Identifying Effects of Multivalued Treatments*

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Abstract

Multivalued treatment models have only been studied so far under restrictive assumptions: ordered choice, or more recently unordered monotonicity. We show how treatment effects can be identified in a more general class of models. Our results rely on two main assumptions: treatment assignment must be a measurable function of threshold-crossing rules; and enough continuous instruments must be available. On the other hand, we do not require any kind of monotonicity condition. We illustrate our approach on several commonly used models; and we also analyze the identifying power of discrete instruments.

KEYWORDS: Identification, selection, multivalued treatments, instruments, monotonicity, multidimensional unobserved heterogeneity.

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1 Introduction

Since the seminal work of Heckman (1979), selection problems have been one of the main themes in both empirical economics and econometrics. One popular approach in the literature is to rely on instruments to uncover the patterns of the self-selection into different levels of treatments, and thereby to identify treatment effects. The main branches of this literature are the local average treatment effect (LATE) framework of Imbens and Angrist (1994) and the local instrumental variables (LIV) framework of Heckman and Vytlacil (2005).

The LATE and LIV frameworks emphasize different parameters of interest and suggest different estimation methods. However, they both focus on binary treatments, and restrict selection mechanisms to be "monotonic". Vytlacil (2002) establishes that the LATE and LIV approaches rely on the same monotonicity assumption. For binary treatment models, it requires that selection into treatment be governed by a single index crossing a threshold.

Many real-world selection problems are not adequately described by single-crossing models. The literature has developed ways of dealing with less restrictive models of assignment to treatment. Angrist and Imbens (1995) analyze ordered choice models. Heckman, Urzua, and Vytlacil (2006, 2008) show how (depending on restrictions and instruments) a variety of treatment effects can be identified in discrete choice models that are additively separable in instruments and errors. More recently, Heckman and Pinto (2015) define an "unordered monotonicity" condition that is weaker than monotonicity for multivalued treatment. They show that given unordered monotonicity, several treatment effects can be identified.

The most generally applicable of these approaches still can only deal with models of treatment that are formally analogous to an additively separable discrete choice model, as proved in Section 6 of Heckman and Pinto (2015). The key condition is that the data contain changes in instruments that create only *one-way flows* in or out of the treatment cells the analyst is interested in. In binary treatment models, this is exactly the meaning of monotonicity: there cannot be both compliers and defiers, so that LATE estimates the average treatment effect on compliers[1]. Things are somewhat more complex in multivalued treatment models. Unless selection only

[1] de Chaisemartin (2015) shows that under a condition weaker than monotonicity, LATE estimates the average treatment effect on a specific subset of the compliers.

depends on one function of the instruments, there will be changes in instruments that generate two-way flows in and out of any treatment cell. Unordered monotonicity requires that we observe *some* changes in instruments that only induce one way-flows.

This is still too restrictive for important applications. Many transfer programs for instance (or many tests in education) rely on several criteria and combine them in complex ways to assign agents to treatment; and agents add their own objectives and criteria to the list. An additively separable discrete choice model may not describe such a selection mechanism. To see this, start from a very simple and useful application: the double hurdle model, which treats agents only if each of *two* indices passes a threshold[2]. While this is a binary treatment model, the existence of two thresholds makes it non-monotonic: if a change in instruments increases a threshold but reduces the other, some agents will move into the treatment group and some will move out of it. Yet it is still an unordered monotonic model: any change in instruments that moves the two thresholds in the same direction only creates one-way flows.

Now let us change the structure of the model slightly: there are still two thresholds, but we only treat agents who are above one threshold and below the other. As we will see in Section 2, *any* change in instruments that moves both thresholds will generate two-way flows, and standard approaches to identification fail. This model of *selection with two-way flows* cannot be represented by a discrete choice model; it is formally equivalent to a discrete choice model with three alternatives in which the analyst only observes partitioned choices (e.g. the analyst only observes whether alternative 2 is chosen or not). Our identification results apply to this variant of the double hurdle model, and to all treatment models generated by a finite family of threshold-crossing rules. In fact, one way to describe our contribution is that it encompasses all additively separable discrete choice models in which the analyst only observes a partition of the set of alternatives.

Our analysis allows selection to be determined by a vector of unobservables, instead of a scalar random variable; and these unobservables can be correlated with potential outcomes. We rely on the control function approach, but we use a vector of control variables to deal with multidimensional unobserved heterogeneity. We establish conditions under which one can identify the probability distribution of unobservables governing the selection mechanism, as well as a generalized version of the marginal treatment effects (MTE) of Heckman and Vytlacil (2005). Furthermore, we

[2]See, e.g. Poirier (1980) for a parametric version of this model.

discuss a few applications to illustrate the usefulness of our approach.

