markets. The outside option for one auction is the equilibrium value of participating in the alternative auction, i.e., outside options are endogenously determined. If frictions lead to uncertain gains from trade in the first auction, then information is not aggregated in either market even if the second auction is frictionless. This is because the two auction markets serve as outside options with positive variance for each other. Our findings are driven by how bidders self-select across options: A large disparity in the payoff variance of the two options implies that optimistic bidders select the option with higher variance while pessimistic bidders select the option with lower variance. Our results suggest a novel mechanism through which market imperfections in one market can have widespread effects across all linked markets.

Keywords: Auctions, Large markets, Information Aggregation.

JEL Codes: C73, D44, D82, D83.

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1. Introduction

Consider a market in which k identical common-value objects of unknown value are sold to n bidders through a sealed bid auction where the highest k bidders are allocated objects and pay a uniform price equal to the highest losing bid. In such an auction, if each bidder has an independent signal about the unknown value of the object, then the equilibrium auction price converges to the object's true value as the number of objects and the number of bidders grow arbitrarily large (see Pesendorfer and Swinkels (1997) for a precise statement of this result). Therefore, the auction price reveals the unknown value of the object and thus aggregates all relevant information dispersedly held by the bidders.

Most previous work on auctions takes the distribution of types who bid in the auction as exogenously given.[1] Yet, in many instances bidders strategically decide whether to trade in a particular market weighing their other alternatives. In other words, the bidder distribution is endogenously determined jointly by the set of available alternatives and bidders' expectations about the relative attractiveness of these alternatives. In order to focus on such situations, we study a model where bidders choose, based on their private information, between an auction (market s) and an outside option (alternative or market r). This framework allows us to highlight the interplay between self-selection into an auction, bidding behavior in the auction and the information content of prices.

In what follows, we assume that there are two states that are equally likely a priori and each object's common-value (V), is equal to one in the good state and zero in the bad state. We first focus on an **exogenous** outside option that pays $u(r|V = v)$ if the value of the object is equal to v. Most past work (for example Pesendorfer and Swinkels (1997)) assumes that a bidder's outside option is equal to zero in each state. In contrast, we assume that $u(r|V = v)$ is an increasing function of v. Under this assumption, we show that if the object to bidder ratio ($\kappa_s := k/n$) exceeds a certain threshold $\kappa^* \in (0, 1)$, then there is no equilibrium where the auction price converges to the object's value as the market grows arbitrarily large.[2] In other words, there is no equilibrium which aggregates information. A corollary to this finding is perhaps more instructive. If the expected payoff of the outside option is equal to zero ($\mathbb{E}[u(r|V)] = \frac{1}{2}u(r|V = 0) + \frac{1}{2}u(r|V = 1) = 0$) but the variance is positive ($Var[u(r|V)] > 0$), then there is no equilibrium where information is aggregated. In this case, a bidder who opts for the outside option faces a loss in the bad state ($V = 0$). However, pessimistic bidders can insure against a loss in the bad state by bidding zero in the auction. In fact in equilibrium more optimistic types choose the outside option, more pessimistic types bid in the auction and such a pattern of self-selection precludes information aggregation. We provide more detailed intuition for these findings using a simple example further below.

Our equilibrium characterization and construction further elucidates the mechanism that leads to prices that fail to aggregate all available information. Moreover, our equilibrium construction allows us to measure the amount of information impounded by the auction price.

[1] Papers by Lauermann and Wolinsky (2014) and Murto and Valimaki (2014) are notable exceptions. We discuss these papers in the literature review.

[2] The cutoff κ^* depends on κ_s, the signal distribution and the value of the outside option but it is independent of the number of number of objects and bidders.

If the outside option is valuable in both states (i.e., $u(r|V = v) \geq 0$ for all v), then there is a unique equilibrium. In this equilibrium, optimistic types select market s but the expected number of bidders is less than the number of objects for sale with positive probability in both states of the world. In this case, information is not aggregated because of a lack of competition even though the pattern of self-selection drives the auction price towards value. Alternatively, if the outside option is valuable only in the good state (i.e., $u(r|V = 0) < 0$ and $u(r|V = 1) > 0$) and the variance of the outside option is not large, then there are many equilibria. In all of these equilibria, the expected number of bidders exceeds the number of objects in both states (i.e., there is sufficient competition) but only the pessimistic types select market s. In this case, information is not aggregated because of the pattern of self-selection even though there is sufficient competition.

Next, we turn to a model where bidders choose between two concurrent auction markets: the **endogenous** outside option for one is the equilibrium value of participating in the other. In addition to the frictionless auction market s, we assume that there is another, possibly frictional auction market r where there are an additional $n\kappa_r = k_r$ units of the same object on sale and $\kappa_s + \kappa_r < 1$. Our goal here is to understand which markets can generate outside option payoff profiles that can disrupt information aggregation. If there are frictions in market r and if the object to bidder ratio in market s exceeds a certain cutoff $\kappa^* \in (0, 1)$, then there is no equilibrium where information is aggregated in either market. Frictions in market r inhibit information aggregation also in the frictionless market because frictions transform market r into an outside option with positive variance for market s. In turn, the distribution of types that select market s forces the price to diverge from value in market s and thus market s also serves as an outside option with positive variance for market r. In contrast if there are no frictions in market r, then information is aggregated in both markets in every equilibrium. Therefore, our findings suggest that institutional differences are key for generating outside options which can hinder information aggregation.

We model frictions in market r as a reserve price $c \geq 0$ (the price that a bidder pays for an object cannot fall below c). The reserve price has various interpretations: (1) It is a *reserve price* set by a single auctioneer selling the k_r goods. (2) The auction is comprised of k_r non-strategic sellers, the reservation value (or the cost) for these sellers is equal to c and each seller requires at least c in order to participate in the auction, i.e., there are *informational frictions* as in Myerson and Satterthwaite (1983).[3] (3) A *government/regulator imposes a minimum price*. Although we focus on a reserve price, other institutional differences can also be detrimental to information aggregation. In particular, if market r is a "pay-as-you-bid" auction (as in Jackson and Kremer (2007)) or an "all-pay" auction (as in Chi et al. (2016)), then market r can generate an outside option for market s which leads to similar consequences as the reserve price that we discuss in detail in the paper.

For the case where market r is perturbed by a small friction ($c > 0$ but small), we construct an equilibrium. In this equilibrium, the expected prices are equalized across the two markets in each state. Therefore, from the perspective of a bidder who wins an object with probability

[3] The auction is in fact a double auction with non-strategic sellers who simply bid their valuations c.

one, the state by state payoff and hence the payoff variance is also equalized across markets.[4] The pattern of self-selection is the main force that equalizes prices. A large disparity in the variance of payoff between the two markets implies that optimistic bidders select the market with higher variance and pessimistic bidders select the option with lower variance, i.e., market selection has a cut-off structure. This is because the losses in the bad state are relatively small in the market with lower variance and hence the more pessimistic bidders opt for this market. However, if market selection has a cutoff structure and an auction attracts the type distribution's upper tail (i.e., the more optimistic types), then we show that prices are driven towards the object's value in this auction and therefore payoff variance is low. In contrast, if an auction attracts the type distribution's lower tail (i.e., the more pessimistic types), then prices diverge from value and payoff variance is high. Thus, an equilibrium is sustained by types self-selecting into markets in a way which equalizes the prices across markets in each state. Although the equilibrium that we construct is for a special case with binary signals, we show that all equilibria resemble the equilibrium that we construct if c is lower than a certain threshold value.

Previous work on information aggregation mainly focuses on homogeneous (or closely-linked) objects that trade in a single centralized frictionless auction market. However, such a centralized market is an exception rather than the rule. Fragmentation, the disperse trading of the same security in multiple markets, is common place: many stocks listed on the New York Stock Exchange trade concurrently on the regional exchanges (see Hasbrouck (1995)). Investors, who participate in a primary treasury bond auction, could purchase a bond with similar cashflow characteristics from the secondary market. Labor markets are linked but also segmented according to industry, geography and skill. Buyers in the market for aluminum or steel can choose between the London Metal Exchange or the New York Mercantile Exchange. Such fragmented markets and exchanges also differ in structure, rules and regulations. In particular, markets are heterogeneous in terms of the frictions that participants face. The results that we present in this paper suggest that selection into markets can have important implications for the information content of prices especially when individuals choose between markets that differ in terms of institutional detail and therefore frictions. In particular, we demonstrate how frictions can disrupt information aggregation not only in the market with frictions but also in frictionless, substitute markets because of how imperfectly informed bidders select across markets.

1.1. An Illustrative Example. Suppose, before bidding in the auction each bidder receives a signal that perfectly reveals the value of the object with probability p and receives an uninformative signal with the remaining probability. A bidder who receives the uninformative signal continues to believe that $V = 1$ with probability $1/2$. In contrast, a bidder who receives the perfectly revealing signal knows the object's value. Information is aggregated in such an auction as the number of bidders n and the number of object $n\kappa$ grow arbitrarily large under the assumption that all bidders participate in the auction: the auction price P converges to 1 and

[4] A bidder who wins an object with probability one in market m will expect to purchase an object at the market's expected price. In this case, the payoff in state v is $v - \mathbb{E}[P_m|V = v]$ and the payoff variance is $Var[V - \mathbb{E}[P_m|V]]$ where P_m is the random variable that represents the price in market m.

0 when $V = 1$ and $V = 0$, respectively. Assuming that all types participate is innocuous since each bidder can guarantee a payoff of at least zero by bidding zero in the auction. Therefore, bidding in this auction is individually rational for each bidder, if a non-participating bidder's payoff equals zero, i.e., if each bidder's outside option is state independent and equal to zero.

Let us continue to assume that the expected payoff of a bidder who does not participate in the auction is equal to zero. However, let us further suppose that the value of this outside option equals $-c$ if $V = 0$ and equals $c > 0$ if $V = 1$. Thus the expected value of the outside option is equal to zero and the variance is equal to $c^2 > 0$. We will argue that information cannot be aggregated with such an outside option in an arbitrarily large auction.

On the way to a contradiction, suppose that information is aggregated in the auction. If information is aggregated in the auction, then the auction price P converges to 1 if $V = 1$. Consider bidder i who has received a perfectly revealing signal. This bidder's payoff from participating in the auction converges to zero because the auction price converges to the value of the object. In contrast, bidder i's outside option is equal to $c > 0$ if $V = 1$. Therefore, any bidder who receives the perfectly informative signal will opt for the outside option if $V = 1$ and will submit a bid equal to 0 in the auction if $V = 0$ in a sufficiently large auction. However, if no bidders other than the uninformed submit non-trivial bids that exceed zero, then all the uninformed bidders would submit a bid equal to $1/2$, their valuation for the object. Thus, the price cannot converge to 1 when $V = 1$ contradicting our initial assumption of information aggregation in the auction. The main result that we present is general and shows that the insight highlighted by this example does not depend on our restriction to two states or to the special signal structure.

Continuing with the example, we now argue that such an outside option can be generated by an alternative auction market r with a reserve price $c = 1/2$. Suppose that the information structure is as in the above discussion and further assume that $p < \kappa_r = \kappa < 1/2$. An uninformed bidder's expected value for the object in either market is equal $1/2$ because both states are equally likely. Uninformed bidders choose market r only if the expected price in this market is equal to c irrespective of the state. This is because the minimum price, c, is equal to the bidder's expected value for the good by assumption. On the other hand, if no uninformed bidder bids in market r, then the expected number of bidders in market r is fewer than the number of goods in either state. This is because the number of goods, $n\kappa_r$, exceeds the expected number of informed bidders, np, by assumption. So, once again market r's expected price is equal to c in both states and completely uninformative. Interestingly, the price in market s does not aggregate information either. This is because the equilibrium value from bidding in market r is identical to the exogenous outside option discussed in the previous paragraph. Thus, information is not aggregated in market s. Again, the findings that we present show that the insight provided by the above example generalizes to a setting with multiple states, multiple markets and arbitrary signal distributions.

1.2. Relation to the literature. Whether prices aggregate information is a central question in economic theory which was first studied in the context of rational expectation models (see for example Grossman and Stiglitz (1976, 1980) and Grossman (1981)). This paper however is most closely related to earlier work which studies information aggregation in large, common-value auctions. Wilson (1977) studied second-price auctions with common value for one object for sale, and Milgrom (1979) extended the analysis to any arbitrary number of objects. Both of these papers show that as the number of bidders gets arbitrarily large, price converges to the true value of the object, but only provided that there are bidders with arbitrarily precise signals about the state of the world. Pesendorfer and Swinkels (1997) further generalized the analysis to the case where there are no arbitrarily precise signals. They showed that prices converge to the true value of a common-value object in all symmetric equilibria if and only if both the number of identical objects and the number of bidders who are not allocated an object grow without bound.[5] Our model is closest to the auction model studied by Pesendorfer and Swinkels (1997).[6]

We make four main contributions to the literature on information aggregation in multi-object common-value auctions. (1) We are the first to study bidding behavior in a multi-object common-value auction where bidders have outside options and the distribution of types is endogenously determined. (2) In this context, we highlight a new mechanism, based on self-selection, that can lead to the failure of information aggregation. (3) We introduce a new method, based on a local limit theorem, a central limit theorem and the theory of moderate deviations, that allows us to analytically solve for equilibria. (4) We show that information is aggregated if people choose between multiple, **frictionless** auction markets. In this case, the argument of Pesendorfer and Swinkels (1997) does not necessarily apply because the equilibrium bidding function need not be strictly increasing. Nevertheless, using the pattern of self-selection across markets, we establish that information is aggregated in all auction markets.

Our paper is closely related to recent work on single-unit common-value auctions by Lauermann and Wolinsky (2014) and Murto and Valimaki (2014).[7] The novel feature of Lauermann and Wolinsky (2014)'s model is that the auctioneer knows the value of the object but must solicit bidders for the auction and soliciting bidders is costly. Therefore, the number of bidders in the auction is endogenously determined by the auctioneer. Our paper differs from Lauermann and Wolinsky (2014) because (1) We study a multi-unit multi-market auction while they study a single-object single-market auction and Pesendorfer and Swinkels (1997)'s analysis implies that the information aggregation properties of a multi-unit auctions differ substantially from the information aggregation properties of an auction with a single object. (2) In our model the distribution of types is determined by the participation decision of the bidders while in

[5]Pesendorfer and Swinkels (2000) generalizes the analysis in Pesendorfer and Swinkels (1997) to a mixed private, common-value environment. Kremer (2002) shows that the information aggregation properties of auctions are more general than the particular mechanisms studied before; he does this by providing a unified approach that uses the statistical properties of certain order statistics. See Hong and Shum (2004) for a calculation of the rate at which price converges to the true value in large common-value auctions. Jackson and Kremer (2007) show that the result of Pesendorfer and Swinkels (1997) does not generalize to an auction with price discrimination. See Kremer and Skrzypacz (2005) for related results concerning the link between information aggregation and the properties of order statistics.

[6]There is extensive work on information aggregation and the role of prices in various other market contexts. For example, see Reny and Perry (2006) and Cripps and Swinkels (2006) for large double auctions; Vives (2011) and Rostek and Weretka (2012) for markets for divisible objects; Rubinstein and Wolinsky (1990), Wolinsky (1990), Golosov et al. (2011), Ostrovsky (2012), Lauermann and Wolinsky (2015) and Lambert et al. (2016) for search markets and markets with dynamic trading.

[7]Also, see Lauermann and Virág (2012) who study the incentives of an auctioneer to release information in a single-unit common-value auction where bidders who do not win an object from the auction receive an outside option.

their paper the auctioneer's solicitation strategy determines the number of bidders. This implies that in our model participation decisions are type dependent while in theirs it is type independent but state dependent. In Murto and Valimaki (2014)'s model, potential bidders must pay a cost to participate in the auction. This creates type dependent participation as in our model. However, in contrast to this paper they focus on a single-object, single-market auction, mainly focus on characterizing equilibria with two bidders and their emphasis is not on information aggregation.

Lauermann and Wolinsky (2014) and Atakan and Ekmekci (2014) also present models where information aggregation can fail in a large common-value auction.[8] In both of these papers, information aggregation fails because there is an atom in the bid distribution (i.e., many types submit the same pooling bid) and the auction price is equal to this atom (pooling bid) with positive probability in both states of the world. In this paper, although the bid distribution may feature atoms, the failure of information aggregation is not caused by these atoms. In fact, we show that information aggregation fails either because the auction is not sufficiently competitive[9] or because, although the auction is competitive, the same set of types determine the price in every state due to the pattern of self-selection.[10] In the equilibrium that we construct, we show that the limit price distribution is atomless over a non-trivial interval of prices which occur with positive probability in every state.

Finally, the literature on competition between mechanism designers or auctioneers is also related to this paper. See for example past work by McAfee (1993), Peters and Severinov (1997) and Burguet and Sákovics (1999). These papers study the competition between auctioneers who compete for bidders by offering different selling mechanisms or by choosing different reserve prices. These papers differ from the analysis here because their focus is on competition between auctioneers in an independent private values framework whereas our focus is on information aggregation in an auction with common but unknown values.

2. The Model

We study an auction with n bidders and $\lfloor \kappa_s n \rfloor = k_s$ identical objects where $\kappa_s \in (0,1)$.[11] Each bidder has unit demand and puts value V on a single object, and value 0 on any further objects. Thus, a bidder who wins a good at price P enjoys utility $V - P$. The value V (or the state) is common across players, but is unknown. The value is drawn from a finite set $\mathcal{V} \subset [0,1]$ according to a common prior $\pi(v) = 1/|\mathcal{V}|$ where v denotes an arbitrary element of $\mathcal{V}$. We focus on a uniform prior which puts equal probability on all value for expositional simplicity only and none of our results depend on this assumption. For much of the paper we assume that $\mathcal{V} = \{0,1\}$. In section 5, we discuss the case where $\mathcal{V}$ is an arbitrary finite subset of $[0,1]$.

[8] Also, see Axelson and Makarov who study a model where the winner of a single-object common-value auction must make an ex-post investment in order to put the object into productive use which is related to Atakan and Ekmekci (2014).

[9] Information aggregation fails because self-selection into the auction implies that the number of bidders who are not allocated objects is finite with positive probability in both states, i.e., Pesendorfer and Swinkels (1997)'s double-largeness conditions fails.

[10] In this case the number of bidders who are not allocated objects is infinite with probability one in both states, i.e., Pesendorfer and Swinkels (1997)'s double-largeness conditions is satisfied, but information aggregation nevertheless fails.

[11] The largest integer not greater than x is denoted by $\lfloor x \rfloor$.

Each bidder receives a signal $\theta \in [0,1]$ according to a continuous, increasing cumulative distribution function $F(\theta|V=v) := F(\theta|v)$ which admits a density function $f(\theta|v)$.[12] Conditional on $V=v$, the signals are identically and independently distributed. We assume that signals contain a bounded amount of information: $\frac{1}{\eta} < f(\theta|v) < \eta$. Also, we assume that the signals satisfy the monotone likelihood ratio property (MLRP) which we formally define below.

Definition 2.1 (MLRP)**.** Signals satisfy strict MLRP if the likelihood ratio

$$l(v,v'|\theta_i=\theta) := \frac{f(\theta|v)}{f(\theta|v')}$$

is increasing in θ for any $v > v'$.

The signals satisfy MLRP if (1) the likelihood ratio, $l(v,v'|\theta_i=\theta)$, is nondecreasing in θ for any $v > v'$; and (2) the signals contain some information, i.e., the signals satisfy the following two conditions:

a. $l(v,v'|\theta_i=1) > l(v,v'|\theta_i=0)$;

b. $l(v,v'|\theta_i=\theta)$ is continuous at $\theta=0$ and $\theta=1$.

After observing their private signal, bidders can choose to submit a bid in the auction market, they can select their outside option, or they can choose neither and receive a payoff equal to zero. We will ignore the agents' option of choosing "neither" since this option is dominated by bidding zero in the auction market.[13] We consider both **exogenous** outside options and **endogenous** outside options. The endogenous outside option value is determined in auction markets that run parallel to the auction under consideration. We will describe the outside options in more detail in the relevant sections.

A bidder does not observe anything beyond her private signal when deciding which bid to choose in the auction. The goods are sold using a closed-bid uniform-price auction where the auction price is equal to the highest losing bid if there are more bidders than there are goods. If the number of bidders is less than or equal to the number of goods, then the auction price is equal to zero. Ties are broken uniformly and randomly.

2.1. Strategies and Equilibrium. After observing their signal, each bidder submits a bid in auction $m \in M$ where M is the finite set of markets and m is an arbitrary element of M.[14]

[12] For any half open interval interval $(\theta', \theta'']$, we use $F((\theta', \theta'']|v) := F(\theta''|v) - F(\theta'|v)$, i.e., slightly abusing notation, we also use F to denote the measure induced (by Lebegue's theorem) by the cumulative distribution function.

[13] Ignoring the option of choosing "neither" is justified because we focus on the symmetric equilibrium of the model. If we allowed for the option of choosing "neither," then this option would not be chosen by any type in any symmetric equilibrium of the model. Therefore, the set of symmetric equilibrium outcomes with this option is the same as the set of symmetric equilibrium outcomes without this option. The reasoning is as follows: if a positive mass of types were to choose "neither" in a symmetric equilibrium, then any bidder who submits a bid equal zero in the auction would win an object with strictly positive probability in state $V=1$. Thus, all types who choose "neither" and receive a payoff equal to zero would rather bid zero in the auction and receive a strictly positive expected payoff.

[14] If the chosen option is an exogenous outside option, then the player's bid has no payoff or equilibrium consequence.

We allow for an arbitrary finite set of options denoted by M even though we focus mainly on the case where $M = \{s, r\}$.

We represent strategies by distributional strategies. A distributional strategy is a measure H on $[0,1]\times M\times[0,\infty)$ with marginal distribution $F(\theta) = \sum_v \pi(v)F(\theta|v)$ on its first coordinate (see Milgrom and Weber (1985)). A symmetric strategy profile is one in which all players use the same distributional strategy H and we refer to a symmetric strategy profile by simply the strategy H. For a given strategy H, we define the **distribution of signals in auction** m as $F_m(\theta) := H([0,\theta] \times \{m\} \times [0,\infty))$, we set $\bar{F}_m(\theta|v) = F_m(1|v) - F_m(\theta|v)$ and we define a **selection function** $a_m : [0,1] \to [0,1]$ as the function such that $F_m(\theta) = \int_0^\theta a_m(\theta)dF(\theta)$.[15] In words, $a_m(\theta)$ is the probability that an agent chooses to bid in auction m when she receives signal θ. A symmetric bidding strategy in market m is pure if there is function $b : [0,1] \to [0,\infty)$ such that $H(\{\theta, m, b(\theta)\}_{\theta\in[0,1]}) = H([0,1] \times \{m\} \times [0,\infty)) = F_m(1)$.[16]

The notation $\Pr^H$ represents the joint probability distribution over states of the world, signal and bid distributions, allocations, market choices, and prices, where this distribution is induced by the symmetric strategy H. We denote by $u^H(m, b|\theta)$ the payoff to type θ from bidding b in auction $m \in M$ if players are using strategy H and we denote by $u^H(\theta)$ type θ's payoff under strategy H. We denote by $Y^n(k)$ the kth highest signal out of n signals. Also, we let $Y_m^n(k)$ denote the kth highest signal in auction m and we set $Y_m^n(k)$ equal to zero if there are fewer than k bidders in auction m.

We focus on the symmetric Nash equilibria of the game Γ and we refer to a symmetric Nash equilibrium simply as an equilibrium.[17] We also find it convenient to define an auction where participation in the auction is exogenously determined by a function $F_m(\cdot)$ that is absolutely continuous with respect to $F(\cdot)$. More precisely, given F_m we define $a_m : [0,1] \to [0,1]$ as the function such that $F_m(\theta) = \int_0^\theta a_m(\theta)dF(\theta)$ and we denote by $\hat{\Gamma}(F_m)$ the auction where each player of type θ is allowed to bid in the auction with probability $a_m(\theta)$ and assigned a payoff equal to zero with the remaining probability $1 - a_m(\theta)$. We say that H is a bidding equilibrium for market m if $H(\cdot \times \{m\} \times \cdot)$ is a Nash equilibrium for the auction $\hat{\Gamma}(F_m)$ where participation is determined by $F_m(\theta) := H([0,\theta] \times \{m\} \times [0,\infty))$. Intuitively, in a bidding equilibrium H, the market selection of players implied by H need not be a best response to the strategy profile but the bidding behavior of players implied by H is a best response to the bidding strategies of others given the exogenously fixed participation decisions.[18]

The lemma further below allows us to work exclusively with pure and nondecreasing bidding strategies. Let $\mathcal{E}(\theta') = \{\theta : l(v, v'|\theta_i = \theta) = l(v, v'|\theta_i = \theta'), \forall v, v' \in \mathcal{V}\}$. Each set $\mathcal{E}(\theta')$ is comprised of types who receive signals which have the same information content. If $\mathcal{E}(\theta')$ is not a singleton, then H may involve a range of bids given a signal in $\mathcal{E}(\theta')$. However, it is easily seen that for any such H, there is another strategy which is pure and increasing on each

[15] Mathematically a is the Radon-Nikodym derivate of F_s with respect to F and is unique up to almost every $\theta \in [0,1]$. The distributional strategy's definition implies that F_s is absolutely continuous with respect to F.

[16] If b represents H, then so does any function that is equal to b at almost every $\theta \in [0,1]$.

[17] Also, we say that H is a **bidding** equilibrium in market s if $H(\cdot \times \{s\} \times \cdot)$ is an Bayesian-Nash equilibrium for the auction Γ_s where the distribution of participating types is given by F_s.

[18] Note that if H is an equilibrium of Γ, then $H(\cdot \times \{s\} \times \cdot)$ is a Nash equilibrium for the auction $\hat{\Gamma}(F_s)$ where participation is determined by $F_s(\theta) := H([0,\theta] \times \{s\} \times [0,\infty))$ and therefore H is also a bidding equilibrium for market s.

$\mathcal{E}(\theta')$, such that this strategy yields the same payoff to the player, and is indistinguishable to any other player. Strategies which differ only in their representation over sets $\mathcal{E}(\theta')$ generate the same joint distribution over values, bids, and equilibrium prices. In what follows we choose a representation of H which is pure and increasing over sets $\mathcal{E}(\theta')$.

Lemma 2.1. *If signals satisfy MLRP, then any bidding equilibrium H* for market m *can be represented by a nondecreasing bidding function b.*

2.2. Information Aggregation, Sequences and Pivotal Types. We study a sequence of games Γ^n where the n^{th} game has n bidders and $\lfloor n\kappa_m \rfloor = k_m$ objects for sale in auction m. Along the sequence games Γ^n, the state space, the good to bidder ratio (κ_m) signal distribution and payoffs (i.e., all primitives except the number of objects and the number of bidders) are kept the same. The behavior in the sequence of games is described by a sequence of strategies $\mathbf{H} = \{H^n\}_n^\infty$. We say that $\mathbf{H}$ is an equilibrium sequence if H^n is an equilibrium of Γ^n for each n.

The random variable P_s^n describes the price in an auction where bidder behave according to strategy H^n. Our definition of information aggregation is given below:[19]

Definition 2.2 (Information Aggregation.)**.** (Pesendorfer and Swinkels (1997) and Kremer (2002)) A sequence of strategies $\mathbf{H}$ **aggregates information** in market s if $P_s^n \xrightarrow{p} V$, i.e., if P_s^n converges in probability to 0 and 1 whenever $V = 0$ and 1, respectively.

Lemma 2.1 showed that bidding is monotone in equilibrium. This implies that we can define a certain type $\theta_s^H(v)$ for each state v such that the expected number of players who submit a bid above this type's bid in state v, i.e, $\bar{F}_s^H(\theta_s^H(v)|v)$, is exactly equal to the number of goods in market s. We refer to $\theta_s^H(v)$ as the pivotal type in state v because the types that determine the auction price are concentrated around $\theta_s^H(v)$ in a large market by the law of large numbers (see Lemma B.1 for a precise statement of this fact). The pivotal types are particular important for our analysis because understanding whether information is aggregated reduces to understanding whether the bids of the types which determine the price converge to v. The pivotal types and some other concepts, which we use frequently, are defined below.

Definition 2.3 (Pivotal types)**.** For any symmetric strategy H and market $m \in M$ define the **pivotal type** $\theta_m^H(v)$ in state v and market m as the type such that

$$\theta_m^H(v) = \max\left\{\theta : \bar{F}_m^H(\theta|v) = \kappa_m\right\}$$

and we set $\theta_m^H(v) = 0$ if the set is empty.[20] Also, let $\theta_m^F(v)$ be the type such that $\bar{F}(\theta_m^F(v)|v) = \kappa_m$, in words, $\theta_m^F(v)$ is the pivotal type in state v if all types where to bid in the auction m.[21]

[19]In the appendix we also study a weaker notion of information aggregation that is termed informativeness (Definition B.2) also used in Kremer (2002) and Atakan and Ekmekci (2014).

[20]The equation $\bar{F}_m^H(\theta|v) = \kappa_m$ can have multiple solutions if F_m^H is flat over a range of θ (it is nondecreasing but not necessarily increasing.) However, the function $\bar{F}_m^H(\theta|v)$ is continuous because it is absolutely continuous with respect to $\bar{F}(\theta|v)$. Hence, the set $\left\{\theta : \bar{F}_m^H(\theta|v) = \kappa_m\right\} \subset [0,1]$ is compact and has a unique maximal element if it is nonempty.

[21]In this case $\bar{F}(\theta|v) = \kappa_m$ has a unique solution because F is increasing.

Definition 2.4 (Value and belief conditional on being pivotal)**.** Define the likelihood of being pivotal as follows:

$$l(Y_m^{n-1}(k) = \theta, \theta_i = \theta) = l(\theta_i = \theta)^2 \left(\frac{F_m^n(\theta|1)}{F_m^n(\theta|0)}\right)^{n-k-1} \left(\frac{\bar{F}_m^n(\theta|1)}{\bar{F}_m^n(\theta|0)}\right)^{k-1}$$

where

$$l(\theta_i = \theta) := \frac{f(\theta|1)}{f(\theta|0)}.$$

Define the value of a type conditional on being pivotal, i.e., $\mathbb{E}[v|Y_s^{n-1}(k) = \theta, \theta_i = \theta]$, as follows:

$$\beta_m^n(\theta) \quad = \quad \frac{l(Y_m^{n-1}(k) = \theta, \theta_i = \theta)}{1 + l(Y_m^{n-1}(k) = \theta, \theta_i = \theta)}. \tag{2.1}$$

Note that the bidding function $\beta_m^n(\theta)$ involves bidding one's value condition being pivotal and is identical to the bidding function described by Wilson (1977), Milgrom (1979) and Pesendorfer and Swinkels (1997).

Definition 2.5 (Sequences and limits)**.** For any sequence of strategies $\{H^n\}$ we will denote each $\theta_m^{H^n}(v)$ simply by $\theta_m^n(v)$, we let $\theta_m(v) = \lim_n \theta_m^n(v)$ and $F_m(\theta|v) = \lim F_m^n(\theta|v)$ whenever such limits exist.[22] For $F_m(\theta|v) = \lim F_m^n(\theta|v)$, we define $a_m : [0,1] \to [0,1]$ as the function such that $F_m(\theta|v) = \int_0^\theta a_m(\theta) dF(\theta|v)$.[23]

3. Information Aggregation with an Exogenous Outside Option

In this section, we focus on the case where each player chooses between bidding in auction s and an exogenously given outside option r. The definition below summarizes the assumptions that we make on the outside option.

Definition 3.1 (Exogenous Outside Options)**.** If a player selects the exogenous outside option, then the player's payoff is equal to $u(r|v) := u(r|V = v)$ which is nondecreasing in v. We define

$$\begin{aligned} \mathbb{E}[u(r|V)] &= \sum_{v \in \mathcal{V}} \frac{u(r|v)}{|\mathcal{V}|} \\ Var[u(r|V)] &= \sum_{v \in \mathcal{V}} \frac{u(r|v)^2}{|\mathcal{V}|} - \mathbb{E}[u(r|V)]^2 \end{aligned}$$

We assume that

$$\mathbb{E}[u(r|V)|\theta_i = 1] = \sum_{v \in \mathcal{V}} u(r|v) \frac{f(1|v)}{\sum_{v \in \mathcal{V}} f(1|v)} > 0,$$

i.e., the outside option is valuable for at least some types sufficiently close to $\theta = 1$.

[22] Such limits always exist possibly along a subsequence. This is because the sequence $\{\theta^n(v)\}$ is a subset of $[0,1]$ and because $\{F_m^n(\theta|v)\}$ is a sequence of nondecreasing, continuous and bounded functions that has a subsequential limit by Helly's theorem.

[23] The fact that each a^n is uniformly bounded implies that $F_s(\theta|v)$ is a absolutely continuous with respect to $F(\theta|v)$ and therefore the function a is well defined.

On the other hand, if a player selects market s and wins an object, then his payoff is equal to $V - P_s$ and if the player chooses the auction but is unsuccessful in winning an object, then his payoff is equal to zero.[24]

The main assumption that we make is that the outside option's state dependent value is positively correlated with the auctioned object's value. This assumption is natural in our setting since it is satisfied by the endogenous outside option that we discuss in the next section and since this assumption generates the more interesting dynamics. However, our analysis easily extends to the case where $u(r|v)$ is nonincreasing and we discuss the unique equilibrium of the auction under this assumption as Corollary 3.2 in subsection 3.2.

This section's main theorem shows that an outside option with positive variance will disrupt information aggregation if the fraction of goods in the auction, κ_s, exceeds a certain cutoff κ^*. The corollary to this theorem then argues that if, in addition to the variance, the expected value of the outside option is also positive, then information is not aggregated in market s in any equilibrium irrespective of the fraction of goods in market s. In the following subsection, we turn to characterizing equilibria in an auction with an exogenous outside option. In particular, in Proposition 3.1 we show that the game has a unique equilibrium if $u(r|0) \geq 0$. In Proposition 3.2, we characterize the properties of all equilibria for the case where $u(r|0) < 0$ if the variance of the outside option is smaller than a cutoff.

3.1. Failure of Information Aggregation. In order to state our main theorem, we first define a cutoff κ^* for the fraction of goods in auction s which we will use in the statement of our theorem.

Definition 3.2. For any $\theta' \in [0, 1)$, let

$$\kappa^*(\theta') = \max_{\theta \in [0,\theta']} \left\{ F(\theta'|1) - F(\theta|1) : F(\theta'|1) - F(\theta|1) \geq F(\theta'|0) - F(\theta|0) \right\}$$

and let $\kappa^*(\theta') = 0$ if the set over which the maximum is taken is empty. Note that $\kappa^*(\theta') < 1$.[25]

To better understand the above definition, consider a strategy H where all types greater than θ' select the outside option r while all types smaller than θ' bid in the auction s. The cutoff $\kappa^*(\theta')$ is defined so that, if the fraction of goods in market s is greater than $\kappa^*(\theta')$, then the pivotal type in state $v = 0$, $\theta_s^H(0)$, exceeds the pivotal type in state $v = 1$, $\theta_s^H(1)$. Such an ordering of pivotal types is ruled out by MLRP if all types participate in the auction. However, given strategy H, MLRP implies that the measure of players selecting the auction is smaller in state $v = 1$ than in state $v = 0$. This implies that $\kappa^*(\theta')$ is always less than one

[24] Note that the outside option is no longer available to bidders in market s who are unable to win an object from the auction. This assumption is motivated by a situation where the outside option is generated in a market that runs concurrently with market s (such as our model with an endogenous outside option). Also, this assumption is sensible if the outside option is viewed as an entry cost into the auction. This assumption might not be satisfied in other conceivable circumstances. However, our conclusions related to the failure of information aggregation (presented as Theorem 3.1 and Corollary 3.1) continue to hold true even if the outside option is available to bidders who bid in auction s but are unable to win an object.

[25] Suppose that the set under consideration is non-empty. Under strict MLRP, there is a unique point where the functions $F(\theta'|1) - F(\theta|1)$ and $F(\theta'|0) - F(\theta|0)$ cross and $\kappa^*(\theta')$ is equal to the value at this point. Under MLRP the two functions may coincide over a range of θs. In such a case $\kappa^*(\theta')$ is equal to the highest value where $F(\theta'|1) - F(\theta|1)$ and $F(\theta'|0) - F(\theta|0)$ are equal.

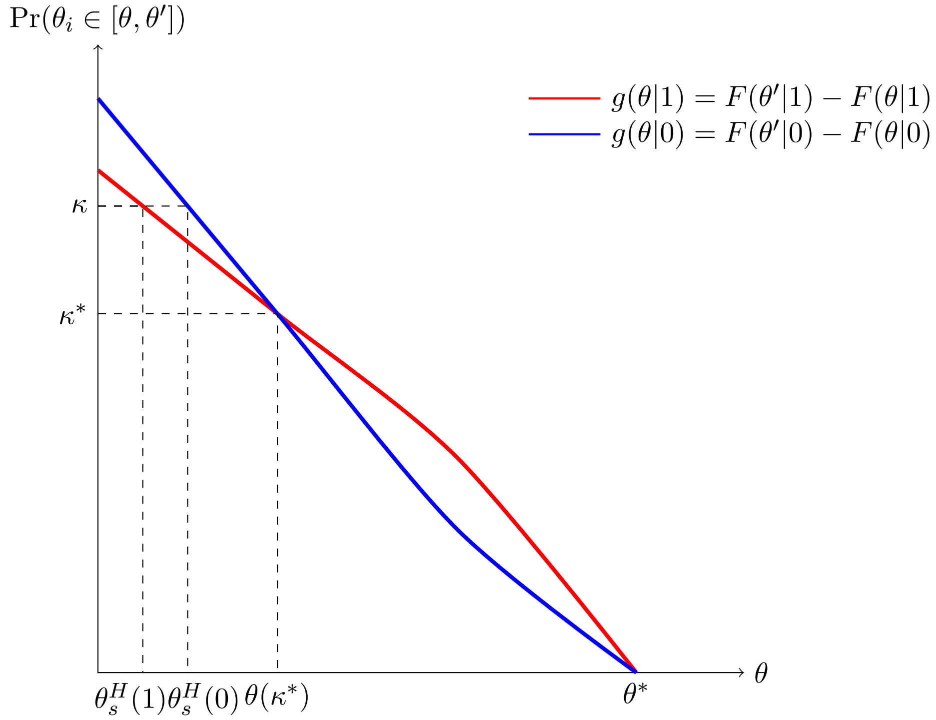


Figure 3.1: The cutoff κ^*. The function $g(\theta|v)$ depicts the fraction of types above θ that bid in auction s in state v given that all types $\theta > \theta'$ take the outside option. Note that MLRP and $\theta' < 1$ implies that $g(0|0) = F(\theta'|0) > g(0|1) = F(\theta'|1)$. Also, note that $g(\theta|1)$ and $g(\theta|0)$ can cross at most once under strict MLRP because the ratio of their slopes $-\frac{f(\theta|1)}{f(\theta|0)}$ is decreasing in θ. We define κ^* as the value of g at the point where the two functions cross. Therefore, if $\kappa > \kappa^*$, then the pivotal type in state $v = 0$ exceeds the pivotal type in state $v = 1$. If the functions never cross, then $g(\theta|0) > g(\theta|1)$ for all $\theta < \theta'$ and we then define $\kappa^* = 0$. In this case $\theta_s^H(0) > \theta_s^H(1)$ for any κ.

and therefore there is an open interval $(\kappa^*(\theta'), 1)$ such that if $\kappa \in (\kappa^*(\theta'), 1)$, then the ordering of the pivotal types is reversed, i.e., $\theta_s^H(0) > \theta_s^H(1)$. In fact, our main theorem argues that information cannot be aggregated in an equilibrium where the order of the pivotal types is reversed. See Figure 3.1 for a graphical depiction of $\kappa^*(\theta')$. Our main theorem also shows that if the fraction of goods is less than the cutoff $\kappa^*(\theta')$ and if the variance of the outside option is lower than a certain cutoff $\bar{u}$ (formally defined by equation (D.1)), then information is aggregated by every equilibrium sequence.

Theorem 3.1. *Suppose that MLRP is satisfied. Suppose that players have access to outside options as defined by Definition 3.1 and let* $\theta' = \inf\{\theta : \mathbb{E}[u(r|V)|\theta_i = \theta] > 0\}$.

i. If $\kappa_s > \kappa^*(\theta')$*, then no equilibrium sequence* $\mathbf{H}$ *aggregates information.*

ii. Let $\bar{u}$ *be the constant defined by equation (D.1) in the appendix. Suppose that strict MLRP is satisfied and that* $Var[u(r|V)] < \bar{u}$.[26] *If* $\kappa_s < \kappa^*(\theta')$*, then any equilibrium sequence* $\mathbf{H}$ *aggregates information.*

[26]The assumption of strict MLRP can be relaxed and replaced by the likelihood ratio function $l(\theta)$ being increasing at the type θ such that $l(\theta) = 1$, i.e., that strict MLRP be satisfied locally around θ such that $l(\theta) = 1$.