We will give a detailed comparison between our paper and the existing literature in Section 6. Let us only mention at this stage a few points in which our paper differs from the literature. Unlike Imbens (2000), Hirano and Imbens (2004), and Cattaneo (2010), we allow for selection on unobservables. Gautier and Hoderlein (2015) study binary treatment when selection is driven by a rule that is linear in a vector of unobservable heterogeneity; this breaks monotonicity in a different way than ours. We focus on the point identification of marginal treatment effects, unlike the research on partial identification (see e.g. Manski (1990), Manski (1997) and Manski and Pepper (2000)). Chesher (2003), Hoderlein and Mammen (2007), Florens, Heckman, Meghir, and Vytlacil (2008), Imbens and Newey (2009), D'Haultfœuille and Février (2015), and Torgovitsky (2015) study models with continuous endogenous regressors, based on control function approaches. Each of these papers develops identification results for various parameters of interest. Our paper complements this literature by considering multivalued (but not continuous) treatments with more general types of selection mechanisms.

Heckman and Vytlacil (2007, Appendix B) and Heckman, Urzua, and Vytlacil (2008) and more recently Heckman and Pinto (2015) and Pinto (2015) are closer to our paper. But they focus on the selection induced by multinomial discrete choice models, whereas our paper allows for more general selection problems.

The paper is organized as follows. Section 2 sets up our framework; it motivates our central assumptions by way of examples. We present and prove our identification results in Section 3. Section 4 applies our results to four important classes of applications, including the models mentioned in this introduction. We also use some of these models to analyze the case in which instruments are discrete-valued in Section 5. Finally, we relate our contributions to the literature in Section 6. Some further results and details of the omitted proofs are collected in Online Appendices.

2 The Model and our Assumptions

We assume throughout that treatments take values in a finite set of treatments $\mathcal{K}$. This set may be naturally ordered, as with different tax rates. But it may not be, as when welfare recipients enroll in different training schemes for instance; this makes no difference to our results. We assume that treatments are exclusive; this involves no loss of generality as treatment values could easily be redefined otherwise. We denote $K = |\mathcal{K}|$ the number of treatments, and we map the set $\mathcal{K}$ into $\{0, \dots, K-1\}$ for notational convenience.

Potential outcomes $\{Y_k : k \in \mathcal{K}\}$ are generated by

$$Y_k = \mu_k(\boldsymbol{X}, U_k),$$

where $\boldsymbol{X}$ is a vector of covariates, U_k is an unobserved random variable, and $\mu_k(\cdot, \cdot)$ is an unknown function of $\boldsymbol{X}$ and U_k for each $k \in \mathcal{K}$. We denote $D_k = 1$ if the k treatment is realized and $D_k = 0$ otherwise. The observed outcome and treatment are $Y := \sum_{k \in \mathcal{K}} Y_k D_k$ and $D := \sum_{k \in \mathcal{K}} k D_k$, respectively.

In addition to the covariates $\boldsymbol{X}$, observed treatment D and outcomes Y, the data contain a random vector $\boldsymbol{Z}$ that will serve as instruments. We will always condition on the value of $\boldsymbol{X}$ in our analysis of identification; and we suppress it from the notation. Observed data consist of a sample $\{(Y_i, D_i, \boldsymbol{Z}_i) : i = 1, \dots, N\}$ of $(Y, D, \boldsymbol{Z})$, where N is the sample size. We will denote the generalized propensity scores by $P_k(\boldsymbol{Z}) := \Pr(D = k|\boldsymbol{Z})$; they are directly identified from the data.

Let G denote a function defined on the support $\mathcal{Y}$ of Y. We focus on identification of $EG(Y_k)$. For example, if we take $G(Y_k) = Y_k$, then the object of interest is the mean of the counterfactual outcome Y_k (conditional on the omitted covariates $\boldsymbol{X}$). Once we identify $EG(Y_k)$ for each k, we also identify the average treatment effect $E(G(Y_k) - G(Y_j))$ between any two treatments k and j. Alternatively, if we let $G(Y_k) = \mathbb{1}(Y_k \leq y)$ for some y, where $\mathbb{1}(\cdot)$ is the usual indicator function, then the object of interest is the marginal distribution of Y_k. This leads to the identification of quantile treatment effects.

One of our aims is to relax the usual monotonicity assumption that underlies LATE and LIV. Consider the following, simple example where $K = 3$, and treatment assignment is driven by a pair of random variables V_1 and V_2 whose marginal distributions are normalized to be $U[0, 1]$.

Example 1 (Selection with Two-Way Flows)**.** Assume that there are two thresholds $Q_1(\boldsymbol{Z})$ and $Q_2(\boldsymbol{Z})$ such that

- $D = 0$ iff $V_1 < Q_1(\boldsymbol{Z})$ and $V_2 < Q_2(\boldsymbol{Z})$,
- $D = 1$ iff $V_1 > Q_1(\boldsymbol{Z})$ and $V_2 > Q_2(\boldsymbol{Z})$,

- $D = 2$ iff $(V_1 - Q_1(\boldsymbol{Z}))$ and $(V_2 - Q_2(\boldsymbol{Z}))$ have opposite signs.

We could interpret Q_1 and Q_2 as minimum grades or scores in a two-part exam or an eligibility test based on two criteria: failing both parts/criteria assigns you to $D = 0$, passing both to $D = 1$, and failing only one to $D = 2$.

If F is the joint cdf of (V_1, V_2), it follows that the generalized propensity scores are

$$
\begin{aligned}
P_0(\boldsymbol{Z}) &= F\left(Q_1(\boldsymbol{Z}), Q_2(\boldsymbol{Z})\right), \\
P_1(\boldsymbol{Z}) &= 1 - Q_1(\boldsymbol{Z}) - Q_2(\boldsymbol{Z}) + F\left(Q_1(\boldsymbol{Z}), Q_2(\boldsymbol{Z})\right), \\
P_2(\boldsymbol{Z}) &= Q_1(\boldsymbol{Z}) + Q_2(\boldsymbol{Z}) - 2F\left(Q_1(\boldsymbol{Z}), Q_2(\boldsymbol{Z})\right).
\end{aligned}
\tag{2.1}
$$

Take a change in the values of the instruments that increases both $Q_1(\boldsymbol{Z})$ and $Q_2(\boldsymbol{Z})$, as represented in Figure 1: both criteria, or both parts of the exam, become more demanding. Then some observations (a) will move from $D = 1$ to $D = 2$, some (b) from $D = 1$ to $D = 0$, and some (c) will move from $D = 2$ to $D = 0$. This violates monotonicity, and even the weaker assumption that generalized propensity scores are monotonic in the instruments. Note also that some observations leave $D = 2$ and some move into $D = 2$: there are *two-way flows* in and out of $D = 2$. Moreover, it is easy to see that *any* change in the thresholds creates such two-way flows; Figure 2 illustrates it for changes in opposite directions, with observations (d) moving from $D = 0$ to $D = 2$, observations (e) moving from $D = 2$ to $D = 1$, observations (f) moving from $D = 1$ to $D = 2$ and observations (g) moving from $D = 2$ to $D = 0$.

Therefore this model violates the weaker requirement of unordered monotonicity of Heckman and Pinto (2015), which we describe in Section 6.3—unless we are only interested in treatment values 0 and 1. $\square$

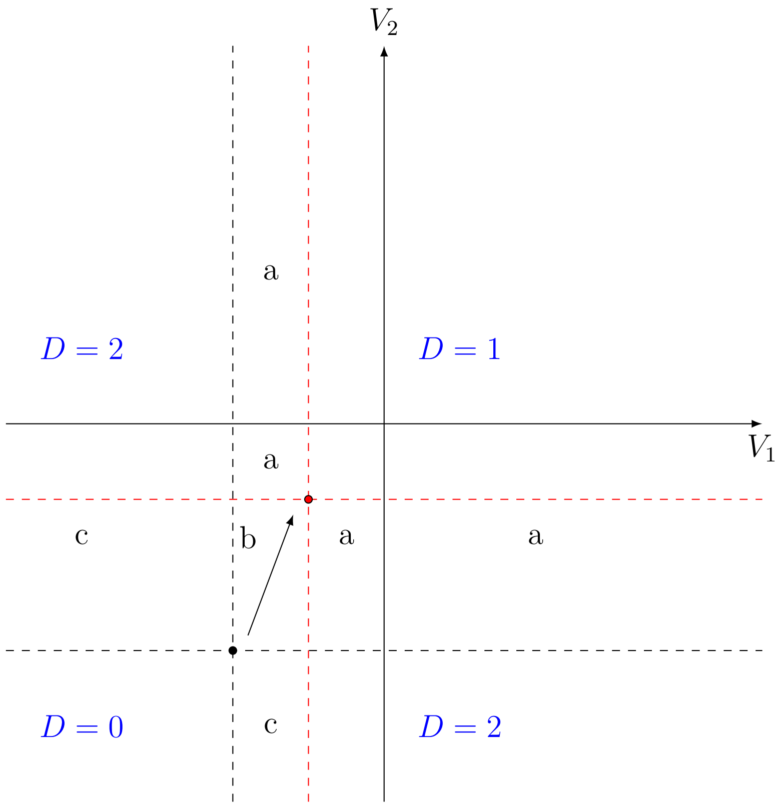
To take a slightly more complicated example, consider the following entry game.