Corollary 3.1. *Suppose that players have access to outside options as defined in Definition 3.1. Assume that strict MLRP is satisfied. If $\mathbb{E}[u(r|V)] \geq 0$ and $Var[u(r|V)] > 0$, then $\kappa^*(\theta') = 0$ and therefore there is no equilibrium sequence* **H** *that aggregates information for any $\kappa > 0$. Alternatively, assume that MLRP is satisfied. If $\mathbb{E}[u(r|V)] > 0$, then $\kappa^*(\theta') = 0$ and therefore there is no equilibrium sequence* **H** *that aggregates information for any $\kappa_s > 0$.*[27]

Remark 3.1. Theorem 3.1 is stated under the assumption that $\mathcal{V} = \{0, 1\}$. In section 5, we discuss the case where $\mathcal{V}$ is any arbitrary finite subset of $[0, 1]$ and we show that the findings presented in Theorem 3.1 and Corollary 3.1 fully generalize.

We now provide some intuition for Theorem 3.1: If information is aggregated in market s, then the equilibrium payoff of submitting any bid b in market s is equal to zero. This is because the price in this market converges to the object's value in each state and therefore no bidder can make a profit in this market. First, suppose that $0 \leq u(r|0) < u(r|1)$. In this case, the expected value of the outside option is positive for all types θ. However, if the payoff from the outside option is positive and if information is aggregated in market s, then no player would choose market s in equilibrium. This implies that information cannot be aggregated in market s under the assumption that $0 \leq u(r|0) < u(r|1)$.

Instead suppose that $u(r|0) < 0 < u(r|1)$. In this case, the value of the outside option is positive for all types that exceed a certain cutoff $\theta' = \inf\{\theta : \mathbb{E}[u(r|V)|\theta_i = \theta] > 0\}$. This is because of a **single crossing property**: We know that $\mathbb{E}[u(r|V)|\theta_i = \theta] > 0$ for a certain subset of types θ because we assume $\mathbb{E}[u(r|V)|\theta_i = 1] > 0$. Moreover, $\mathbb{E}[u(r|V)|\theta_i = \theta] > 0$ for all $\theta > \theta'$ because all types $\theta > \theta'$ put more weight on state $v = 1$ than θ' as a consequence of MLRP. If information is aggregated in market s, then all types above θ' would select the outside option because the value of the outside option is positive for these types while the equilibrium value from bidding in market s is zero for all types. For example, if $\mathbb{E}[u(r|V)] = 0$, then θ' is the type such that $l(\theta_i = \theta') = 1$. In this case the value of the outside option is negative for all $\theta < \theta'$ because $l(\theta_i = \theta) < 1$ implies that $\mathbb{E}[u(r|V)|\theta_i = \theta] < \mathbb{E}[u(r|V)] = 0$ and positive for all $\theta > \theta'$ because $l(\theta_i = \theta) > 1$ implies that implies that $\mathbb{E}[u(r|V)|\theta_i = \theta] > \mathbb{E}[u(r|V)] = 0$

The final step in the argument concludes that all types that exceed θ' opting for option r and information aggregation in market s taken together are incompatible with bidders behaving according to a monotone bidding function as we claimed that they must in Lemma 2.1 if $\kappa > \kappa^*(\theta')$. More precisely, if all types that exceed θ' select the outside option then we find that $\theta_s(0) > \theta_s(1)$ under our assumption that $\kappa > \kappa^*(\theta')$ (see figure 3.1). Our assumption of information aggregation in market s implies that $\lim_{n\to\infty} b_s^n(\theta_s^n(1)) = 1$ and $\lim_{n\to\infty} b_s^n(\theta_s^n(0)) = 0$. However, this provides the contradiction that proves the result. The findings that $\lim_{n\to\infty} b_s^n(\theta_s^n(1)) = 1$ and $\lim_{n\to\infty} b_s^n(\theta_s^n(0)) = 0$, and $\theta_s(0) > \theta_s(1)$ together contradict that the bidding function is nondecreasing for all n.

Returning to the example where $\mathbb{E}[u(r|V)] = 0$ we find that only types with $l(\theta_i = \theta) < 1$ would select market s. However, if $l(\theta_i = \theta) < 1$ for all types in market s, then we have that $\theta_s(0) > \theta_s(1)$. To see this, note that $\theta_s(0)$ is the type such that $\int_{\theta_s(0)}^{\theta^*} f(\theta|0)d\theta = \kappa$. But, we

[27] To see that $\kappa^* = 0$. Note that if $\mathbb{E}[u(r|V)] \geq 0$ and $Var[u(r|V)] > 0$, then $\mathbb{E}[u(r|V)|\theta_i = \theta] > 0$ for all types for which $l(\theta_i = \theta) > 1$; and if $\mathbb{E}[u(r|V)] > 0$ and $Var[u(r|V)] > 0$, then $\mathbb{E}[u(r|V)|\theta_i = \theta] > 0$ for all types for which $l(\theta_i = \theta) \geq 1$. Therefore, $F(\theta'|1) - F(\theta|1) < F(\theta'|0) - F(\theta|0)$ for all $\theta < \theta'$ and hence $\kappa^* = 0$.

know $\int_{\theta_s(0)}^{\theta^*} f(\theta|1)d\theta < \int_{\theta_s(0)}^{\theta^*} f(\theta|0)d\theta$ because $l(\theta_i = \theta) = \frac{f(\theta|1)}{f(\theta|0)} < 1$ for all $\theta < \theta^*$. Therefore, we conclude that $\int_{\theta_s(0)}^{\theta^*} f(\theta|1)d\theta < \kappa$ which establishes $\theta_s(1) < \theta_s(0)$.

3.2. Equilibrium Construction and Characterization. Theorem 3.1 showed that information is not aggregated in market s. However, the proof of the theorem was by contradiction and therefore did not provide much insight into actual equilibrium behavior. Propositions 3.1 and 3.2 aim to describe equilibrium behavior in an auction with an outside option in greater detail. Proposition 3.3 further below constructs an equilibrium for the case where the outside option is valueless for a subset of types.

3.2.1. *An Outside Option which is Valuable for all Types.* In Proposition 3.1 below, we assume that $u(r|0) > 0$, i.e., we focus on the case where the outside option's value is strictly positive for all types. In this case, there is a unique equilibrium. In this equilibrium, all types above a certain cutoff, i.e., the more optimistic types, select the auction market s and bid according to a strictly increasing bidding function. In particular, each type submits a bid which is equal to that type's value conditional on the event that they win an object from the auction at a price equal to their own bid, i.e, $\beta_s^n(\theta) = \mathbb{E}[V|Y_s^{n-1}(k) = \theta, \theta_i = \theta]$ defined in equation 2.1. This bidding function is identical to the bidding function of Pesendorfer and Swinkels (1997).

Proposition 3.1. *Assume that MLRP is satisfied and that players have access to outside options as defined by Definition 3.1. If $u(r|0) > 0$, then the auction Γ^n has a unique equilibrium H^n for all n. In the unique equilibrium sequence $\mathbf{H}$, the following properties are satisfied:*

i. There is a certain cutoff type $\hat{\theta}^n$ such that all types $\theta < \hat{\theta}^n$ opt for the outside option.

ii. All types $\theta > \hat{\theta}^n$ bid in the auction according the increasing bidding function $\beta_s^n(\theta)$.

iii. All bids converge to one, i.e., $\beta_s^n(\theta) \to 1$ for all $\theta > \hat{\theta} = \lim \hat{\theta}^n$.

iv. Moreover, $\lim |F_s^n(1|1) - \kappa|\sqrt{n} = x < \infty$ and $F_s(1|0) < \kappa$.

v. If $V = 0$, the price converges to zero almost surely. If $V = 1$, then the price converges to a random variable which is equal to zero with probability $q > 0$ and equal to one with the remaining probability. Moreover, $q = \mathbb{E}[u(r|V)|\theta_i = \hat{\theta}]\frac{l(\theta_i=\hat{\theta})}{1+l(\theta_i=\hat{\theta})}$ if the function $l(\theta_i = \theta)$ is continuous at $\theta_i = \hat{\theta}$.[28]

It is worthwhile highlighting some properties of the equilibrium identified by the above proposition: If a bidder bids in the auction, then the bidder behaves as he does in the equilibrium of the auction without an outside option (item *ii*). Therefore, the bid of the pivotal type in state $V = 1$ and also the bids of all types that exceed $\theta_s(1)$ must converge to one. This further implies that if there are many bidders in the auction, then the payoff of these bidders would converge to zero. Hence, only types above the pivotal type $\theta_s(1)$ participate

[28]In the appendix we study a weaker notion of information aggregation that is termed informativeness. A sequence of equilibria in a large auction is termed informative (Definition B.2), if an outside observer can learn the state asymptotically by simply observing the equilibrium price. Note that the equilibrium price is not informative either. This is because the price is equal to zero with positive probability in both states.

in the auction but all types who participate bid aggressively (Items *iii* and *iv*). In fact, the number of bidders in the auction is less than the number of goods with probability one in state $V = 0$ and with probability $q > 0$ in state $V = 1$.[29] Bidders who participate in auction s are compensated for foregoing the outside option by occasionally winning a good at a price equal to zero in state $V = 1$. Information aggregation fails because there are fewer bidders than there are goods with positive probability, i.e., information aggregation fails because of a lack of competition for objects.[30,31] Moreover, although information aggregation fails, a price close to one is fully revealing because such a price occurs with positive asymptotic probability only if $V = 1$.

Remark 3.2. If $u(r|0) = u(r|1) > 0$, then the auction that we study is equivalent to an auction with costly entry. However, the analysis here differs from Murto and Valimaki (2014)'s analysis of a common-value auction with costly entry. In contrast to this paper, Murto and Valimaki (2014) assume that the object's value is positive in the bad state and exceeds the cost of entry. Under this assumption, information cannot be aggregated since if it were, then no bidder would have an incentive to pay the entry cost. However, our equilibrium characterization does not cover the case considered by Murto and Valimaki (2014). This is because under their assumption bidders would have an incentive to pay the entry cost and bid in the auction if the expected number of bidders is less than the number of objects on auction in the bad state as we find in Proposition 3.1.

Throughout this paper we assume that the value of the exogenous outside option is positively correlated with the object on auction. We maintain this assumption because the endogenous outside options that we discuss in the next section generate such correlations. However, the characterization that we provided in Proposition 3.1 also covers the case where the outside option is negatively correlated with the object on auction. The following corollary discusses this case.

Corollary 3.2. *Assume that MLRP is satisfied and that the value of the exogenous outside option is $u(r|v)$ is nonincreasing in v. If $\mathbb{E}[u(r|V)|\theta_i = \theta_s^F(1)] > 0$, then the unique equilibrium of the auction is the equilibrium described by Proposition 3.1. Therefore, the equilibrium sequence $\mathbf{H}$ does not aggregate information. If instead $\mathbb{E}[u(r|V)|\theta_i = \theta_s^F(1)] < 0$, then the unique equilibrium of the auction is described by Proposition 3.1 item i and ii. In this case, the equilibrium sequence $\mathbf{H}$ aggregates information.*

3.2.2. *An Outside Option which is Valuable for a Subset of Types.* The following proposition focuses on the case where the outside option is valuable for some bidders and possibly valueless

[29] The probability q can be viewed as the amount of information not contained in the auction price. Note that item v implies that as the value of the outside option becomes small for all types, q converges to zero. In other words, if the auction is perturbed by a outside option whose value is small, then the information lost in the auction is also small.

[30] The failure of information aggregation can be related to a failure of the double largeness concept introduced by Pesendorfer and Swinkels (1997). Pesendorfer and Swinkels (1997) showed that information is aggregated if and only if the number of goods and the number of losers not allocated goods converge to infinity. They termed this double largeness. In the unique equilibrium described by Lemma 3.1 double largess fails with probability one in state $v = 0$ and it fails with positive probability in state $V = 1$.

[31] A similar indeterminacy also arises in the take-over model presented in Ekmekci and Kos (2016) although the decisions are binary in their model.

for others ($u(r|0) < 0$). In this case, there are a large number of equilibria and it is difficult to describe all equilibria succinctly. However, if the outside option's value is small for all types, i.e., if the variance of the outside option $Var[u(r|V)]$ is less than a certain cutoff value $\bar{u} > 0$ (which depends on the signal distribution and κ_s), then it is possible to characterize equilibria. The exact expression for $\bar{u}$ is given in the appendix as equation (D.1). In the next proposition, we focus on the case where $Var[u(r|V)] < \bar{u}$ and we outline the properties of all equilibria under this restriction. In the next subsection, we construct an equilibrium with the properties described in the proposition below.

Proposition 3.2. *Assume that MLRP is satisfied and that players have access to outside options as defined by Definition 3.1. Suppose that $u(r|0) < 0$, $\kappa > \kappa^*(\theta')$ and $Var[u(r|V)] < \bar{u}$ where θ' and $\bar{u}$ are defined in Theorem 3.1 and equation (D.1), respectively. Then, along every equilibrium sequence* $\mathbf{H}$*,*

i. $F_s(1|v) \geq \kappa$ *for* $v = 0, 1$*. Moreover, if* $\mathbb{E}[u(r|V)|\theta_i = 0] < 0$*, then* $F_s(1|v) > \kappa$ *for* $v = 0, 1$*.*

ii. $\lim \sqrt{n}|F_s^n(\theta_s^n(1)|v) - F_s^n(\theta_s^n(0)|v)| < \infty$ *for* $v = 0, 1$*, i.e., the distance between the pivotal types is at the order of* $1/\sqrt{n}$*.*[32]

It is worthwhile highlighting some of the properties of the equilibria described by Proposition 3.2: Theorem 3.1 shows that information is not aggregated in market s under the hypothesis of Proposition 3.2. Proposition 3.2 further shows that the expected number of bidders who are not allocated a good converges to infinity in both states. Therefore, the failure of information aggregation is not caused by a lack of competition as in Proposition 3.1.[33] Proposition 3.2 also argues that the pivotal types in state $v = 0$ and $v = 1$ are arbitrarily close to each other. This is a consequence of the way in which types self-select into market s in equilibrium. Information aggregation fails because the auction clears at the bids of the same set of types in all states and this is exactly because the pivotal types are arbitrarily close to each other in equilibrium. Interestingly and in contrast to Proposition 3.1, this configuration of pivotal types implies that there are no fully revealing prices: any set of prices that occur with positive probability in state $V = 0$ also occur with positive probability in state $V = 1$ and vice versa. Therefore, the posterior belief of an outside observer about the state will remain strictly bounded away from zero and one after observing the price in market s.

A central finding of Proposition 3.2 is that the pivotal types converge to each other as n grows large. The argument for this is as follows: If the pivotal types do not converge to each other, then two alternative limiting outcomes are possible: (1) The pivotal types do not submit the same pooling bid and therefore their bids converge to two distinct values, i.e.,

[32] In the appendix we study a weaker notion of information aggregation that is termed informativeness. Our finding that $F_s(1|v) > \kappa$ for $v = 0, 1$ and the fact that information is not aggregated in market s together imply by applying Lemma B.3 that the price in market s is not informative according to Definition B.2 either.

[33] The finding $F_s^n(1|v) > \kappa$ for $v = 0, 1$ implies that the double largeness condition of Pesendorfer and Swinkels (1997) is satisfied in equilibria. This condition would have implied information aggregation if there were no selection effects. However, information is not aggregated in spite of the fact that double largeness is satisfied in equilibrium. Thus, Lemma 3.2 shows that the information aggregation failure is caused purely through selection in instances where the outside option is only valuable for a subset of agents.

$\lim b^n(\theta_s^n(1)) > \lim b^n(\theta_s^n(0))$. However, then information is aggregated which is ruled out by Theorem 3.1. (2) The pivotal types submit the same pooling bid. In this case, a key argument shows that the pooling price must remain larger than a uniform and positive lower bound in order for pooling to be possible. A high pooling price implies that in state $V = 0$ the bidders' loss at the pooling bid exceeds their loss if they choose outside option r. This however implies that market selection has a cutoff structure: optimistic types above a threshold choose market s and all other more pessimistic types choose option r. But such a selection implies that information is aggregated in market s which is again incompatible with Theorem 3.1. Ruling out these two alternative outcomes leaves only the equilibrium configuration where that pivotal types must be arbitrarily close to each other.

Summarizing our characterization result: Propositions 3.1 and 3.2 outlined two different ways in which information aggregation fails in equilibrium: if $u(r|0) \geq 0$, then information aggregation fails because of a lack of competition. On the other hand, if $u(r|0) < 0$ and $Var[u(r|V)] < \bar{u}$, then information aggregation fails because the pivotal types are arbitrarily close to each other.

The case where $u(r|0) < 0$ and the outside option's variance is large, i.e., $Var[u(r|V)] > \bar{u}$, is not covered by these two propositions. In this case, there are a large number of other equilibrium configurations which lead to the failure of information aggregation. One of these configurations that does not appear in the previous two propositions is a situation where the pivotal types do not converge to each other as in Proposition 3.2 but information aggregation nevertheless fails because the pivotal types pool at the same bid. We discuss such an equilibrium in Example D.1 presented in appendix.

3.2.3. *Equilibrium Construction with Binary Signals.* In this subsection, we construct an equilibrium with the properties described by Proposition 3.2. For the construction we focus on a setup with a binary signal structure. In particular, we assume the following:

Definition 3.3. All signals $\theta \in [0, \frac{1}{2}] = \mathcal{E}(0)$ provide the same bad news about the state while all signals $\theta \in [\frac{1}{2}, 1] = \mathcal{E}(1)$ provide the same good news. Agents receive a good signal, i.e., a signal $\theta \in [\frac{1}{2}, 1]$ with probability f_h in state $V = 1$ and probability $f_l < f_h$ in state $V = 0$. Therefore, $l(\theta_i = \theta) = f_h/f_l$ for $\theta \in [\frac{1}{2}, 1]$ and $l(\theta_i = \theta) = (1 - f_h)/(1 - f_l)$ for $\theta \in [0, \frac{1}{2}]$.

The following proposition describes the equilibrium that we construct.

Proposition 3.3. *Assume that the signals satisfy Definition 3.3 and agents have access to an exogenous outside option with $u(r|0) = -c$ and $u(r|1) = c$. There exists a constant c^* such that if $c \in (0, c^*)$, then there exists an N such that for all $n > N$, there is an equilibrium H^n with the following properties:*

i. There is a cutoff type $\theta_1^n > \frac{1}{2}$ such that types $\theta > \theta_1^n$ select market r and all other types select market s.

ii. There is a cutoff type $\theta_p^n \in (0, 1/2)$ and a pooling bid b_p^n such that $b_s^n(\theta) = b_p^n$ if $\theta < \theta_p^n$, i.e., all types below θ_p^n submit the pooling bid and all types $\theta > \theta_p^n$ bid according to the increasing bidding function $\beta_s^n(\theta)$ defined in equation 2.1.

iii. The cutoff type θ_p^n is the unique $\theta \in (0, 1/2)$ that satisfies the following equality:

$$l(Y_s^{n-1}(k) = \theta, \theta_i = \theta) = l(Y_s^{n-1}(k) \leq \theta, \theta_i = \theta) \tag{3.1}$$

iv. The price in market s converges to a random variable $P_s(v)$ for $v = 0, 1$. The random variables $\{P_s(v)\}_{v=0,1}$ both have atomless distributions over the interval $(b_p, 1]$ with full support and atoms at b_p where $b_p = \lim b_p^n$.

Remark 3.3. Although we assume a binary signal structure in Proposition 3.3 for simplicity, our equilibrium construction is more general. In fact, an identical equilibrium can be constructed using the steps that we follow if there are finitely many signals and if κ_s is sufficiently large.

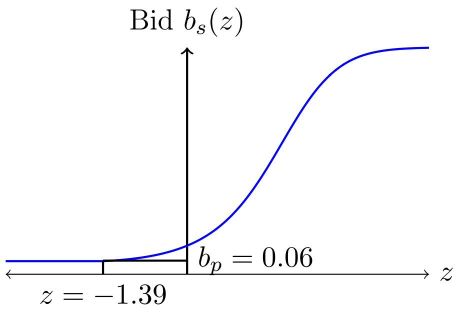
The equilibrium constructed in Proposition 3.3 provides further insight into equilibrium behavior under the assumptions of Proposition 3.2. The proposition describes both equilibrium bidding behavior as well as the pattern of sorting across options. In particular, equilibrium bidding has the following properties: (1) the bidding function b_s^n is strictly increasing for $\theta > \theta_p^n$. Therefore, types $\theta > \theta_p^n$ who bid in market s submit a bid equal to their valuation conditional on being pivotal. (2) The bidding function has a pooling region. In fact, all types $\theta \leq \theta_p^n$ submit a bid equal to the pooling bid b_p^n, i.e., the pooling region extends from 0 to θ_p^n. (3) Pooling starts at the unique type such that the belief of type θ_p^n conditional on observing that the price is equal to the pooling bid (i.e., $l(Y_s^{n-1}(k) \leq \theta_p^n, \theta_i = \theta_p^n)$) is equal to this type's belief conditional on the price being equal to $\beta^n(\theta_p^n) > b_p^n$, (i.e., $l(Y_s^{n-1}(k) = \theta, \theta_i = \theta)$) (4) At the limit, the auction clears at a price equal to the pooling price with strictly positive probability in both states. In fact, $\lim \sqrt{n}|F_s(\theta_s^n(v)|v) - F_s(\theta_p^n|v)| < \infty$, i.e., the type θ_p^n is arbitrarily close to the pivotal type in both states. (5) The limit price distributions have full support over the set $[b_p, 1]$ where

$$\lim b_p^n = \lim \frac{l(Y_s^{n-1}(k) \leq \theta_p^n, \theta_i = \theta_p^n)}{1 + l(Y_s^{n-1}(k) \leq \theta_p^n, \theta_i = \theta_p^n)} = \lim \frac{l(Y_s^{n-1}(k) = \theta_p^n, \theta_i = \theta_p^n)}{1 + l(Y_s^{n-1}(k) = \theta_p^n, \theta_i = \theta_p^n)}$$
$$= \lim \beta^n(\theta_p^n) \in (0, 1)$$

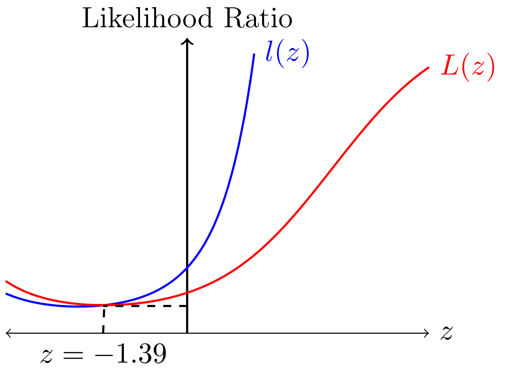
in both states. Therefore, a continuum of prices are possible in both states.

To better understand the equilibrium price distribution note that all types $\theta > \theta_s(0)$ submit a bid equal to one at the limit and all $\theta < \theta_s(0)$ submit the pooling bid at the limit. However, the equilibrium price is determined by θ^n which are within a finite number of standard deviations of $\theta_s^n(0)$, i.e., types θ^n such that $z(\theta) = \lim \sqrt{n}|F_s^n(\theta^n|0) - F_s^n(\theta_s^n(0)|0)| < \infty$. Note that $z(\theta) < \infty$ implies $\lim \sqrt{n}|F_s^n(\theta^n|1) - F_s^n(\theta_s^n(1)|1)| < \infty$. Therefore, the same set of types are within a finite number of standard deviations of $\theta_s^n(1)$ and hence determine the price when $v = 1$. Figure 3.2a focuses in on such types and depicts bidding behavior for types which are within z standard deviations of $\theta_s(0)$ and therefore $\theta_s(1)$. Also, applying a central limit theorem allows us to characterize the distribution of the price in closed-form. Figure 3.3 depicts the cumulative price distribution for a particular parametric example.

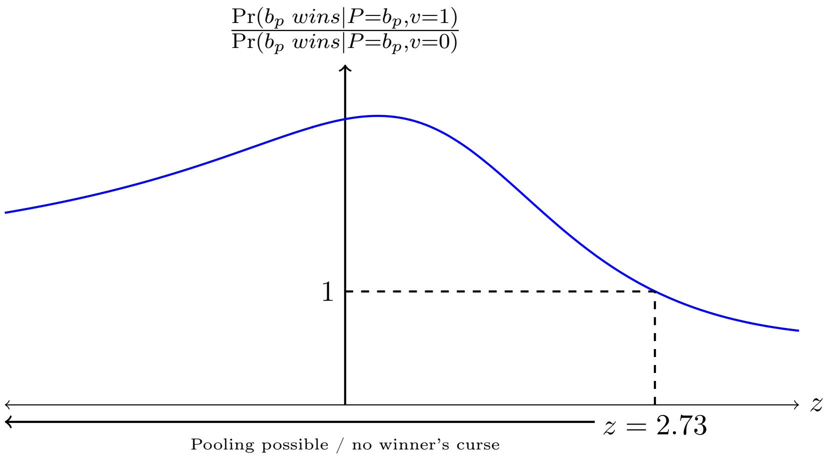
The proposition also shows that selection in equilibrium has a cutoff structure: types above the cutoff θ_1^n select the outside option while the remaining types select market s. Proposition



(a) Bidding Function: Types with $z < -1.39$ submit the pooling bid and all types with $z > -1.39$ bid according to the strictly increasing function $\beta(z) = \lim \beta^n(\theta^n)$.



(b) Likelihood Functions and the determination of θ_p^n: $l(z) = \lim l(Y_s^{n-1}(k) = \theta^n, \theta_i = \theta^n)$, $L(z) = \lim l(Y_s^{n-1}(k) \leq \theta^n, \theta_i = \theta^n)$ and the two curves cross at a unique $z = -1.39$.



(c) Loser's and Winner's Curse at the Pooling bid: The y-axis is the relative likelihood of winning at the pooling bid if all bidders below z submit the pooling bid. For $z > 2.73$ there is winner's and loser's curses at pooling (i.e., $\lim \frac{\Pr(b_p\ wins|P=b_p,V=1)}{\Pr(b_p\ wins|P=b_p,V=0)} > 1$) therefore pooling cannot be sustained. For $z < 2.73$ there is no winner's or loser's curse ($\lim \frac{\Pr(b_p\ wins|P=b_p,V=1)}{\Pr(b_p\ wins|P=b_p,V=0)} < 1$) and hence pooling can be sustained.

Figure 3.2: Equilibrium Construction: In this example $f_h = 0.6$, $f_l = 0.4$ and $\kappa = 0.5$. Types are on the x-axis and measured by their distance, in standard deviations, from the pivotal type $\theta_s(0)$, i.e., $z = \lim \sqrt{\frac{n}{\kappa(1-\kappa)}}(F_s^n(\theta^n|0) - F_s^n(\theta_s^n(0)|0))$.

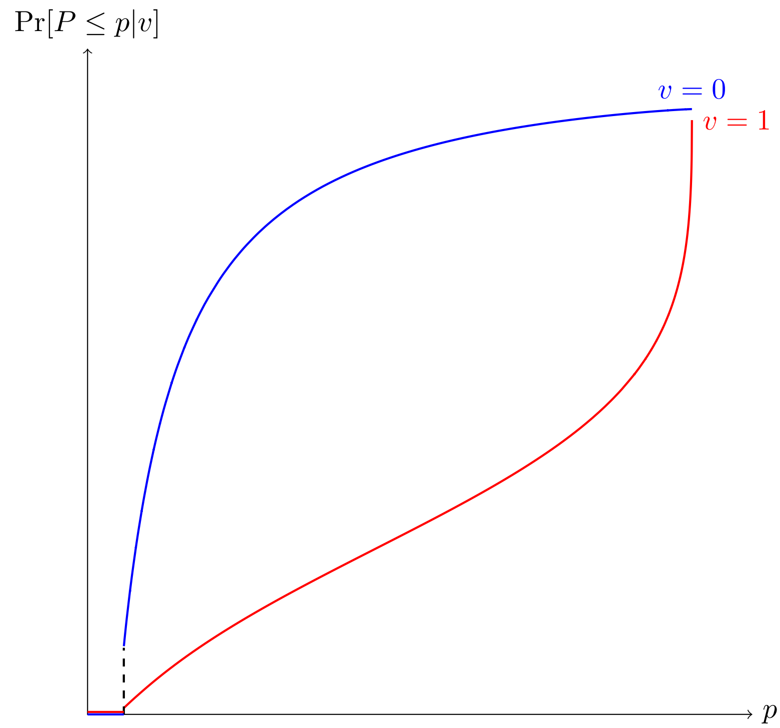


Figure 3.3: Limit Cumulative Price Distribution of $V = 0$ and $V = 1$.

3.2 established that the pivotal types converge to each other at the limit in any equilibrium, i.e., $\theta_s(1) = \theta_s(0)$. However, $\theta_s(1) = \theta_s(0)$ implies that the limit of the cutoff types, θ_1^n, is equal to $\frac{1+\kappa}{2}$ and the pivotal types converge to $\frac{1-\kappa}{2}$. Therefore, in this equilibrium a portion of the agents who receive high signal, i.e., those above $\frac{1+\kappa}{2}$, select market r while others with the high signal select market s. Hence, agents who receive the high signal are indifferent between the two options, i.e., $u^n(s, b_s^n(\theta)|\theta)$ converges to $u(r|\theta = 1)$ for all types $\theta \in [1/2, 1]$. On the other hand, the payoff of the agents who receive the low signal ($\theta \in [0, 1/2]$) converges to zero. For these agents, the outside option is valueless by assumption. In market s these types do not obtain any payoff either. This is because these types are willing to submit the pooling bid and the probability of winning an object at the pooling bid converges to zero.

A key difficulty in the equilibrium construction arises from the fact that $l(Y_s^{n-1}(k) = \theta, \theta_i = \theta)$ is decreasing for all $\theta < \theta_s(0)$ for sufficiently large n (see figure 3.2b). Lemma 2.1 implies that the bidding function must be nondecreasing. Moreover, if the bidding function is increasing for some θ, then the bid of this θ must be equal to $\beta_s^n(\theta) = \frac{l(Y_s^{n-1}(k)=\theta,\theta_i=\theta)}{1+l(Y_s^{n-1}(k)=\theta,\theta_i=\theta)}$ (this fact is well known and follows from Pesendorfer and Swinkels (1997)). However, if the likelihood function is decreasing at θ, then the bidding function cannot be equal to $\beta_s^n(\theta)$ because if it were to equal $\beta_s^n(\theta)$, then it would be decreasing contradicting Lemma 2.1. Thus we conclude that there must be pooling in the range of types where $l(Y_s^{n-1}(k) = \theta, \theta_i = \theta)$ is decreasing.

In constructing the equilibrium, we choose the type where pooling ends, i.e., type θ_p^n, so that this type is exactly indifferent between submitting the pooling bid and outbidding the pooling bid just slightly. In a key step of the construction, we argue that this choice of pooling region ensures that $l(Y_s^{n-1}(k) = \theta, \theta_i = \theta)$ is increasing (and therefore $\beta_s^n(\theta)$) beyond the pooling region. See figure 3.2b for a depiction. When constructing the equilibrium we also argue that the pooling region that we choose ensures that there is no winner's curse at pooling, i.e., winning an object from pooling is more likely when $v = 1$. See figure 3.2c for a depiction.

In order to construct an equilibrium, we show that there is a value for $\lim \sqrt{n}|F_s^n(\theta_s^n(0)|0) - F_s^n(\theta_s^n(1)|0)|$ which ensures that the high types are indifferent between market s and the outside option r for sufficiently large n. In particular, we argue if $\lim \sqrt{n}|F_s^n(\theta_s^n(0)|0) - F_s^n(\theta_s^n(1)|0)|$ is sufficiently large, then (1) the pooling price is small, (2) the expected price is close to zero if $V = 0$ and the expected close to 1 if $V = 1$. In this case the high types' payoff is close to zero in market s. On the other hand, if $\lim \sqrt{n}|F_s^n(\theta_s^n(0)|1) - F_s^n(\theta_s^n(1)|1)|$ is sufficiently small, then we show that the expected price in state $V = 1$ is small and the high types' profit in market s dominates the outside option. Therefore, we are able to apply the intermediate value theorem to conclude that there is a value for $\lim \sqrt{n}|F_s^n(\theta_s^n(0)|0) - F_s^n(\theta_s^n(1)|0)|$ which leaves the high types exactly indifferent between the two options for all n sufficiently large.

We use results from moderate deviations theory (See, for example, Lesigne (2005)) in order to construct this equilibrium. In particular, a central limit theorem and a local limit theorem imply that the density and the cumulative distribution of the random variable $\sqrt{n}(F_s^n(Y^{n-1}(k)|v) - F_s^n(\theta_s^n(v)|v))$ are approximated well by the normal density and the normal cumulative distribution for sufficiently large n. Moreover, the pivotal types that set the price are arbitrarily close to each other in the equilibrium that we construct, or more precisely, $\lim \sqrt{n}|F_s^n(\theta_s^n(1)|v) - F_s^n(\theta_s^n(0)|v)| < \infty$. We use these key facts to approximate the functions $l(Y_s^{n-1}(k) \leq \theta, \theta_i = \theta)$, $l(Y_s^{n-1}(k) = \theta, \theta_i = \theta)$ and therefore the increasing bidding function $\beta_s^n(\theta)$. Moderate deviations theory implies that $l(Y_s^{n-1}(k) = \theta^n, \theta_i = \theta^n)$ converges to the likelihood ratio of two normal densities and $l(Y_s^{n-1}(k) \leq \theta, \theta_i = \theta)$ converges to the likelihood ratio of two normal cumulative distributions which have the same variance but have means which differ by $\lim \sqrt{n}|F_s^n(\theta_s^n(1)|0) - F_s^n(\theta_s^n(0)|0)|$. Also, moderate deviations theory allows us to approximate the relative probability of winning at the pooling bid using using the reciprocal Mill's ratio for the normal distribution (see Sampford (1953) for a discussion of the reciprocal Mill's ratio and figure 3.2c for a depiction). Moreover, moderate deviations theory further implies that the convergence of the above mentioned functions is uniform over all sequences $\{\theta^n\}$ such that $\lim \sqrt{n}|F_s^n(\theta^n|0) - F_s^n(\theta_s^n(0)|0)| < \infty$. We then use these insights to solve for an equilibrium at the limit and then argue that there is an equilibrium for all n above a cutoff N because the convergence of all functions that are relevant to constructing an equilibrium is uniform over all sequences $\{\theta^n\}$ such that $\lim \sqrt{n}|F_s^n(\theta^n|0) - F_s^n(\theta_s^n(0)|0)| < \infty$.

4. Information Aggregation with an Endogenous Outside Option

In this section, we focus on the case where each player chooses between two actual concurrent auction markets. Our aim is to demonstrate how frictions in one market can endogenously generate the outside option payoff profile which can hinder information aggregation in another frictionless market.

We assume that there are $\lfloor n\kappa_r \rfloor$ objects for sale in market r in additional to the $\lfloor n\kappa_s \rfloor$ objects that are on auction in market s and $\kappa_s + \kappa_r < 1$. The objects for sale in the two markets are identical. The auction format in market r is identical to the auction format in market s except for a reserve price $c \geq 0$. The following definition summarizes our assumption on the reserve price.

Definition 4.1 (Endogenous Outside Options)**.** The uniform price in market r is equal to the maximum of c and the highest losing bid in market r if there are more bidders than objects and equal to c otherwise. We assume that the reserve price is known by all players and that $c < \Pr(v = 1|\theta_i = 1)$, i.e., we assume that there are some types who would want to purchase the object at the reserve price.

This section's main theorem shows that market r, which has a reserve price $c > 0$, serves as an outside option with positive variance for market s. Therefore, if the fraction of goods in market s, i.e., κ_s, exceeds a certain cutoff $\kappa^*(\theta^*)$ which we describe further below, then the logic of Theorem 3.1 implies that information is not aggregated in market s in any equilibrium sequence. The theorem also considers two additional cases: (1) the reserve price in market r is equal to zero and (2) the fraction of goods $\kappa_s < \kappa^*(\theta^*)$. In such cases the theorem argues that information is aggregated in market s. In the next subsection we turn to characterizing equilibrium behavior and constructing an equilibrium when the reserve price in market r is not large.

Before stating our main theorem, we first need to define a certain cutoff type θ^* which we will use in the theorem.

Definition 4.2. Recall that $\theta_r^F(1)$ is the type such that $1 - F(\theta_r^F(1)|1) = \kappa_r$.[34] Let

$$\theta^* = \max\left\{\theta_r^F(1), \inf\{\theta : \Pr(V = 1|\theta_i = \theta) > c\}\right\}.$$

To better understand the definition, consider a hypothetical sequence of strategies $\mathbf{H}$ where all types greater than θ^* select auction r while all types lower than θ^* bid in auction s. Moreover, suppose that in the sequence $\mathbf{H}$ all types who select market r bid according to a strictly increasing bidding function. Given such a sequence $\mathbf{H}$, the type θ^* is defined as the smallest type who makes a positive profit in market r in an arbitrarily large auction. Note that θ^* must be at least as large as $\theta_r^F(1)$ because only those types greater than $\theta_r^F(1)$ can actually win an object in the auction in state $v = 1$. Also, note that any type $\theta > \theta_r^F(1)$ will make a profit in market r only if $\Pr(V = 1|\theta_i = \theta) > c$ because any such type will win an object with probability one given $\mathbf{H}$ in both states and will pay a price which is at least equal to c. The definition of this lowest type is more delicate than the similar definition used in Theorem 3.1 because in order to define the lowest type who makes a profit in market r we need to consider the fact that profits depend on the equilibrium bid distribution.

Theorem 4.1. *Suppose that MLRP is satisfied. Suppose that the outside option r is the auction market described by Definition 4.1 and let $\kappa^*(\theta^*) < 1$ denote the cutoff defined by Definition 3.2.*

i. If $c > 0$ and if $\kappa_s > \kappa^*(\theta^*)$, then no equilibrium sequence $\mathbf{H}$ aggregates information in either market.

[34] In words, $\theta_r^F(1)$ is the pivotal type in state 1 if all types where to bid in the auction r.

ii. If $c = 0$, then any equilibrium sequence $\mathbf{H}$ aggregates information in both markets.

iii. Let $\bar{c}$ denote the cutoff defined in equation (E.1) in the appendix. Suppose that strict MLRP is satisfied and that $0 < c < \bar{c}$. If $\kappa_s < \kappa^*(\theta^*)$, then any equilibrium sequence $\mathbf{H}$ aggregates information in market s.

Item i of Theorem 4.1 shows that information aggregation fails in market r if there is an alternative market r with a reserve price. The failure of information aggregation in market r is straightforward: price cannot converge to value if $V = 0$ by assumption because the reserve price is positive. The price cannot converge to value if $V = 1$ either. This is because if it did, then any bidder who bought a good from this market would make a loss. The failure of information aggregation in market s is driven by two main forces identified in Theorem 3.1 and Lemma 3.2: (1) Market r provides an outside option with positive variance for market s, (2) The fact that $\kappa_s > \kappa^*(\theta^*)$ implies that the value of participating in market r is non-negative for a sufficiently large portion of the bidders. Item ii of the theorem further argues that without the reserve price, i.e., if market r is also frictionless, then information aggregation is restored in both markets. Therefore, market r cannot generate the outside option payoff profile that results in a failure of information aggregation without a reserve price. Finally, the theorem argues that if the faction of goods in market s is less than the cutoff value $\kappa^*(\theta^*)$, then information is aggregated in market s for all small enough values of the reserve price.

Remark 4.1. This theorem is one example of an alternative market which provides an outside option that can hinder information aggregation. There are many other institutional configurations that could result in a similar outcome. Suppose that the alternative market r uses a different auction format and in particular suppose that (1) Market r is pay-as-you-bid (discriminatory price) auction as in Jackson and Kremer (2007), where all bidders who win an object from the auction pay their own bid, or (2) Market r is an all-pay-auction as in Chi et al. (2016). The payoff distribution in these alternative auction formats have similar properties to the payoff distribution in market r as described by Theorem 4.1: payoffs are negative in state $V = 0$ and positive in state $V = 1$. Therefore, our analysis suggests that information would not be aggregated in market s because of the existence of an outside option which would lead to only certain types selecting auction market s.

Remark 4.2. Theorem 4.1 assumes that the set of options available to bidders is $M = \{s, r\}$. In the appendix we show that if there is a market r with a possible reserve price $c \geq 0$ and an arbitrary number of markets without reserve prices, then the conclusions of Theorem items i and ii continue to hold, that is information is not aggregated in any market $M \setminus r$ if $c > 0$ and if $\sum_{m \in M \setminus} \kappa_m$ exceeds a certain threshold and information is aggregated in all market if $c = 0$.

Intuitively, information aggregation fails in market s under the assumptions of item i even though this market is frictionless because bidders with lower signals, i.e., more pessimistic bidders, self select into market s. This, in turn, implies that there are many more bidders who are willing to pay at least the bid of $\theta_s(1)$ in the bad state, i.e., the demand for goods is higher in the bad state. The fact that the demand is high at the bid of $\theta_s(1)$ exactly when people do not value the goods implies that market s cannot clear properly. Thus, information is not aggregated in this market because this selection effect overwhelms competitive forces.

In order to provide more precise intuition suppose that c is less than $\Pr(V = 1|\theta_r^F(1))$ so that $\theta^* = \theta_r^F(1)$. On the way to a contradiction, assume that information is aggregated in market s: that is, the price converges to one in state 1 and zero in state 0. Therefore, the payoff of any type that bids in market s is equal to zero.

We first argue that the market selection function has a cutoff structure with all types that exceed θ^* opting for market r: At the limit, bidders face a choice between market s, where their payoff is equal to zero, and market r, where their payoff is strictly negative if $v = 0$. The choice is essentially identical to the choice that bidders faced in the case where they had an exogenously given outside option as in Theorem 3.1. Therefore, the argument for Theorem 3.1 implies a single crossing property, i.e., all types that exceed θ^* opt for market r.