Example 2 (Entry Game)**.** Two firms $j = 1, 2$ are considering entry into a new market. Firm j has profit π_j^m if it becomes a monopoly, and $\pi_j^d < \pi_j^m$ if both firms enter. The static Nash equilibria are simple:

- if both $\pi_j^m < 0$, then no firm enters;

- if $\pi_j^m > 0$ and $\pi_k^m < 0$, then only firm j enters;

Figure 1: Example 1



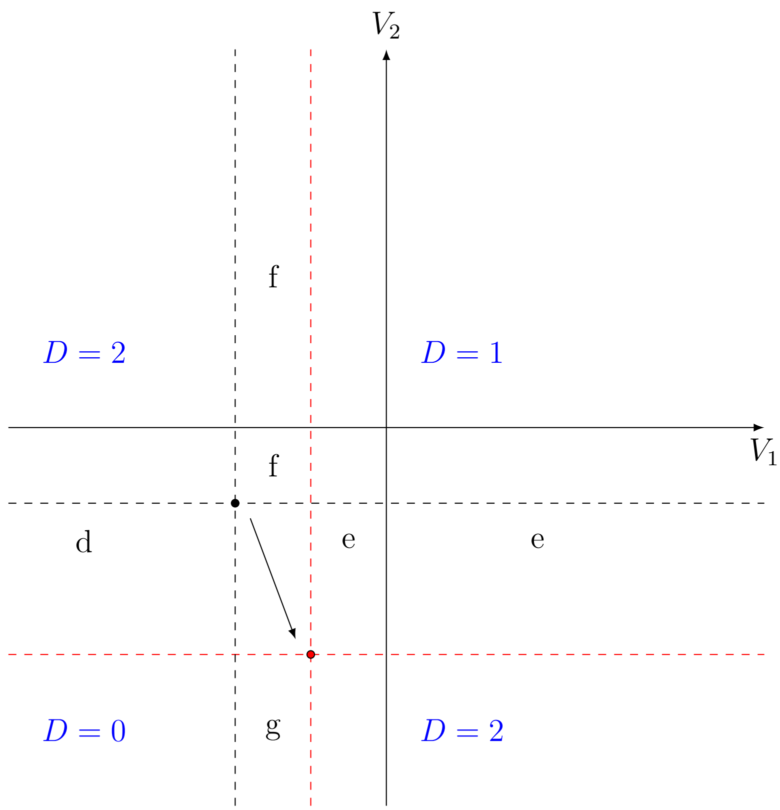
- if both $\pi_j^d > 0$, then both firms enter;
- if $\pi_j^d > 0$ and $\pi_k^d < 0$, then only firm j enters;
- if $\pi_j^m > 0 > \pi_j^d$ for both firms, then there are two symmetric equilibria, with only one firm operating.

Now let $\pi_j^m = V_j - Q_j(\boldsymbol{Z})$ and $\pi_j^d = \bar{V}_j - \bar{Q}_j(\boldsymbol{Z})$, and suppose we only observe the number $D = 0, 1, 2$ of entrants. Then

- $D = 0$ iff $V_1 < Q_1(\boldsymbol{Z})$ and $V_2 < Q_2(\boldsymbol{Z})$
- $D = 2$ iff $\bar{V}_1 > \bar{Q}_1(\boldsymbol{Z})$ and $\bar{V}_2 > \bar{Q}_2(\boldsymbol{Z})$
- $D = 1$ otherwise.

This is very similar to the structure of Example 1; in fact it coincides with it in the degenerate case when for each firm, π_m^j and π_d^j have the same sign with probability one. $\square$

Figure 2: Example 1 (bis)



2.1 The Selection Mechanism

These two examples motivate the weak assumption we impose on the underlying selection mechanism. In the following we use $\boldsymbol{J}$ to denote the set $\{1, \ldots, J\}$.