The final step in the argument concludes that all types that exceed θ^* opting for market r and information aggregation in market s taken together are incompatible with bidders behaving according to a monotone bidding function as we claimed that they must in Lemma 2.1. More precisely, if all types that exceed θ^* select market r and if $\kappa > \kappa^*(\theta^*)$, then we find $\theta_s(0) > \theta_s(1)$ (see figure 3.1). Our assumption of information aggregation in market s implies that $\lim_{n\to\infty} b_s^n(\theta_s^n(1)) = 1$ and $\lim_{n\to\infty} b_s^n(\theta_s^n(0)) = 0$. However, this provides the contradiction that proves the result. The findings that $\lim_{n\to\infty} b_s^n(\theta_s^n(1)) = 1$ and $\lim_{n\to\infty} b_s^n(\theta_s^n(0)) = 0$, and $\theta_s(0) > \theta_s(1)$ together contradict that the bidding function is increasing for all n.

The argument for item *ii* depends on an intermediate result that shows that if $\theta_m(1) > \theta_m(0)$ in a particular market and if the measure of types bidding in this market when $V = 1$, is greater or equal to the measure of bidders when $V = 0$, i.e, $F_m(1|1) \geq F_m(1|0)$, then information is aggregated in market m. This intermediate result implies that information must be aggregated in at least one market. This is because if, say, information is not aggregated in market s and if $F_r(1|1) < F_r(1|0)$, then $F_s(1|1) > F_s(1|0)$. Also, using MLRP we observe that $F_s(1|1) > F_s(1|0)$ implies $\theta_s(1) > \theta_s(0)$. But then $F_s(1|1) > F_s(1|0)$ and $\theta_s(1) > \theta_s(0)$ together imply that information is aggregated in market s.

If information is aggregated in market s, then the payoff to bidders in market s is equal to zero. The fact that information is not aggregated in market r means that the price is greater than zero with positive probability if $V = 0$. Thus, the payoff in market r is strictly negative if $V = 0$. But then a single-crossing property implies that all types that exceed $\theta_r(1)$ would opt for market r. Consequently, applying MLRP we establish that $1 - F(\theta_r(1)|1) > 1 - F(\theta_r(1)|0)$. This, however, contradicts the fact that $\theta_r(1) \leq \theta_r(0)$ because $1 - F(\theta_r(1)|1) = \kappa_r$ implies that $1 - F(\theta_r(1)|0) < \kappa_r$ and thus $\theta_r(1) > \theta_r(0)$.

4.1. Equilibrium Characterization and Construction. In this subsection, we provide a characterization for all equilibria under the assumption that $\kappa > \kappa^*(\theta^*)$ and the reserve price is below a certain cutoff $\bar{c}$. The cutoff $\bar{c}$ is a function of the signal distribution $f(\cdot|v)$ and the fractions of goods in the two markets κ_s and κ_r. An explicit formula for the cutoff $\bar{c}$ can be found in the appendix as equation (E.1). We then construct an equilibrium that displays the properties described by the characterization in an environment where there are two distinct signals.

Proposition 4.1 (Characterization)**.** *Suppose that MLRP is satisfied and that the outside* *option r is the auction market described by Definition 4.1 Let $\bar{c}$ denote the cutoff defined in equation (E.1). Assume that $\kappa_s > \kappa^*(\theta^*)$ and that $c < \bar{c}$. For any sequence of equilibria,*

i. In market r, $\lim \sqrt{n}|F_r^n(1|1) - \kappa_r| < \infty$ and $F_r(1|0) < \kappa_r$.

ii. If $V = 0$, then the price in market r converges to c with probability one. If $V = 1$, then the price in market r converges to a random variable which is equal to zero with probability $q > 0$ and equal to one with the remaining probability.

iii. The expected prices are equal across states and markets. In particular, $\lim \mathbb{E}[P_s^n|V = 0] = \lim \mathbb{E}[P_r^n|V = 0] = c$ and $\lim \mathbb{E}[P_r^n|V = 1] = \lim \mathbb{E}[P_s^n|V = 1]$.

iv. Expected payoffs $\lim u^n(\theta) = 0$ for all $\theta < \min\{\theta_s(1), \theta_r(1)\}$ and

$$\lim u^n(\theta) = \Pr[V = 1|\theta_i = \theta](1 - \lim \mathbb{E}[P_r^n|V = 1]) - c \Pr[V = 0|\theta_i = \theta]$$

for all $\theta > \min\{\theta_s(1), \theta_r(1)\}$.

v. In market s, the distance between the pivotal types converges to zero at rate $\sqrt{n}$, more precisely, $\lim \sqrt{n}|F_s(\theta_s^n(1)|1) - F_s(\theta_s^n(0)|1)| < \infty$.

This characterization shows that equilibrium behavior in market r is identical to the equilibrium we described in Proposition 3.1 while equilibrium behavior in market s is similar to the equilibrium that we described in Proposition 3.2. In market r, equilibrium bidding is strictly increasing and information aggregation fails because of a lack of competition, i.e., because the number of bidders in the market is less than the number of objects in both states of the world with positive probability. In market s, in contrast, the number of bidders exceeds the number of objects with probability one in both states. However, information aggregation fails because the bids of same set of types determine the auction price in both states and this is a result of the pattern of self selection across markets.

Remark 4.3. Price converging to value is a strong form of information aggregation. In fact, since the reserve price exceeds the object's value in state $v = 0$, information aggregation in market r is ruled out by assumption. In the appendix we study a weaker notion of information aggregation that is termed informativeness also used by Kremer (2002) and Atakan and Ekmekci (2014). A sequence of equilibria is termed informative (Definition B.2), if an outside observer can learn the state asymptotically by simply observing the equilibrium price.[35] The results presented in Proposition 4.1 imply that the prices are not informative either. Proposition 4.1, item *i* implies that $F_s^n(1|v) > \kappa_s$ for $v = 0, 1$. However, $F_s^n(1|1) > 0$ and the fact that information is not aggregated in market s together imply by applying Lemma B.3 that the price in market s is not informative according to Definition B.2. Also, Proposition 4.1, item *iii*, i.e., the fact that the price is equal to zero with positive probability in both states, implies that the price in market r is not informative either.

[35] A reserve price does not necessarily rule out an equilibrium sequence from being informative according to Definition B.2.

Remark 4.4. Theorem 4.1 showed that an outside observer would not learn the state with certainty after observing the prices in the two markets separately. The above characterization theorem further implies that an outside observer could not deduce the state with certainty even if she observed the price in both markets. This is because the price is equal to the reserve price with strictly positive probability in both states and the support of the price distribution in market s is identical across both states.[36]

4.1.1. *Equilibrium Construction with Binary Signals.* In this subsection, we construct an equilibrium with the properties described by Proposition 4.1. For the construction, we again focus on a setup with a binary signal structure as in Proposition 3.3.

Proposition 4.2 (Construction). *Suppose that MLRP is satisfied and that the outside option r is the auction market described by Definition 4.1 Assume that the signals satisfy Definition 3.3. There exists a constant c^* such that if $\kappa_s > \frac{f_h - \kappa_r}{f_h}$ and if $c \in (0, c^*)$, then there exists an N such that for all $n > N$, there is an equilibrium H^n with the following properties:*

i. All types $\theta \in [0,1]$ are indifferent between the two markets. Moreover, there are cutoffs $1 > \theta_1^n > \frac{1}{2}$ and $0 < \theta_0^n < \frac{1}{2}$ such that types $\theta \in (\theta_0^n, \theta_1^n)$ select market s and the remaining types select market r.

ii. In market s, there is a cutoff type $\theta_p^n \in (\theta_0^n, 1/2)$ which satisfies equation (3.1) and a pooling bid b_p^n such that all types $\theta \in (\theta_0^n, \theta_p^n)$, submit the pooling bid b_p^n and all types $\theta \geq \theta_p^n$ bid according to the increasing bidding function $\beta_s^n(\theta)$ defined in equation (2.1).

iii. Types in market r bid according to the increasing function $\beta_r^n(\theta)$ defined in equation (2.1) and $\lim_n \beta_r^n(\theta) = 1$ for all types θ that submit a bid in market r.

iv. The price in market s converges to a random variable $P_s(v)$ for $v = 0, 1$. The random variables $\{P_s(v)\}_{v=0,1}$ both have continuous distributions over the interval $(b_p, 1]$ with full support and atoms at b_p where $b_p = \lim b_p^n$.

Behavior in market r was described precisely by Proposition 4.1. The equilibrium construction above provides further insight into behavior in market s. In particular, the construction shows that (1) the bidding function b_s^n is strictly increasing for $\theta > \theta_p^n$. Therefore, types $\theta > \theta_p^n$ who bid in market s submit a bid equal to their valuation conditional on being pivotal. (2) The bidding function has a pooling region. In fact, all types $\theta \leq \theta_p^n$ submit a bid equal to the pooling bid b_p^n, i.e., the pooling region extends from 0 to θ_p^n. (3) At the limit, the auction clears at a price equal to the pooling price with strictly positive probability in both states. In fact, $\lim \sqrt{n}|F_s(\theta_s^n(v)|v) - F_s(\theta_p^n|v)| < \infty$, i.e., pooling starts at a type which is arbitrarily close to the pivotal type in both states. Therefore, the bottom of the price distribution mimics the price distribution in market r (4) The limit price distributions have full support over the set

[36]To be more precise, the probability $\lim \frac{\Pr(Y_s^n(k)=\theta^n|V=1,(\kappa_r - F_r^n(1|v=1))\sqrt{n}\geq y)}{\Pr(Y_s^n(k)=\theta^n|V=0,(\kappa_r - F_r^n(1|v=0))\sqrt{n}\geq y)} \in (0,\infty)$ for any $y > 0$ and any sequence of θ^n such that $\sqrt{n}|F_s^n(\theta^n|v) - F_s^n(\theta_s^n(v)|v)| < \infty$ because of the central limit theorem. Note that the likelihood ratio is the probability that the price is equal to the bid of θ^n given that the auction clears at the reserve price in market r. Moreover, the pivotal type is the limit of a sequence of θ^n such that $\sqrt{n}|F_s^n(\theta^n|v) - F_s^n(\theta_s^n(v)|v)| < \infty$ with probability one for $v = 0, 1$.

$[b_p, 1]$ where $b_p < 1$ in both states. Therefore, a continuum of prices are possible in both states. In fact, it is possible to calculate the price distributions by applying a version of the central limit theorem. We provide more detail about the limit price distributions in the appendix.

5. Multiple States

In this section we generalize Theorem 3.1 and Corollary 3.1 to the case where V is an arbitrary finite set. In order to do so, we first begin by introducing a cutoff level for the fraction of objects in market s which is analogous to the κ^* introduced in Definition 3.2.

Definition 5.1. For $\theta' < 1$, let

$$\kappa_v^*(\theta') = \max_{\theta \in [0,\theta']} \left\{ F(\theta'|v) - F(\theta|v) : F(\theta'|v) - F(\theta|v) \geq F(\theta'|v') - F\left(\theta|v'\right) \text{ for all } v' < v \right\}$$

where $\kappa_v^*(\theta') = 0$ if the set over which the maximum is taken is empty. Let $\kappa^*(\theta') = \min_{v \in \mathcal{V}} \kappa_v^*(\theta')$.

To better understand the above definition, consider a strategy H where all types greater than θ' select the outside option r while all types lower than θ' bid in the auction s. The cutoff $\kappa_v^*(\theta')$ is defined so that, if the fraction of goods in market s is greater than κ^*, then the pivotal type in state v, $\theta_s^H(v)$, is less than the pivotal type in state v', $\theta_s^H(v')$ for some pair of states $v' < v$. Such an ordering of pivotal types is ruled out by MLRP if all types participate in the auction. However, given strategy H, MLRP implies that the measure of players selecting the auction is smaller in state v than in state v'. This implies that $\kappa_v^*(\theta')$ is always less than one and such a reversal of ordering occurs between state v and v' whenever $\kappa_s > \kappa_v^*(\theta')$. The cutoff $\kappa^*(\theta')$ is defined as the minimum over all $\kappa_v^*(\theta')$. Therefore there is an open interval $(\kappa^*(\theta'), 1)$ such that if $\kappa_s \in (\kappa^*(\theta'), 1)$, then there is $v > v'$ such that the ordering of the pivotal types is reversed in these states, i.e., $\theta_s^H(v') > \theta_s^H(v)$. In fact, our main theorem will again argue that information cannot be aggregated if the order of the pivotal types is reversed between any two states in equilibrium.

Theorem 5.1. *Assume that $\mathcal{V} \subset [0, 1]$ is a finite set of states and MLRP is satisfied. Suppose that players have access to outside options as defined by Definition 3.1 and let $\theta' = \inf\{\theta : \mathbb{E}[u(r|V)|\theta_i = \theta] > 0\}$. If $\kappa_s > \kappa^*(\theta')$, then no equilibrium sequence $\mathbf{H}$ aggregates information.*

Proof. Given Definition 5.1, the proof is identical to the proof of Theorem 3.1. □

Corollary 5.1. *Suppose that players have access to outside options as defined in Definition 3.1. Assume that strict MLRP is satisfied. If $\mathbb{E}[u(r|V)] \geq 0$ and $Var[u(r|V)] > 0$, then $\kappa^* = 0$ and therefore there is no equilibrium sequence $\mathbf{H}$ that aggregates information for any $\kappa > 0$. Alternatively, assume that MLRP is satisfied. If $\mathbb{E}[u(r|V)] > 0$, then $\kappa^* = 0$ and therefore there is no equilibrium sequence $\mathbf{H}$ that aggregates information for any $\kappa > 0$.*

Proof. The proof is in the main text below. □

We will now provide the argument for Corollary 5.1. If information is aggregated in market s, then we will argue below that we reach a contradiction. Suppose that strict MLRP is satisfied and assume that $\mathbb{E}[u(r|V)] = 0$ and $Var[u(r|V)] > 0$. Consider any type $\hat{\theta}$ such that

$l(v, v'|\theta_i = \hat{\theta}) = f(\hat{\theta}|v)/f(\hat{\theta}|v') > 1$ for any two states $v > v'$, i.e., $\hat{\theta}$ is a signal that is more likely in higher states than in lower states. For example, $\theta = 1$ is such a type. Note that $\mathbb{E}[u(r|V)|\theta_i = \hat{\theta}] > 0$ because $\hat{\theta}$ puts uniformly more weight on higher states than on lower states as compared to the prior (i.e., the posterior of $\hat{\theta}$ first order stochastically dominates the prior). Therefore, if information is aggregated in market s any such type $\hat{\theta}$ would opt for the outside option r. Now pick the highest type θ' who selects market s. Note that for this type θ', there exists a pair of states $v > v'$ such that $l(v, v'|\theta_i = \theta') \leq 1$ because otherwise this type would select the outside option. Also, note that for any type $\theta < \theta'$ we have $l(v, v'|\theta_i = \theta) < 1$ because of strict MLRP and hence we find that $l(v, v'|\theta_i = \theta) < 1$ for all types who select market s. However, if $l(v, v'|\theta_i = \theta) < 1$ for all types in market s, then we have that $\theta_s(v') > \theta_s(v)$. To see this, note that $\theta_s(v')$ is the type such that $\int_{\theta_s(v')}^{\theta'} f(\theta|v')d\theta = \kappa$. But, we know $\int_{\theta_s(v')}^{\theta'} f(\theta|v)d\theta < \int_{\theta_s(v')}^{\theta'} f(\theta|v')d\theta$ because $l(v, v'|\theta_i = \theta) = \frac{f(\theta|v)}{f(\theta|v')} < 1$ for all $\theta < \theta'$. Therefore, we conclude that $\int_{\theta_s(v')}^{\theta'} f(\theta|v)d\theta < \kappa$ which shows that $\theta_s(v) < \theta_s(v')$. However, if information is aggregated in market s, then $\lim_{n\to\infty} b_s^n(\theta_s^n(v)) = v$ and $\lim_{n\to\infty} b_s^n(\theta_s^n(v')) = v'$. However, this provides the contradiction that proves the result. The findings that $\lim_{n\to\infty} b_s^n(\theta_s^n(v)) = v$, $\lim_{n\to\infty} b_s^n(\theta_s^n(v')) = v'$ and $\theta_s(v') > \theta_s(v)$ together with $v > v'$ contradict that the bidding function is nondecreasing for all n.

6. Discussion and Conclusion

The results that we presented in the paper argued that the price in a large, uniform-price common-value auction may not aggregate all available information if bidders have access to an outside option whose value is correlated with the common-value object on auction. We showed that exogenous as well as endogenous outside options can hinder information aggregation because of the pattern of type dependent market selection that such outside options generate. However, our discussion so far has not touched upon a number of issues that we will comment on in conclusion. In particular, in this section we discuss (1) Equilibrium existence in the model that we study, (2) Welfare properties of equilibria, (3) The amount of information impounded by prices.

6.1. Equilibrium Existence. To the best of our knowledge, equilibrium existence in the framework that we consider is not guaranteed by any result already in the literature. However, if we restrict bids in all markets to a finite grid $B = \{0, \Delta, 2\Delta, ..., \infty\}$ where $\Delta > 0$ is the fineness of the grid so that a symmetric distributional strategy H^Δ is a probability measure over $[0, 1] \times M \times B$, then equilibrium existence follows immediately from Milgrom and Weber (1985). Also, all our information aggregation and information non-aggregation findings continue to hold if we use the following definition: Let $\{\Gamma^{n,\Delta}\}$ denote a sequence of auction with n bidders where the fineness of the grid is Δ and suppose that behavior in these auction are described by a sequence of strategies $\mathbf{H} = \{H^{n,\Delta}\}$. The sequence $\mathbf{H}$ **aggregates information** in market m if $\lim_{n\to\infty} \lim_{\Delta\to 0} P_m^{n,\Delta} = V$ where convergence refers to convergence in probability.

6.2. Efficiency. Our focus throughout the paper has been on the informational efficiency of prices and we have not commented on equilibrium welfare. Typically, in common-value auctions, welfare considerations are not particularly interesting because such auctions are efficient as long as all objects are allocated to bidders. This is also true in auction that we study:

given the set of agents that participate in auction s, any allocation of objects to bidders where all objects are allocated in state $V = 1$ is efficient. However, the market selection strategy that players use in equilibrium has welfare implication. In fact, if information is aggregated in market s, then every equilibrium outcome of the model that we study is asymptotically efficient as the market grows arbitrarily large. On the other hand, if information is not aggregated in market s, then the equilibria characterized in Propositions 3.2 and 4.1 remain inefficient even as the market grows large.

In particular, assume an exogenous outside option with $u(r|0) < 0 < u(r|1) < 1$ and consider a planner, with no knowledge of the state or the realized signals, who chooses a market selection strategy $\mu : [0, 1] \to \{s, r\}$. That is, the planner chooses s or r for each bidder only as a function of that bidder's type θ. Recall that θ', defined in Theorem 3.1, is the smallest type θ such that $\mathbb{E}[u(r|0)|\theta_i = \theta] > 0$. Assume that $F(\theta'|1) > \kappa_s$. In this case, the planner's would optimally choose $\mu(\theta) = r$ for all $\theta > \theta'$ and $\mu(\theta) = s$ for all $\theta \leq \theta'$. The findings that we presented as Proposition 3.2 implies that all equilibria are inefficient under the hypothesis of this proposition. In fact, the equilibrium that we construct in Proposition 3.3 is inefficient because there is over entry into market s. Prices in market s remain bounded away from the object's value because bidders refrain from bidding the price higher in order to avoid winning an object at a high price when $V = 0$. The fact that price remains bounded away from the object's value in market s implies that this market remains profitable for optimistic types. However, this implies that inefficiently many optimistic bidders choose market s instead of market r. In contrast, under the hypotheses of Theorem 3.1 item *ii*, information is aggregated in market s, profits in market s are dissipated and types $\theta \geq \theta'$ opt for the outside option in every equilibria. Therefore such equilibrium outcomes are asymptotically efficient.[37]

The situation is similar for the case of an endogenous outside option. Recall that θ^* is the type introduced by Definition 3.2. Under the hypotheses of Proposition 4.1, a planner would assign all types $\theta \geq \theta^*$ to market r and the remainder to market s. However, Proposition 4.1 shows that this is not how types sort themselves across the two markets in equilibrium and therefore the equilibrium outcomes are asymptotically inefficient. On the other hand, under the hypotheses of Theorem 3.1 item *iii*, information is aggregated in market s and types $\theta \geq \theta^*$ opt for market r in every equilibria. Therefore such equilibrium outcomes are asymptotically efficient.

6.3. Continuity. Our results that show that the equilibrium price does not aggregate information under certain assumptions. However, we should stress that these results do not imply that prices contain little or no information as the market grows large. In fact, our equilibrium characterization and construction results imply that if the variance of the outside option is small, then the information that is not incorporated into the equilibrium price is also small. In other words, the equilibrium price in market s is close to the object's value if the market is perturbed by a "small" outside option. To be more price, in the case of an exogenous outside option, Proposition 3.3 implies that $\lim_{c\to 0} \lim_{n\to\infty} \mathbb{E}[P_s^{n,c}|V] = V$ where $P_s^{n,c}$ denotes the equilibrium price when $u(r|1) = -u(r|0) = c$. Moreover, in the case of an endogenous outside

[37]Under the hypotheses of Proposition 3.1, the unique equilibrium identified in this proposition is asymptotically efficient.

option, Proposition 4.1 implies that $\lim_{c\to 0}\lim_{n\to}\mathbb{E}[P_m^{n,c}|V]=V$ for $m=r,s$ where $P_m^{n,c}$ denotes the price when the reserve price in market r is equal to c. Therefore, equilibrium price converge to full information aggregation in a continuous way as the variance of the outside option converges to zero.

A. The Appendix: Preliminary Results

We first begin by stating a number of statistical results (with the relevant references) that we will apply in our arguments.

Proposition A.1. *(Chernoff's Inequality) Suppose that $X \sim bi(n,p)$, i.e., X is a binomial random variable with probability of success equal to p, then for any $z \geq 0$*

$$\begin{aligned} \Pr(X \geq np + z) = 1 - Bi(np+z;n,p) &\leq \exp\left(-\frac{z^2}{2(np+\frac{z}{3})}\right), \\ \Pr(X \leq np - z) = Bi(np-z;n,p) &\leq \exp\left(-\frac{z^2}{2np}\right). \end{aligned}$$

Proof. See Janson et al. (2011, Theorem 2.1). □

In the proofs, we use in the following version of Chernoff's Inequality. For any $\delta \in (0,1)$,

$$\begin{aligned} \Pr(X \geq (1+\delta)np) \leq \exp\left(-\frac{\delta^2 np}{2+\delta}\right), \\ \Pr(X \leq (1-\delta)np) \leq \exp\left(-\frac{\delta^2 np}{2}\right). \end{aligned}$$

Proposition A.2. *(Local Limit Theorem and Moderate Deviations Estimate for the Binomial Distribution) For $X \sim bi(n,p)$, set*

$$\Pr(X = k) = bi(k;n,p) = \binom{n}{k} p^k (1-p)^{n-k} = \frac{1+\delta_n(p)}{\sqrt{2\pi p(1-p)n}} \phi\left(\frac{k-np}{\sqrt{p(1-p)n}}\right)$$

where ϕ denotes the standard normal density. Then for $t < 2/3$, we have

$$\lim_{n\to\infty} \sup_{p:|np-k|<n^t} \delta_n(p) = 0,$$

i.e., the binomial density converges to the normal density uniformly over the set of p and k such that $|np-k| < n^t$for $t < \frac{2}{3}$.

Proof. See Lesigne (2005, Proposition 8.2). □

Proposition A.3. *(Central Limit Theorem) Suppose, $X \sim bi(n,p)$, set $a_n = \frac{k-np}{\sqrt{np(1-p)}}$ and suppose that $\lim a_n = a$. Then,*

$$\Pr(X \leq k) = Bi(k;n,p) \to \Phi(a)$$

where Φ denotes the standard normal cumulative distribution. Moreover, the convergence is uniform over all $a \in (-\infty,\infty)$.

Proof. See Lesigne (2005, Proposition 8.3). □

Proposition A.4. *(Properties of the Mill's Ratio) Define the reciprocal Mill's ratio (or the hazard function) for the standard normal as $h(z) = \frac{\phi(z)}{1-\Phi(z)}$. Then the following inequalities* *are satisfied*

$$h'(z) = h(z)(h(z) - z) \in (0,1)$$
$$h''(z) = h(z)((h(z) - z)(2h(z) - z) - 1) > 0$$

Proof. See Sampford (1953). □

B. Bidding Equilibria

Recall that we say H is a bidding equilibrium for market m if $H(\times, \{m\}, \times)$ is a Nash equilibrium for auction $\hat{\Gamma}(F_m)$ where $F_m(\theta) = H([0,\theta] \times \{m\} \times [0,\infty))$. Fix a strategy H and recall that $u^H(m, b|\theta)$ denotes the payoff to type θ from submitting b to market m. Note that if $m = r$ and if r is an exogenous outside option, then $u^H(r, b|\theta) = \mathbb{E}[u(r|V)|\theta_i = \theta]$.

Proof of Lemma 2.1: Bidding is nondecreasing in any bidding equilibrium. The argument for this lemma closely follows Pesendorfer and Swinkels (1997, Lemmata 3-7). Fix a bidding equilibrium H for market m.

Suppose there is a positive probability under H of a bid strictly above $V = 1$. Then there is a positive probability that $k + 1$ bids, and thus the price is strictly larger than 1. But any bid that wins with positive probability at a price above 1 is strictly worse than a bid of 1 and we can conclude that bids are always less than 1.

Suppose there is a positive probability under H of a bid strictly below $V = 0$. Then a bid of 0 wins with strictly greater probability if $V = 1$ than any bid strictly below 0 and makes strictly positive profit if $V = 1$. We can then conclude that bids are always greater than 0.

Take any $b' < b < 1$, let $Y(b, b')$ denote the event where a player wins an object with bid b and does not win an object with bid b' and suppose $\Pr(Y(b, b')) > 0$. Take any $\theta > \theta'$ such that $\theta \notin \mathcal{E}(\theta')$. Note that $\theta > \theta'$ and $\theta \notin \mathcal{E}(\theta')$ implies that $l(\theta_i = \theta') < l(\theta_i = \theta)$. We argue if $u(b|\theta') \geq u(b'|\theta')$, then $u(b|\theta) > u(b'|\theta)$ where the reference to market m in the function u has been suppressed for simplicity. Writing the utility that type θ' enjoys from submitting bid b we obtain

$$u(b|\theta') = u(b'|\theta') + \Pr(Y(b, b')|\theta')(\mathbb{E}[V - P|Y(b, b'), \theta']).$$

Observing that

$$\begin{aligned}\Pr(Y(b, b')|\theta')(\mathbb{E}[V - P|Y(b, b'), \theta']) &= \Pr(V = 0|\theta') \Pr(Y(b, b')|0)(0 - \mathbb{E}[P|Y(b, b'), 0])) \\ &\quad + \Pr(V = 1|\theta') \Pr(Y(b, b')|1)(1 - \mathbb{E}[P|Y(b, b'), 1])\end{aligned}$$

we obtain

$$\begin{aligned}u(b|\theta') - u(b'|\theta') = \Pr(V = 1|\theta') \Pr(Y(b, b')|1)(1 - \mathbb{E}[P|Y(b, b'), 1])+ \\ \Pr(V = 0|\theta') \Pr(Y(b, b')|0)(0 - \mathbb{E}[P|Y(b, b'), 0])) \geq 0.\end{aligned}$$

Note that $u(b|\theta') - u(b'|\theta') \geq 0$ and $\Pr(Y(b, b')) > 0$ together imply that $\Pr(Y(b, b')|1)(1 - \mathbb{E}[P|Y(b, b'), 1]) > 0$. Because $l(\theta_i = \theta') < l(\theta_i = \theta)$ implies that $\frac{\Pr(1|\theta)}{\Pr(0|\theta)} > \frac{\Pr(1|\theta')}{\Pr(0|\theta')}$ and because

$\Pr(Y(b,b')|1)(1-\mathbb{E}[P|Y(b,b'),1]) > 0$ we obtain the following inequality

$$\Pr(1|\theta)\Pr(Y(b,b')|1)(1-\mathbb{E}[P|Y(b,b'),1]) + \Pr(0|\theta)\Pr(Y(b,b')|0)(0-\mathbb{E}[P|Y(b,b'),0])) > 0.$$

However, this implies that $u(b|\theta) - u(b'|\theta) > 0$.

Suppose $\theta^* \in [0,1]$ and $b^* \in [0,1]$. The argument above immediately implies that if $H([0,\theta^*] \times [b^*,1]) > 0$, then $H((\theta^*,1] \times [0,b^*)) = 0$. The conclusion of the lemma then follows directly from Lemma 6 in Pesendorfer and Swinkels (1997). □

B.1. Information Aggregation in Bidding Equilibria. In this subsection, we study information aggregation in bidding equilibrium. The following lemma shows that the bids of the pivotal types determine the auction clearing price in any bidding equilibria of a sufficiently large auction.

Lemma B.1. *Assume that MLRP is satisfied. If* $\mathbf{H} = \{H^n\}$ *is a convergent sequence of bidding equilibria for market* m*, then for every* $\epsilon > 0$*,*

$$\lim \Pr(P_m^n \in [b_m^n(\theta_m^n(v) - \epsilon), b_m^n(\theta_m^n(v) + \epsilon)] | V = v) \quad = \quad 1$$

Proof. The law of large numbers implies that $\lim \Pr(Y_m^n(\kappa n + 1) \geq \theta_m^n(v) - \epsilon | v) = 1$ for every $\epsilon > 0$. However, if $Y_m^n(\kappa n + 1) \geq \theta_m^n(v) - \epsilon$, then $p_m^n = b_m^n(Y_m^n(\kappa n+1)) \geq b_m^n(\theta_m^n(v) - \epsilon)$ because b_m^n is nondecreasing by Lemma 2.1. Therefore, $\Pr(p_m^n \geq b_m^n(\theta_m^n(v) - \epsilon) | V = v) \geq \Pr(Y_m^n(\kappa n + 1) \geq \theta_m^n(v) - \epsilon | V = v)$ and taking limits proves the first part of the claim. We establish $\lim \Pr(Y_m^n(\kappa n + 1) \leq \theta_m^n(\omega) + \epsilon | V = v) = 1$ using the same idea. □

We now introduce some notation. Given strategy H, denote by $\Pr(b \ wins | P^n = b, V = v, \theta_i = \theta)$ the conditional probability that bidder i wins an object with a bid equal to b given that the auction price is equal to b, the state is equal to v and bidder i receives a signal equal to θ. Our assumptions that the signals are conditionally independent given the state v and the strategy H is symmetric together imply that $\Pr(b \ wins | P^n = b, V = v, \theta_i = \theta) = \Pr(b \ wins | P^n = b, V = v)$. This is because once one conditions on the state, the individual signal of bidder i, $\theta_i = \theta$, does not provide any additional information about the signals and therefore the bids of other players (conditional independence). Moreover, this probability is independent of the identity of the bidder that we consider, i.e., bidder i, because we focus on symmetric strategies.

Definition B.1. We say that there is **pooling by pivotal types** in market m if there is a sequence of pooling bids b_p^n with $\underline{\theta}_p^n = \inf\{\theta : b_m^n(\theta) = b_p^n\} < \theta_p^n = \sup\{\theta : b_m^n(\theta) = b_p^n\}$ such that $\sqrt{n}(F_m^n(\theta_p^n|v) - F_m^n(\underline{\theta}_p^n|v)) \to \infty$, $\lim \Pr(P_m^n = b_p^n|v) > 0$ for $v = 0, 1$, $\underline{\theta}_p := \lim \underline{\theta}_p^n \leq \min\{\theta_m(1), \theta_m(0)\}$ and $\theta_p := \lim \theta_p^n \geq \max\{\theta_m(1), \theta_m(0)\}$. We say that there is **no pooling by pivotal types** in market m if no such sequence exists.

Lemma B.2. *Assume that MLRP is satisfied and fix a convergent sequence of bidding equilibria* $\mathbf{H} = \{H^n\}$ *for market* m*. Suppose that there is pooling by pivotal types in market* m

and suppose $\theta_p > \underline{\theta}_p$. If $\theta_m(v) \in [\underline{\theta}_p, \theta_p]$, then

$$\begin{aligned}
\lim \Pr(b^n \text{ } wins|P^n = b^n, V = v) &= \frac{\kappa_m - (F_m(1|v) - F_m(\theta_p|v))}{F_m(\theta_p|v) - F_m(\underline{\theta}_p|v)} \\
&= \frac{\kappa_m - \bar{F}_m(\theta_p|v)}{F_m(\theta_p|v) - F_m(\underline{\theta}_p|v)}
\end{aligned}$$

Therefore, if $\theta_m(v) \in (\underline{\theta}_p, \theta_p]$, then

$$\begin{aligned}
\lim \Pr(P^n = b^n, b^n \text{ } wins|V = v) &= \frac{\kappa_m - (F_m(1|v) - F_m(\theta_p|v))}{F_m(\theta_p|v) - F_m(\underline{\theta}_p|v)} \\
&= \frac{\kappa_m - \bar{F}_m(\theta_p|v)}{F_m(\theta_p|v) - F_m(\underline{\theta}_p|v)}
\end{aligned}$$

Proof. We show if $\theta_m(v) \in [\underline{\theta}_p, \theta_p)$, then

$$\lim \Pr(b^n \text{ } wins|P^n = b^n, v) = \frac{\kappa_m - (F_m(1|v) - F_m(\theta_p|v))}{F_m(\theta_p|v) - F_m(\underline{\theta}_p|v)}.$$

Note that $\lim \Pr(Y_m^{n-1}(\lfloor n\kappa \rfloor) \in (\theta_m(v) - \epsilon_1, \theta_m(v) + \epsilon_1)|P^n = b^n, V = v) = 1$ for every $\epsilon_1 > 0$. Let y^n denote the random variable equal to the fraction of bidders who submit a bid equal to b^n including bidder i. Let g^n denote random variable equal to the fraction of goods allocated to the bidders bidding b^n. Note that $\mathbb{E}[g^n/y^n|P^n = b^n, V = v] = \Pr(b^n \text{ } wins|P^n = b^n, V = v)$. Also let

$$x = \frac{\kappa_m - (F_m(1|v) - F_m(\theta_p|v))}{F_m(\theta_p|v) - F_m(\underline{\theta}_p|v)}.$$

For every $\epsilon > 0$, there is some $\epsilon_1 > 0$ such that:

$$\lim \Pr\left(g^n/y^n \in (x - \epsilon, x + \epsilon)\,|Y_m^{n-1}(\lfloor n\kappa \rfloor) \in (\theta_m(v) - \epsilon_1, \theta_m(v) + \epsilon_1), V = v\right) = 1.$$

This follows from the observation that for each of the $\lfloor n\kappa_m \rfloor$ people who received signals above some θ, the number of people who received a signal above θ_p^n has a binomial distribution and the LLN applies because $\theta_p^n \to \theta_p$ and delivers that the fraction of people receiving signals above θ_p^n converges to the κ times the the probability that a bidder receives a signal above θ_p divided by the probability that a bidder receives a signal above θ. Note that $\lim \Pr(Y_m^{n-1}(\lfloor n\kappa_m \rfloor) \in (\theta_m(v) - \epsilon_1, \theta_m(v) + \epsilon_1)|P^n = b^n, V = v) = 1$ for every $\epsilon_1 > 0$. Therefore,

$$\lim \Pr\left(g^n/y^n \in (x - \epsilon, x + \epsilon)\,|P^n = b^n, V = v\right) = 1.$$

The fact that $\lim \Pr\left(g^n/y^n \in (x - \epsilon, x + \epsilon)\,|P^n = b^n, V = v\right) = 1$ implies that $\lim E[g^n/y^n|P^n = b^n, V = v] = x = \lim \Pr(b^n \text{ } wins|P^n = b^n, V = v)$ which establishes the result.

We now consider the case where $\theta_m(v) = \theta_p$. We argue that

$$\lim \frac{\Pr(P^n = b^n, b^n \text{ } wins|V = v)}{\Pr(P^n = b^n|V = v)} = \lim \Pr(b^n \text{ } wins|V = v, p^n = b^n) = 0.$$

Notice that $\lim \Pr(Y_m^{n-1}(\lfloor n\kappa_m \rfloor) \geq \theta_m(v) - \epsilon | V = v, P^n \in [b(\underline{\theta}_p^n), b(\theta_p^n)]) = 1$ for every $\epsilon > 0$. Hence, for every $\epsilon > 0$ the following two equations are satisfied

$$\begin{aligned} \lim \Pr(y_n > F_m(\theta_p|v) - F_m(\underline{\theta}_p|v) - \epsilon | V = v, P^n = b^n) &= 1 \\ \lim \Pr(g_n < \epsilon | V = v, P^n = b^n) &= 1 \end{aligned}$$

Therefore,

$$\Pr(b^n \; wins|v, p^n = b^n) < \frac{\epsilon \Pr\left(y^n > F_m(\theta_p|v) - F_m(\underline{\theta}_p|v) - \epsilon, g^n < \epsilon | V = v, P^n = b^n\right)}{F_m(\theta_p|v) - F_m(\underline{\theta}_p|v) - \epsilon} + \Pr(g^n \geq \epsilon | V = v, P^n = b^n)$$

Since $\lim \Pr(g^n \geq \epsilon | v, p^n = b^n) = 0$ and since the inequality is true for all $\epsilon > 0$, we conclude that $\lim \Pr(win|V = v, P^n = b^n) = 0$.

If $\theta_m(v) \in (\underline{\theta}_p, \theta_p)$, then $\lim \Pr(P^n = b^n|v) = 1$ by LLN. Using the fact that $\lim \Pr(P^n = b^n|V = v) = 1$ and applying Bayes' rule we find that

$$\begin{aligned} \lim \Pr(P^n = b^n, b^n \; wins|V = v) &= \lim \Pr(P^n = b^n|V = v) \times \lim \Pr(b^n \; wins|P^n = b^n, V = v) \\ &= \lim \Pr(b^n \; wins|P^n = b^n, V = v) = \frac{\kappa_m - (F_m(1|v) - F_m(\theta_p|v))}{F_m(\theta_p|v) - F_m(\underline{\theta}_p|v)}, \end{aligned}$$

if $\theta_m(v) \in (\underline{\theta}_p, \theta_p)$. On the other hand, if $\theta_m(v) = \theta_p$, then Bayes' rule and the LLN implies that

$$\lim \Pr(P^n = b^n, b^n \; wins|V = v) = 0 = \frac{\kappa_m - (F_m(1|v) - F_m(\theta_p|v))}{F_m(\theta_p|v) - F_m(\underline{\theta}_p|v)},$$

completing the proof. □

B.2. Information Aggregation and Informative price. Below we introduce a weaker definition which was also used by Atakan and Ekmekci (2014). In particular, we say that a sequence of equilibria in a large auction is informative, if an outside observer can learn the state asymptotically by simply observing the equilibrium price. In other words, this definition does not require that price converge to value but requires only that the price identify the state. The precise definition is as follows:

Definition B.2. (Kremer (2002) and Atakan and Ekmekci (2014)) Let

$$l(v', v|P_s^n = p) := \frac{\Pr(V = v'|P_s^n = p)}{\Pr(V = v|P_s^n = p)}$$

A sequence of strategies $\mathbf{H}$ is **informative** if for any $v \neq v'$ and any $\epsilon > 0$,

$$\lim_{n\to\infty} \Pr\left(P_s^n \in \left\{p : l(v', v|P_s^n = p) \leq \epsilon\right\} |V = v\right) = 1.$$

That is, the random variable $l(v', v|P_s^n)$ converges in probability to zero in state v. We denote this by $l(v', v|P_s^n) \xrightarrow{p} 0$.[38]

[38] Kremer (2002) terms an auction informative if $\mathbb{E}[V|P_s^n] - V \xrightarrow{p} 0$. This definition by Kremer (2002) and

Remark B.1. Note that a reserve price $c > 0$ in market r rules out an equilibrium sequence from aggregating information in market r according to Definition 2.2. However, such a reserve price does not necessarily rule out an equilibrium sequence from being informative according according to Definition B.2.[39]

The following lemma relates Definition 2.2 and Definition B.2 in the current context with an auction with endogenous participation. The lemma shows that if a sequence aggregates information, then it is also informative. This part of the lemma follows from Kremer (2002). The second part of the lemma also provides a partial converse: if the expected number of bidders participating in the auction exceeds the number of goods, then informativeness also implies information aggregation.

Lemma B.3. *Fix an equilibrium sequence* $\mathbf{H}$.

i. If $\mathbf{H}$ *aggregates information, then* $\mathbf{H}$ *is informative.*

ii. If $\mathbf{H}$ *is informative and* $F_m^n(1|1) > \kappa$*, then* $\mathbf{H}$ *aggregates information in market* m.

The following proves Lemma B.3 and also establishes some results that we will use in our subsequent proofs. In particular, item i of the lemma shows that if a sequence is informative, then the expected number of people who receive signals in between the signals of the two pivotal types must converge to infinity at the order of $\sqrt{n}$. This implies that for any price that one observes in state v one can reject the hypothesis that this price was the bid of the $k+1$st highest type in state $v' = v$. The proof applies the central limit theorem for quantiles and shows that if the κ_mth quantile in each state, i.e., the pivotal types, are not sufficiently separated, then one cannot reject the hypothesis that the market clears at the bid of the $k+1$st highest type in state v even though the state is v'. Item ii of the lemma shows that if there are many bidders between the pivotal types and the pivotal types do not pool (as in Definition B.1), then an equilibrium sequence is informative. Hence, item ii is a partial converse of item i. Item iii of the lemma argues that if the expected prices do not converge to zero, then informativeness implies information aggregation. Intuitively, if bidders are paying a positive price for a good in state $V = 1$, then it must be that the bidders are competing for goods in state $V = 1$. However, if they know that they are in state $V = 1$ conditional on winning an object, then the bidders will be willing to compete the auction price up to the value of the object. Under the hypothesis that the price is informative, a sufficient condition for competition to result in information aggregation is $F_m(1|1) > \kappa$, that is, for there to be many bidders who are unable to buy an object in the auction.

Lemma B.4 (Proof of Lemma B.3 and related results)**.** *Assume that MLRP is satisfied and fix a convergent sequence of bidding equilibria* $\mathbf{H} = \{H^n\}$ *for market* m.

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the above definition are equivalent.

[39] For another contrast between the two definitions in the context of a pay-as-you-bid auction (also referred to as a discriminatory price auction), see Jackson and Kremer (2007). As noted by Jackson and Kremer (2007), in a pay-as-you-bid auction, prices do not converge to value, however, in such an auction format any equilibrium sequence is informative.

i. If the sequence $\mathbf{H}$ *is informative, then* $\sqrt{n}|F_m^n[\theta_m^n(v)|v] - F_m^n[\theta_m^n(v')|v]| \to \infty$ *for all* $v' \neq v$ *and* $P_m^n \xrightarrow{p} 0$ *in state* $v = 0$.

ii. If $\sqrt{n}|F_m^n[\theta_m^n(v)|v] - F_m^n[\theta_m^n(v')|v]| \to \infty$ *for all* $v' \neq v$ *and there is no pooling by pivotal types, then the sequence* $\mathbf{H}$ *is informative.*

iii. If $\mathbf{H}$ *is informative and* $\lim_n \max\{\mathbb{E}[P_m^n|V=0], \mathbb{E}[P_m^n|V=1]\} > 0$*, then* $\mathbf{H}$ *aggregates information. Therefore, if* $\mathbf{H}$ *is informative and* $F_m(1|1) > \kappa$*, then* $\mathbf{H}$ *aggregates information.*

Proof. Fix an equilibrium bidding sequence $\mathbf{H}$ which is informative according to definition B.2.

Claim. $\sqrt{n}|F_m^n(\theta_m^n(v)|v) - F_m^n(\theta_m^n(v')|v)| \to \infty$.

Proof. Suppose without loss of generality that $\bar{F}_m^n(\theta_m^n(v)|v) \leq \bar{F}_m^n(\theta_m^n(v')|v)$ and assume that $\sqrt{n}(\bar{F}_m^n(\theta_m^n(v)|v) - \kappa) \to \infty$. We return to the case of $\sqrt{n}(\bar{F}_m^n(\theta_m^n(v)|v) - \kappa) \to x < \infty$ at the end of the claim's proof. Suppose, on the way to a contradiction that $\sqrt{n}|\bar{F}_m^n(\theta_m^n(v)|v) - \bar{F}_m^n(\theta_m^n(v')|v)| = x$. The fact that $f(\theta|v)/f(\theta|v') \in (1/\eta, \eta)$ implies that $\sqrt{n}|\bar{F}_m^n(\theta_m^n(v)|v') - \bar{F}_m^n(\theta_m^n(v')|v')| = x'$ for some finite x'. Set $\sigma = \sqrt{\kappa(1-\kappa)}$. Pick $y > 0$ and let

$$A^n = \left\{p : p = b^n(\theta), |\bar{F}_m^n(\theta|v) - \kappa| \leq \sigma y/\sqrt{n}\right\},$$

i.e., A_n is the set of bids submitted by types in $\{\theta' : |\bar{F}_m^n(\theta'|v) - \kappa| \leq \sigma y/\sqrt{n}\}$. Note that the central limit theory implies that

$$\lim \Pr\left(p^n = p \in A^n | V = v\right) \geq 2\Phi\left(y\right) - 1,$$

the inequality does not necessarily hold with equality because types other than types in $\{\theta' : |\bar{F}_m^n(\theta'|v) - \kappa| \leq \sigma y/\sqrt{n}\}$ may also choose a bid in A^n also. Therefore,

$$\lim_n \Pr\left(p^n = p \in A^n | V = v'\right) \quad \geq \quad \Phi\left(y' - x'\right) + \Phi\left(y' + x'\right) - 1 > 0$$

where $y' = \eta_1 y$ (this follows from the fact that $f(\theta|v)/f(\theta|v') \in (1/\eta, \eta)$). Hence,

$$\lim \Pr\left(p^n = p \in A^n | V = v\right) > 0 \text{ and } \lim \Pr\left(p^n = p \in A^n | V = v'\right) > 0.$$

In other words, the auction price is in the set A_n with strictly positive probability in both states. We now derive a contradiction to information aggregation by showing that the likelihood ratio, conditional on any price in the set A^n, is uniformly bounded away from 0 and ∞. Proposition A.2 implies that

$$\lim_n \frac{\Pr(\bar{F}_m^n(Y^n(k+1)|v) - \bar{F}_m^n(\theta^n(v)|v) = z\frac{\sigma}{\sqrt{n}}|V=v)}{\Pr(\bar{F}_m^n(Y^n(k+1)|v') - \bar{F}_m^n(\theta^n(v)|v') = z'\frac{\sigma}{\sqrt{n}}|V=v')} = \frac{\phi\left(z\right)}{\phi(z'+x')}.$$

For each $z \in [0, y]$ and each $z' \in [0, y']$

$$0 < \frac{\min_{z\in[0,y]} \phi(z)}{\max_{z'\in[0,y']} \phi(z'+x')} < \frac{\phi(z)}{\phi(z'+x')} < \frac{\max_{z\in[0,y]} \phi(z)}{\min_{z'\in[0,y']} \phi(z'+x')} < \infty.$$

For any $p \in A^n$, if p is not part of an atom that extends beyond $z \in [0, y]$ eventually along the sequence, then

$$\frac{\min_{z\in[0,y]} \phi(z)}{\max_{z'\in[0,y']} \phi(z'+x')} < \lim \frac{\Pr(P^n = p|v)}{\Pr(P^n = p|v')} < \frac{\max_{z\in[0,y]} \phi(z)}{\min_{z'\in[0,y']} \phi(z'+x')}$$

and if p is part of an atom that extends beyond $z \in [0, y]$ infinitely often, then

$$\begin{aligned} 0 < \min_{z\in\left[-x-\frac{y}{2},y\right]} \left\{\Phi(z) - \Phi(z - \frac{y}{2})\right\} &\leq \lim \frac{\Pr(P^n = p|V = v)}{\Pr(P^n = p|V = v')} \\ &\leq \frac{1}{\min_{z'\in\left[-x-\frac{y'}{2},y'\right]} \left\{\Phi(z') - \Phi(z' - \frac{y'}{2})\right\}} < \infty. \end{aligned}$$

However, the inequalities above together with the fact that $\lim \Pr(P_m^n = p \in A^n|V = v) > 0$ contradicts that the sequence $\mathbf{H}$ is informative.

We now return to the case of$\sqrt{n}(\bar{F}_m^n(\theta_m^n(v)|v) - \kappa) \to x < \infty$. In this case, the central limit theorem implies that the probability that the number or goods in the auction exceeds the number of bidders is strictly positive in state v and therefore the probability that the auction price is the minimum price (i.e, $p^n = c = 0$) is strictly positive in state v. More formally, $0 < \lim_n \Pr(Y^n(k+1) = 0|V = v) = \Phi(x)$. However, applying the central limit theorem once again implies that if $\lim_n \frac{\sqrt{n}}{\sigma}|\bar{F}_m^n(\theta_m^n(v)|v') - \bar{F}_m^n(\theta_m^n(v')|v')| = y'$, then $\lim_n \Pr(Y^n(k+1) = 0|V = v') = \Phi(x' + y') > 0$ and therefore $\lim_n \Pr(P^n = 0|V = v') \geq \Phi(x' + y') > 0$. ◻

Claim. If $\sqrt{n}|F_m^n[\theta_m^n(v)|v] - F_m^n[\theta_m^n(v')|v]| \to \infty$ for all $v' \neq v$ and there is no pooling by pivotal types, then the sequence $\mathbf{H}$ is informative.

Proof. Suppose without loss of generality that $\theta_m^n(v) < \theta_m^n(v')$. Set

$$\begin{aligned} s^n(v) &= \sup\{\theta : b^n(\theta) = b^n(\theta_m^n(v))\}, \\ s^n(v') &= \inf\{\theta : b^n(\theta) = b^n(\theta_m^n(v'))\} \end{aligned}$$

and s^n be the type such that $F_m^n(s^n|0) - F_m^n(s^n(v)|v) = F_m^n(s^n(v')|0) - F_m^n(s^n|v)$. Note that $\sqrt{n}|F_m^n[\theta_m^n(v)|v] - F_m^n[\theta_m^n(v')|v]| \to \infty$ for all $v' \neq v$ and the fact that there is no pooling by pivotal types together imply that $\sqrt{n}|F_m^n[s^n|v] - F_m^n[\theta_m^n(v)|v]| \to \infty$ and $\sqrt{n}|F_m^n[s^n|v'] - F_m^n[\theta_m^n(v')|v']| \to \infty$. Therefore, $\lim \Pr(P_s^n \leq b^n(s^n)|V = v') = 0$, $\lim \Pr(P_s^n \geq b^n(s^n)|V = v) = 0$ and moreover $\lim \Pr(P_s^n \geq b^n(s^n)|V = v') = 1$, $\lim \Pr(P_s^n \leq b^n(s^n)|V = v) = 1$. Hence, the sequence is informative. ◻

Claim. Suppose that $v > v'$. If $\lim_n \max\{\mathbb{E}[P^n|V = v'], \mathbb{E}[P^n|V = v]\} > 0$, then $\theta^n(v) -$

$\theta^n(v') > 0$ for sufficiently large n. Therefore, $\sqrt{n}(F_m^n(\theta_m^n(v)|v) - F_m^n(\theta_m^n(v')|v)) \to \infty$.

Proof. Assume to the contrary that $\theta_m^n(v') - \theta_m^n(v) \geq 0$ for all n. Above we established that $\sqrt{n}|\bar{F}_m^n(\theta_m^n(v)|v) - \bar{F}_m^n(\theta_m^n(v')|v)| \to \infty$. Therefore, $\theta_m^n(v') - \theta_m^n(v) > 0$ for all n sufficiently large. Set

$$\begin{aligned} s^n(v) &= \sup \{\theta : b^n(\theta) = b^n(\theta_m^n(v))\} \\ s^n(v') &= \inf \left\{\theta : b^n(\theta) = b^n(\theta_m^n(v'))\right\}. \end{aligned}$$

An argument analogous to the previous claim's implies that $\sqrt{n}\left|\bar{F}_m^n(s^n(v)|v) - \bar{F}_m^n(\theta_m^n(v')|v)\right| \to \infty$ and $\sqrt{n}\left|\bar{F}_m^n(\theta_m^n(v)|v) - \bar{F}_m^n(s^n(v')|v)\right| \to \infty$.

There are two cases to consider: (1)$\sqrt{n}\left|\bar{F}_m^n(s^n(v)|v) - \bar{F}_m^n(s^n(v')|v)\right| \to \infty$ and (2) $\sqrt{n}\left|\bar{F}_m^n(s^n(v)|v) - \bar{F}_m^n(s^n(v')|v)\right| = x < \infty$.

Informativeness of the sequence, $\theta_m^n(v') - \theta_m^n(v) > 0$, and the monotonicity of the bidding function together imply that $b^n(\theta_m^n(v')) > b^n(\theta_m^n(v))$ for all n sufficiently large. Consider a bidder who submits a bid equal to[40] $b^n = b^n\left(\frac{s^n(v)+s^n(v')}{2}\right)$ if we are in case 1 and $b^n = \frac{b^n(\theta_m^n(v))+b^n(\theta_m^n(v'))}{2}$ if we are in case 2. Note that in both cases $b^n(\theta_m^n(v')) > b^n > b^n(\theta_m^n(v))$ and the probability of such a bidder winning an object in states v and v' converge to one and zero, respectively.[41] Therefore, the utility, at the limit, from submitting such a sequence of bids is as follows

$$\lim_n u^n(m, b^n|\theta_i = \theta) = v - \lim \mathbb{E}[P^n|V = v]$$

for any θ. However, the utility of a type who wins an object with positive probability in state v' is strictly worse than the above expression because $0 < \max\{\lim \mathbb{E}[P^n|V = v], \lim \mathbb{E}[P^n|V = v']\} \leq \lim \mathbb{E}[P^n|V = v']$. This however leads to a contradiction. □

Claim. If $\sqrt{n}(F_m^n(\theta_m^n(1)|v) - F_m^n(\theta_m^n(0)|v)) \to \infty$, then $\lim \Pr(P^n \leq 1 - \delta|V = 1) = 0$ and $\lim \Pr(P^n \geq \delta|V = 0) = 0$ for all $\delta > 0$. Therefore, if $\lim_n \max\{\mathbb{E}[P^n|V = 0], \mathbb{E}[P^n|V = 1]\} > 0$, then **H** aggregates information.

Proof. Let

$$A^n = \{\theta \in [s^n(0), s^n(1)] : F_m^n(\theta|0) - F_m^n(s^n(0)|0) = F_m^n(s^n(1)|0) - F_m^n(\theta|0), b^n(\theta) \geq \delta\}$$

Assume that $\lim_n \Pr[P^n = b^n(\theta), \theta \in A^n|V = 0] > 0$. Note that any $\theta \in A^n$ such that $\theta \neq s^n(1)$ wins with probability converging to zero if $V = 1$. Therefore, the profit of any such a type θ is strictly negative for sufficiently large n leading to a contradiction.

Let $A^n = \{\theta : b^n(\theta) \leq 1 - \delta\}$. Assume to the contrary that $\lim_n \Pr[p^n = b^n(\theta), \theta \in A^n|V = 1] > 0$. For type $\theta_m^n(0)$ bidding one does strictly better than bidding $b^n(\theta_m^n(0))$. This is because

[40] If any one type θ' deviates from equilibrium, this deviation does not change the limiting price distribution nor the probability of winning an object at the limit because these depend on the order statistics and $\lim_n \Pr(Y^n(n\kappa + 1) \leq \theta) = \lim_n \Pr(Y^{n-1}(n\kappa) \leq \theta)$ for any θ.

[41] In case 2, the probability of such a bidder winning an object in state h converges to one because $b^n > b^n(\theta)$ for any $\theta < s^n(1)$ and because $\sqrt{n}\,|s^n(1) - s^n(0)| = x$ and $\sqrt{n}\,|\theta^n(1) - s^n(0)| \to \infty$ together imply $\sqrt{n}\,|\theta^n(1) - s^n(1)| \to \infty$. The probability of such a type winning an object in state l converges to zero because $b^n < b^n(\theta)$ for any $\theta > s^n(0)$ and because $\sqrt{n}\,|\theta^n(0) - s^n(0)| \to \infty$.

the bid $b^n(\theta_m^n(0))$ wins with positive probability in state 0 at a price equal to zero (from the above paragraph) and never wins in state 1 and therefore makes zero profit at the limit. In contrast, a bid equal to one wins and makes positive profit equal to δ with strictly positive probability in state 1 and makes exactly zero profit in state 0.

The previous claim implies that if

$$\lim_n \max\{\mathbb{E}\left[P^n|V=0\right], \mathbb{E}\left[P^n|V=1\right]\} > 0,$$

then $\sqrt{n}(F_m^n(\theta_m^n(1)|v) - F_m^n(\theta_m^n(0)|v)) \to \infty$. Combining this with the above finding we conclude that if $\lim_n \max\{\mathbb{E}\left[P^n|V=0\right], \mathbb{E}\left[P^n|V=1\right]\} > 0$, then $\mathbf{H}$ aggregates information. $\square$

We complete the proof by noting that if $F_m^n(1|1) > \kappa$, then

$$\lim_n \max\{\mathbb{E}\left[P^n|V=1\right], \mathbb{E}\left[P^n|V=0\right]\} > 0.$$

Because if $\lim_n \max\{\mathbb{E}\left[P^n|V=1\right], \mathbb{E}\left[P^n|V=0\right]\} = 0$, then any type who does not win an object with probability one has an incentive to submit a bid equal to one. Moreover $F_m^n(1|1) > \kappa$ implies that there are many types who do not win an object with probability one. $\square$

In the definition below, we define $\hat{\theta}_m$ as the smallest type which wins a good in auction m with positive probability if $V=0$ at the limit as n grows large, i.e., this type is the smallest "active" type in state $v=0$. The definition is somewhat involved as we need to define type $\hat{\theta}_m$ at the limit of a sequence of strategies as n grows arbitrarily large and the pivotal type $\theta_m(0)$ could be pooling with other types along such a sequence.

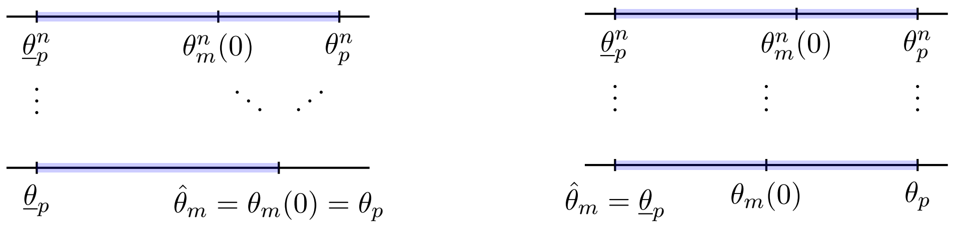
Definition B.3. Assume that MLRP is satisfied. Fix a sequence of symmetric distributional strategies $\{H^n\}$ that can be represented by increasing bidding functions b_m^n. If $F_m(1|0) \geq \kappa_m$, let

$$\begin{aligned}
\theta_m^n(\epsilon) &:= \inf\{\theta : H^n([0,1] \times m \times (b_m^n(\theta), 1]|0) < \kappa_m - \epsilon\} \\
\hat{\theta}_m(\epsilon) &:= \limsup \theta_m^n(\epsilon) \\
\hat{\theta}_m &:= \inf_{\epsilon>0} \theta_m(\epsilon).
\end{aligned}$$

If $F_m(1|0) < \kappa_m$, let $\hat{\theta}_m = \inf\{\theta : F_m(\theta|0) > 0\}$ and $\hat{\theta}_m = 1$ if the set is empty.

Suppose that $F_m(1|0) \geq \kappa_m$. The definition above simply selects type $\hat{\theta}_m = \theta_m(0)$ if the bidding function b_m^n is strictly increasing at $\theta_m^n(0)$ for n sufficiently large. The definition has more bite if, on the other hand, $\theta_m^n(0)$ submits a pooling bid. If $\theta_m^n(0)$ submits a pooling bid, then there are types $\underline{\theta}_p^n \leq \theta_m^n(0) \leq \theta_p^n$ who submit the same bid as $\theta_m^n(0)$. There are two cases to consider: In the first case $\theta_m(0) = \lim \theta_p^n$. Then the definition selects $\hat{\theta}_m = \theta_m(0)$. In the second case, if $\theta_m(0) < \lim \theta_p^n$, then the definition selects $\hat{\theta}_m = \lim \underline{\theta}_p^n$. Also, see figure B.1 for an illustration.

The following lemma explores information aggregation in a certain market m where there are more bidders than goods in state $v=1$, that is, the case where $F_m(1|1) > \kappa_m$. The lemma



(a) The pivotal type converges to the upper-bound of the pooling region.

(b) The pivotal type remains in the interior of the pooling region.

Figure B.1: Implications of Definition B.3. Suppose that $b_m^n(\underline{\theta}_p^n) = b_m^n(\theta_m^n(0)) = b_m^n(\theta_p^n)$ along some sequence. In subfigure (a), $\theta_m(0) = \lim \theta_p^n$ and therefore $\hat{\theta}_m = \theta_m(0)$. In subfigure (b) $\theta_m(0) < \lim \theta_p^n$ and therefore $\hat{\theta}_m = \lim \underline{\theta}_p^n$.

argues that if there are more "active" players in state 1 than in 0, i.e., if there are more bidders above $\hat{\theta}_m$ when $V = 1$ than when $V = 0$ (more precisely, if $\bar{F}_m(\hat{\theta}_m|1) > \bar{F}_m(\hat{\theta}_m|0)$), then pooling by pivotal types cannot be sustained. Moreover, if $\bar{F}_m(\hat{\theta}_m|1) > \bar{F}_m(\hat{\theta}_m|0)$, then the pivotal types are ordered and distinct, that is, $\bar{F}_m(\hat{\theta}_m|1) > \bar{F}_m(\hat{\theta}_m|0)$ implies that $\theta_m(1) > \theta_m(0)$. The lemma thus concludes that information is aggregated in market m under the assumption that $\bar{F}_m(\hat{\theta}_m|1) > \bar{F}_m(\hat{\theta}_m|0)$ because the pivotal types do not pool and because they are distinct. The lemma also provides two sufficient conditions for $\bar{F}_m(\hat{\theta}_m|1) > \bar{F}_m(\hat{\theta}_m|0)$ that are repeatedly used in our subsequent analysis.

Lemma B.5. *Assume MLRP. Fix a convergent sequence of bidding equilibria* $\mathbf{H} = \{H^n\}$ *for market* m*. Suppose* $F_m(1|1) > \kappa_m$*. If* $\bar{F}_m(\hat{\theta}_m|1) > \bar{F}_m(\hat{\theta}_m|0)$, where $\hat{\theta}_m$ is the type introduced in Definition B.3, *then there is no pooling by pivotal types (see Definition B.1). Therefore, if* $\bar{F}_m(\hat{\theta}_m|1) > \bar{F}_m(\hat{\theta}_m|0)$*, then* $\mathbf{H}$ *aggregates information (by Lemma B.4 item (ii)). Moreover,*

i. If $F_m(1|1) \geq F_m(1|0)$ *and* $\theta_m(1) > \theta_m(0)$ *or*

ii. If $\lim a_m^n(\theta) = 1$ *for all* $\theta > \hat{\theta}_m$,

then $\bar{F}_m(\hat{\theta}_m|1) > \bar{F}_m(\hat{\theta}_m|0)$.

Proof. The hypotheses that $F_m(1|1) \geq F_m(1|0)$ and $\theta_m(1) > \theta_m(0)$ or the hypothesis that $\lim a_m^n(\theta) = 1$ for all $\theta > \hat{\theta}_m$ both separately imply that $\bar{F}_m(\hat{\theta}_m|1) > \bar{F}_m(\hat{\theta}_m|0)$. These are straightforward consequences of MLRP. Also, note that if $\bar{F}_m(\hat{\theta}_m|1) > \bar{F}_m(\hat{\theta}_m|0)$, then $\theta_m(1) > \theta_m(0)$. This is because $\theta_m(0) \geq \hat{\theta}_m$ by definition, $\bar{F}_m(\theta|1) > \bar{F}_m(\theta|0)$ for any $\theta > \hat{\theta}_m$ by MLRP and thus $\bar{F}_m(\theta_m(0)|1) > \bar{F}_m(\theta_m(0)|0) = \kappa$ showing that $\theta_m(1) > \theta_m(0)$.

We show that pooling by pivotal types is incompatible with equilibrium behavior if $\bar{F}_m(\hat{\theta}_m|1)$ exceeds $\bar{F}_m(\hat{\theta}_m|0)$ because bidders are subject to the loser's curse at the pooling bid for n large enough. Existence of a loser's cruse means that bidders have a profitable deviation to some bid above b^n. For details, see Lemma 7 and Corollary 3 in Pesendorfer and Swinkels (1997). More precisely, we will show that

$$\lim \frac{\Pr(p^n = b^n, b^n\ wins|V=1)}{\Pr(p^n = b^n, b^n\ wins|V=0)} < \lim \frac{\Pr(p^n = b^n|V=1)}{\Pr(p^n = b^n|V=0)} < \lim \frac{\Pr(p^n = b^n, b^n\ loses|V=1)}{\Pr(p^n = b^n, b^n\ loses|V=0)}.$$

Pooling by pivotal types implies that $\underline{\theta}_p \leq \theta_m(0) < \theta_m(1) \leq \theta_p$. There are three cases to consider: (1) $\underline{\theta}_p < \theta_m(0)$ and $\theta_p > \theta_m(1)$; (2) $\theta_p = \theta_m(1)$ and $\underline{\theta}_p < \theta_m(0)$; and (3) $\underline{\theta}_p = \theta_m(0)$. If $\theta_p = \theta_m(1)$ and $\underline{\theta}_p < \theta_m(0)$, then Lemma B.2 implies that $\lim \Pr(p^n = b^n, b^n \ wins|V = 1) = 0$ and $\lim \Pr(p^n = b^n, b^n \ wins|V = 0) > 0$ showing that the above inequality is satisfied. If $\underline{\theta}_p = \theta_m(0)$, then B.2 implies that $\lim \Pr(p^n = b^n, b^n \ wins|V = 0) = 1$ and $\lim \Pr(p^n = b^n, b^n \ wins|V = 1) < 1$ again showing that the above inequality is satisfied.

We now turn to the case of $\underline{\theta}_p < \theta_m(0)$ and $\theta_p > \theta_m(1)$. Note that in this case $\hat{\theta}_m = \underline{\theta}_p$. In this case, the following equalities are satisfied:

$$\lim \Pr(P^n = b^n|V = 1) = \lim \Pr(P^n = b^n|V = 0) = 1, \tag{B.1}$$

$$\lim \Pr(P^n = b^n, b^n \quad loses|V = v) = \frac{\bar{F}_m(\underline{\theta}_p|v) - \kappa}{F_m(\theta_p|v) - F_m(\underline{\theta}_p|v)}, \tag{B.2}$$

by Lemma B.2. We argue that

$$\lim \frac{\Pr(p^n = b^n, b^n \ loses|V = 1)}{\Pr(p^n = b^n, b^n \ loses|V = 0)} > 1,$$

i.e., that the following inequality is satisfied:

$$\frac{\bar{F}_m(\underline{\theta}_p|1) - \kappa}{\bar{F}_m(\underline{\theta}_p|0) - \kappa} > \frac{F_m(\theta_p|1) - F_m(\underline{\theta}_p|1)}{F_m(\theta_p|0) - F_m(\underline{\theta}_p|0)}.$$

Our assumption that $\bar{F}_m(\underline{\theta}_p|1) - \kappa \geq \bar{F}_m(\underline{\theta}_p|0) - \kappa > 0$ implies that

$$\frac{\bar{F}_m(\underline{\theta}_p|1) - \kappa}{\bar{F}_m(\underline{\theta}_p|0) - \kappa} \geq \frac{\bar{F}_m(\underline{\theta}_p|1)}{\bar{F}_m(\underline{\theta}_p|0)}. \tag{B.3}$$

Moreover, MLRP implies that $\frac{g(\theta)f(\theta|1)}{g(\theta)f(\theta|0)}$ is increasing in θ for any function $g(\theta)$. Hence, we conclude that

$$\frac{\bar{F}_m(\underline{\theta}_p|1)}{\bar{F}_m(\underline{\theta}_p|0)} = \frac{\int_{\underline{\theta}_p}^1 a_m(\theta)f(\theta|1)d\theta}{\int_{\underline{\theta}_p}^1 a_m(\theta)f(\theta|0)d\theta} \geq \frac{\int_{\underline{\theta}_p}^{\theta_p} a_m(\theta)f(\theta|1)d\theta}{\int_{\underline{\theta}_p}^{\theta_p} a_m(\theta)f(\theta|0)d\theta} = \frac{F_m(\theta_p|1) - F_m(\underline{\theta}_p|1)}{F_m(\theta_p|0) - F_m(\underline{\theta}_p|0)}. \tag{B.4}$$

Therefore, combining inequalities (B.3) and (B.4) establishes the claim. □

C. The Market Selection Lemmata.

In this section we prove two central lemmata which characterize market selection. The first lemma and its corollary below characterizes market selection for any n (not necessarily at the limit). Under the assumption that there is a minimum price in market r and the payoff in market s is equal to zero for all types, the corollary shows that the upper tail of the type distribution selects market r.

Lemma C.1 (Single Crossing Lemma). *Assume that MLRP is satisfied. Suppose that* $a_m^H(\theta') >$

0 *for some type* θ' *in an equilibrium* H. *If*

$$u^H(m, b(\theta')|V = 0) < \min_{b,m' \neq m} u^H(m', b|V = 0),$$

then $a_m^H(\theta) = 1$ *for all* $\theta > \theta'$ *such that* $\theta \notin \mathcal{E}(\theta')$.

Proof. Fix an equilibrium H. For the remainder of the proof we suppress reference to equilibrium H.

We begin by noting that $u(m, b|\theta, v) = u(m, b|v)$ for any b, θ and v. Writing down the profits for any type θ and any bid b we obtain $u(m, b|\theta) = u(m, b|V = 0)\Pr(V = 0|\theta) + u(m, b|V = 1)\Pr(V = 1|\theta)$. Therefore,

$$\frac{u(m, b(\theta')|\theta') - u(m', b|\theta')}{\Pr(V = 0|\theta')} = \\ u(m, b(\theta')|V = 0) - u(m', b|V = 0) + l(\theta_i = \theta')(u(m, b(\theta')|V = 1) - u(m', b|V = 1)).$$

Moreover,

$$\frac{u(m, b(\theta')|\theta) - u(m', b|\theta)}{\Pr(V = 0|\theta')} - \frac{u(m, b(\theta')|\theta') - u(m', b|\theta')}{\Pr(V = 0|\theta')} = \\ (l(\theta_i = \theta) - l(\theta_i = \theta'))(u(m, b(\theta')|V = 1) - u(m', b|V = 1))$$

for any two types θ and θ'.

Our initial assumption that $a_m(\theta') > 0$ implies

$$u(m, b(\theta')|\theta') - u(m', b|\theta') \geq 0$$

for all b and all $m' \neq m$. Fix a b and $m' \neq m$. Our assumption that $u(m, b(\theta')|V = 0) < \min_{b,m' \neq m} u(m', b|V = 0)$ implies that

$$u(m, b(\theta')|V = 0) - u(m', b|V = 0) < 0.$$

Combining the inequalities above we conclude that $u(m, b(\theta')|V = 1) - u(m', b|V = 1) > 0$. Therefore, if $\theta > \theta'$ and $\theta \notin \mathcal{E}(\theta')$, then

$$\frac{u(m, b(\theta')|\theta) - u(m', b|\theta)}{\Pr(V = 0|\theta)} - \frac{u(m, b(\theta')|\theta') - u(m', b|\theta')}{\Pr(V = 0|\theta')} = \\ (l(\theta_i = \theta) - l(\theta_i = \theta'))(u(m, b(\theta')|V = 1) - u(m', b|V = 1)) > 0$$

because $l(\theta_i = \theta) > l(\theta_i = \theta')$ by MLRP. Hence,

$$\frac{u(m, b(\theta')|\theta) - u(m', b|\theta)}{\Pr(V = 0|\theta')} > \frac{u(m, b(\theta')|\theta') - u(m', b|\theta')}{\Pr(V = 0|\theta')} \geq 0$$

proving that $u(m, b(\theta')|\theta) - u(m', b|\theta) > 0$. Since b and $m' \neq m$ are arbitrary, we can conclude that $a_m(\theta) = 1$. $\square$

Corollary C.1. *Assume that MLRP is satisfied. Suppose that players have access to outside options as defined by Definition 3.1. Fix an equilibrium H. If $u^H(s, b(\theta')|V = 0) < u(r|V = 0)$ and if $a_s(\theta') > 0$ for some θ', then $a_s(\theta) = 1$ for all $\theta > \theta'$.*

Suppose that $\hat{\theta}_m$ denotes the smallest type which wins a good in auction m with positive probability in state 0, i.e., the type defined by Definition B.3. Also, suppose that the expected price in state 0 converges to zero in market $m' \neq m$ while the expected price in state 0 and market m remains bounded away from zero. Under these assumptions, the lemma presented below argues that all types $\theta > \hat{\theta}_m$ would select market m with probability one in equilibrium.

Lemma C.2. *Assume that MLRP is satisfied. Suppose that for an equilibrium sequence* $\mathbf{H}$ *we have that* $\lim \mathbb{E}(P^n_{m'}|0) = 0$, $\lim \mathbb{E}(P^n_m|0) > 0$ *for* $m \neq m'$, *then* $\lim a^n_m(\theta) = 1$ *for any* $\theta > \hat{\theta}_m$.

Proof. Without loss of generality suppose that $m' = s$ and $m = r$. The fact that $\lim_n \mathbb{E}(P^n_s|V = 0) = 0$ implies that $\lim u^n(s, b|V = 0) = 0$ for any b.

Pick $\theta > \hat{\theta}_r$. The fact that $\theta > \hat{\theta}_r$ implies that there exists $\epsilon > 0$ such that $\hat{\theta}_r \leq \hat{\theta}_r(\epsilon) < \theta$. Note that $\lim \hat{\theta}^n_r(\epsilon) \leq \hat{\theta}_r(\epsilon) < \theta$ for any convergent subsequence. Pick $\theta^n \in [\hat{\theta}^n_r(\epsilon/2), \hat{\theta}^n_r(\epsilon)]$ such that $a^n_r(\theta^n) > 0$. The probability that $P^n_r \leq b^n_r(\theta^n)$ converges to one in state 0. The probability that θ^n wins an object in state 0 and market m converges to at least $\epsilon/2$ (see Lemma B.2 for this lower bound). Therefore,

$$\lim u(r, b^n_r(\theta^n)|V = 0) \leq \frac{\epsilon}{2} \lim \mathbb{E}(0 - P^n_r|V = 0) < 0.$$

Let $\theta' = \lim \theta^n$ be any possibly sub-sequential limit and note $\theta' < \theta$. For such θ', we find

$$\lim u(r, b^n_r(\theta^n)|\theta') = \lim u(r, b^n_r(\theta^n)|\theta^n) \geq \lim u(s, b|\theta^n) = \lim u(s, b|\theta')$$

for any b because $a^n_r(\theta^n) > 0$. Also,

$$\lim u(r, b^n_r(\theta^n)|V = 0) < \lim u^n(s, b|V = 0) = 0$$

for any b. The argument in Lemma C.1 then implies that $\lim u(r, b^n_r(\theta^n)|\theta) > \lim u(s, b|\theta)$ for any b and any type $\theta > \theta'$ such that $\theta \notin \mathcal{E}(\theta')$. Therefore, $a_r(\theta) = 1$ for type $\theta \notin \mathcal{E}(\theta')$ and $\theta > \theta'$. Also, $a^n_r(\theta') > 0$ implies that $a^n_r(\theta) = 1$ for each $\theta \in \mathcal{E}(\theta')$. This is because we focus without loss of generality on a pure and increasing representation of strategies over each $\mathcal{E}(\theta')$. □

Lemma C.3. *Suppose that MLRP is satisfied. Suppose that there is a type $\theta^* < 1$ such that $\lim a^n_r(\theta) = 1$ for all $\theta > \theta^*$. If $\kappa_s > \kappa^*(\theta^*)$, then either $\theta_s(0) > \theta_s(1)$ or $\kappa_s > F_s(1|1)$. Alternatively, if $\kappa_s < \kappa^*(\theta^*)$ and $F_s(1|1) > \kappa_s$, then $\theta_s(0) \leq \theta_s(1)$. Moreover, if strict MLRP is satisfied, then $\theta_s(0) < \theta_s(1)$.*

Proof. We argue that $\lim a^n_r(\theta) = 1$ for all $\theta > \theta^*$, $\kappa_s \leq F_s(1|1)$ and $\kappa_s > \kappa^*(\theta^*)$, together imply that $\theta_s(0) - \theta_s(1) > 0$. Let L_1 denote the set of measurable functions $\alpha : [0, 1] \to [0, 1]$

and consider the following optimization problem:

$$W(\kappa_s, \theta^*) = \max_{\alpha \in L_1} \frac{\int_{[0,\theta^*]} \alpha(\theta) dF(\theta|1)}{\int_{[0,\theta^*]} \alpha(\theta) dF(\theta|0)}$$
$$s.t. \int_{[0,\theta^*]} \alpha(\theta) dF(\theta|1) \geq \kappa_s$$

Choose the unique θ' such that $F(\theta^*|1) - F(\theta'|1) = \kappa_s$. MLRP implies that the maximized value

$$W(\kappa_s, \theta^*) = \frac{F(\theta^*|1) - F(\theta'|1)}{F(\theta^*|0) - F(\theta'|0)}.$$

If $\kappa_s > \kappa^*(\theta^*)$, then $\kappa_s = F(\theta^*|1) - F(\theta'|1) > \kappa^*(\theta^*)$. Moreover, the inequality $F(\theta^*|0) - F(\theta'|0) > F(\theta^*|1) - F(\theta'|1)$ is satisfied by the definition of $\kappa^*(\theta^*)$. Therefore, $\kappa_s > \kappa^*(\theta^*)$ implies that the maximized value $W(\kappa_s, \theta^*) < 1$.

The fact that $\lim a_r^n(\theta) = 1$ for all $\theta > \theta^*$ implies that no type θ greater than θ^* will bid in market s at the limit. Therefore, $\lim F_s^n(1|v) - F_s^n(\theta^*|v) = F_s(1|v) - F_s(\theta^*|v) = 0$. Let $a_s(\theta)$ be such that $F_s(\theta|v) = \int_{[0,\theta]} a_s(\theta) dF(\theta|v)$ and note that such a function exists because F_s is absolutely continuous with respect to F. Choosing a function $\alpha(\theta)$ which is equal to zero for all $\theta \leq \theta_s(1)$ and equal to $a_s(\theta)$ for all $\theta > \theta_s(1)$ we obtain

$$\begin{aligned}
\lim \frac{F_s^n(1|1) - F_s^n(\theta_s^n(1)|1)}{F_s^n(1|0) - F_s^n(\theta_s^n(1)|0)} &= \frac{F_s(1|1) - F_s(\theta_s(1)|1)}{F_s(1|0) - F_s^n(\theta_s(1)|0)} \\
&= \frac{\int_{[\theta_s(1),\theta^*]} a_s(\theta) dF(\theta|1)}{\int_{[\theta_s(1),\theta^*]} a_s(\theta) dF(\theta|0)} \\
&= \frac{\kappa_s}{\int_{[\theta_s(1),\theta^*]} a_s(\theta) dF(\theta|0)} \\
&= \frac{\int_{[0,\theta^*]} \alpha(\theta) dF(\theta|1)}{\int_{[0,\theta^*]} \alpha(\theta) dF(\theta|0)} \\
&\leq W(\kappa_s, \theta^*) < 1
\end{aligned}$$

where the third equality is satisfied by the definition of $\theta_s(1)$, the fourth equality is satisfied by the definition of the function α and the final inequality is satisfied because the function α is feasible in the maximization problem. Therefore, $\int_{[\theta_s(1),\theta^*]} a_s(\theta) dF(\theta|0) > \kappa$ and hence $\theta_s(0) > \theta_s(1)$. However, $\theta_s(0) > \theta_s(1)$ implies that $\theta_s^n(0) - \theta_s^n(1) > 0$ for all n sufficiently large.

We now argue that if $\kappa_s < \kappa^*(\theta^*)$ and $F_s(1|1) > \kappa_s$, then $\theta_s(0) \leq \theta_s(1)$. Suppose that θ' is such that $F(1|1) - F(\theta'|1) = y + x$ where $y = 1 - F(\theta^*|1)$. Consider the following minimization problem

$$W(y, x) = \min_{\alpha \in L_1} \frac{\int_{[\theta',1]} \alpha(\theta) dF(\theta|1)}{\int_{[\theta',1]} \alpha(\theta) dF(\theta|0)}$$
$$s.t. \int_{[\theta',1]} \alpha(\theta) dF(\theta|1) = x$$

MLRP implies that the optimized value of the above program is as follows:

$$W(y,x) = \frac{F(\theta^*|1) - F(\theta'|1)}{F(\theta^*|0) - F(\theta'|0)}$$

Also, the definition of $\kappa^*(\theta^*)$ implies that if $x < \kappa^*(\theta^*)$, then $W(y,x) \geq 1$. Moreover, if strict MLRP is satisfied, then $W(y,x) > 1$. Therefore, if $\kappa_s < \kappa^*(\theta^*)$, then $W(y,\kappa_s) \geq 1$ where the inequality is strict if strict MLRP holds. However, Let $a_s(\theta)$ be such that $F_s(\theta|v) = \int_{[0,\theta]} a_s(\theta)dF(\theta|v)$ and note that such a function exists because F_s is absolutely continuous with respect to F. We choose a function $\alpha(\theta)$ which is equal to zero for all $\theta \leq \theta_s(1)$ and equal to $a_s(\theta)$ for all $\theta > \theta_s(1)$. Observing that $\theta' \leq \theta_s(1)$ and assuming that strict MLRP is satisfied we obtain

$$\begin{aligned}
\frac{F_s(1|1) - F_s(\theta_s(1)|1)}{F_s(1|0) - F_s(\theta_s(1)|0)} &= \frac{\int_{[\theta_s(1),1]} a_s(\theta)dF(\theta|1)}{\int_{[\theta_s(1),1]} a_s(\theta)dF(\theta|0)} \\
&= \frac{\int_{[\theta',1]} \alpha(\theta)dF(\theta|1)}{\int_{[\theta',1]} \alpha(\theta)dF(\theta|0)} \\
&\geq W(y,\kappa_r) > 1
\end{aligned}$$

where the last inequality follows from the same logic as above. Thus we have $\theta_s(1) > \theta_s(0)$ if strict MLRP is satisfied and $\theta_s(1) \geq \theta_s(0)$ if weak MLRP is satisfied. □

D. Exogenous Outside Option Proofs

In this section we provide proofs for all our findings that relate to exogenous outside options except for the equilibrium construction which we do in subsection F.2.

D.1. Information Aggregation Failure.

Proof of Theorem 3.1, item i. On the way to a contradiction suppose that an equilibrium sequence $\mathbf{H}$ aggregates information in auction s.

If $\mathbf{H}$ aggregates information, then $u^n(s,b|\theta) \to 0$ for all θ and all b. Let θ' be the type defined in the theorem, i.e., $\theta' = \inf\{\theta : \mathbb{E}[u(r|V)|\theta_i = \theta] > 0\}$. The fact that $u^n(s,b|\theta) \to 0$ for all θ and all b implies that all types $\theta > \theta'$ would opt for the outside option. This is because $\mathbb{E}[u(r|V)|\theta_i = \theta] > 0$ for all $\theta > \theta'$ by MLRP. However, the fact that $\kappa_s > \kappa^*(\theta')$ implies that $\theta_s(0) > \theta_s(1)$ or $F_s(1|1) < \kappa_s$ because of Lemma C.3.

Note that if $F_s(1|1) < \kappa_s$, then information is not aggregated because the price is equal to zero with probability one whenever $v = 1$ because there are more goods than the expected number of bidders.

We now show $\theta_s(0) > \theta_s(1)$ is not compatible with information aggregation. Pick $\epsilon > 0$ such that $\theta_s(0) - \epsilon > \theta_s(1) + \epsilon$. We know that $\lim \Pr(P_s^n \geq b_s^n(\theta_s(0) - \epsilon)|V = 0) = 1$ and $\lim \Pr(P_s^n \leq b_s^n(\theta_s(1) + \epsilon)|V = 1) = 1$ by Lemma B.1. However, $\lim b_s^n(\theta_s(1) + \epsilon) = 1 > \lim b_s^n(\theta_s(0) - \epsilon) = 0$ because $P_s^n \to V$. However, $\lim b_s^n(\theta_s(1) + \epsilon) = 1 > \lim b_s^n(\theta_s(0) - \epsilon) = 0$ contradicts Lemma 2.1, i.e., contradicts the bidding function being nondecreasing. □

Proof of Theorem 3.1, item ii. Let $x := \kappa - \bar{F}(\theta_s^F(1)|0)$. Note that $\theta_s^F(1)$ is the pivotal type under the assumption that all types are in a market. Therefore, x is independent of n and any equilibrium under consideration. Moreover, the constant $x > 0$ due to MLRP. Let

$$\bar{u} := \frac{l(\theta_i = 0)}{1 + 2l(\theta_i = 0)} x \tag{D.1}$$

Fix an equilibrium sequence $\mathbf{H}$. We will argue below that if $2\sqrt{Var[u(r|V)]} < \bar{u}$, then $\mathbf{H}$ aggregates information.

In any equilibrium, the set of types that choose r is a subset of $\{\theta \geq \theta'\}$. This is because $\mathbb{E}[u(r|V)|\theta_i = \theta] < 0$ for any $\theta < \theta'$. Below we argue that our maintained assumption of $\kappa_s < \kappa^*(\theta')$ implies that $F_s(1|1) > \kappa_s$. Also, strict MLRP, $\kappa_s < \kappa^*(\theta')$ and $F_s(1|1) > \kappa_s$ together imply that $\theta_s(1) > \theta_s(0)$ by Lemma C.3. Claims D.1 - D.3 in the proof of Proposition 3.2 (further below in the next subsection) imply that there cannot be pooling by pivotal types if $2\sqrt{Var[u(r|V)]} < \bar{u}$. However, $\theta_s(1) > \theta_s(0)$ and no pooling by pivotal types together imply that information is aggregated by Lemma B.4, item *ii*.

We now show that $\kappa_s < \kappa^*(\theta')$ implies $F_s(1|1) > \kappa_s$. The fact that $\kappa_s < \kappa^*(\theta')$ and the definition of $\kappa^*(\theta')$ together imply that there is a certain type $\theta'' < \theta'$ such that

$$\kappa^*(\theta') = F(\theta'|1) - F(\theta''|1) = F(\theta'|0) - F(\theta''|0) > \kappa_s.$$

As we mentioned above, the set of types that choose r is a subset of $\{\theta \geq \theta'\}$ and hence all types $\theta < \theta'$ choose market s. This implies that

$$\bar{F}_s(\theta''|1) > \kappa_s$$

which establishes that $F_s(1|1) > \kappa_s$. □

D.2. Characterization.

Lemma D.1 (Another Single Crossing Lemma)**.** *Assume that MLRP is satisfied, players have access to outside options as defined by Definition 3.1 and* $u(r|V = 0) > 0$. *If* $a(\theta') > 0$, *then* $a(\theta) = 1$ *for all* $\theta > \theta'$.

Proof. Let $\underline{\theta} = \sup(\theta : \bar{F}_s(\theta|v) = \bar{F}_s(0|v))$. Pick any $\theta' > \underline{\theta}$ with $a(\theta') > 0$. First, we note that $b(\theta') > 0$ since there cannot be pooling at a bid equal to zero. This is because if there was pooling at zero any type pooling at zero would have an incentive to outbid the pooling bid and win with probability one conditional on the price being equal to the pooling bid.

Let Y denote the event that the price is less than or equal to $b(\theta')$ and bidder i who bids $b(\theta')$ wins an object. Note that $\mathbb{E}[P_s|Y, v] > 0$ because $F_s(\theta'|v) - F_s(\underline{\theta}|v) > 0$ by the definitions of θ' and $\underline{\theta}$ and because $b(\theta) > 0$ for any $\theta \in (\underline{\theta}, \theta']$. The fact that type θ' bids $b(\theta')$ in the auction instead of choosing the outside option implies the following inequality

$$\begin{aligned} u(s, b(\theta')|\theta') - \mathbb{E}[u(r|V)|\theta'] = \\ &- \Pr(V = 0|\theta')(\Pr(Y|V = 0)(\mathbb{E}[P_s|Y, V = 0] + u(r|0))+ \\ &\quad (\Pr(Y|V = 1)(1 - \mathbb{E}[P_s|Y, V = 1]) - u(r|1)) \Pr(V = 1|\theta') \geq 0. \end{aligned}$$

Note that because $-(\Pr(Y|V=0)\mathbb{E}[P_s|Y,V=0]+u(r|0))<0$ we must have $(\Pr(Y|V=1)(1-\mathbb{E}[P_s|Y,V=1])-u(r|1))>0$ for the above inequality to be satisfied. MLRP implies that if $\theta>\theta'$ and $\theta\notin\mathcal{E}(\theta')$, then

$$\begin{aligned}
0 \le &-\Pr(V=0|\theta')(\Pr(Y|V=0)\mathbb{E}[P_s|Y,V=0]+u(r|0))+\\
&\qquad(\Pr(Y|V=1)(1-\mathbb{E}[P_s|Y,V=1])-u(r|1))\Pr(V=1|\theta')\\
&< -\Pr(V=0|\theta)(\Pr(Y|V=0)\mathbb{E}[P_s|Y,V=0]+u(r|V=0))+\\
&\qquad(\Pr(Y|V=1)(1-\mathbb{E}[P_s|Y,V=1])-u(r|1))\Pr(V=1|\theta)\\
&\qquad\qquad= u(s,b(\theta')|\theta)-\mathbb{E}[u(r|V)|\theta]
\end{aligned}$$

Therefore, if any θ' bids in the auction, then any $\theta>\theta'$ with $\theta\notin\mathcal{E}(\theta')$ would also choose to bid in the auction. This establishes that entry into the auction has a cutoff structure. □

Lemma D.2. *Assume that MLRP is satisfied and players have access to outside options as defined by Definition 3.1. Suppose that $a(\theta')>0$ implies $a(\theta)=1$ for all $\theta>\theta'$, i.e., entry into the auction has a cutoff structure. Then the bidding function $b(\theta)$ is strictly increasing and $b(\theta)=\beta_s^n(\theta)$.*

Proof. This follow immediately from the arguments that can be found in Pesendorfer and Swinkels (1997). □

Proof of Proposition 3.1. We argue that there is a type $\hat\theta^n$ such that $\mathbb{E}[v|Y^{n-1}(k)\le\theta,\theta]<\mathbb{E}[u(r|V)|\theta]$ for all $\theta<\hat\theta^n$ and $\mathbb{E}[V|Y^{n-1}(k)\le\theta,\theta]>\mathbb{E}[u(r|V)|\theta]$ for all $\theta>\hat\theta^n$. In particular, we will show that the function

$$f(\theta)=\frac{\mathbb{E}[V|Y^{n-1}(k)\le\theta,\theta]-\mathbb{E}[u(r|V)|\theta]}{\Pr(0|\theta)}=l(\theta)(\Pr(Y^{n-1}(k)\le\theta|1)-u(r|1))-u(r|0)$$

can cross zero at most once. Let $\hat\theta^n=\inf\{\theta: f(\theta)\ge 0\}$ and set $\hat\theta^n=1$ if the set is empty. Note that if $f(\theta')\ge 0$ for some θ', then $f(\theta)>0$ for all $\theta>\theta'$ because $l(\theta)$ is non decreasing and $\Pr(Y^{n-1}(k)\le\theta|1)$ is strictly increasing in θ. Therefore, $f(\theta)>0$ for all $\theta>\hat\theta^n$. A similar argument implies $f(\theta)<0$ for all $\theta<\hat\theta^n$; proving the result.

Claim. Recall that we showed in Lemma D.1 that entry has a cutoff structure. We will now argue that this cutoff $\underline{\theta}$ is equal to $\hat\theta^n$, i.e., all types above $\underline{\theta}=\hat\theta^n$ bid in the auction and all types below $\underline{\theta}=\hat\theta^n$ take the outside option.

Proof. Case *i*. Suppose that $\underline{\theta}>\hat\theta^n$ and therefore that $\mathbb{E}[V|Y^{n-1}(k)\le\underline{\theta},\underline{\theta}]>\mathbb{E}[u(r|V)|\underline{\theta}]$. If $\theta<\underline{\theta}$, then $a(\theta)=0$ for all θ and thus $u(s,b=0|\theta)=\mathbb{E}[V|Y^{n-1}(k)\le\underline{\theta},\theta]\le\mathbb{E}[u(r|V)|\theta]$ for all $\theta<\underline{\theta}$. Then we have

$$\mathbb{E}[V|Y^{n-1}(k)\le\theta,\theta]=\Pr(Y^{n-1}(k)\le\theta|V=1)\Pr(V=1|\theta)<\mathbb{E}[u(r|V)|\theta]$$

for all $\theta<\underline{\theta}$ because $\Pr(Y^{n-1}(k)\le\theta|V=1)$ is strictly increasing in θ. This however implies that $\mathbb{E}[V|Y^{n-1}(k)\le\theta,\theta]<\mathbb{E}[u(r|V)|\theta]$ for all $\theta<\underline{\theta}$ contradicting the definition of $\hat\theta^n$.

Case ii. Suppose that $\underline{\theta} < \hat{\theta}^n$ and therefore that $\mathbb{E}[V|Y^{n-1}(k) \leq \underline{\theta}, \underline{\theta}] < \mathbb{E}[u(r|V)|\underline{\theta}]$. If $\theta' > \underline{\theta}$, then $a(\theta') = 1$ and thus $u(s, b(\theta')|\theta') \geq \mathbb{E}[u(r|V)|\theta']$. Moreover, $u(s, b(\theta')|\theta) \geq \mathbb{E}[u(r|V)|\theta]$ for $\theta \geq \theta'$ because of MLRP (see Lemma D.1). Therefore, $\lim_{\theta' \downarrow \underline{\theta}} u(s, b(\theta')|\theta) = \mathbb{E}[V|Y^{n-1}(k) \leq \underline{\theta}, \theta] \geq \mathbb{E}[u(r|V)|\theta]$ for all $\theta > \underline{\theta}$. Consequently,

$$\mathbb{E}[V|Y^{n-1}(k) \leq \theta, \theta] = \Pr(Y^{n-1}(k) \leq \theta|V = 1) \Pr(V = 1|\theta) > \mathbb{E}[u(r|V)|\theta]$$

for all $\theta > \underline{\theta}$ because $\Pr(Y^{n-1}(k) \leq \theta|V = 1)$ is strictly increasing in θ. This however implies that $\mathbb{E}[V|Y^{n-1}(k) \leq \theta, \theta] > \mathbb{E}[u(r|V)|\theta]$ for all $\theta > \underline{\theta}$ contradicting the definition of $\hat{\theta}^n$. □

Note that the claim above and Lemma D.1 together prove item i of the proposition. Moreover, item ii of the proposition follows from Lemma D.2.

We now prove items iii and iv of the proposition. In particular, we show that $\lim \sqrt{n}(\kappa - F_s^n(1|1)) = \lim \sqrt{n}(\kappa - \bar{F}_s^n(\hat{\theta}^n|1)) = x \in (-\infty, \infty]$. Therefore $\kappa > \lim F_s^n(1|0)$ by MLRP. A direct calculation shows that there is a sufficiently small $\epsilon > 0$ such that if $\theta > \theta_s(1) - \epsilon$, then $\lim \beta_s^n(\theta) = 1$. If $\sqrt{n}(\kappa - \bar{F}_s^n(\hat{\theta}^n|1)) \to -\infty$, then the price would converge in probability to 1 if $v = 1$. This is because $\lim \Pr[Y_s^{n-1}(k) \geq \max\{\hat{\theta}^n, \theta_s(1) - \epsilon\}|1] = 1$ because $\lim \sqrt{n}(\kappa - \bar{F}_s^n(\hat{\theta}^n|1)) \to -\infty$ and $\lim \beta_s^n(\max\{\hat{\theta}^n, \theta_s(1) - \epsilon\}) = 1$. However, then no type would choose to bid in the auction because the payoff in the auction converges to zero. Therefore, $\lim \sqrt{n}(\kappa - \bar{F}_s^n(\hat{\theta}^n|1)) = x \in (-\infty, \infty]$ and the probability that there are more goods than bidders in market s in state $v = 1$ converges to a positive constant, i.e., the probability that the price is equal to zero if $v = 1$ is positive. Also, the fact that $\lim \hat{\theta}^n \geq \theta_s(1)$ implies that $\lim \beta_s^n(\theta) = 1$ for all $\theta > \lim \hat{\theta}^n$.

We now prove item v. If $v = 1$, then the price converges in probability to a binary random variable which is either equal to one or equal to zero. This is because all bids converge to 1 hence there are only two possible values that the price can take at the limit: the price is equal to one if the auction clears at the bid of any bidder and the price is equal to zero if there are fewer bidders than there are objects. Moreover, the price is equal to zero almost surely if $v = 0$. The calculation for the value of q ensures that the type $\hat{\theta}$ is indifferent between the two options and verifying the formula is straightforward. □

Proof of Corollary 3.2. Suppose that $u(r|0) > 0$. Note that Lemmata D.1 and D.2 do not depend on the assumption that $u(r|V = v)$ is nondecreasing in v and therefore continue to remain valid if $u(r|V = v)$ is nonincreasing v. Suppose instead that $u(r|0) \leq 0$. Then no type would select the outside option in equilibrium and therefore the two lemmata are again satisfied trivially. Therefore, the corollary follows directly from the proof given for Proposition 3.1 above. □

Proof of Proposition 3.2. Fix an equilibrium sequence **H**. Let $x := \kappa - \bar{F}(\theta_s^F(1)|0)$. Note that $\theta_s^F(1)$ is the pivotal type under the assumption that all types are in a market. Therefore, x is independent of n and the equilibrium under consideration. Moreover, the constant $x > 0$ due to MLRP. Recall that

$$\bar{u} := \frac{l(\theta_i = 0)}{1 + 2l(\theta_i = 0)} x$$

Assume that $\lim \sqrt{n}|\bar{F}_s^n(\theta_s^n(1)|v) - \bar{F}_s^n(\theta_s^n(0)|v)| \to \infty$. We will argue below that if $2\sqrt{Var[u(r|V)]} < \bar{u}$, then pooling by pivotal types (see Definition B.1) cannot be sustained. However, if there is no pooling by pivotal types and if $\lim \sqrt{n}|\bar{F}_s^n(\theta_s^n(1)|1) - \bar{F}_s^n(\theta_s^n(0)|1)| \to \infty$, then information is aggregated by Lemma B.4. Therefore, once we conclude that pooling cannot be sustained, this conclusion and Theorem 3.1's finding that information is not aggregated together imply that $\lim \sqrt{n}|\bar{F}_s^n(\theta_s^n(1)|1) - \bar{F}_s^n(\theta_s^n(0)|1)| < \infty$. Suppose that there is a pooling region. As in Definition B.1, let $\underline{\theta}_p^n = \inf\{\theta : b_m^n(\theta) = b_p^n\} < \theta_p^n = \sup\{\theta : b_m^n(\theta) = b_p^n\}$, $b_p = \lim b_p^n$, and $\lim \theta_i^n = \theta_i$ for $i = 0, 1$.

Claim D.1. Suppose that $\lim \sqrt{n}|\bar{F}_s^n(\theta_s^n(1)|v) - \bar{F}_s^n(\theta_s^n(0)|v)| \to \infty$. If $\lim \Pr(P_s^n \leq b_p^n|V = 1) = 1$ and $\lim \Pr(P_s^n < b_p^n|V = 0) = 0$, then there cannot be pooling by pivotal types.

Proof. If $\lim \Pr(P_s^n \leq b_p^n|V = 1) = 1$, then the limit of the pooling price must satisfy the following inequality because otherwise all types who choose the outside option would instead prefer to bid just above the pooling bid and win an object with probability one instead of choosing the outside option:

$$\begin{aligned}
(1-b_p)\frac{f(1|\theta)}{f(1|\theta)+f(0|\theta)} - b_p\frac{f(0|\theta)}{f(1|\theta)+f(0|\theta)} &\leq u(r|1)\frac{f(1|\theta)}{f(0|\theta)+f(1|\theta)} + u(r|0)\frac{f(0|\theta)}{f(0|\theta)+f(1|\theta)} \\
(1-b_p)l(\theta_i = \theta) - b_p &\leq u(r|1)l(\theta_i = \theta) + u(r|0) \\
&\leq u(r|1)l(\theta_i = \theta)
\end{aligned}$$

Note that $|u(r|v)| \leq 2\sqrt{Var[u(r|V)]}$ for each v. Therefore,

$$b_p \geq (1 - 2\sqrt{Var[u(r|V)]})\frac{l(\theta_i = \theta)}{1 + l(\theta_i = \theta)} \geq (1 - 2\sqrt{Var[u(r|V)]})\frac{l(\theta_i = 0)}{1 + l(\theta_i = 0)}.$$

However, this lower bound on the pooling price implies that $b_p > -u(r|0)$ if $2\sqrt{Var[u(r|V)]} < \bar{u} < \frac{l(\theta_i=0)}{1+2l(\theta_i=0)}$ because

$$b_p \geq (1 - 2\sqrt{Var[u(r|V)]})\frac{l(\theta_i = 0)}{1 + l(\theta_i = 0)} > 2\sqrt{Var[u(r|V)]} > -u(r|0).$$

Note that any type $\theta > \theta_p$ that bids in market s wins an object with probability one and pays a price at least equal to b_p if $V = 0$. However, then we find that all $\theta > \theta_p$ would select market s by Corollary C.1 because $b_p > -u(r|V = 0)$. We now look at two cases to show that pooling by pivotal types is not possible under our initial assumptions that $\lim \Pr(P_s^n \leq b_p^n|V = 1) = 1$ and $\lim \Pr(P_s^n < b_p^n|V = 0) = 0$.

Case 1. $\lim \sqrt{n}|\bar{F}_s^n(\theta_p^n|v) - \bar{F}_s^n(\theta_s^n(0)|v)| < \infty$. If $\lim \sqrt{n}|\bar{F}_s^n(\theta_p^n|v) - \bar{F}_s^n(\theta_s^n(0)|v)| < \infty$, then the type $\hat{\theta}_s$ defined in Definition B.3 is equal to θ_p (intuitively, $\hat{\theta}_s$ is the smallest type that wins an object in state 0 in market s). Moreover, any type $\theta > \theta_p$ selects market s by the argument in the previous paragraph. This however, leads to a contradiction because if all $\theta > \theta_p$ select market s and if $\lim |\bar{F}_s^n(\theta_p^n|v) - \bar{F}_s^n(\theta_s^n(0)|v)| = 0$, then $\theta_s(1) > \theta_s(0)$ by MLRP and hence $\theta_s(1) > \theta_p$ contradicting that $\lim \Pr(P_s^n \leq b_p^n|V = 1) = 1$.

Case 2. $\lim \sqrt{n}|\bar{F}_s^n(\theta_p^n|v) - \bar{F}_s^n(\theta_s^n(0)|v)| = \infty$. In this case, we argue below that the probability of winning a good at pooling is at least $x > 0$ if $v = 0$. Type $\hat{\theta}_s$ is equal to $\underline{\theta}_p$ and all types above $\hat{\theta}_s$ win an object with probability at least x in market s at a price which is at least equal to b_p. Then however, all $\theta > \hat{\theta}_s$ would select market s by Corollary C.1 because $b_p x > -u(r|V=0)$. This would however imply that information is aggregated in the auction by Lemma B.5 which leads us to a contradiction.

Continuing with Case 2, we now argue that the expected fraction of objects remaining at pooling if $v=0$ is at least x and therefore the probability of winning an object by bidding the pooling bis at least x if $v=0$. Note that $\lim \Pr(P_s^n \leq b_p^n|V=1) = 1$ implies that $\kappa_s - \bar{F}_s(\theta_p|V=1) \geq 0$ by the LLN. The fact that all types $\theta > \theta_p$ select market s implies that $\kappa_s - \bar{F}_s(\theta_p|V=1) = \kappa_s - \bar{F}(\theta_p|V=1) \geq 0$. Therefore, we find $\theta_p \geq \theta_s^F(1)$. MLRP implies that $\kappa_s - \bar{F}(\theta_p|V=1) > \kappa_s - \bar{F}(\theta_p|v=0)$. Also, $\theta_p \geq \theta_s^F(1)$ implies that $\kappa_s - \bar{F}(\theta_p|V=0) \geq \kappa_s - \bar{F}(\theta_s^F(1)|V=0) = x$ hence the expected fraction of goods left over to pooling is at least x. The expected fraction of bidders who submit the pooling bid is at most 1. Therefore, the fact that the probability of winning at pooling is at least x follows from the argument in Lemma B.2. □

Claim D.2. Suppose that $\lim \sqrt{n}|\bar{F}_s^n(\theta_s^n(1)|v) - \bar{F}_s^n(\theta_s^n(0)|v)| \to \infty$. If there is pooling by pivotal types, then $\lim \Pr(P_s^n \leq b_p^n|V=1) = 1$.

Proof. We prove this claim by looking at two mutually exclusive cases.

Case 1: $\lim \sqrt{n}|\bar{F}_s^n(\theta_p^n|1) - \bar{F}_s^n(\theta_s^n(1)|1)| \to \infty$. If $\lim \sqrt{n}|\bar{F}_s^n(\theta_p^n|1) - \bar{F}_s^n(\theta_s^n(1)|1)| \to \infty$, then $\lim \Pr(P_s^n \leq b_p^n|V=1) = 1$ because of our maintained assumption (in the definition of pooling by pivotal types) that $\lim \Pr(P_s^n = b_p^n|1) > 0$.

Case 2: $\lim \sqrt{n}|\bar{F}_s^n(\theta_p^n|1) - \bar{F}_s^n(\theta_s^n(1)|1)| = \Delta < \infty$. Note $\lim \sqrt{n}|\bar{F}_s^n(\theta_p^n|1) - \bar{F}_s^n(\theta_s^n(1)|1)| = \Delta < \infty$ and $\lim \sqrt{n}|\bar{F}_s^n(\theta_s^n(0)|1) - \bar{F}_s^n(\theta_s^n(1)|1)| = \infty$ imply that $\lim \sqrt{n}|\bar{F}_s^n(\theta_p^n|1) - \bar{F}_s^n(\theta_s^n(0)|1)| = \infty$ and hence that $\lim \Pr(P_s^n < b_p^n|V=1) = 0$. Therefore, $\liminf \sqrt{n}(\bar{F}_s^n(\theta_s^n(0)|0) - \bar{F}_s^n(\theta_p^n|0)) \leq \liminf \sqrt{n}(\kappa_s - \bar{F}_s^n(\theta_p^n|0)) = +\infty$. Also, $\limsup \sqrt{n}(\kappa_s - \bar{F}_s^n(\theta_p^n|1)) \leq a$ for some finite constant a.[42]

We now prove that

$$\lim \frac{\Pr(b_p^n \text{ } wins|P^n = b_p^n, V=1)}{\Pr(b_p^n \text{ } wins|P^n = b_p^n, V=0)} = 0$$

and therefore any bidder would prefer slightly outbidding the pooling bid. Let X^n denote the total number of bidders who submit the pooling bid, let $\bar{X}_v^n = \mathbb{E}[X^n|V=v]$, and let G^n denote the total number of goods left to be allocated to bidders who submit the pooling bid, i.e, G^n is equal to k minus the number of bidders who submit a bid greater than the pooling bid.

Below we provide an upper bound for the posterior likelihood ratio that conditions on

[42]Note that $\lim \sqrt{n}(\bar{F}_s^n(\theta_s^n(0)|0) - \bar{F}_s^n(\theta_p^n|0)) = +\infty$ implies that $\lim \sqrt{n}(\bar{F}_s^n(\theta_s^n(0)|v) - \bar{F}_s^n(\theta_s^n(1)|v)) = +\infty$. Therefore, $\lim \bar{F}_s^n(\theta_s^n(1)|1) \geq \kappa_s$. Because if $\lim \bar{F}_s^n(\theta_s^n(1)|1) < \kappa_s$, then $\theta_s^n(1) = \theta_s^n(0) = 0$ for all n sufficiently large contradicting that $\lim \sqrt{n}(\bar{F}_s^n(\theta_s^n(0)|v) - \bar{F}_s^n(\theta_s^n(1)|v)) = +\infty$.

winning an object at a bid equal to the pooling bid:

$$\begin{aligned}
& \lim \frac{\Pr(b_p^n\ wins|P_s^n = b_p^n, V = 1)}{\Pr(b_p^n\ wins|P_s^n = b_p^n, V = 0)} && \text{(D.2)}\\
= \quad & \lim \frac{\mathbb{E}[G^n/X^n|P_s^n = b_p^n, V = 1]}{\mathbb{E}[G^n/X^n|P_s^n = b_p^n, V = 0]} && \text{(D.3)}\\
\leq \quad & \lim \frac{\mathbb{E}[\frac{G^n}{(1-\delta)\bar{X}_1^n}|P_s^n = b_p^n, V = 1, X^n \geq (1-\delta)\bar{X}_1^n] + \Pr[X^n \leq (1-\delta)\bar{X}_1^n|P_s^n = b_p^n, V = 1]}{\mathbb{E}[\frac{G^n}{(1+\delta)\bar{X}_0^n}|P_s^n = b_p^n, V = 0, X^n \leq (1+\delta)\bar{X}_0^n] \Pr[X^n \leq (1+\delta)\bar{X}_0^n|P_s^n = b_p^n, V = 0]} && \text{(D.4)}\\
= \quad & \lim \frac{\mathbb{E}[\frac{G^n}{(1-\delta)\bar{X}_1^n}|P_s^n = b_p^n, V = 1] + \Pr[X^n \leq (1-\delta)\bar{X}_1^n|P_s^n = b_p^n, V = 1]}{\mathbb{E}[\frac{G^n}{(1+\delta)\bar{X}_0^n}|P_s^n = b_p^n, V = 0] \Pr[X^n \leq (1+\delta)\bar{X}_0^n|P_s^n = b_p^n, V = 0]} && \text{(D.5)}\\
\leq \quad & \lim \frac{\mathbb{E}[G^n|P_s^n = b_p^n, V = 1] + (1-\delta)\bar{X}_0^n e^{-\frac{\delta^2 \bar{X}_1^n}{2}}}{\mathbb{E}[G^n|P_s^n = b_p^n, V = 0]} \left(\frac{(1+\delta)\bar{X}_0^n}{(1-\delta)\bar{X}_1^n}\right) && \text{(D.6)}\\
\leq \quad & \frac{(1+\delta)}{(1-\delta)l(\theta_i = 0)} \lim \frac{\mathbb{E}[G^n|P_s^n = b_p^n, V = 1] + (1-\delta)\bar{X}_1^n e^{-\frac{\delta^2 \bar{X}_1^n}{2}}}{\mathbb{E}[G^n|P_s^n = b_p^n, V = 0]} && \text{(D.7)}\\
= \quad & \frac{(1+\delta)}{(1-\delta)l(\theta_i = 0)} \lim \frac{\mathbb{E}[G^n|V = 1]/\sqrt{n}}{\Pr[P_s^n = b_p^n|V = 1]\mathbb{E}[G^n|P_s^n = b_p^n, V = 0]/\sqrt{n}} && \text{(D.8)}\\
= \quad & 0 && \text{(D.9)}
\end{aligned}$$

The numerator in inequality D.4 is obtained by observing that $X^n \geq (1-\delta)\bar{X}_1^n$ in each term in the expectation, $\Pr[X^n \geq (1-\delta)\bar{X}_1^n|P_s^n = b_p^n, V = 1] \leq 1$ and $\mathbb{E}[\frac{G^n}{X^n}|P_s^n = b_p^n, V = 1, X^n \leq (1-\delta)\bar{X}_1^n] \leq 1$. The reasoning for the denominator in inequality D.4 is similar. Note that for any $\delta \in (0,1)$, $\Pr[X^n \leq (1-\delta)\bar{X}_1^n|V = 1] \leq e^{-\frac{\delta^2 \bar{X}_1^n}{2}}$ and $\Pr[X^n \geq (1+\delta)\bar{X}_0^n|V = 1] \leq e^{-\frac{\delta^2 \bar{X}_0^n}{2+\delta}}$ by Chernoff's Inequality (see Lemma A.1). Therefore, $\lim \Pr[X^n \geq (1-\delta)\bar{X}_1^n|V = 1] = 1$ and $\lim \Pr[X^n \leq (1+\delta)\bar{X}_0^n|V = 1] = 1$. Equality D.5 is obtained from D.4 by noticing that $\lim \Pr[X^n \geq (1-\delta)\bar{X}_1^n|P_s^n = b_p^n, V = 1] = 1$ because $\lim \Pr[X^n \geq (1-\delta)\bar{X}_1^n|V = 1] = 1$ and $\lim \Pr[P_s^n = b_p^n|V = 1] > 0$. Therefore,

$$\lim \frac{\mathbb{E}[\frac{G^n}{(1-\delta)\bar{X}_1^n}|P_s^n = b_p^n, V = 1]}{\mathbb{E}[\frac{G^n}{(1-\delta)\bar{X}_1^n}|P_s^n = b_p^n, V = 1, X^n \geq (1-\delta)\bar{X}_1^n]} = 1$$

for the term in the numerator and the reasoning for the denominator is similar. Inequality D.6 follows from inequality D.5 via Chernoff's inequality introduced in Lemma A.1. Inequality D.7 follows because we have $\frac{(1+\delta)\bar{X}_0^n}{(1-\delta)\bar{X}_1^n} \leq \frac{1}{l(\theta_i=0)}$ by MLRP. Inequality D.8 follows from D.7 since we have

$$\begin{aligned}
\lim \mathbb{E}[G^n|V = 1] = \lim \Pr[P_s^n < b_p^n|V = 1]\mathbb{E}[G^n|P_s^n < b_p^n, V = 1] \\
+ \lim \Pr[P_s^n = b_p^n|V = 1]\mathbb{E}[G^n|P_s^n = b_p^n, V = 1] \\
+ \lim \Pr[P_s^n > b_p^n|V = 1]\mathbb{E}[G^n|P_s^n > b_p^n, V = 1] \\
= \lim \Pr[P_s^n = b_p^n|V = 1]\mathbb{E}[G^n|P_s^n = b_p^n, V = 1]
\end{aligned}$$

because $\mathbb{E}[G^n|P_s^n > b_p^n, V = 1] = 0$ by definition and $\Pr[P_s^n < b_p^n|V = 1] \to 0$ because of our initial contradiction assumption $\lim \sqrt{n}|\bar{F}_s^n(\theta_p^n|1) - \bar{F}_s^n(\theta_s^n(1)|1)| = \Delta < \infty$. Dividing both the numerator and denominator by $\sqrt{n}$ and evaluating the limit $\lim(1-\delta)\frac{\bar{X}_1^n}{\sqrt{n}}e^{-\frac{\delta^2 \bar{X}_1^n}{2}} = 0$ delivers inequality D.8. Finally, we reach the conclusion in equation D.9 because we have $\mathbb{E}[G^n|V = 1]/\Pr[P_s^n = b_p^n|V = 1]\sqrt{n} \leq a < \infty$ since $\limsup \sqrt{n}(\kappa - \bar{F}_s^n(\theta_p^n|1)) \leq a$ for some a and because $\lim \Pr[P_s^n = b_p^n|V = 1] > 0$ by assumption. Moreover, we now argue that $\mathbb{E}[G^n|P_s^n = b_p^n, V = 0]/\sqrt{n} \to \infty$. This is because $\liminf \sqrt{n}(\kappa - \bar{F}_s^n(\theta_p^n|0)) = +\infty$ implies that $\lim \Pr[G^n \geq A\sqrt{n}|V = 0] = 1$ for all $A > 0$ and therefore $\lim \Pr[G^n \geq A\sqrt{n}|P_s^n = b_p^n, V = 0] = 1$ for all $A > 0$. Hence, we find $\lim \mathbb{E}[G^n|P_s^n = b_p^n, V = 0]/\sqrt{n} \geq A$ for any $A > 0$. $\square$

Claim D.3. Suppose that $\lim \sqrt{n}|\bar{F}_s^n(\theta_s^n(1)|v) - \bar{F}_s^n(\theta_s^n(0)|v)| \to \infty$. If there is pooling by pivotal types, then $\lim \Pr(P_s^n < b_p^n|v = 0) = 0$. This claim and the previous two claims together imply that there cannot be pooling by pivotal types. Therefore, $\lim \sqrt{n}|\bar{F}_s^n(\theta_s^n(1)|1) - \bar{F}_s^n(\theta_s^n(0)|1)| < \infty$.

Proof. We will prove this claim by looking at two mutually exclusive cases.

Case 1:$\lim \sqrt{n}|\bar{F}_s^n(\underline{\theta}_p^n|0) - \bar{F}_s^n(\theta_s^n(0)|0)| \to \infty$. In this case $\lim \sqrt{n}|\bar{F}_s^n(\underline{\theta}_p^n|0) - \bar{F}_s^n(\theta_s^n(0)|0)| \to \infty$ and our initial assumption that $\lim \Pr(P_s^n = b_p^n|0) > 0$ together imply that $\lim \Pr(P_s^n < b_p^n|0) = 0$.

Case 2: $\lim \sqrt{n}|\bar{F}_s^n(\underline{\theta}_p^n|0) - \bar{F}_s^n(\theta_s^n(0)|0)| \to \Delta < \infty$. Note that $\lim \sqrt{n}|\bar{F}_s^n(\underline{\theta}_p^n|0) - \bar{F}_s^n(\theta_s^n(0)|0)| \to \Delta < \infty$ and $\lim \sqrt{n}|\bar{F}_s^n(\theta_s^n(0)|1) - \bar{F}_s^n(\theta_s^n(1)|1)| \to \infty$ imply that $\lim \sqrt{n}|\bar{F}_s^n(\underline{\theta}_p^n|1) - \bar{F}_s^n(\theta_s^n(1)|1)| \to \infty$. Therefore, $\liminf \sqrt{n}|\bar{F}_s^n(\underline{\theta}_p^n|1) - \bar{F}_s^n(\theta_s^n(1)|1)| = \liminf \sqrt{n}(\bar{F}_s^n(\underline{\theta}_p^n|1) - \kappa_s) = +\infty$. Also, $\limsup \sqrt{n}(\bar{F}_s^n(\underline{\theta}_p^n|0) - \kappa_s) \leq a$ for some positive a. Also, note $\lim \Pr[P_s^n = b_p^n|1] = 1$ because $\lim \Pr[P_s^n \leq b_p^n|1] = 1$ by the previous claim and because $\lim \sqrt{n}|\bar{F}_s^n(\underline{\theta}_p^n|1) - \bar{F}_s^n(\theta_s^n(1)|1)| \to \infty$ and $\lim \Pr[P_s^n = b_p^n|1] > 0$ together imply $\lim \Pr[P_s^n < b_p^n|1] = 0$.

We now prove that

$$\lim \frac{\Pr(b_p^n \ loses|P_s^n = b_p^n, V = 0)}{\Pr(b_p^n \ loses|P_s^n = b_p^n, V = 1)} = 0.$$

Let U^n denote the number of bidders who submit the pooling bid who do not get allocated an object, i.e., $U^n = X^n - G^n$. Below we provide an upper bound for the posterior likelihood ratio that conditions on winning an object at a bid equal to the pooling bid:

$$\begin{aligned}
& \lim \frac{\mathbb{E}[\frac{U^n}{X^n}|P_s^n = b_p^n, V=0]}{\mathbb{E}[\frac{U^n}{X^n}|P_s^n = b_p^n, V=1]} \\
= \ & \lim \frac{\Pr(b_p^n \ loses|P_s^n = b_p^n, V=0)}{\Pr(b_p^n \ loses|P_s^n = b_p^n, V=1)} && \text{(D.10)} \\
\leq \ & \lim \frac{\mathbb{E}[\frac{U^n}{(1-\delta)\bar{X}_0^n}|P_s^n = b_p^n, V=0, X^n \geq (1-\delta)\bar{X}_0^n] + \Pr[X^n \leq (1-\delta)\bar{X}_0^n|P_s^n = b_p^n, V=0]}{\mathbb{E}[\frac{U^n}{(1+\delta)\bar{X}_1^n}|P_s^n = b_p^n, V=1, X^n \leq (1+\delta)\bar{X}_1^n]\Pr[X^n \leq (1+\delta)\bar{X}_1^n|P_s^n = b_p^n, V=1]} && \text{(D.11)} \\
\leq \ & \lim \frac{\mathbb{E}[U^n|P_s^n = b_p^n, V=0] + (1-\delta)\bar{X}_0^n e^{-\frac{\delta^2 \bar{X}_0^n}{2}}}{\mathbb{E}[U^n|P_s^n = b_p^n, V=1]} \left(\frac{(1+\delta)\bar{X}_1^n}{(1-\delta)\bar{X}_0^n}\right) && \text{(D.12)} \\
\leq \ & \frac{(1+\delta)l(\theta_i=0)}{(1-\delta)} \lim \frac{\mathbb{E}[U^n|P_s^n = b_p^n, V=0] + (1-\delta)\bar{X}_0^n e^{-\frac{\delta^2 \bar{X}_0^n}{2}}}{\mathbb{E}[U^n|P_s^n = b_p^n, V=1]} && \text{(D.13)} \\
= \ & \frac{(1+\delta)l(\theta_i=0)}{(1-\delta)} \lim \frac{\mathbb{E}[U^n|P_s^n = b_p^n, V=0]/\sqrt{n}}{\mathbb{E}[U^n|P_s^n = b_p^n, V=1]/\sqrt{n}} \\
= \ & 0
\end{aligned}$$

The numerator in inequality (D.11) is obtained by observing that $X^n \geq (1-\delta)\bar{X}_0^n$ in each term in the expectation, $\Pr[X^n \geq (1-\delta)\bar{X}_0^n|P_s^n = b_p^n, V=1] \leq 1$ and $\mathbb{E}[\frac{U^n}{X^n}|P_s^n = b_p^n, V = 0, X^n \leq (1-\delta)\bar{X}_0^n] \leq 1$. The reasoning for the denominator in inequality D.11 is similar. Inequality D.12 is obtained via Chernoff's inequality using a logic analogous to the argument for $\lim \Pr(P_s^n \leq b_p^n|V=1) = 1$. Inequality (D.13) follows because we have $\frac{(1+\delta)\bar{X}_1^n}{(1-\delta)\bar{X}_0^n} \geq \frac{1}{l(\theta_i=0)}$ by MLRP. Finally, we reach the conclusion in equation D.9 because we have $\liminf \sqrt{n}(\bar{F}_s^n(\underline{\theta}_p^n|1) - \kappa_s) = +\infty$ and $\lim \Pr(P_s^n = b_p^n|V=1) = 1$ together imply that

$$\frac{\lim \mathbb{E}[U^n|P_s^n = b_p^n, V=1]}{\sqrt{n}} = \frac{\lim \mathbb{E}[U^n|V=1]}{\sqrt{n}} \geq \liminf \frac{n(\bar{F}_s^n(\underline{\theta}_p^n|1) - \kappa_s)}{\sqrt{n}} = +\infty$$

showing that the denominator diverges. Moreover, $\limsup \sqrt{n}(\bar{F}_s^n(\underline{\theta}_p^n|0) - \kappa_s) \leq a$ for some a implies that $\lim \Pr[U^n \leq \sqrt{n}a|0] = 1$. However, $\lim \Pr[U^n \leq \sqrt{n}a|V=0] = 1$ and $\Pr[P_s^n = b_p^n|V=0] > 0$ together imply that $\lim \Pr[U^n \leq \sqrt{n}a|V=0] = 1$. Therefore, we find that $\lim \mathbb{E}[U^n|P_s^n = b_p^n, V=1]/\sqrt{n} \leq a$ showing that the numerator converges to at most a and thus establishing the bound. □

Claim D.4. $F_s(1|v) \geq \kappa_s$ for $v = 0, 1$.

Proof. Assume not, i.e., that $F_s(1|1) < \kappa_s$. Note that $F_s(1|1) < \kappa_s$ implies that $F_s(1|0) < \kappa_s$ because the pivotal types are arbitrarily close by the argument above. However, if $F_s(1|v) < \kappa_s$ for $v = 0, 1$, then all types would opt for market s because $u(r|1) < \bar{u} < 1$ and because the price in market s is equal to zero with probability one in both states. □

Claim D.5. If $\mathbb{E}[u(r|V)|\theta_i = 0] < 0$, then $F_s(1|v) > \kappa_s$ for $v = 0, 1$.

Proof. Suppose to the contrary that $\mathbb{E}[u(r|V)|\theta_i = 0] < 0$ and $F_s(1|v) = \kappa_s$ for some v. If $F_s(1|v) = \kappa_s$ for some v, then $F_s(1|v) = \kappa_s$ for all v because $\theta_s(1) = \theta_s(0)$ by the argument further above. However, if $F_s(1|v) = \kappa_s$, then any type θ such that $F_s[\theta|v] > F_s[\theta_s(v)|v] = 0$ who bids in the auction wins an object with probability one in both states. To see this note that $F_s[\theta|v] > F_s[\theta_s(v)|v]$ implies that $\lim \Pr[P_s^n \leq b_s^n(\theta)|V = v] = 1$. Moreover, $\lim \Pr[b_s^n(\theta)\ wins|P_s^n \leq b_s^n(\theta), V = v] = 1$. This follows because the winning chances are smallest if this type submits a pooling bid but even in this case $F_s(1|v) = \kappa_s$ and Lemma B.2 together imply that $\lim \Pr[b_s^n(\theta)\ wins|P_s^n \leq b_s^n(\theta), V = v] = 1$. Therefore, such a type wins an object with certainty at a price equal to $\lim \mathbb{E}[P_s^n|v]$ in state $v = 0, 1$. However, if $\lim \mathbb{E}[P_s^n|0] < -u(r|0)$, then all $\theta' < \theta$ would select market s by Corollary C.1. However, this is incompatible with $F_s(1|v) = \kappa_s$, $v = 0, 1$. Similarly, if $\lim \mathbb{E}[P_s^n|V = 0] > -u(r|0)$, then all $\theta' > \theta$ would select market s by Corollary C.1. This is again incompatible with $F_s(1|v) = \kappa_s$, $v = 0, 1$. Therefore, $\lim \mathbb{E}[P_s^n|0] = -u(r|0)$. However, $\lim \mathbb{E}[P_s^n|0] = -u(r|0)$ implies that $\lim \mathbb{E}[1 - P_s^n|V = 1] = u(r|1)$ in order to ensure that there are types willing to select the options as assumed. However, if $\lim \mathbb{E}[P_s^n|V = 0] = -u(r|0)$ and $\lim \mathbb{E}[1 - P_s^n|V = 1] = u(r|1)$, then any type $\theta > 0$ such that $\mathbb{E}[u(r|1)|\theta_i = \theta] < 0$ has a negative equilibrium payoff. This is because any such type wins an object for sure in market s at a price equal to $\lim \mathbb{E}[P_s^n|V = v]$ in state $v = 0, 1$ and $\lim \mathbb{E}[v - P_s^n|\theta_i = \theta] = \mathbb{E}[u(r|V)|\theta_i = \theta] < 0$. $\square$

$\square$

D.3. Example where All Types Pool. The following example presents an example where the pivotal types *do* not converge to each other but information aggregation nevertheless fails because the pivotal types submit the same pooling bid. In the equilibrium that we construct, the failure of information aggregation is particularly severe as the auction clears at the same pooling price with probability one in both states. However, as we pointed out in Proposition 3.2, the existence of this type of pooling equilibria relies on $Var[u(r|V)]$ remaining large and such equilibria no longer exist when $2\sqrt{Var[u(r|V)]} < \bar{u}$. The mechanics of this example works is similar to the construction in Lauermann and Wolinsky (2014).

Example D.1. Assume that the signals satisfy Definition 3.3 (i.e., the signals are binary) and agents have access to an exogenous outside option with $u(r|0) = -c$ and $u(r|1) = c$. Below we construct an equilibrium sequence with the following properties:

i. There is a cutoff type θ^n such that all types $[0, \theta^n)$ bid in the auction and all types $[\theta^n, 1]$ take the outside option and moreover $\lim F(\theta^n|v) > \kappa$ for $v = 0, 1$

ii. All types submit the same pooling bid b^n, i.e., $b^n(\theta) = b^n$ for all $\theta \in [0, \theta^n)$,

iii. Let $b = \lim b^n$. The auction price P is equal to b for both $V = 0$ and $V = 1$ almost surely.

For the example assume that $\frac{f_h}{f_l} = 2$, $\frac{1-f_h}{1-f_l} = \frac{1}{2}$ and that $\kappa = \frac{1}{3}$. Let $c = 0.9$. Fix $b^n \in (0.44, 0.5)$. Suppose that all $\theta \leq 1/2$ bid the pooling bid b^n in the auction and all $\theta > 1/2$ opt for the outside option. There are two constraints that need to be satisfied in order for this to be an equilibrium. Below we argue that these constraints are satisfied for sufficiently large n

i. The first constraint requires that bidders do not want to outbid the pooling bid. Let $b = \lim b^n$. The probability of winning an object at the pooling bid converges to $1/2$ if $v = 0$ and 1 if $V = 1$ because the mass of people submitting the pooling bid converges to $2/3$ if $v = 0$ and $1/3$ if $V = 1$. Therefore, the pooling bid converges to a value which satisfies the following equality

$$-\frac{2}{3}\frac{1}{2}b + \frac{1}{3}(1-b) \geq 0.$$

This implies that the pooling price $b \leq 1/2$. Note that no bidder would want to outbid the pooling bid because they win an object with probability one at the pooling bid.

ii. The second constraint ensures that people self-select as described above. The value we have chosen for the outside option ensures that bidders who receive the high signal prefer the outside option to bidding in the auction. This is because their payoff from the outside option is equal to 0.3. In contrast, if they bid in the auction, the receive a payoff of

$$-\frac{1}{3}\frac{1}{2}b + \frac{2}{3}(1-b) = \frac{2}{3} - \frac{5}{6}b$$

at the limit when n is large. Note that $\frac{2}{3} - \frac{5}{6}b > 0.3$ if $b > 0.44$ as we have assumed. Also, note that types who receive the low signal would never choose the outside option because the expected value is negative for them.

iii. Note that this example is not a knife-edge example and can be sustained for many parameter value as long as $Var[u(r|V)]$ is sufficiently large. In particular, if $\kappa = 1/3 - \epsilon$, then a similar equilibrium could be sustained for sufficiently small ϵ.

E. Endogenous Outside Option Proofs

In this section we provide proofs for all our findings that relate to endogenous outside options except for the equilibrium construction which is in Section F.

E.1. Information Aggregation and Non-Aggregation Results

Proof of Theorem 4.1, Item i. Fix an equilibrium sequence $\mathbf{H}$. Assume, on the way to a contradiction, that information is aggregated in market s along this sequence. Note that $\lim_n \mathbb{E}\left[P_r^n|V=0\right] \geq c > 0$ by assumption. The fact that information is aggregated in market s implies that $\lim_n \mathbb{E}\left[P_s^n|V=0\right] = 0$ and that $\lim_n \mathbb{E}\left[P_s^n|V=1\right] = 1$. Note that if $\theta > \hat{\theta}_r$, then $\lim a_r^n(\theta) = 1$ by Lemma C.2 because $\lim_n \mathbb{E}\left[P_r^n|V=0\right] > 0$ and $\lim_n \mathbb{E}\left[P_s^n|V=0\right] = 0$. Let θ^* denote the type as introduced by Definition 4.2. Note that $\hat{\theta}_r \leq \theta^*$ because if $\hat{\theta}_r > \theta^*$, then $\lim_n \mathbb{E}\left[P_r^n|V=1\right] = c$ because $\theta_r^F(1) \leq \theta^*$ by Definition 4.2. However, if $\lim_n \mathbb{E}\left[P_r^n|V=1\right] = c$, then $\lim u^n(r, b_r^n(\theta)|\theta) > 0 = \lim u^n(s, b|\theta)$ for all b and all $\theta \in (\theta^*, \hat{\theta}_r)$ contradicting that $\hat{\theta}_r \leq \theta^*$.

Suppose that $F_s(1|1) \geq \kappa_s$. Lemma C.3 shows that if $\lim a_r^n(\theta) = 1$ for all $\theta > \theta^*$ and if $\kappa_s > \kappa^*(\theta^*)$, then $\theta_s^n(0) - \theta_s^n(1) \geq 0$ for all n sufficiently large. If $F_s(1|1) \geq \kappa_s$, then $\theta_s^n(0) - \theta_s^n(1) \geq 0$ for all n sufficiently large however contradicts our initial assumption that information is aggregated in market s. This is because information aggregation in market s

implies, by Lemma B.4, that $\theta_s^n(1) - \theta_s^n(0) > 0$ for all n sufficiently large. On the other hand if $F_s(1|1) < \kappa_s$, then $P_s^n \to 0$ in state $v = 1$ showing that information is not aggregated. $\square$

Proof of Theorem 4.1, Item ii. For any equilibrium sequence pick a subsequence $\mathbf{H}$ such that the sequence of numbers $\mathbb{E}[P_m^n|V = v]$ have a limit for each $m \in M$ and each v. We will show that for any such convergent sequence we have $\mathbb{E}[P_m^n|V = v] \to v$ for all $m \in M$ and all $v = 0, 1$ which, in turn, implies that $P_m^n \xrightarrow{p} V$ for $m \in M$.

Step 1. The equality $\lim \mathbb{E}[P_m^n|V = 0] = 0$ and the inequality $\lim \mathbb{E}[P_{m'}^n|V = 0] > 0$ cannot be jointly satisfied.

Let $B \subset M$ be the set of markets such that $\lim \mathbb{E}[P_{m'}^n|V = 0] > 0$. The fact $\lim \mathbb{E}[P_{m'}^n|V = 0] > 0$ implies that $F_{m'}(1|0) \geq \kappa_{m'}$ for all $m' \in B$ because otherwise $P_{m'}^n \xrightarrow{p} 0$ by the law of large numbers. The facts that $\lim \mathbb{E}[P_m^n|V = 0] = 0$ for $m \notin B$ and $\lim \mathbb{E}[P_{m'}^n|V = 0] > 0$ for $m' \in B$ together imply that $\lim a_B^n(\theta) = 1$ for all $\theta > \hat{\theta} := \min_{m' \in B} \hat{\theta}_{m'}$ by Lemma C.2. The fact that $\lim a_{m'}^n(\theta) = 1$ for all $\theta > \hat{\theta}$ implies that

$$F_B(1|1) - F_B(\hat{\theta}|1) = F(1|1) - F(\hat{\theta}|1) > F(1|0) - F(\hat{\theta}|0) = F_B(1|0) - F_B(\hat{\theta}|0).$$

This, in turn, implies that there is at least one market $m' \in M$ such that $F_{m'}(1|1) - F_{m'}(\hat{\theta}|1) > F_{m'}(1|0) - F_{m'}(\hat{\theta}_{m'}|0)$. For this market, $F_{m'}(1|1) - F_{m'}(\hat{\theta}|1) > F_{m'}(1|0) - F_{m'}(\hat{\theta}_{m'}|0)$ implies that $F_{m'}(1|1) > \kappa_{m'}$ because $F_{m'}(1|0) \geq \kappa_{m'}$. However, if $F_{m'}(1|1) - F_{m'}(\hat{\theta}|1) > F_{m'}(1|0) - F_{m'}(\hat{\theta}_{m'}|0)$ and if $F_{m'}(1|1) > \kappa_{m'}$, then we conclude, by Lemma B.5, that $P_{m'}^n \xrightarrow{p} 0$ if $v = 0$. This contradicts $\lim \mathbb{E}[P_{m'}^n|V = 0] > 0$.

Step 2. The inequality $\lim \mathbb{E}[P_m^n|V = 0] > 0$ for all $m \in M$ is not possible.

The inequalities imply that $F_m(1|0) \geq \kappa_m$ for all $m \in M$. Also, MLRP implies that there is at least one market where $F_{m'}(1|1) \geq F_{m'}(1|0)$ and $\theta_{m'}(1) > \theta_{m'}(0)$. The facts that $F_{m'}(1|0) \geq \kappa_{m'}$ and $\theta_{m'}(1) > \theta_{m'}(0)$ together imply that $F_{m'}(1|1) > \kappa_{m'}$. For this market, however, Lemma B.5, implies that $P_m^n \xrightarrow{p} 0$ in state $V = 0$ contradicting the assumption that $\lim \mathbb{E}[P_m^n|V = 0] > 0$.

Step 3. The above two steps have established that $\lim \mathbb{E}[P_m^n|0] = 0$ for all $m \in M$ in any equilibrium. We now show that $\lim \mathbb{E}[P_m^n|V = 0] = 0$ for all $m \in M$ implies that $P_m^n \xrightarrow{p} V$ for all $m \in M$.

The fact that $\lim \mathbb{E}[P_m^n|0] = 0$ implies that $\lim u^n(m, b_m^n(\theta)|\theta_i = \theta, V = 0) = 0$ for all θ. Moreover, $\lim u^n(m, b_m^n(1)|\theta_i = 1, V = 1) = 1 - \lim \mathbb{E}[P_m^n|V = 1]$ for each $m \in M$. Each type θ can mimic type 1's bidding strategy and obtain a payoff identical to type 1 if $V = 1$. Such a type would obtain a payoff equal to zero in the low state at the limit regardless of the strategy that she uses. Therefore,

$$\lim u^n(m, b_m^n(\theta)|\theta_i = \theta, V = 1) \geq \lim u^n(m, b_m^n(1)|\theta_i = \theta, V = 1)$$

for each θ. However, a symmetric argument from the perspective of type 1 implies that $\lim u^n(m, b_m^n(\theta)|\theta_i = 1, V = 1) \leq \lim u^n(m, b_m^n(1)|\theta_i = 1, V = 1)$ for each θ. These two inequalities and noting that $\lim u^n(m, b|\theta_i = \theta, V = 1) = \lim u^n(m, b|V = 1)$ for each θ shows $\lim u^n(m, b_m^n(\theta)|V = 1) = \lim u^n(m, b_m^n(1)|V = 1)$ for each θ. Therefore, if there is a positive mass of bidders in any two markets $m' \neq m$, then we must have that $\lim u^n(m|\theta_i = \theta, V = 1) = 1 - \lim \mathbb{E}[P^n_m|V = 1] = 1 - \lim \mathbb{E}[P^n_{m'}|V = 1]$ for all θ.

In any equilibrium total expected utility plus the total revenue must be at most equal to the total available surplus, that is,

$$\lim \int_{[0,1]} u^n(\theta)dF(\theta) + \sum_m \frac{\lim \mathbb{E}[P^n_m|V = 1]}{2} \min\{F_m(1|1), \kappa_m\} \leq \sum_m \min\{F_m(1|1), \kappa_m\}\frac{1}{2}.$$

Noting that $\lim 2u^n(\theta)f(\theta) = f(\theta|1) \lim u(m, b_m(\theta)|V = 1) = f(\theta|1)(1 - \lim \mathbb{E}[P^n_m|V = 1])$ and substituting into the above we obtain

$$\begin{aligned}
\int_{[0,1]} (1 - \lim \mathbb{E}[P^n_m|V = 1])dF(\theta) &\leq \sum_m \min\{F_m(1|1), \kappa_m\}(1 - \lim \mathbb{E}[P^n_m|V = 1]) \\
1 - \lim \mathbb{E}[P^n_m|V = 1] &\leq \sum_m \min\{F_m(1|1), \kappa_m\}(1 - \lim \mathbb{E}[P^n_m|V = 1])
\end{aligned}$$

Using the fact that $1 - \lim \mathbb{E}[P^n_m|V = 1] = 1 - \lim \mathbb{E}[P^n_{m'}|1]$ to simplify we obtain $(1 - \lim \mathbb{E}[P^n_m|V = 1])(1 - \sum_m \min\{F_m(1|1), \kappa_m\}) \leq 0$ showing that $1 - \lim \mathbb{E}[P^n_m|V = 1] = 0$ for any market that attracts bidders.

If one of the markets, say r, has no bidders, then following the logic above, we obtain $1 - \lim \mathbb{E}[P^n_m|V = 1] = 0$ for all markets m which attract a positive mass of bidders. This, however, contradicts that market r has no bidders because each bidder can submit a bid equal to zero in market r and obtain a payoff equal to $1 > 0$ in state $V = 1$. $\square$

Proof of Theorem 4.1, Item iii. Let $x_v = F[\theta''|v] - F[\theta'|v] > 0$ for $v = 0, 1$ where $1 - F[\theta''|1] = \kappa_s + \kappa_r$ and $1 - F[\theta'|0] = \kappa_s + \kappa_r$. If $\kappa_s < \kappa^*(\theta^*)$ and if $c < \bar{c}$, then information is aggregated in market s where

$$\bar{c} = x_0 \underline{b} > 0$$

and where

$$\underline{b} = \frac{l(\theta)(1 - \frac{\kappa_s}{\kappa_s + x_1})}{(1 - x_0) + l(\theta)(1 - \frac{\kappa_s}{\kappa_s + x_1})} > 0.$$

Assume that information is not aggregated in market s. We will prove this claim by looking at three distinct cases.

Case 1. Suppose that $F_s(1|0) > \kappa_s$ and $\theta_s(0) \geq \theta_s(1)$. This implies that $F_r(1|1) > F_r(1|0)$ and $\theta_r(1) > \theta_r(0)$. Also, $\kappa_s < \kappa^*$ implies that $F_r(1|1) > \kappa_r$ by Lemma C.3. However, $F_r(1|1) > F_r(1|0)$, $\theta_r(1) > \theta_r(0)$ and $F_r(1|1) > \kappa_r$ together imply that $\lim \mathbb{E}[P^n_r|1] = 1$ by Lemma B.5. This however, is not possible because any type $\theta > \theta_r(1)$ who selects market r would have strictly negative profits because $\lim \mathbb{E}[P^n_r|V = 0] \geq c > 0$.

Case 2. Suppose that $F_s(1|0) \leq \kappa_s$. We must also have $F_s(1|1) \leq \kappa_s$ because if $F_s(1|1) > \kappa_s$, then information would be aggregated in market s by Lemma B.5.

First, suppose $F_s(1|1) \leq F_s(1|0) \leq \kappa_s$. If $F_s(1|1) \leq F_s(1|0) \leq \kappa_s$, then $F_r(1|1) \geq F_r(1|0)$, $\theta_r(1) > \theta_r(0)$ and $F_r(1|1) > \kappa_r$ which in turn implies that $\lim \mathbb{E}[P^n_r|V = 1] = 1$ by Lemma B.5. However, we argued that this is not possible above.

Alternatively, suppose $F_s(1|0) < F_s(1|1) \leq \kappa_s$. If $F_s(1|0) < F_s(1|1) \leq \kappa_s$, then $\lim \mathbb{E}[P^n_s|V =$

$0] = 0$. We observe that $\lim \mathbb{E}[P_s^n|0] = 0$ and $\lim \mathbb{E}[P_r^n|0] \geq c_r$ together imply that $\lim a_r^n(\theta) = 1$ for all $\theta > \theta^*$ by Lemma C.2. Also, $F_r(1|1) > \kappa_r$ because $F_s(1|1) \leq \kappa_s$. This line of reasoning implies that $\lim \mathbb{E}[P_r^n|V = 1] = 1$ by Lemma B.5 because $\lim a_r^n(\theta) = 1$ for all $\theta > \theta^*$ and $F_r(1|1) > \kappa_r$. However, we argued that $\lim \mathbb{E}[P_r^n|V = 1] = 1$ is not compatible with equilibrium behavior as was argued further above.

Case 3. Suppose that $F_s(1|0) > \kappa_s$ and $\theta_s(0) < \theta_s(1)$. In order for information to not aggregate in market s it must be that $F_s(1|0) > F_s(1|1)$. This implies that $F_r(1|1) > F_r(1|0)$ and $\theta_r(1) > \theta_r(0)$. This implies that $F_r(1|1) \leq \kappa_r$ because otherwise we would have $\lim \mathbb{E}[P_r^n|V = 1] = 1$ by Lemma B.5.

Information non-aggregation implies that there is pooling by pivotal types (see definition B.1)

Step 1. $\lim \Pr(b^n\ wins, P_s^n = b^n|V = 0) \geq x_0$ and $\lim \Pr(b^n\ wins, P_s^n = b^n|V = 1) \leq \frac{\kappa_s}{\kappa_s + x_1}$. These bounds follow from Lemma B.2.

Step 2. The pooling bid $b \geq \underline{b}$

For each θ who submits the pooling bid, submitting the pooling bid must be at least as profitable as submitting a bid slightly higher than the pooling bid, i.e., submitting a bid $b^n + \epsilon$ for any $\epsilon > 0$ (also see the analogous argument provided for Claim D.1). Therefore:

$$-\lim \Pr(b^n\ wins, P^n = b^n|V = 0)b + \lim \Pr(b^n\ wins, P^n = b^n|V = 1)(1 - b)l(\theta) \geq -b + (1 - b)l(\theta).$$

The pooling bid $b \leq 1$. Therefore, for the above inequality hold we must have $0 \leq b$. Therefore, using the facts that $\lim \Pr(b^n\ wins, P^n = b^n|V = 0) \geq x_0$ and $\lim \Pr(b^n\ wins, P^n = b^n|V = 1) \leq \frac{\kappa_s}{\kappa_s + x_1}$ we obtain

$$\begin{aligned} -bx_0 + \frac{\kappa_s}{\kappa_s + x_1}(1 - b)l(\theta) &\geq -b + (1 - b)l(\theta) \\ b &\geq \frac{l(\theta)(1 - \frac{\kappa_s}{\kappa_s + x_1})}{(1 - x_l) + l(\theta)(1 - \frac{\kappa_s}{\kappa_s + x_1})} \end{aligned}$$

Note that $1 > b > 0$.

Step 3. If $-bx_0 < -c$, i.e., if $c < \bar{c}$, then all $\theta > \hat{\theta}_s$ select market s. This is because $\lim u(s, b^n|V = 0) \leq -bx_0$. Also, $\lim u(r, b'|V = 0) = -c < 0$ for any $b' \geq c$ and $\lim u(r, b'|V = 0) = 0$ for any $b' < c$. However, then Lemma C.2 implies that $\theta > \hat{\theta}_s = \lim \underline{\theta}_p^n$ select market s because there is always a type $\theta' \in [\theta, \hat{\theta}_s]$ who submits the pooling bid. However, if $a_s(\theta) = 1$ for all $\theta > \hat{\theta}_s$, then information is aggregated in market s by Lemma B.5. □

E.2. Information Aggregation Failure with Multiple Markets.

Theorem E.1. *Suppose that MLRP is satisfied. Let κ^* denote the cutoff defined by Definition 3.2. If $c > 0$ and if $\sum_{m \in M \setminus r} \kappa_m > \kappa^*$, then information is not aggregated in any market $m \in M$ in any sequence of equilibria $\mathbf{H}$.*

Proof. Fix an equilibrium sequence $\mathbf{H}$. Let $I \subset M$ denote the set markets where information is aggregated along the equilibrium sequence and let $N \subset M$ denote the set of markets such that $\lim \mathbb{E}[P_m^n|0] > 0$. Thus $I \cap N = \emptyset$. Assume, on the way to a contradiction, that $I \neq \emptyset$, i.e., assume that information is aggregated in some market. Let $\hat{\theta} := \min_{m \in N} \hat{\theta}_m$. Note that $r \in N$.

If $\theta > \hat{\theta}$, then $\lim a^n_{M \setminus I}(\theta) = 1$ by Lemma C.2 because $\lim \mathbb{E}[P_m^n|V=0] > 0$ for all $m \in N$ and $\lim \mathbb{E}[P_m^n|V=0] = 0$ for all $m \in I$.[43] We show below that this selection implies that $\theta_I(0) > \theta_I(1)$. However, if $\theta_I(0) > \theta_I(1)$, then there exists $m \in I$ such that $\theta_m(0) > \theta_m(1)$. However, information aggregation in this market m implies that $\theta_s^n(1) - \theta_s^n(0) > 0$ for all n sufficiently large by Lemma B.4 which contradicts that $\theta_m(0) > \theta_m(1)$.

We now show that $\theta_I(0) > \theta_I(1)$. Because no $\theta > \hat{\theta}$ bids in any of the markets in I, we find using the reasoning in Lemma C.3 that the following inequality is satisfied:

$$\frac{F_I(\hat{\theta}|1) - F_I(\theta_I(1)|1)}{F_I(\hat{\theta}|0) - F_I(\theta_I(1)|0)} \leq W(\sum\nolimits_I \kappa_m, \hat{\theta}).$$

Let $\Delta_1 = \hat{\theta}_r - \hat{\theta}$ and $\Delta_2 = F(\hat{\theta}_r|1) - F(\hat{\theta}|1)$. We have $W(\hat{\theta}, \sum_I \kappa_m) = W(\hat{\theta}_r - \Delta_1, \sum_{m \neq r} \kappa_m - \Delta_2) \leq W(\hat{\theta}_r, \sum_{m \neq r} \kappa_m)$ where the inequality is satisfied because any choice α for the program $W(\hat{\theta}_r - \Delta_1, \sum_{m \neq r} \kappa_m - \Delta_2)$ is also feasible for $W(\hat{\theta}_r, \sum_{m \neq r} \kappa_m)$. Our initial assumption that $\sum_{m \in M} \kappa_s > \kappa^*$ implies that $W(\hat{\theta}_r, \sum_{m \neq r} \kappa_m) < 1$. $\square$

E.3. Characterization.

Proof of Proposition 4.1 . Let $x := \kappa_s - \bar{F}(\theta_s^F(1)|0)$. Note that $\theta_s^F(1)$ is the pivotal type under the assumption that all types are in market s. Therefore, x is independent of n and the equilibrium under consideration. Moreover, the constant $x > 0$ due to MLRP. Define

$$\bar{c} := \min \left\{ x \frac{l(0)(1 - \kappa_r - \kappa_s)}{1 + l(0)(1 - \kappa_r - \kappa_s)}, \Pr[v = 1 | \theta_i = \theta_s^F(1)] \right\}. \tag{E.1}$$

Claim E.1. We show that $F_r(1|0) < F_r(1|1) \leq \kappa_r$. This step in conjunction with Claim E.5 establishes item i.

Proof. In the two paragraphs below we will argue that $F_r(1|0) < F_r(1|1)$. If $F_r(1|0) < F_r(1|1)$, then we must have $F_r(1|1) \leq \kappa_r$. This is because $F_r(1|0) < F_r(1|1)$ and $F_r(1|1) > \kappa_r$ together imply that $P_r^n \to 1$ if $v = 1$ by Lemma B.5. But this is not possible because all the bidders in market r would then earn negative profits.

First, suppose that $F_r(1|0) > F_r(1|1)$. This implies that $F_s(1|0) < F_s(1|1)$. There are two cases: $F_s(1|1) > \kappa_s$ and $F_s(1|1) \leq \kappa_s$. If $F_s(1|1) > \kappa_s$, then $P_s^n \to 0$ when $v = 0$ by Lemma B.5 and if $F_s(1|1) \leq \kappa_s$, then again $P_s^n \to 0$ when $v = 0$ because $F_s(1|0) < \kappa_s$. However, if $P_s^n \to 0$ when $v = 0$, then $a_r(\theta) = 1$ for all $\theta > \hat{\theta}_r$ by Lemma C.2. However, if $F_r(1|0) < \kappa_r$, then $a_r(\theta) = 1$ for all $\theta > \hat{\theta}_r$ implies that $F_r(1|1) > F_r(1|0)$ which contradicts our initial assumption. On the other hand if $F_r(1|0) \geq \kappa_r$, then $F_r(1|1) > \kappa_r$ However, $F_r(1|1) > \kappa_r$ and $a_r(\theta) = 1$ for all $\theta > \hat{\theta}_r$ together imply that $P_r^n \to 1$ if $v = 1$ by Lemma B.5 which is not possible.

[43] As before, for a set of markets $B \subset M$ we define $F_B(\theta) := H([0, \theta] \times B \times [0, \infty))$ and $a_B(\theta)$ as the Radon-Nikodym derivative of F_B with respect to F.

Second, suppose that $F_r(1|0) = F_r(1|1)$. There are two cases to consider: $F_r(1|1) > \kappa_r$ and $F_r(1|1) \leq \kappa_r$. If $F_r(1|1) > \kappa_r$, then $P_r^n \to 1$ when $v = 1$ by Lemma B.5 which is not possible. Alternatively, If $F_r(1|1) \leq \kappa_r$, then $F_s(1|1) > \kappa_s$. However, $F_s(1|0) = F_s(1|1)$ and $F_s(1|1) > \kappa_s$ together imply by Lemma B.5 that $P_s^n \to 0$ if $v = 0$. However, as argued previous, if $P_s^n \to 0$ when $v = 0$ and if $F_r(1|0) \leq \kappa_r$, then almost all types in market r win an object when $v = 0$ at a price which is at least c. Therefore, $a_r(\theta) = 1$ for all $\theta > \hat{\theta}_r$ by Lemma C.2. Thus, we conclude that $F_r(1|1) > F_r(1|0)$ because $a_r(\theta) = 1$ for all $\theta > \hat{\theta}_r$. However, this contradicts that $F_r(1|0) = F_r(1|1)$ as we initially assumed. □

Claim E.2. We now argue that $\lim \sqrt{n}|\bar{F}_s^n(\theta_s^n(1)|v) - \bar{F}_s^n(\theta_s^n(0)|v)| \in (0, \infty)$ for each $v = 0, 1$. This step proves item *vi*.

Proof. Suppose instead that $\lim \sqrt{n}|\bar{F}_s^n(\theta_s^n(1)|v) - \bar{F}_s^n(\theta_s^n(0)|v)| \to \infty$ for some v and note that this implies $\lim \sqrt{n}|\bar{F}_s^n(\theta_s^n(1)|v) - \bar{F}_s^n(\theta_s^n(0)|v)| \to \infty$ for each $v = 0, 1$. We will argue below that if $c < \bar{c}$, then pooling by pivotal types (see Definition B.1) cannot be sustained. However, if there is no pooling by pivotal types and if $\lim \sqrt{n}|\bar{F}_s^n(\theta_s^n(1)|1) - \bar{F}_s^n(\theta_s^n(0)|1)| \to \infty$, then information is aggregated by Lemma B.4. Therefore, once we conclude that pooling by pivotal types cannot be sustained, this conclusion and Theorem 4.1's finding that information is not aggregated together imply that $\lim \sqrt{n}|\bar{F}_s^n(\theta_s^n(1)|1) - \bar{F}_s^n(\theta_s^n(0)|1)| < \infty$.

In the proof of Proposition 3.2 we showed that if there is pooling by pivotal types, then $\lim \Pr(P_s^n \leq b_p^n|V = 1) = 1$ and $\lim \Pr(P_s^n < b_p^n|V = 0) = 0$. Below we will show that if $\lim \Pr(P_s^n \leq b_p^n|V = 1) = 1$ and $\lim \Pr(P_s^n < b_p^n|V = 0) = 0$, then there cannot be pooling by pivotal types and thereby establish that $\lim \sqrt{n}|\bar{F}_s^n(\theta_s^n(1)|1) - \bar{F}_s^n(\theta_s^n(0)|1)| < \infty$.

□

Claim E.3. If $\lim \Pr(P_s^n \leq b_p^n|V = 1) = 1$ and $\lim \Pr(P_s^n < b_p^n|V = 0) = 0$, then

$$b_p \geq \frac{l(0)(1 - \kappa_r - \kappa_s)}{1 + l(0)(1 - \kappa_r - \kappa_s)}.$$

Proof. Suppose that there is pooling by pivotal types and $\lim \Pr(P_s^n \leq b_p^n|V = 1) = 1$ and $\lim \Pr(P_s^n < b_p^n|V = 0) = 0$. Let $\lim b_p^n = b_p$. We first argue that the pooling price

$$b_p \geq \frac{l(0)(1 - \kappa_r - \kappa_s)}{1 + l(0)(1 - \kappa_r - \kappa_s)}.$$

For any type θ who submits the pooling bid, the pooling bid must be at least as profitable as submitting a bid slightly higher than the pooling bid, i.e., a bid $b_p^n + \epsilon$ for any $\epsilon > 0$. Therefore:

$$\begin{aligned}
&\Pr(b_p^n\ wins|V = 1, P_s^n = b_p^n) \Pr(P_s^n = b_p^n|V = 1)(1 - b_p^n)l(\theta) \\
&\qquad - \Pr(b_p^n\ wins|V = 0, P_s^n = b_p^n) \Pr(P_s^n = b_p^n|V = 0)b_p^n \geq \\
&\qquad\qquad - b_p^n \Pr(P_s^n = b_p^n|V = 0) + (1 - b_p^n) \Pr(P_s^n = b_p^n|V = 1)l(\theta),
\end{aligned}$$

where we ignore ϵ as this constant is arbitrarily small. Rearranging we find that

$$\begin{aligned}\frac{b_p^n}{1-b_p^n} &\geq \frac{\Pr(b_p^n\ loses|V=1,P_s^n=b_p^n)\Pr(P_s^n=b_p^n|V=1)}{\Pr(b_p^n\ loses|V=0,P_s^n=b_p^n)\Pr(P_s^n=b_p^n|V=0)}l(\theta)\\ &\geq \frac{\Pr(b_p^n\ loses,P_s^n=b_p^n|V=1)}{\Pr(b_p^n\ loses,P_s^n=b_p^n|V=0)}l(\theta=0)\geq l(\theta=0)\lim\Pr(b_p^n\ loses,P_s^n=b_p^n|V=1)\end{aligned}$$

There are two possibilities: either $F_s(\underline{\theta}_p|1)=0$ or $F_s(\underline{\theta}_p|1)>0$.

Suppose that $F_s(\underline{\theta}_p|1)=0$. In this case, Lemma B.2 and $F_s(\underline{\theta}_p|1)=0$ together imply that

$$\begin{aligned}\lim\Pr(b_p^n\ loses,P_s^n=b_p^n|V=1) &= 1-\frac{\kappa_s-(F_s(1|1)-F_s(\theta_p|1))}{F_s(\theta_p|1)-F_s(\underline{\theta}_p|1)}=\frac{F_s(1|1)-\kappa_s}{F_s(\theta_p|1)}\\ &\geq F_s(1|1)-\kappa_s\geq 1-\kappa_r-\kappa_s\end{aligned}$$

where the last inequality is satisfied because $F_r(1|1)\leq\kappa_r$ by the previous step further above. Therefore, we find

$$b_p\geq\frac{(1-\kappa_r-\kappa_s)l(0)}{1+(1-\kappa_r-\kappa_s)l(0)}$$

Suppose instead that $F_s(\underline{\theta}_p|1)>0$. This implies that

$$-\lim\Pr(b_p^n\ wins,P_s^n\leq b_p^n|V=0)b_p^n+\Pr(b_p^n\ wins,P_s^n\leq b_p^n|V=1)(1-b_p^n)l(0)\leq 0,$$

rearranging we find that

$$\frac{b_p}{1-b_p}\geq\lim\frac{\Pr(b_p^n\ wins,P_s^n\leq b_p^n|V=1)}{\Pr(b_p^n\ wins,P_s^n\leq b_p^n|V=0)}l(0)$$

However, the inequality we obtained further above for any type θ submitting the pooling bid showed

$$\frac{b_p}{1-b_p}\geq\lim\frac{\Pr(b_p^n\ loses,P_s^n=b_p^n|V=1)}{\Pr(b_p^n\ loses,P_s^n=b_p^n|V=0)}l(0)=\lim\frac{\Pr(b_p^n\ loses,P_s^n\leq b_p^n|V=1)}{\Pr(b_p^n\ loses,P_s^n\leq b_p^n|V=0)}l(0)$$

However,

$$\max\{\frac{\Pr(b_p^n\ wins|V=1,P_s^n\leq b_p^n)}{\Pr(b_p^n\ wins|V=0,P_s^n\leq b_p^n)},\frac{\Pr(b^n\ loses|V=1,P_s^n\leq b^n)}{\Pr(b^n\ loses|V=0,P_s^n\leq b^n)}\}\frac{\Pr(P_s^n\leq b^n|V=1)}{\Pr(P_s^n\leq b^n|V=0)}\geq 1.$$

Therefore, if $F_s(\underline{\theta}_p|1)>0$, then $b_p\geq\frac{l(0)}{1+l(0)}>\frac{l(0)(1-\kappa_r-\kappa_s)}{1+l(0)(1-\kappa_r-\kappa_s)}$.

The previous argument provided a lower bound on the pooling price. We now argue that it is not possible to pool at such a high pooling price if $c<\bar{c}$.

□

Claim E.4. Suppose that $c<\bar{c}$. If $\lim\Pr(P_s^n\leq b_p^n|V=1)=1$ and $\lim\Pr(P_s^n<b_p^n|V=0)=0$, then there is no pooling by pivotal types.

Proof. Note that any type $\theta>\theta_p$ that bids in market s wins an object with probability one and pays a price at least equal to b_p if $v=0$. However, then we find that all $\theta>\theta_p$ would select market s by Lemma C.1 because $b_p > c$. We now look at two cases to show that pooling by pivotal types is not possible if $\lim \Pr(P_s^n \leq b_p^n|V=1) = 1$ and $\lim \Pr(P_s^n < b_p^n|V=0) = 0$.

Case 1. $\lim \sqrt{n}|\bar{F}_s^n(\theta_p^n|v) - \bar{F}_s^n(\theta_s^n(0)|v)| < \infty$. If $\lim \sqrt{n}|\bar{F}_s^n(\theta_p^n|v) - \bar{F}_s^n(\theta_s^n(0)|v)| < \infty$, then the type $\hat{\theta}_s$ defined in Definition B.3 is equal to θ_p. Moreover, any type $\theta > \theta_p$ selects market s by the argument in the previous paragraph. This however, leads to a contradiction because if all $\theta > \theta_p$ select market s and if $\lim |\bar{F}_s^n(\theta_p^n|v) - \bar{F}_s^n(\theta_s^n(0)|v)| = 0$, then $\theta_s(1) > \theta_s(0)$ by MLRP and hence $\theta_s(1) > \theta_p$ contradicting that $\lim \Pr(P_s^n \leq b_p^n|V=1) = 1$.

Case 2. $\lim \sqrt{n}|\bar{F}_s^n(\theta_p^n|v) - \bar{F}_s^n(\theta_s^n(0)|v)| = \infty$. In this case, we argue below that the probability of winning a good at pooling is at least $x > 0$ if $v = 0$. Type $\hat{\theta}_s$ is equal to $\underline{\theta}_p$ and all types above $\hat{\theta}_s$ win an object with probability at least x in market s at a price which exceeds b_p. Then however, all $\theta > \hat{\theta}_s$ would select market s by Lemma C.1 because $b_p x > \bar{c} > c$. This would however imply that information is aggregated in the auction by Lemma B.5 which leads us to a contradiction.

Continuing with Case 2, we argue that the expected fraction of objects remaining at pooling if $V = 0$ is at least x and therefore, the probability of winning an object from pooling is at least x if $V = 0$. Note that $\lim \Pr(P_s^n \leq b_p^n|V=1) = 1$ implies that $\kappa - \bar{F}_s(\theta_p|V=1) \geq 0$. The fact that all types $\theta > \theta_p$ select market s implies that $\kappa - \bar{F}_s(\theta_p|1) = \kappa - \bar{F}(\theta_p|1) \geq 0$. Therefore, we find $\theta_p \geq \theta_s^F(1)$. MLRP implies that $\kappa - \bar{F}(\theta_p|1) > \kappa - \bar{F}(\theta_p|0)$. Also, $\theta_p \geq \theta_s^F(1)$ implies that $\kappa_s - \bar{F}(\theta_p|0) \geq \kappa_s - \bar{F}(\theta_s^F(1)|0) = x$ hence the expected fraction of goods left over to pooling is at least x. The expected fraction of bidders who submit the pooling bid is at most 1. Therefore, the fact that the probability of winning at pooling is at least x follows from the argument in Lemma B.2. □

Claim E.5. We show $\lim \sqrt{n}|F_r^n(1|1) - \kappa_r| < \infty$. This claim completes the proof of item i.

Proof. In Claim E.1 we showed that $F_r(1|0) < F_r(1|1) \leq \kappa_r$. If $\lim \sqrt{n}|F_r^n(1|1) - \kappa_r| = \infty$, then $\Pr(P_r = c|v) = 1$ for $v = 0, 1$. Therefore any type can make strictly positive profit by bidding c in market r because $c < \bar{c}$ implies that $(1-c)l(0) - c > 0$. Let $\theta_p^n = \sup\{\theta : b_s^n(\theta) = b_s^n(\theta_s(1))\}$, $\underline{\theta}_p^n = \inf\{\theta : b_s^n(\theta) = b_s^n(\theta_s(1))\}$ and $\theta_i = \lim \theta_i^n$, $i = 0, 1$. Note that if $\bar{F}_s(\theta|1) > \bar{F}_s(\underline{\theta}_p|1)$ for any type with $a_s(\theta) > 0$, then $\lim u^n(s, b_s^n(\theta)|1) = 0$. The profit of such a type is zero, i.e., $\lim u^n(s, b_s^n(\theta)|1) = 0$. We show this in the next paragraph. However, such a type can make strictly positive profit by bidding $c+\epsilon$ for $\epsilon > 0$ sufficiently small. Therefore, $\bar{F}_s(\theta|1) \leq \bar{F}_s(\underline{\theta}_p|1)$ for all θ with $a_s(\theta) > 0$ and hence $\bar{F}_s(\underline{\theta}_p|1) = \bar{F}_s(0|1)$. Therefore, $\theta_s(0) > \underline{\theta}_p$. Note that $\bar{F}_s(\underline{\theta}_p|1) = \bar{F}_s(0|1)$, $k_s < \bar{F}_s(0|1)$ and $\theta_p \geq \theta_s(1)$ together imply that $\bar{F}_s(\underline{\theta}_p|1) > \bar{F}_s(\theta_p|1)$. Similarly, if $\bar{F}_s(\theta_p|1) \geq \kappa_s$, then $\lim \Pr(b_s^n(\theta_s(1))\ wins, P_s^n = b_s^n(\theta_s(1))|1) = 0$ by Lemma B.2. Again we find that $\lim u_s(b_s^n(\theta)|1) = 0$ for any $\theta \in [\underline{\theta}_p, \theta_p]$ however any such type can make strictly positive profit by bidding in market r. Therefore, $\bar{F}_s(\theta_p|1) < \kappa_s$ which implies that $\lim \Pr(P_s^n = b_s^n(\theta_s(1))|V=v) = 1$ for all $v = 0, 1$. This implies that there is pooling by pivotal types. However, in Claim E.4, we argued that there is no such sequence of pooling intervals.

We now argue if $\bar{F}_s(\theta'|1) > \bar{F}_s(\underline{\theta}_p|1)$ for any type with $a_s(\theta') > 0$, then $\lim u^n(s, b_s^n(\theta')|1) = 0$. Note that $b_s^n(\theta') < b_s^n(\theta_s(1))$ for all n because $\bar{F}_s(\theta'|1) > \bar{F}_s(\underline{\theta}_p|1)$. Let $\tilde{\theta}^n = \sup\{\theta : b_s^n(\theta) = b_s^n(\theta')\}$ and $\tilde{\theta} = \lim \tilde{\theta}^n$. If $\bar{F}_s(\tilde{\theta}|1) > \bar{F}_s(\underline{\theta}_p|1)$, then $\lim \Pr(b_s^n(\theta')\ wins, p^n = b_s^n(\theta')|1) = 0$

by the law of large numbers. Otherwise, if $\bar{F}_s(\tilde{\theta}|1) = \bar{F}_s(\underline{\theta}_p|1)$, then $\lim \Pr(b_s^n(\theta')\ wins, p^n = b_s^n(\theta')|1) = 0$ by Lemma B.2. ☐

Claim E.6. The profits are equal across states and markets. In particular, $\lim \mathbb{E}[P_r^n|V=1] = \lim \mathbb{E}[P_s^n|V=1]$ and $\lim \mathbb{E}[P_r^n|V=0] = \lim \mathbb{E}[P_r^n|V=0] = c$. This claim establishes item *iv*.

Proof. If $c < \mathbb{E}[P_r^n|V=0]$, then $a_r(\theta) = 1$ for all $\theta > \hat{\theta}_r$ by Lemma C.2. This implies that $\theta_s(0) > \theta_s(1)$ because $\kappa_s > \kappa^*(\theta^*)$ and because $\hat{\theta}_r = \theta^*$. However, this contradicts step 2 above.

Further below we show any type $\theta > \theta_s(1)$ who submits a bid in market s wins an object with probability one if $v = 0$. If $c > \mathbb{E}[P_r^n|V=0]$, then $a_s(\theta) = 1$ for all $\theta > \theta_s(0)$ by Lemma C.2. However, this implies that $\theta_s(1) > \theta_s(0)$ again contradicting step 2 further above.

However, $\lim \mathbb{E}[P_r^n|V=0] = \lim \mathbb{E}[P_r^n|V=0] = -c$ implies that we must have $\lim \mathbb{E}[P_r^n|V = 1] = \lim \mathbb{E}[P_s^n|V=1]$ in order to have any type bid in the two markets.

We now complete the proof by showing that for any type $\theta' > \theta_s(0)$,

$$\lim \Pr(b_s^n(\theta')\ wins, P_s^n \leq b_s^n(\theta')|V=0) = 1.$$

Suppose that $\theta' > \theta_s(0)$ and $\lim \Pr(b_s^n(\theta')\ wins, p_s^n \leq b_s^n(\theta')|V=0) < 1$. The law of large numbers implies that $\lim \Pr(P_s^n \leq b_s^n(\theta_s^n(0))|V=0) = 1$. Moreover, monotonicity of bidding implies that $b_s^n(\theta') \geq b_s^n(\theta_s^n(0))$. Therefore, it must be the case that $b_s^n(\theta') = b_s^n(\theta_s^n(0))$ for all n sufficiently large. Let

$$\underline{\theta}_p^n = \inf\{\theta : b_s^n(\theta) = b_s^n(\theta_s(0))\} \text{ and } \theta_p^n = \sup\{\theta : b_s^n(\theta) = b_s^n(\theta_s(0))\}.$$

Our assumption that $\lim \Pr(b_s^n(\theta')\ wins, p_s^n \leq b_s^n(\theta')|V=0) < 1$ implies that $\underline{\theta}_p < \theta_s(0)$ because otherwise all bidders in the pooling region would win a good with probability one by Lemma B.2. Hence there must be a sequence of pooling regions $(\underline{\theta}_p^n, \theta_p^n)$ such that $\underline{\theta}_p < \theta_s(0) = \theta_s(1) < \theta' \leq \theta_p$. However, Claim E.4 above established that such a sequence does not exist. ☐

Claim E.7. For any type θ that bids in market r we have $b_r^n(\theta) \to 1$. The argument for this claim is straightforward if bidding is atomless in market r. The argument below shows that $b_r^n(\theta) \to 1$ in any equilibrium sequence even if there are atoms in the bidding function.

Proof. For any $\epsilon > 0$, pick θ^n such that $\Pr(Y_r^{n-1}(n\kappa_r) \in (0, \theta^n)|V=1) \leq \epsilon$ and recall that we use the convention that $Y_r^{n-1}(n\kappa_r) = 0$ if there are fewer than $n\kappa_r + 1$ bidders in market r. We argue that $\lim b_r^n(\theta) = 1$ for any $\theta > \lim \theta^n$. Note that $\lim \Pr(P_r^n \geq b_r^n(\theta^n)|V=1) > 0$ and that $\lim \sqrt{n}(\bar{F}_r^n(\theta^n|v) - \kappa_r) = a$ for some a. If we let

$$\delta = \frac{\kappa_r}{F_r^n(1|0)} - 1$$

then, applying Chernoff's Inequality we find

$$\Pr(P_r^n \geq b_r^n(\theta^n)|V=0) \leq e^{-\frac{\delta^2 n F_r^n(1|0)}{2+\delta}}.$$

Therefore,

$$\Pr(P_r^n = b_r^n(\theta^n), b_r^n(\theta^n)\ loses|V=0) \leq \Pr(b_r^n(\theta^n)\ loses|V=0)$$
$$\leq \Pr(P_r^n \geq b_r^n(\theta^n)|V=0) \leq e^{-\frac{\delta^2 n F_r^n(1|0)}{2+\delta}}$$

Any type θ^n in this sequence can ensure winning an object by submitting a bid equal to one in the auction. Therefore, we obtain the following inequality which has type θ^n's equilibrium payoff on the left hand side and θ^n's payoff from submitting a bid equal to one on the right hand side:

$$\begin{gathered}(1-\mathbb{E}[P_r^n|b_r^n(\theta^n)\ wins, V=1])\Pr(b_r^n(\theta^n)\ wins|V=1)l(\theta^n) - c\Pr(b_r^n(\theta^n)\ wins|V=0) \geq \\ (1-\mathbb{E}[P_r^n|b_r^n(\theta^n)\ wins, V=1])\Pr(b_r^n(\theta^n)\ wins|V=1)l(\theta^n) - c\Pr(b_r^n(\theta^n)\ wins|V=0) + \\ (1-\mathbb{E}[P_r^n|b_r^n(\theta^n)\ loses, V=1])\Pr(b_r^n(\theta^n)\ loses|V=1)l(\theta^n) - \Pr(b_r^n(\theta^n)\ loses|V=0)\end{gathered}$$

Therefore,

$$\begin{aligned}(1-\mathbb{E}[P_r^n|b_r^n(\theta^n)\ loses, V=1])\Pr(b_r^n(\theta^n)\ loses|V=1)l(\theta^n) - \Pr(b_r^n(\theta^n)\ loses|V=0) &\leq 0. \\ (1-\mathbb{E}[P_r^n|b_r^n(\theta^n)\ loses, V=1]) - \frac{e^{-\frac{\delta^2 n F_r^n(1|0)}{2+\delta}}}{\Pr(b_r^n(\theta^n)\ loses|V=1)} &\leq 0\end{aligned}$$

We consider two cases: (1) $\lim \Pr(P_r^n = b_r^n(\theta^n)|V=1) = 0$ and (2) $\lim \Pr(P_r^n = b_r^n(\theta^n)|V=1) > 0$.

Suppose that $\lim \Pr(P_r^n = b_r^n(\theta^n)|V=1) = 0$. Then $\lim \Pr(P_r^n > b_r^n(\theta^n)|V=1) = \lim \Pr(P_r^n \geq b_r^n(\theta^n)|V=1) > 0$ and $\Pr(b_r^n(\theta^n)\ loses|V=1) = \lim \Pr(P_r^n \geq b_r^n(\theta^n)|V=1) > 0$. Therefore, using the inequality above we conclude $\lim \mathbb{E}[P_r^n|b_r^n(\theta^n)\ loses, V=1] = \lim \mathbb{E}[P_r^n|P_r^n \geq b_r^n(\theta^n), V=1] = 1$. Thus, $\lim b_r^n(\theta) = 1$ for almost all $\theta > \lim \theta^n$.

Instead, suppose that $\lim \Pr(P_r^n = b_r^n(\theta^n)|V=1) > 0$. If $\lim \Pr(P_r^n = b_r^n(\theta^n)|V=1) > 0$, then further below we argue that $\Pr(P_r^n = b_r^n(\theta^n), b_r^n(\theta^n)\ loses|V=1) \geq \frac{A}{\sqrt{n}}$ for some constant A for all sufficiently large n. Therefore,

$$\begin{aligned}0 &\geq (1-b_r^n(\theta^n))\Pr(P_r^n = b_r^n(\theta^n), b_r^n(\theta^n)\ loses|V=1)l(\theta^n) - \Pr(b_r^n(\theta^n)\ loses|V=0) \\ 0 &\geq (1-b_r^n(\theta^n)) - \frac{e^{-\frac{\delta^2 n F_r^n(1|0)}{2+\delta}}}{\frac{A}{\sqrt{n}}}\end{aligned}$$

for sufficiently large n. Taking limits, we find that $\lim b_r^n(\theta^n) = 1$ and thus $\lim b_r^n(\theta) = 1$ for all $\theta \geq \lim \theta^n$.

We now show that if $\lim \Pr(P_r^n = b_r^n(\theta^n)|V=1) > 0$, then $\Pr(P_r^n = b_r^n(\theta^n), b_r^n(\theta^n)\ loses|V=1) \geq \frac{A}{\sqrt{n}}$, for some constant A, for all sufficiently large n. Let $\underline{\theta}_p^n = \inf\{\theta : b_r^n(\theta) = b_r^n(\theta^n)\}$ and $\theta_p^n = \sup\{\theta : b_r^n(\theta) = b_r^n(\theta^n)\}$. Let θ_*^n be such that

$$\Pr[Y_r^n(n\kappa_r) \in [\theta_*^n, \theta_p^n]|V=1] = \frac{\Pr(P_r^n = b_r^n(\theta^n)|V=1)}{2} = \frac{\Pr[Y_r^n(n\kappa_r) \in [\underline{\theta}_p^n, \theta_p^n]|V=1]}{2}$$

Let X be the random variable which is equal to the expected number of bidders with types in the interval $[\underline{\theta}_p^n, \theta_*^n]$ who bid in market r. We claim that

$$\Pr(P_r^n = b_r^n(\theta^n), b_r^n(\theta^n)\ loses|V=1) \geq \frac{\Pr(P_r^n = b_r^n(\theta^n)|1)}{2} \frac{\mathbb{E}[X|Y_r^n(n\kappa_r) \in [\theta_*^n, \theta_p^n], V=1]}{n}.$$

This inequality is satisfied because $Y_r^n(n\kappa_r) \in [\theta_*^n, \theta_p^n]$ implies that $P_r^n = b_r^n(\theta^n)$; conditional on the event $Y_r^n(n\kappa_r) \in [\theta_*^n, \theta_p^n]$, at least X bidders, who submitted the same bid as θ^n, are not allocated objects, and there are at most n bidders who submit the same bid as θ^n.

Our assumption of $\lim \Pr(P_r^n = b_r^n(\theta^n)|V=1) > 0$ implies that $\lim \sqrt{n} F_r^n[[\underline{\theta}_p^n, \theta_p^n]|V = 1] > 0$ and thus $\lim \sqrt{n} F_r^n[[\underline{\theta}_p^n, \theta_*^n)|V=1] > 0$. Note that

$$\mathbb{E}[X|Y_r^n(n\kappa_r) \in [\theta_*^n, \theta_p^n], V=1] \geq n(1-\kappa_r)A^n$$

where $A^n = \frac{F_r^n[[\underline{\theta}_p^n, \theta_*^n)|1]}{1-F_r^n[[\theta_*^n, 1]|1]}$. Let $A = \frac{1}{2}\lim \frac{\Pr(P_r^n = b_r^n(\theta^n)|V=1)}{2}\sqrt{n}A^n$. Note that $A > 0$ because $\lim \sqrt{n} F_r^n[[\underline{\theta}_p^n, \theta_*^n)|1] > 0$. Thus,

$$\lim \frac{\mathbb{E}[X|Y_r^n(n\kappa_r) \in [\theta_*^n, \theta_p^n], V=1]}{\sqrt{n}} \geq \lim(1-\kappa_r)\sqrt{n}A^n > 0$$

The inequality above and $\lim \Pr(P_r^n = b_r^n(\theta^n)|V=1) > 0$ together imply that

$$\mathbb{E}[X|Y_r^n(n\kappa_r) \in [\theta_*^n, \theta_p^n], V=1] \frac{\Pr(P_r^n = b_r^n(\theta^n)|V=1)}{2} \geq A\sqrt{n}$$

for all sufficiently large n. Therefore, we conclude that $\Pr(P_r^n = b_r^n(\theta^n), b_r^n(\theta^n)\ loses|V=1) \geq A/\sqrt{n}$ for all n sufficiently large. □

Claim E.8. The price in market r converges to c almost surely if $V=0$ and converges to a random variable $P_r(1)$ if $V=1$. The random variable $P_r(1)$ is equal to c with probability $q>0$ and is equal to 1 with the remaining probability.

Proof. The fact that the price converges to c almost surely if $V=0$ follows from the law of large numbers and the fact that $F_r(1|0) < \kappa_r$. Also, note that $\lim \sqrt{n}|F_r^n(1|1) - \kappa_r| < \infty$ implies that $P_r(1)$ is equal to c with probability $q > 0$. With the remainder of the probability, i.e., with probability $1-q$, the auction clears at the bid of some type θ. However, the previous claim showed that $b_r^n(\theta) \to 1$ for all θ. Therefore, the auction price is equal to 1 with probability $1-q$. □

□

F. Equilibrium Construction.

F.1. Endogenous Outside Option. In what follows we construct an equilibrium with the properties outlined in Proposition 4.2. We first define the market selection and the bidding strategies in the two markets below. After we defined these objects, we argue that there is an equilibrium where all types behave according to these strategies.

In what follows we consider sequences $\{\theta_0^n, \theta_1^n, \theta_p^n\}$ such that types $(\theta_0^n, \theta_1^n]$ select market s, types $\theta \notin (\theta_0^n, \theta_1^n)$ select market r and types (θ_0^n, θ_p^n) submit the pooling bid. Suppose that (θ_0^n, θ_1^n) bid in market s, let

$$\Delta_s^n := \sqrt{\frac{n}{(1-\kappa_s)\kappa_s}} \frac{2(f_h - f_l)(\theta_1^n - \frac{1}{2} - \frac{\kappa_s}{2})}{(1-f_l)} \tag{F.1}$$

i.e,

$$\Delta_s^n = \sqrt{\frac{n}{(1-\kappa_s)\kappa_s}} (\bar{F}_s^n(\theta_s^n(0))|1) - \bar{F}_s^n(\theta_s^n(1)|1)) = \sqrt{\frac{n}{(1-\kappa_s)\kappa_s}} (\bar{F}_s^n(\theta_s^n(0))|1) - \kappa_s)$$

Let

$$\Delta_r^n \quad = \quad \sqrt{\frac{n}{(1-\kappa_r)\kappa_r}} (2f_h(1-\theta_1^n) + 2(1-f_h)\theta_0^n - \kappa_r), \tag{F.2}$$

i.e., $\Delta_r^n = \sqrt{\frac{n}{(1-\kappa_r)\kappa_r}} (\bar{F}_r^n(1|1) - \kappa_r)$. Note that since the mapping between $\{\theta_0^n, \theta_1^n\}$ and $\{\Delta_r^n, \Delta_s^n\}$ is linear and one-to-one we use the two sequences $\{\theta_0^n, \theta_1^n\}$ and $\{\Delta_r^n, \Delta_s^n\}$ interchangeably. We use $\Delta_m = \lim \Delta_m^n$ and $\theta_i = \lim \theta_i^n$, $i = 0, 1, p$.

Also we define the bidding function in market r as

$$b_r^n(\theta) = \beta_r^n(\theta) \tag{F.3}$$

for $\theta \leq \theta_0^n$ and $\theta \geq \theta_1^n$; and we define the bidding function in market s as

$$b_s^n(\theta) = \begin{cases} \beta_s^n(\theta) & \text{if } \theta \in [\theta_p^n, \theta_1^n) \\ b_p^n & \text{if } \theta \in [\theta_0^n, \theta_p^n) \end{cases} \tag{F.4}$$

where β_m is the function introduced in definition 2.4.

Lemma F.1. *For any vector* $(\underline{\Delta}_s, \bar{\Delta}_s, \underline{\Delta}_r, \bar{\Delta}_r) \in \mathbb{R}^4$ *and* $\epsilon > 0$, *any sequence* $(\Delta_s^n, \Delta_r^n) \in [\underline{\Delta}_s, \bar{\Delta}_s] \times [\underline{\Delta}_r, \bar{\Delta}_r]$ *with* $\lim \Delta_m^n = \Delta_m$, *if* $\theta \in (\theta_0^n, \theta_1^n)$ *select market* s *and* $\theta \notin (\theta_0^n, \theta_1^n)$ *select market* r, *then there exists* $N(\underline{\Delta}_s, \bar{\Delta}_s, \underline{\Delta}_r, \bar{\Delta}_r, \epsilon)$ *such that for all* $n > N(\underline{\Delta}_s, \bar{\Delta}_s, \underline{\Delta}_r, \bar{\Delta}_r, \epsilon)$ *the following are true:*

i. There is a unique type $\theta_p^n(\Delta_s^n, \Delta_r^n)$ *such that*

$$l(Y_s^{n-1}(\kappa_s n) = \theta_p^n, \theta_i = \theta_p^n) = l(Y_s^{n-1}(\kappa_s n) \in (\theta_0^n, \theta_p^n], \theta_i = \theta_p^n).$$

Moreover, $\theta_p^n(\Delta_s^n, \Delta_r^n)$ *is a continuous function of* Δ_s^n *and* Δ_r^n.

(a) If we define

$$z_0(\theta_p(\Delta_s, \Delta_r)) = \lim \sqrt{n} \frac{(\bar{F}_s^n(\theta_p^n(\Delta_s^n, \Delta_r^n)|0) - \kappa_s)}{\sqrt{\kappa_s(1-\kappa_s)}},$$

then $z_0(\theta_p(\Delta_s, \Delta_r))$ *is increasing in* Δ_s.

(b) There is loser's curse at the pooling interval $(\theta_0^n, \theta_p^n]$, i.e.,

$$\frac{\Pr(b_p^n \ wins | V = 1, Y_s^{n-1}(\kappa_s n) \in (\theta_0^n, \theta_p^n])}{\Pr(b_p^n \ wins | V = 0, Y_s^{n-1}(\kappa_s n) \in (\theta_0^n, \theta_p^n])} > 1.$$

ii. In market s,

$$\frac{\partial}{\partial \theta} l(Y_s^{n-1}(\kappa_s n) = \theta, \theta_i = \theta) > 0$$

for all $\theta \geq \theta_p^n$. In market r,

$$\frac{\partial}{\partial \theta} l(Y_r^{n-1}(\kappa_r n) = \theta, \theta_i = \theta) > 0$$

for all θ.

iii. There exists a unique pooling bid

$$b_p^n(\Delta_s^n, \Delta_r^n) < \frac{l(Y_s^{n-1}(\kappa_s n) \in (\theta_0^n, \theta_p^n], \theta_i = \theta_p^n)}{1 + l(Y_s^{n-1}(\kappa_s n) \in (\theta_0^n, \theta_p^n], \theta_i = \theta_p^n)}$$

such that bidding according to function $b_s^n(\theta)$ is an equilibrium in market s. The pooling bid $b_p^n(\Delta_s^n, \Delta_r^n)$ is a continuous function of (Δ_s^n, Δ_r^n). Moreover,

$$\lim b_p^n(\Delta_s^n, \Delta_r^n) = \lim \frac{l(Y_s^{n-1}(\kappa_s n) \in (\theta_0^n, \theta_p^n], \theta_i = \theta_p^n)}{1 + l(Y_s^{n-1}(\kappa_s n) \in (\theta_0^n, \theta_p^n], \theta_i = \theta_p^n)}.$$

iv. If bidders bid according to $b_s^n(\theta)$ and $b_r^n(\theta)$ in markets s and r, respectively, then

$$|u^n(m, b_m^n(\theta)|\theta, \Delta_s^n, \Delta_r^n) - u(m|\theta, \Delta_s, \Delta_r)| < \epsilon$$

for all θ and m where the functions $u(m|\theta, \Delta_s, \Delta_r)$ are defined as below. Let

$$u(s|\theta, \Delta_s, \Delta_r) \quad := \quad \begin{cases} 0 & \text{if } \theta \leq \frac{1}{2} \\ \frac{f_h - f_l}{(f_h + f_l)(1 - f_h)} \int_{-\infty}^{\infty} \frac{l(x)}{1 + l(x)} d\Phi(x, -z_0(\theta_p)) & \text{otherwise} \end{cases}$$

where

$$l(x) = \phi(\frac{1 - f_h}{1 - f_l} x - \Delta_s) / \phi(x)$$

$\Phi(x, -z_0(\theta_p))$ is a censored standard normal distribution that is censored below $-z_0(\theta_p)$, i.e., the density is equal to the standard normal $\phi(x)$ for $x > -z_0(\theta_p)$ but has an atom of size $\Phi(x)$ at $x = -z_0(\theta_p)$; and let

$$u(r|\theta, \Delta_s, \Delta_r) := \begin{cases} \frac{1 - f_h}{2 - f_h - f_l}(1 - \Phi(\Delta_r))(1 - c_r) - \frac{1 - f_l}{2 - f_h - f_l} c_r & \text{if } \theta \leq \frac{1}{2} \\ \frac{f_h}{f_h + f_l}(1 - \Phi(\Delta_r))(1 - c_r) - \frac{f_l}{f_h + f_l} c_r & \text{otherwise} \end{cases}$$

Proof. Item i is proved in Lemma F.5 and F.6. Item ii is proved in Lemma F.7.

For item iii note that if the pooling bid is equal to

$$\frac{l(Y_s^{n-1}(\kappa_s n) \in (\theta_0^n, \theta_p^n], \theta_i = \theta_p^n)}{1 + l(Y_s^{n-1}(\kappa_s n) \in (\theta_0^n, \theta_p^n], \theta_i = \theta_p^n)}$$

then type θ_p^n would strictly prefer to bid at the pooling bid as we have

$$\frac{\Pr(b_p^n\ wins|V = 1, Y_s^{n-1}(\kappa_s n) \in (\theta_0^n, \theta_p^n])}{\Pr(b_p^n\ wins|V = 0, Y_s^{n-1}(\kappa_s n) \in (\theta_0^n, \theta_p^n])} > 1$$

by construction. If on the other hand the pooling bid is equal to zero, then θ_p^n would rather bid just above the pooling bid and win with probability one if the price is equal to the pooling bid. Therefore, there exists a bid that leaves θ_p^n indifferent between bidding right above the pooling bid and bidding the pooling bid. This bid is unique because it is the solution to a single linear equation. Moreover, $b_p^n(\Delta_s^n, \Delta_r^n)$ is continuous because $\theta_p^n(\Delta_s^n, \Delta_r^n)$ is continuous by item i.

For types $\theta \geq \theta_p^n$ bidding according to function $b_s^n(\theta)$ is compatible with equilibrium because $\frac{\partial}{\partial\theta} l(Y_s^{n-1}(\kappa_s n) = \theta, \theta_i = \theta) > 0$ for all $\theta \geq \theta_p^n$ by item ii. Also, the pooling bid converges to

$$\lim \frac{l(Y_s^{n-1}(\kappa_s n) \in (\theta_0^n, \theta_p^n], \theta_i = \theta_p^n)}{1 + l(Y_s^{n-1}(\kappa_s n) \in (\theta_0^n, \theta_p^n], \theta_i = \theta_p^n)}$$

because the probability of winning and therefore the profit at pooling converges to zero.

Item iv. The fact that $\lim \Delta_m^n = \Delta_m$ for $m \in \{r, s\}$ implies that there are fewer bidders than goods in market r with probability one in state 0 by the law of large numbers. Therefore, the price in state $V = 0$ is equal to c_r with probability one. The fact that agents bid according to $b_r^n(\theta)$, $\lim \Delta_r^n = \Delta_r$, and the central limit theorem together imply that the price in state $V = 1$ is equal to 1 with probability $\Phi(\Delta_r)$ and equal to c_r with probability $1 - \Phi(\Delta_r)$. These imply that the limit utilities are given by the function $u(r, b_r(\theta)|\theta, \Delta_s, \Delta_r)$. Moreover, the central limit theorem implies that $u^n(r, b_r^n(\theta)|\theta, \Delta_s^n, \Delta_r^n)$ converges to $u(r, b_r(\theta)|\theta, \Delta_s, \Delta_r)$ uniformly for any $(\Delta_s^n, \Delta_r^n) \in [\underline{\Delta}_s, \bar{\Delta}_s] \times [\underline{\Delta}_r, \bar{\Delta}_r]$.

In market s, the probability that the price is equal to the pooling price is equal to $\Phi(-\frac{1-f_h}{1-f_l} z(\theta_p) - \Delta_s)$ and $\Phi(-z(\theta_p))$ in states $V = 1$ and $V = 0$, respectively. Also, the probability that the price occurs at the bid of a type x standard deviations away from $\theta_s(0)$ is equal to $\phi(\frac{1-f_h}{1-f_l} x - \Delta_s)$ and $\phi(x)$ in states $V = 1$ and $V = 0$, respectively. Both of these assertions follow from the central limit theorem. The bids $\beta_s^n(\theta^n)$ of a sequence of types θ^n, who are each x standard deviations away from $\theta_s^n(0)$, converge uniformly to $\frac{l(x)}{1+l(x)}$ by Proposition A.2. Therefore, the expected price in state $V = 0$ at the limit is given by

$$p_s(\Delta_s) := \int_{-\infty}^{\infty} \frac{l(x)}{1 + l(x)} d\Phi(x, -z(\Delta_s)). \tag{F.5}$$

Types $\theta < \frac{1}{2}$ are indifferent between the pooling bid and a bid strictly larger than the pooling bid. Also, types $\theta_s(1) < \theta < \frac{1}{2}$ win an object with probability one in both states by the law of large numbers. Moreover, payoff at pooling converges to zero and this implies that

$$\lim u^n(s, b_s^n(\theta)|\theta, \Delta_s^n, \Delta_r^n) = \frac{1-f_h}{2-f_h-f_l}(1 - \lim \mathbb{E}[P_s^n|V=1]) - \frac{1-f_l}{2-f_h-f_l}p_s(\Delta_s) = 0$$

if $\theta < \frac{1}{2}$. Therefore solving for $1 - \lim \mathbb{E}[P_s^n|V=1]$ delivers

$$1 - \lim \mathbb{E}[P_s^n|V=1] = \frac{1-f_l}{1-f_h}p_s(\Delta_s).$$

Moreover,

$$\lim u^n(s, b_s^n(\theta)|\theta, \Delta_s^n, \Delta_r^n) = \frac{f_h}{f_h+f_l}(1 - \lim \mathbb{E}[P_s^n|V=1]) - \frac{f_l}{f_h+f_l}p_s(\Delta_s)$$

for any $\theta > \frac{1}{2}$ and substituting $\frac{1-f_l}{1-f_h}p_s(\Delta_s)$ for $(1 - \lim \mathbb{E}[P_s^n|V=1])$ implies that

$$\lim u^n(s, b_s^n(\theta)|\theta, \Delta_s^n, \Delta_r^n) = \frac{f_h - f_l}{f_h + f_l(1-f_h)}p_s(\Delta_s) = u(s|\theta, \Delta_s, \Delta_r).$$

Furthermore, the central limit theorem implies that the convergence is uniform for any $(\Delta_s^n, \Delta_r^n) \in [\underline{\Delta}_s, \bar{\Delta}_s] \times [\underline{\Delta}_r, \bar{\Delta}_r]$. $\square$

We now use the Lemma F.4 to prove that there is a sufficiently large N such that the equilibrium described in Proposition 4.2.

Proof of Proposition 4.2. The proof will proceed as follows: we will first find values (Δ_s^*, Δ_r^*) which ensures that all types are indifferent between the two markets at the limit. Then, we pick a rectangle $[\underline{\Delta}_s, \bar{\Delta}_s] \times [\underline{\Delta}_r, \bar{\Delta}_r]$ that contains (Δ_s^*, Δ_r^*) and we show that there is (Δ_s^n, Δ_r^n) in this rectangle such that all types are indifferent between the two markets for all n larger than a cutoff N given the market selection defined by (Δ_s^n, Δ_r^n).

We now find (Δ_s^*, Δ_r^*). Pick Δ_r^* as the unique value that satisfies the following equation

$$\frac{1-f_h}{2-f_h-f_l}(1-\Phi(\Delta_r^*))(1-c) \quad = \quad \frac{1-f_l}{2-f_h-f_l}c.$$

This equality ensures that the low types make zero profit in market r. The equality above has a solution if $\frac{1-c_r}{c_r} > \frac{1-f_l}{1-f_h}$, i.e., if c smaller than a threshold. Moreover, the solution to the equation is unique because Φ is strictly increasing.

Pick Δ_s^* as the unique value that satisfies the following equation (see equation (F.5) for the limit price's formula)

$$\lim \mathbb{E}[P_s^n|V=0] = p_s(\Delta_s^*) = \int \frac{l(x)}{1+l(x)}d\Phi(x, -z(\Delta_s^*)) = c.$$

This equality in conjunction with the equality that defined Δ_r^* ensures that all types are indifferent between the two markets. The function $\int \frac{l(x)}{1+l(x)}d\Phi(x, -z(\Delta))$ is decreasing in Δ and therefore the equation above has a unique solution for all c smaller than a threshold. To see this note that $l(x)$ is a monotone increasing function by Lemma F.7. Moreover, if

$\Delta > \Delta'$, then $z(\Delta) > z(\Delta')$ and therefore $\Phi(x, -z(\Delta'))$ first order stochastically dominates $\Phi(x, -z(\Delta))$. This implies that

$$\int \frac{l(x)}{1+l(x)} d\Phi(x, -z(\Delta)) < \int \frac{l(x)}{1+l(x)} d\Phi(x, -z(\Delta')).$$

We now construct the rectangle $[\underline{\Delta}_s, \bar{\Delta}_s] \times [\underline{\Delta}_r, \bar{\Delta}_r]$. Pick $\bar{\Delta}_s > \Delta_s^*$ such that $p_s(\bar{\Delta}_s) = c/2$ and pick $\underline{\Delta}_s < \Delta_s^*$ such that $p_s(\underline{\Delta}_s) = 2c$. Also, pick $\bar{\Delta}_r$ and $\underline{\Delta}_r$ sufficiently close to Δ_r^* such that

$$\frac{f_h}{f_h+f_l}(1-\Phi(\underline{\Delta}_r))(1-c) - \frac{f_l}{f_h+f_l}c \quad < \quad 2c\frac{f_h - f_l}{(f_h+f_l)(1-f_h)}, \tag{F.6}$$

$$\frac{f_h}{f_h+f_l}(1-\Phi(\bar{\Delta}_r))(1-c) - \frac{f_l}{f_h+f_l}c \quad > \quad \frac{c}{2}\frac{f_h - f_l}{(f_h+f_l)(1-f_h)}. \tag{F.7}$$

We now show that there is (Δ_s^n, Δ_r^n) in the rectangle that leaves all types indifferent between the two markets. Define the correspondence

$$\Gamma := \Gamma_s^n \times \Gamma_r^n : [\underline{\Delta}_s, \bar{\Delta}_s] \times [\underline{\Delta}_r, \bar{\Delta}_r] \Longrightarrow [\underline{\Delta}_s, \bar{\Delta}_s] \times [\underline{\Delta}_r, \bar{\Delta}_r]$$

such that

$$\Gamma_s^n(\Delta_s', \Delta_r') \quad = \quad \begin{cases} \bar{\Delta}_s & \text{if } u^n(s, b_s^n(1)|\theta_i = 1, \Delta_s', \Delta_r') > u^n(r, b_r^n(1)|\theta_i = 1, \Delta_s', \Delta_r') \\ [\underline{\Delta}_s, \bar{\Delta}_s] & \text{if } u^n(s, b_s^n(1)|\theta_i = 1, \Delta_s', \Delta_r') = u^n(r, b_r^n(1)|\theta_i = 1, \Delta_s', \Delta_r') \\ \underline{\Delta}_s & \text{if } u^n(s, b_s^n(1)|\theta_i = 1, \Delta_s', \Delta_r') < u^n(r, b_r^n(1)|\theta_i = 1, \Delta_s', \Delta_r') \end{cases}$$

$$\Gamma_r^n(\Delta_s', \Delta_r') \quad = \quad \begin{cases} \bar{\Delta}_r & \text{if } u^n(r, b_r^n(0)|\theta_i = 0, \Delta_s', \Delta_r') > u^n(s, b_s^n(0)|\theta_i = 0, \Delta_s', \Delta_r') \\ [\underline{\Delta}_r, \bar{\Delta}_r] & \text{if } u^n(r, b_r^n(0)|\theta_i = 0, \Delta_s', \Delta_r') = u^n(s, b_s^n(0)|\theta_i = 0, \Delta_s', \Delta_r') \\ \underline{\Delta}_r & \text{if } u^n(r, b_r^n(0)|\theta_i = 0, \Delta_s', \Delta_r') < u^n(s, b_s^n(0)|\theta_i = 0, \Delta_s', \Delta_r') \end{cases}$$

where $b_s^n(1)$ and $b_r^n(1)$ are the bids submitted by type $\theta_i = 1$, i.e., highest bids submitted, in markets s and r respectively. Similarly $b_s^n(1)$ and $b_r^n(1)$ are the bids submitted by type $\theta_i = 0$ in markets s and r respectively. These bidding functions are formally defined in equations (F.3) and (F.4).

Note that the correspondence Γ is upper-hemi continuous, compact and convex valued. Therefore, the correspondence has a fixed point by Kakutani's Theorem. Also, in the claim just below we argue that there exists an N such that for all $n > N$ any fixed point of the correspondence occurs in the interior of the domain, i.e., $\Delta_s^n \in (\underline{\Delta}_s, \bar{\Delta}_s)$ and $\Delta_r^n \in (\underline{\Delta}_r, \bar{\Delta}_r)$ and therefore,

$$u^n(s|\theta_i = \theta, \Delta_s^n, \Delta_r^n) = u^n(r|\theta_i = \theta, \Delta_s^n, \Delta_r^n)$$

which implies that the fixed point is an equilibrium.

Claim F.1. There is an N such that for all $n > N$ any fixed point of the correspondence occurs in the interior of the domain.

Proof. Note that there exits an $N(\underline{\Delta}_s, \bar{\Delta}_s, \underline{\Delta}_r, \bar{\Delta}_r, \epsilon)$ such that

$$\left|u^n(b^n_m(\theta)|\theta_i = \theta, \bar{\Delta}_s, \Delta_r) - u(m|\theta_i = 1, \bar{\Delta}_s, \Delta_r)\right| < \epsilon$$

for all $n > N(\underline{\Delta}_s, \bar{\Delta}_s, \underline{\Delta}_r, \bar{\Delta}_r, \epsilon)$ by the Lemma (F.1), item *iv*.

In order to show that the fixed point occurs in the interior, we show that it does not occur at an edge of the rectangle. We show this by focusing on four cases which correspond to the four edges of the rectangle.

Case 1: Suppose that $(\underline{\Delta}_s, \Delta_r) \in \Gamma^n(\bar{\Delta}_s, \Delta_r)$ and note that

$$\begin{aligned} u(s|\theta_i = 1, \underline{\Delta}_s, \Delta_r) = 2c\frac{f_h - f_l}{(f_h + f_l)(1 - f_h)} \quad &> \quad u(r|\theta_i = 1, \underline{\Delta}_s, \Delta_r) \\ &= \quad \frac{f_h}{f_h + f_l}(1 - \Phi(\Delta_r))(1 - c) - \frac{f_l}{f_h + f_l}c \end{aligned}$$

for all $\Delta_r \in [\underline{\Delta}_r, \bar{\Delta}_r]$. This follows from (F.6). Therefore, $u^n(s, b^n_s(1)|\theta_i = 1, \underline{\Delta}_s, \Delta_r) > u^n(r, b^n_r(1)|1, \underline{\Delta}_s, \Delta_r)$ for $\Delta_r \in [\underline{\Delta}_r, \bar{\Delta}_r]$ and all $n > N(\underline{\Delta}_s, \bar{\Delta}_s, \underline{\Delta}_r, \bar{\Delta}_r, \epsilon)$ for sufficiently small ϵ by the Lemma (F.1), item *iv*. However, this implies that $\Gamma^n_s(\underline{\Delta}_s, \Delta_r) = \bar{\Delta}_s$ a contradiction.

Case 2: Suppose that $(\bar{\Delta}_s, \Delta_r) \in \Gamma^n(\bar{\Delta}_s, \Delta_r)$ and note that

$$\begin{aligned} u(s|1, \bar{\Delta}_s, \Delta_r) = \frac{c}{2}\frac{f_h - f_l}{(f_h + f_l)(1 - f_h)} \quad &< \quad u(r|1, \bar{\Delta}_s, \Delta_r) \\ &= \quad \frac{f_h}{f_h + f_l}(1 - \Phi(\Delta_r))(1 - c) - \frac{f_l}{f_h + f_l}c \end{aligned}$$

for all $\Delta_r \in [\underline{\Delta}_r, \bar{\Delta}_r]$. This follows from equality (F.7). Therefore, $u^n(s, b^n_s(1)|\theta_i = 1, \bar{\Delta}_s, \Delta_r) < u^n(r, b^n_r(1)|\theta_i = 1, \bar{\Delta}_s, \Delta_r)$ for $\Delta_r \in [\underline{\Delta}_r, \bar{\Delta}_r]$ and all $n > N(\underline{\Delta}_s, \bar{\Delta}_s, \underline{\Delta}_r, \bar{\Delta}_r, \epsilon)$ for sufficiently small ϵ by the Lemma (F.1), item *iv*. However, this implies that $\Gamma^n_s(\bar{\Delta}_s, \Delta_r) = \underline{\Delta}_s$ a contradiction.

Case 3: Suppose that $(\Delta_s, \underline{\Delta}_r) \in \Gamma^n(\Delta_s, \underline{\Delta}_r)$ and note that $0 = u(s|\theta_i = 0, \Delta_s, \underline{\Delta}_r)$ for all $(\Delta_s, \Delta_r) \in [\underline{\Delta}_s, \bar{\Delta}_s] \times [\underline{\Delta}_r, \bar{\Delta}_r]$ by construction. The fact that $\underline{\Delta}_r < \Delta^*_r$ implies that $u(r|\theta_i = 0, \Delta_s, \underline{\Delta}_r) > 0$. Therefore, $u(r|\theta_i = 0, \Delta_s, \underline{\Delta}_r) > u(s|\theta_i = 0, \Delta_s, \underline{\Delta}_r) = 0$. Hence,

$$u^n(r, b^n_r(0)|\theta_i = 0, \Delta_s, \underline{\Delta}_r) > u^n(s, b^n_s(0)|\theta_i = 0, \Delta_s, \underline{\Delta}_r)$$

for $\Delta_s \in [\underline{\Delta}_s, \bar{\Delta}_s]$ and all $n > N(\underline{\Delta}_s, \bar{\Delta}_s, \underline{\Delta}_r, \bar{\Delta}_r, \epsilon)$ for sufficiently small ϵ by the Lemma (F.1), item *iv*. However, this implies that $\Gamma^n_r(\Delta_s, \underline{\Delta}_r) = \bar{\Delta}_r$ a contradiction.

Case 4: suppose that $(\Delta_s, \bar{\Delta}_r) \in \Gamma^n(\Delta_s, \bar{\Delta}_r)$. The fact that $\bar{\Delta}_r > \Delta^*_r$ implies that $u(r|\theta_i = 0, \Delta_s, \bar{\Delta}_r) < 0$. Therefore, $u(r|\theta_i = 0, \Delta_s, \bar{\Delta}_r) < u(s|\theta_i = 0, \Delta_s, \bar{\Delta}_r) = 0$. Hence,

$$u^n(r, b^n_r(0)|\theta_i = 0, \Delta_s, \bar{\Delta}_r) < u^n(s, b^n_s(0)|\theta_i = 0, \Delta_s, \bar{\Delta}_r)$$

for $\Delta_s \in [\underline{\Delta}_s, \bar{\Delta}_s]$ and all $n > N(\underline{\Delta}_s, \bar{\Delta}_s, \underline{\Delta}_r, \bar{\Delta}_r, \epsilon)$ for sufficiently small ϵ by the Lemma (F.1), item *iv*. However, this implies that $\Gamma^n_r(\Delta_s, \bar{\Delta}_r) = \underline{\Delta}_r$ a contradiction. □

□

F.2. Exogenous Outside Option. We now prove Proposition 3.3. This proposition can be proved by applying Lemma F.1 where we take $\theta_0^n = 0$. If $\theta_0^n = 0$, then all the conclusions of Lemma F.1 with respect to market s remain valid because Δ_s^n is independent of θ_0^n and depends only on θ_1^n.

Proof of Proposition 3.3. Suppose that $[0, \theta_1^n]$ bid in market s where $\theta_1^n > 1/2$. We will show that we can choose this type so that that all types $\theta \in [1/2, 1]$ are indifferent between the two markets. Also, all types $\theta \in [0, 1/2]$ prefer market s as their payoff from option r is negative.

The payoff of a type $\theta \in [1/2, 1]$ from submitting a bid in market s is arbitrarily close to

$$u(m, b_m(\theta)|\theta_i = \theta, \Delta_s) = \int_{-\infty}^{\infty} \frac{l(x)}{1 + l(x)} d\Phi(x, -z(\Delta_s)) \frac{f_h - f_l}{(f_h + f_l)(1 - f_h)}$$

by Lemma F.1. If Δ_s is large then this payoff is close to zero because the price in the low state is close to zero. Moreover, if Δ_s is small then

$$\int_{-\infty}^{\infty} \frac{l(x)}{1 + l(x)} d\Phi(x, -z(\Delta_s)) \frac{f_h - f_l}{(f_h + f_l)(1 - f_h)}$$

is a positive number which strictly exceeds the expected payoff of a type θ in market r. Therefore, there is a Δ_s which leaves the types $\theta \in [1/2, 1]$ indifferent between the two markets by the intermediate value theorem because the function under consideration is continuous. Also, there exists an N such that for all $n > N$, there exists a Δ_s^n such that types in $\theta \in [1/2, 1]$ are indifferent between the two markets. This is because $u^n(m, b_m^n(\theta)|\theta_i = \theta, \Delta_s^n)$ converges to $u(m, b_m(\theta)|\theta_i = \theta, \Delta_s)$ uniformly by Lemma F.1, the function $u^n(m, b_m^n(\theta)|\theta_i = \theta, \Delta_s^n)$ is continuous in Δ_s^n and hence the intermediate value theorem delivers results. □

F.3. Intermediate Results. In this subsection we fix $(\underline{\Delta}_s, \bar{\Delta}_s, \underline{\Delta}_r, \bar{\Delta}_r) \in \mathbb{R}^4$ and consider sequences $\{\theta_0^n, \theta_1^n, \theta_p^n\}$ such that types $(\theta_0^n, \theta_1^n]$ select market s, types $\theta \notin (\theta_0^n, \theta_1^n)$ select market r and types (θ_0^n, θ_p^n) submit the pooling bid. Note that since the mapping between $\{\theta_0^n, \theta_1^n\}$ and $\{\Delta_r^n, \Delta_s^n\}$ is linear and one-to-one we use the two sequences $\{\theta_0^n, \theta_1^n\}$ and $\{\Delta_r^n, \Delta_s^n\}$ interchangeably. Along the sequences, we assume that $\Delta_m^n \in [\underline{\Delta}_m, \bar{\Delta}_m]$. We define the type θ^{*n} such that

$$\bar{F}_s^n(\theta^{*n}|1) = \bar{F}_s^n(\theta^{*n}|0)$$
$$(1 - f_h)(\theta_s^n(0) + \Delta_s^n - \theta^{*n}) = (1 - f_l)(\theta_s^n(0) - \theta^{*n})$$

In words, there is the same mass of types above type θ^{*n} in market s in both states. We assume that $\{\theta_0^n, \theta_1^n, \theta^n\}$ converges and therefore θ^{*n} converges.

For any θ^n we define:

$$z_v^n(\theta^n) = \frac{\sqrt{n}(\bar{F}(\theta^n|v) - \kappa_s)}{\sqrt{\bar{F}(\theta^n|v)(1 - \bar{F}(\theta^n|v))}}.$$

In the sequences that we consider, we assume that $z_v^n(\theta^n) < 2z_v^n(\theta^{*n})$, i.e., we assume that θ^n is greater or equal to $\theta^{*n} - a/\sqrt{n}$ for some finite and positive a. Moreover, we assume that $z_v^n(\theta^n) > -\infty$. In what follows we say a function $x^n(\Delta_r, \Delta_s, \theta)$ converges uniformly to $x(\Delta_r, \Delta_s, \theta)$ if for every ϵ there is an N such that for all $n > N$ we have

$$|x^n(\Delta_r, \Delta_s, \theta) - x(\Delta_r, \Delta_s, \theta)| \leq \epsilon$$

for all $(\Delta_s, \Delta_r) \in [\underline{\Delta}_s, \bar{\Delta}_s] \times [\underline{\Delta}_r, \bar{\Delta}_r]$ and all $-\infty < z_v^n(\theta) < 2z_v^n(\theta^*)$. We note that there is an N and a positive constant A such that $\Pr(Y_s^{n-1}(k) \in (\theta_0, \theta)|v) \geq A > 0$. This is because $\lim z_v^n(\theta^{*n})$ is uniformly bounded below by some finite z^* and $\Pr(Y_s^{n-1}(k) \in (\theta_0, \theta)|v) \geq \Phi(z^*) > 0$.

Let X^n denote the total number of bidders who submit the pooling bid and let

$$\bar{X}_v^n := \mathbb{E}[X^n|V = v] = n(1 - f_v)(\theta^n - \theta_0^n).$$

We let G^n denote the total number of goods left to be allocated to bidders who submit the pooling bid, i.e, G^n is equal to k minus the number of bidders who submit a bid greater than the pooling bid and let

$$\bar{G}_v^n := \mathbb{E}[G^n|V = v, Y_s^{n-1}(k) \in (\theta_0^n, \theta^n)].$$

Definition F.1. Let

$$h^n(\theta; \Delta_r^n, \Delta_s^n, v) \quad := \quad \frac{bi(n\kappa - 1; n-1, \bar{F}_s(\theta|v))}{Bi(n\kappa - 1; n, \bar{F}_s(\theta|v)) - Bi(n\kappa; n, F_s(1|v))}$$

$Bi(k; n, p)$ and $bi(k; n, p)$ represent the CDF and PDF for the binomial distribution with k successes out of n tries where the success probability is equal to p. Note that $F_s^n(1|v) = F((\theta_0^n, \theta_1^n]|v)$ and $\bar{F}_s(\theta|v) = F((\theta, \theta_1^n]|v)$.

The function h^n is continuously differentiable in θ, Δ_r^n and Δ_s^n. This is because (1) the cutoffs θ_1^n and θ_0^n are linear functions and hence continuously differentiable functions of Δ_r^n and Δ_s^n; (2) The cumulative distribution function $F(\cdot|v)$ is continuously differentiable at any $\theta \neq \frac{1}{2}$; and (3) The functions $Bi(k; n, p)$ and $bi(k; n, p)$ are continuously differentiable in p.

Lemma F.2. *The expected number of goods allocated to types who submit a pooling bid conditional on the price equaling the pooling is as follows:*

$$\frac{\bar{G}_v^n}{\sqrt{np_v(1-p_v)}} = h^n(\theta^n; v)\sqrt{np_v(1-p_v)} - z_v^n(\theta^n)/\epsilon^n \geq 0$$

where we have suppressed reference to (Δ_r^n, Δ_s^n), ϵ^n is a function that converges uniformly to 1 and $p_v = \bar{F}_s(\theta_p^n|v)$.

Proof. Let $\lambda_s(v) = F_s(1|v)$, $\epsilon^n = Bi(k-1; n, p_v)/(Bi(k-1; n, p_v) - Bi(k; n, \lambda_s(v)))$. The expected number of goods available to types who pool

$$\bar{G}_v^n = \frac{\sum_{i=0}^{k-1} C(n,i) p_v^i (1-p_v)^{n-i}(k-i)}{\sum_{i=0}^{k-1} C(n,i) p_v^i (1-p_v)^{n-i} - \sum_{i=0}^{k} C(n,i)\lambda_s(v)^i(1-\lambda_s(v))^{n-i}}$$

We simplify the numerator as follows:

$$
\begin{aligned}
& \sum_{i=0}^{k-1} C(n,i)p_v^i(1-p_v)^{n-i}(k-i) \\
= \; & k\sum_{i=0}^{k-1} C(n,i)p_v^i(1-p_v)^{n-i} - \sum_{i=0}^{k-1} iC(n,i)p_v^i(1-p_v)^{n-i} \\
= \; & k\sum_{i=0}^{k-1} C(n,i)p_v^i(1-p_v)^{n-i} - pn\sum_{i=1}^{k-1} C(n-1,i-1)p_v^{i-1}(1-p_v)^{n-i} \\
= \; & k\sum_{i=0}^{k-1} C(n,i)p_v^i(1-p_v)^{n-i} - pn\sum_{i=0}^{k-2} C(n-1,i)p_v^{i}(1-p_v)^{n-i-1} \\
= \; & kBi(k-1;n,p_v) - pn\left(Bi(k-1;n,p_v) - (1-p_v)bi(k-1;n-1,p_v)\right)
\end{aligned}
$$

Where the last equality follows because:

$$
\begin{aligned}
Bi(k-1;n,p_v) &= \sum_{i=0}^{k-1} C(n,i)p_v^i(1-p_v)^{n-i} \\
&= \sum_{i=0}^{k-2} C(n-1,i)p_v^i(1-p_v)^{n-i-1} + (1-p_v)C(n-1,k-1)p_v^{k-1}(1-p_v)^{n-k} \\
&= Bi(k-2;n,p_v) + (1-p_v)bi(k-1;n-1,p_v)
\end{aligned}
$$

Therefore, we find

$$
\begin{aligned}
\bar{G}_v^n &= np_v(1-p_v)\left(h^n(\theta_p^n;v) - \frac{p_v-\kappa_s}{p_v(1-p_v)}\frac{1}{\epsilon^n}\right) \\
&= np_v(1-p_v)h^n(\theta^n;v) - \frac{z_v^n(\theta^n)}{\epsilon^n}\sqrt{np_v(1-p_v)}
\end{aligned}
$$

The above derivation implies that $\sqrt{np_v(1-p_v)}h^n(\theta^n;v) - \frac{z_v^n(\theta^n)}{\epsilon^n} \geq 0$ for all (θ^n, v). □

Lemma F.3. *The following converges uniformly*

$$
\sqrt{p_v(1-p_v)}\left(h^n(\theta^n;v)\sqrt{np_v(1-p_v)} - z_v^n(\theta^n)/\epsilon^n\right) \to \sqrt{\kappa_s(1-\kappa_s)}\left(h(z_v) - z_v\right)
$$

where

$$
h(z_v) = \frac{\phi(z_v)}{1-\Phi(z_v)}
$$

is the reciprocal Mill's ratio (or the hazard function) for the standard normal (see by Sampford (1953) for details) and $z_v = \lim z_v^n(\theta^n)$.

Proof. Follows immediately from Proposition A.2 and from the fact that $\epsilon^n \to 1$ uniformly. □

Lemma F.4. *We have the following*

$$\frac{\Pr(b_p^n \text{ } wins|V=1, Y_s^{n-1}(k) \in (\theta_0^n, \theta^n])}{\Pr(b_p^n \text{ } wins|V=0, Y_s^{n-1}(k) \in (\theta_0^n, \theta^n])} \to \left(\frac{1-f_l}{1-f_h}\right)\left(\frac{h(z_1)-z_1}{h(z_0)-z_0}\right) \tag{F.8}$$

$$\frac{l(Y_s^{n-1}(k)=\theta^n, \theta_i=\theta^n)}{l(Y_s^{n-1}(k) \in (\theta_0^n, \theta^n], \theta_i=\theta^n)} \to \frac{(1-f_h)}{(1-f_1)}\frac{h(z_1)}{h(z_0)} \tag{F.9}$$

$$\frac{\frac{d}{d\theta}\ln h^n(\theta^n; \Delta_r^n, \Delta_s^n, 1)}{\frac{d}{d\theta}\ln h^n(\theta^n; \Delta_r^n, \Delta_s^n, 0)} \to \frac{(1-f_h)}{(1-f_1)}\frac{h(z_1)-z_1}{h(z_0)-z_0} \tag{F.10}$$

where the convergence is uniform.

Proof. We first show that equation (F.9) is satisfied. Note that

$$l(Y_s^{n-1}(k)=\theta^n, \theta_i=\theta^n) = \left(\frac{1-f_h}{1-f_l}\right)^2 \frac{bi(k-1;n-2, \bar{F}_s(\theta^n|1))}{bi(k-1;n-2, \bar{F}_s(\theta^n|0))}$$

and

$$l(Y_s^{n-1}(k) \in (\theta_0^n, \theta^n], \theta_i=\theta^n) = \frac{1-f_h}{1-f_l}\left(\frac{Bi(k-1;n-1, \bar{F}_s^n(\theta^n|1)) - Bi(k;n-1, F_s^n(1|1))}{Bi(k-1;n-1, \bar{F}_s^n(\theta^n|0)) - Bi(k;n-1, F_s^n(1|0))}\right)$$

Therefore,

$$\frac{l(Y_s^{n-1}(k)=\theta^n, \theta_i=\theta^n)}{l(Y_s^{n-1}(k) \in (\theta_0^n, \theta^n], \theta_i=\theta^n)} = \left(\frac{1-f_h}{1-f_l}\right)\frac{h^n(\theta^n, 1)}{h^n(\theta^n, 0)}.$$

Thus, equation (F.9) follows from Lemma F.3.

We now show that equation (F.10) is satisfied. Through direct computation (and suppressing reference to Δ_s and Δ_r) we find

$$\frac{d\ln h^n(\theta; v)}{dp} = -\frac{1-f_v}{h^n(\theta; v)}\times$$
$$\left(\frac{nbi(k-1;n-1,p)^2}{(Bi(k-1;n,p)-Bi(k;n,F_s^n(1|v))^2} - \frac{pn-k}{p(1-p)}\frac{bi(k-1;n-1,p)}{(Bi(k-1;n,p)-Bi(k;n,F_s^n(1|v)))}\right)$$

where we define $p = \bar{F}_s^n(\theta|v)$ and use the facts that $\frac{d}{d\theta}\bar{F}_s^n(\theta|v) = -(1-f_v)$, $\frac{d}{dp}bi(k-1;n-1,p) = -\frac{pn-k}{p(1-p)}bi(k-1;n-1,p)$, $\frac{d}{dp}Bi(k;n,F_s^n(1|v)) = 0$, and $\frac{d}{dp}(Bi(k-1;n,p)-Bi(k;n,F_s^n(1|v))) = -nbi(k-1;n-1,p)$. Rearranging we find

$$\frac{d\ln h^n(\theta; v)}{d\theta} \quad = \quad -(1-f_v)n\left(h^n(\theta; v) - \frac{\bar{F}_s^n(\theta|v)-\kappa_s}{\bar{F}_s^n(\theta|v)(1-\bar{F}_s^n(\theta|v))}\right)$$

Hence, equation (F.10) again follows from Lemma F.3.

We now turn to equation (F.8). In order to show that equation (F.8) is satisfied, below we establish the following intermediate result:

$$\Pr(b_p^n \text{ } wins|V=v, Y_s^{n-1}(k) \in (\theta_0^n, \theta^n]) = \frac{\bar{G}_v^n}{\bar{X}_v^n}\delta^n = \frac{\bar{G}_v^n}{n(\theta^n-\theta_0^n)(1-f_v)}\delta^n$$

where $\delta^n \to 1$ uniformly. Therefore, once we establish this intermediate result, equation (F.8) follows from Lemma F.2, Lemma F.3 and the fact that $\delta^n \to 1$ uniformly.

We now prove that

$$\Pr(b_p^n\ wins | V = v, Y_s^{n-1}(k) \in (\theta_0^n, \theta^n]) = \frac{\bar{G}_v^n}{\bar{X}_v^n}\delta^n$$

where $\delta^n \to 1$ uniformly.

For any positive constant $\delta < 1$, let $q = \Pr[X^n \leq (1-\delta)\bar{X}_v^n | Y_s^{n-1}(k) \in (\theta_0^n, \theta^n], V = v]$. We have $\Pr[X^n \leq (1-\delta)\bar{X}_v^n | V = v] \leq e^{-\frac{\delta^2 \bar{X}_v^n}{2}}$ by Chernoff's Inequality (see Lemma A.1) and

$$\Pr[X^n \leq (1-\delta)\bar{X}_v^n | V = v, Y_s^{n-1}(k) \in (\theta_0^n, \theta^n]] \Pr(Y_s^{n-1}(k) \in (\theta_0^n, \theta^n] | V = v) \leq \Pr[X^n \leq (1-\delta)\bar{X}_v^n | V = v]$$

Therefore,

$$\Pr[X^n \leq (1-\delta)\bar{X}_v^n | V = v, Y_s^{n-1}(k) \in (\theta_0^n, \theta^n]] \quad \leq \quad Ae^{-\frac{\delta^2 \bar{X}_v^n}{2}}$$

where $1/\Pr(Y_s^{n-1}(k) \in (\theta_0^n, \theta^n] | V = v) \geq \Phi(C) = A$ for sufficiently large n. Below we provide an upper bound for $\Pr(b_p^n\ wins | V = v, Y_s^{n-1}(k) \in (\theta_0^n, \theta^n])$:

$$\begin{aligned}
& \Pr(b_p^n\ wins | V = v, Y_s^{n-1}(k) \in (\theta_0^n, \theta^n]) \\
= \ & \mathbb{E}[\frac{G^n}{X^n} | Y_s^{n-1}(k) \in (\theta_0^n, \theta^n], V = v] \\
\leq \ & \mathbb{E}[\frac{G^n}{(1-\delta)\bar{X}_v^n} | Y_s^{n-1}(k) \in (\theta_0^n, \theta^n], V = v, X^n \geq (1-\delta)\bar{X}_v^n](1-q) + q && \text{(F.11)} \\
\leq \ & \mathbb{E}[\frac{G^n}{(1-\delta)\bar{X}_v^n} | Y_s^{n-1}(k) \in (\theta_0^n, \theta^n], V = v, X^n \geq (1-\delta)\bar{X}_v^n] + Ae^{-\frac{\delta^2 \bar{X}_v^n}{2}} && \text{(F.12)} \\
\leq \ & \frac{\mathbb{E}[G^n | Y_s^{n-1}(k) \in (\theta_0^n, \theta^n], V = v]}{(1-\delta)\bar{X}_v^n (1 - Ae^{-\frac{\delta^2 \bar{X}_v^n}{2}})} + Ae^{-\frac{\delta^2 \bar{X}_v^n}{2}} && \text{(F.13)} \\
= \ & \frac{\bar{G}_v^n}{n(\theta^n - \theta_0^n)(1 - f_v)} \left(\frac{1}{(1-\delta)(1 - Ae^{-\frac{\delta^2 \bar{X}_v^n}{2}})} + \frac{n(\theta^n - \theta_0^n)Ae^{-\frac{\delta^2 \bar{X}_v^n}{2}}}{\bar{G}_v^n} \right)
\end{aligned}$$

Inequality F.11 is obtained by observing that $X^n \geq (1-\delta)\bar{X}_1^n$ in each term in the expectation, $\Pr[X^n \geq (1-\delta)\bar{X}_v^n | Y_s^{n-1}(k) \in (\theta_0^n, \theta^n], V = v] \leq 1$ and $\mathbb{E}[\frac{G^n}{X^n} | Y_s^{n-1}(k) \in (\theta_0^n, \theta^n], V = v, X^n \leq (1-\delta)\bar{X}_v^n] \leq 1$. For inequality F.13 we use the facts that

$$\mathbb{E}[G^n | Y_s^{n-1}(k) \in (\theta_0^n, \theta^n], V = v, X^n \geq (1-\delta)\bar{X}_v^n](1-q) \leq \mathbb{E}[G^n | Y_s^{n-1}(k) \in (\theta_0^n, \theta^n], V = v]$$

and $1 - q \geq 1 - Ae^{-\frac{\delta^2 \bar{X}_v^n}{2}}$.

Below we provide a lower bound for $\Pr(b_p^n\ wins | V = v, Y_s^{n-1}(k) \in (\theta_0^n, \theta^n])$. For any positive constant $\delta < 1$, let $q = \Pr[X^n > (1+\delta)\bar{X}_v^n | Y_s^{n-1}(k) \in (\theta_0^n, \theta^n], V = v]$. We have $\Pr[X^n > (1+\delta)\bar{X}_v^n | V = v] \leq e^{-\frac{\delta^2 \bar{X}_v^n}{2+\delta}}$ by Chernoff's Inequality (see Lemma A.1). Therefore,

$$\Pr[X^n > (1+\delta)\bar{X}_v^n | V = v, Y_s^{n-1}(k) \in (\theta_0^n, \theta^n]] \leq \frac{e^{-\frac{\delta^2 \bar{X}_v^n}{2+\delta}}}{\Pr(Y_s^{n-1}(k) \in (\theta_0^n, \theta^n] | V = v)} \leq Ae^{-\frac{\delta^2 \bar{X}_v^n}{2+\delta}}$$

because $1/\Pr(Y_s^{n-1}(k) \in (\theta_0^n, \theta^n]|V = v) \geq \Phi(C) = A$ for all n sufficiently large. Using a similar logic as in the upper bound we obtain the following

$$\begin{aligned}
& \Pr(b_p^n \; wins|V = v, Y_s^{n-1}(k) \in (\theta_0^n, \theta^n]) \\
\geq \; & \mathbb{E}[G^n|Y_s^{n-1}(k) \in (\theta_0^n, \theta^n], V = v, X^n \leq (1+\delta)\bar{X}_v^n] \frac{(1 - Ae^{-\frac{\delta^2 \bar{X}_v^n}{2+\delta}})}{(1+\delta)\bar{X}_v^n} \\
\geq \; & (1 - Ae^{-\frac{\delta^2 \bar{X}_v^n}{2+\delta}}) \frac{\mathbb{E}[G^n|Y_s^{n-1}(k) \in (\theta_0^n, \theta^n], V = v]}{(1+\delta)\bar{X}_v^n} - \frac{nAe^{-\frac{\delta^2 \bar{X}_v^n}{2+\delta}}}{(1+\delta)\bar{X}_v^n} \\
= \; & \frac{\bar{G}_v^n}{n(\theta^n - \theta_0^n)(1 - f_v)} \left(1 - Ae^{-\frac{\delta^2 \bar{X}_v^n}{2+\delta}}\right) \left(\frac{1}{(1+\delta)(1 - e^{-\frac{\delta^2 \bar{X}_v^n}{2+\delta}})} + \frac{nAe^{-\frac{\delta^2 \bar{X}_v^n}{2+\delta}}}{\bar{G}_v^n}\right) \qquad \text{(F.14)}
\end{aligned}$$

Therefore,

$$\begin{aligned}
& \frac{\bar{G}_v^n}{n(\theta^n - \theta_0^n)(1 - f_v)} \left(1 - Ae^{-\frac{\delta^2 \bar{X}_v^n}{2+\delta}}\right) \left(\frac{1}{(1+\delta)(1 - e^{-\frac{\delta^2 \bar{X}_v^n}{2+\delta}})} + \frac{nAe^{-\frac{\delta^2 \bar{X}_v^n}{2+\delta}}}{\bar{G}_v^n}\right) \\
& \qquad \leq \Pr(b_p^n \; wins|V = v, Y_s^{n-1}(k) \in (\theta_0^n, \theta^n]) \\
& \qquad \leq \frac{\bar{G}_v^n}{n(\theta^n - \theta_0^n)(1 - f_v)} \left(\frac{1}{(1-\delta)(1 - Ae^{-\frac{\delta^2 \bar{X}_v^n}{2}})} + \frac{n(\theta^n - \theta_0^n)Ae^{-\frac{\delta^2 \bar{X}_v^n}{2}}}{\bar{G}_v^n}\right)
\end{aligned}$$

Hence there exists a δ^n such that

$$\Pr(b_p^n \; wins|V = v, Y_s^{n-1}(k) \in (\theta_0^n, \theta^n]) = \frac{\bar{G}_v^n}{n(\theta^n - \theta_0^n)(1 - f_v)} \delta^n$$

where

$$\begin{aligned}
& \left(1 - Ae^{-\frac{\delta^2 \bar{X}_v^n}{2+\delta}}\right) \left(\frac{1}{(1+\delta)(1 - e^{-\frac{\delta^2 \bar{X}_v^n}{2+\delta}})} + \frac{nAe^{-\frac{\delta^2 \bar{X}_v^n}{2+\delta}}}{\bar{G}_v^n}\right) \leq \delta^n \\
& \qquad \leq \left(\frac{1}{(1-\delta)(1 - Ae^{-\frac{\delta^2 \bar{X}_v^n}{2}})} + \frac{n(\theta^n - \theta_0^n)Ae^{-\frac{\delta^2 \bar{X}_v^n}{2}}}{\bar{G}_v^n}\right)
\end{aligned}$$

Moreover, for any arbitrary $\delta \in (0, 1)$, the lower-bound converges uniform to $1/(1+\delta)$ and the upper-bound converges uniformly to $1/(1-\delta)$. Therefore, $\delta^n \to 1$ uniformly. □

Lemma F.5. *There exist an N such that*

$$l(Y_s^{n-1}(k) = \theta_p^n, \theta_i = \theta_p^n) = l(Y_s^{n-1}(k) \in (\theta_0^n, \theta_p^n], \theta_i = \theta_p^n)$$

has a unique solution $\theta_p^n(\Delta_s, \Delta_r)$ for all $n > N$, Moreover, $\theta_p^n(\Delta_s, \Delta_r)$ is a continuous function of Δ_s and Δ_r; and $z_0(\theta_p(\Delta_s, \Delta_r))$ is increasing in Δ_s.

Proof. The equality

$$l(Y_s^{n-1}(k) = \theta_p^n, \theta_i = \theta_p^n) = l(Y_s^{n-1}(k) \in (\theta_0^n, \theta_p^n], \theta_i = \theta_p^n)$$

can be expressed as follows:

$$g^n(\theta_p^n; \Delta_s^n, \Delta_r^n) = \ln \frac{1 - f_h}{1 - f_l} + \ln h^n(\theta_p^n; \Delta_s^n, \Delta_r^n, 1) - \ln h^n(\theta_p^n; \Delta_s^n, \Delta_r^n, 0) \quad = \quad 0.$$

Recall that θ^{*n} is such that $\bar{F}_s^n(\theta^{*n}|1) = \bar{F}_s^n(\theta^{*n}|0)$. We will show that $g^n(\theta; \Delta_s^n, \Delta_r^n)$ is increasing in θ, that $g^n(\theta^{*n}) < 0$ and that $g^n(\theta^{''n}) > 0$ for sufficiently large θ''^n. Also, the function $g^n(\theta; \Delta_s^n, \Delta_r^n)$ is clearly continuously differentiable in all three of its arguments (see Definition F.1). Therefore, by the implicit function theorem (see Rudin (1964), Theorem 9.28), we will conclude that there is a unique $\theta_p^n(\Delta_s, \Delta_r)$ which satisfies the required equation and $\theta_p^n(\Delta_s, \Delta_r)$ is continuous in both of its arguments. This will establish the first part of the lemma.

Lemma F.4 implies that

$$\frac{\frac{d}{d\theta} \ln h^n(\theta^n; 0)}{\frac{d}{d\theta} \ln h^n(\theta^n; 1)} \to \frac{(1 - f_l)}{(1 - f_h)} \frac{(h(z_0) - z_0)}{(h(z_1) - z_1)}$$

uniformly. The fact that $h'(z) < 1$ (see Sampford (1953)) implies that $h(z) - z$ is decreasing in z. Because $z_1 > z_0$ for any sequence of $\theta^n \geq \theta^{*n}$ and because $h(z) - z$ is decreasing in z we have that

$$\frac{(1 - f_l)}{(1 - f_h)} \frac{(h(z_0) - z_0)}{(h(z_1) - z_1)} > \frac{(1 - f_l)}{(1 - f_h)} > 1.$$

Therefore, uniform convergence implies that for any $A > 0$, there exists an N such that

$$\lim \frac{d}{d\theta} \ln h^n(\theta^n; 1) = -(h(z_1) - z_1)(1 - f_h) > \lim \frac{d}{d\theta} \ln h^n(\theta^n; 0) = -(1 - f_l)(h(z_0) - z_0)$$

for all $\theta^n \in [\theta^{*n}, \theta'' = \frac{A}{\sqrt{n}} + \theta^{*n}]$ if $n > N$. This establishes that $g^n(\theta; \Delta_s^n, \Delta_r^n)$ is increasing for all $\theta \in [\theta^{*n}, \theta'' = \frac{A}{\sqrt{n}} + \theta^{*n}]$ if $n > N$.

We now show that the constant $A > 0$ and N can be chosen sufficiently large such that for all $n > N$ we have $\ln \frac{1-f_h}{1-f_l} + \ln h^n(\theta''; 1) > \ln h^n(\theta''; 0)$ where $\theta'' = \frac{A}{\sqrt{n}} + \theta^{*n}$.

To see this, note that if θ'' is large, i.e., if A is large, then $z_v < 0$ and $z_1 - z_0$ is large. Therefore,

$$\frac{h^n(\theta''; 1)}{h^n(\theta''; 0)} \to \frac{h(z_1(\theta''))}{h(z_0(\theta''))} = \frac{1 - \Phi(z_1)}{1 - \Phi(z_0)} e^{-\frac{(z_1 - z_0)(z_1 + z_0)}{2}}$$

is large. This is because $\frac{1-\Phi(z_1)}{1-\Phi(z_0)}$ is close to one if z_v are large and negative. Moreover, $e^{-\frac{(z_1-z_0)(z_1+z_0)}{2}}$ is large because the exponent is large (since $z_1 - z_0$ is large) and positive (since $z_1 - z_0 > 0$ and $z_1 + z_0 < 0$). Hence, $\ln \frac{1-f_h}{1-f_l} + \ln h^n(\theta''; 1) > \ln h^n(\theta''; 0)$ for all $n > N$ for appropriately chosen A.

We now show that $z_0(\theta_p(\Delta_s, \Delta_r))$ is increasing in Δ_s. The type z_0 solves the following equation

$$\ln \rho + \ln h(z_0) - \ln h(z_1) = 0$$

where $z_1 = \rho z_0 + \Delta_s$ and $\rho = \frac{1-f_h}{1-f_l} < 1$. Therefore,

$$\frac{dz_0}{d\Delta_s} = \frac{h(z_1) - z_1}{h(z_0) - z_0 - \rho(h(z_1) - z_1)},$$

The numerator is positive by Sampford (1953); and the denominator is positive because $h(z_0) - z_0 > h(z_1) - z_1$ (since $h'(z) < 1$ and $z_0 < z_1$) and because $\rho < 1$. $\square$

Lemma F.6. *There exist an N such that for all $n > N$, there is loser's curse at the pooling interval $(\theta_0^n, \theta_p^n]$, i.e.,*

$$\frac{\Pr(b_p^n\ wins|V = 1, Y_s^{n-1}(k) \in (\theta_0^n, \theta_p^n])}{\Pr(b_p^n\ wins|V = 0, Y_s^{n-1}(k) \in (\theta_0^n, \theta_p^n])} > 1.$$

Proof. Note that

$$\frac{\Pr(b_p^n\ wins|V = 1, Y_s^{n-1}(k) \in (0, \theta_p^n])}{\Pr(b_p^n\ wins|V = 0, Y_s^{n-1}(k) \in (0, \theta_p^n])} \to \frac{1-f_l}{1-f_h}\frac{(h(z_1) - z_1)}{(h(z_0) - z_0)}$$

uniformly by Lemma F.3. However,

$$\frac{1-f_l}{1-f_h}\frac{(h(z_1) - z_1)}{(h(z_0) - z_0)} = \frac{h(z_1)}{h(z_0)}\frac{(h(z_1) - z_1)}{(h(z_0) - z_0)}$$

because $\frac{1-f_l}{1-f_h} = \frac{h^n(\theta_p^n,1)}{h^n(\theta_p^n,0)}$ by construction (see Lemma F.5) for each $n > N$ and because $\frac{1-f_l}{1-f_h} = \frac{h^n(\theta_p^n,1)}{h^n(\theta_p^n,0)} \to \frac{h(z_1)}{h(z_0)}$ uniformly. However,

$$\frac{h(z_1)}{h(z_0)}\frac{(h(z_1) - z_1)}{(h(z_0) - z_0)} > 1$$

because $h(z)(h(z) - z) = h'(z)$, because $h'(z)$ is increasing in z by Sampford (1953) and because $z_1 > z_0$. $\square$

Lemma F.7. *There exist an N such that for all $n > N$*

$$\frac{\partial}{\partial\theta} l(Y_s^{n-1}(k) = \theta, \theta_i = \theta) > 0$$

for all $\theta \geq \theta_p^n$ and

$$\frac{\partial}{\partial\theta} l(Y_r^{n-1}(k) = \theta, \theta_i = \theta) > 0$$

for all θ. Moreover, if $z_0^n(\theta^n) \to z_0$, then

$$\sqrt{n}\frac{\partial}{\partial\theta}\sqrt{n}l(Y_s^{n-1}(k) = \theta^n, \theta_i = \theta^n) \to \frac{d}{dz}l(z_0) = \frac{d}{dx}(\frac{\phi(\frac{1-f_h}{1-f_l}z_0 - \Delta_s)}{\phi(z_0)})$$

and $l'(z) > 0$ for all $z > z_0(\theta_p)$.

Proof. Calculating $(\frac{\partial}{\partial\theta}\ln l)/n$ explicitly we find

$$\frac{\frac{\partial}{\partial\theta}l(Y_s^{n-1}(k) = \theta, \theta_i = \theta)}{nl(Y_s^{n-1}(k) = \theta, \theta_i = \theta)} = (1-\kappa_s)(\frac{f(\theta|1)}{1-\bar{F}_s^n(\theta|1)} - \frac{f(\theta|0)}{1-\bar{F}_s^n(\theta|0)}) + \kappa_s(\frac{f(\theta|0)}{\bar{F}_s^n(\theta|0)} - \frac{f(\theta|1)}{\bar{F}_s^n(\theta|1)})$$

Note that

$$\frac{f(\theta|0)}{\bar{F}_s^n(\theta|0)} - \frac{f(\theta|1)}{\bar{F}_s^n(\theta|1)} \geq 0$$

by MLRP.

Suppose first that $\theta > 1/2$. If $\theta > 1/2$, then $f(\theta|1) = f_h > f(\theta|0) = f_l$ and $\bar{F}_s^n(\theta|1) > \bar{F}_s^n(\theta|0)$. Therefore,

$$\frac{f(\theta|1)}{1-\bar{F}_s^n(\theta|1)} - \frac{f(\theta|0)}{1-\bar{F}_s^n(\theta|0)} > 0.$$

Suppose alternatively that $\theta \leq \frac{1}{2}$. If $\theta \leq \frac{1}{2}$, then $f(\theta|v) = 1 - f_v$, $\bar{F}_s^n(\theta|v) = 2f_v(\theta_1^n - \frac{1}{2}) + 2(1-f_v)(\frac{1}{2} - \theta)$ and $1 - \bar{F}_s^n(\theta|v) = F_r^n(1|v) + 2(1-f_v)(\theta - \theta_0^n)$. Therefore,

$$(\frac{\partial}{\partial\theta}\ln l)/n = -\frac{A}{(1-\bar{F}(\theta|1))(1-\bar{F}(\theta|0))} + \frac{B}{\bar{F}(\theta|0)\bar{F}(\theta|1)}$$

where

$$\begin{aligned} A &= (1-\kappa_s)((1-f_l)F_r^n(1|1) - (1-f_h)F_r^n(1|0)), \\ B &= \kappa_s(2f_h(\theta_1^n - \frac{1}{2})(1-f_l) - 2f_l(\theta_1^n - \frac{1}{2})(1-f_h)). \end{aligned}$$

The constant B is positive because $f_h/f_l > (1-f_h)/(1-f_l)$. If $\frac{F_r^n(1|1)}{F_r^n(1|0)} \leq \frac{1-f_h}{1-f_l}$, then $A \leq 0$ and hence the derivative is positive.

If alternatively, $\frac{F_r^n(1|1)}{F_r^n(1|0)} > \frac{1-f_h}{1-f_l}$, then $A > 0$. The fact that $\frac{A}{(1-\bar{F}(\theta|0))(1-\bar{F}(\theta|0))}$ is strictly decreasing in θ and $\frac{B}{\bar{F}(\theta|0)\bar{F}(\theta|1)}$ is strictly increasing in θ implies that there is θ' such that for all $\theta > \theta'$ we have $\frac{\partial \ln l}{n} > 0$. Rewriting the derivative we find that

$$(\frac{\partial}{\partial\theta}\ln l)/n = (1-f_h)\frac{\bar{F}(\theta|1) - \kappa_s}{\bar{F}(\theta|1)(1-\bar{F}(\theta|1))} - (1-f_l)\frac{\bar{F}(\theta|0) - \kappa_s}{\bar{F}(\theta|0)(1-\bar{F}(\theta|0))}.$$

Hence, θ' is the unique type such that $(\frac{\partial}{\partial\theta}\ln l)/n = 0$, that is

$$\frac{1-f_h}{1-f_l} = \frac{z_0^n(\theta')}{z_1^n(\theta')}\sqrt{\frac{\bar{F}(\theta'|0)(1-\bar{F}(\theta'|0))}{\bar{F}(\theta'|1)(1-\bar{F}(\theta'|1))}}.$$

Note that

$$\lim \frac{z_0^n(\theta'^n)}{z_1^n(\theta'^n)} \sqrt{\frac{\bar{F}(\theta'^n|0)(1-\bar{F}(\theta'^n|0))}{\bar{F}(\theta'^n|1)(1-\bar{F}(\theta'^n|1))}} = \frac{z_0(\theta')}{z_1(\theta')} = \frac{1-f_h}{1-f_l}.$$

We now show $\theta_p^n > \theta'^n$ by showing that

$$\frac{h(z_1)}{z_1} < \frac{h(z_0)}{z_0}.$$

This inequality is satisfied because $z_1 > z_0$ and because

$$\begin{aligned}
\frac{d}{dz}(\frac{h(z)}{z}) &= zh'(z) - h(z) \\
&< h(z)h'(z) - h(z) \\
&= h(z)(h'(z) - 1) < 0
\end{aligned}$$

where the first inequality follows from $h(z) > z$ and the final inequality follows from $h'(z) < 1$. Hence,

$$\frac{h^n(\theta'^n, 0)}{h^n(\theta'^n, 1)} > \frac{1-f_h}{1-f_l}$$

for all $n > N$. Therefore, $\theta_p^n > \theta'^n$ for all $n > N$ by Lemma F.5.

For market r calculating the derivative explicitly we find

$$(\frac{\partial}{\partial\theta}\ln l)/n = (1-f_h)\frac{\bar{F}_r^n(\theta|1) - \kappa_r}{\bar{F}_r^n(\theta|1)(1-\bar{F}_r^n(\theta|1))} - (1-f_l)\frac{\bar{F}_r^n(\theta|0) - \kappa_r}{\bar{F}_r^n(\theta|0)(1-\bar{F}_r^n(\theta|0))}.$$

The argument further above implies that if $\frac{\partial}{\partial\theta} l(Y_r^{n-1}(k) = \theta, \theta_i = \theta) > 0$ for some θ, then the derivative is positive for all $\theta' > \theta$. However, noting that $\lim \frac{\partial}{\partial\theta} l(Y_r^{n-1}(k) = 0, \theta_i = 0) > 0$ delivers the result. □

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