Assumption 2.1 (Selection Mechanism)**.** *There exist a finite number J, a vector of unobserved random variables $\boldsymbol{V} := \{V_j : j \in \boldsymbol{J}\}$, and a vector of known functions $\{\boldsymbol{Q}_j(\boldsymbol{Z}) : j \in \boldsymbol{J}\}$ such that, equivalently:*

(i) the treatment variable D is measurable with respect to the σ-field generated by the events

$$E_j(\boldsymbol{Z}) := \{V_j < Q_j(\boldsymbol{Z})\} \text{ for } j \in \boldsymbol{J};$$

(ii) each event $\{D = k\} = \{D_k = 1\}$ is a member of this σ-field;

(iii) for each k, there exists a function g_k that is measurable with respect to this σ-field such that $D_k = 1$ iff $g_k(\boldsymbol{V}, \boldsymbol{Q}(Z)) = 0$.

Moreover, every treatment value k has positive probability for all $\boldsymbol{Z}$.

Note that the fact that every observation belongs to one and only one treatment group imposes further constraints; we will not need to spell them out at this stage, but we will show later how they can be used for overidentification tests.

In this notation, the validity of the instruments translates into:

Assumption 2.2 (Conditional Independence of Instruments)**.** *Y_k and $\boldsymbol{V}$ are independent of $\boldsymbol{Z}$ for each $k = 0, \ldots, K-1$.*

2.2 Atoms and Indices

To describe the class of selection mechanisms defined in Assumption 2.1 more concretely, we focus on a treatment value k. We define $S_j(\boldsymbol{Z}) := \mathbb{1}(V_j < Q_j(\boldsymbol{Z}))$ for $j = 1, \ldots, J$. Any element of the σ-field generated by $\{E_j(\boldsymbol{Z}) : j = 1, \ldots, J\}$ can be written uniquely as a finite union of the 2^J disjoint sets

$$F_1 \cap \ldots \cap F_J,$$

where F_j is either E_j or its complement $\bar{E}_j$. We will call them the *atoms* of the σ-field. Note that any such atom has an indicator function of the form

$$\prod_{j=1}^{J} T_j,$$

where T_j is either S_j (when $F_j = E_j$) or $(1 - S_j)$ (when $F_j = \bar{E}_j$). The event $\{D = k\}$ is a finite union of such atoms, defined by a subset N_k of $\{1, \ldots, 2^J\}$. For any atom $n \in N_k$, denote M_n the subset of indices $j = 1, \ldots, J$ for which $F_j = \bar{E}_j$ in atom n. Then the indicator function Π_n of atom n is

$$\Pi_n = \prod_{j \notin M_n} S_j \prod_{l \in M_n} (1 - S_l).$$

The highest degree term of Π_n is

$$(-1)^{p_n} \prod_{j=1}^{J} S_j,$$

where $p_n = |M_n|$.

To illustrate, suppose that $J = 4$ and take atom n to be $E_1 \cap \bar{E}_2 \cap \bar{E}_3 \cap E_4$: its M_n subset is $\{2, 3\}$, its $p_n = 2$, and its indicator function is

$$\Pi_n = S_1 S_4 (1 - S_2)(1 - S_3) = S_1 S_4 \left(1 - (S_2 + S_3) + S_2 S_3\right).$$

Now consider $\{D = k\}$ as the union of these $|N_k|$ atoms. Since they are disjoint, its indicator function is simply the sum of their indicator functions. By construction, it is a multivariate polynomial in $\boldsymbol{S} \equiv (S_1, \ldots, S_J)$. Consider any subset $(j_1, \ldots, j_m)$ of indices in $\boldsymbol{J}$. Then it is easy to see that the coefficient of the product

$$\prod_{l=1,\ldots,m} S_{j_l}$$

in the indicator function of treatment value k is

$$\sum_{n \in N_k} (-1)^{|(j_1,\ldots,j_m) \cap M_n|}.$$

The highest degree term of this polynomial will play a central role in our analysis. Note that if we choose $(j_1, \ldots, j_m) = \boldsymbol{J}$, then $(j_1, \ldots, j_m) \cap M_n = M_n$ for any atom n. It follows that the coefficient of the full product $\prod_{j=1}^{J} S_j$ is

$$a_k = \sum_{n \in N_k} (-1)^{p_n}.$$

We call this number the *index* of treatment k. It can be any integer between $-|N_k|$ and $|N_k|$, including zero. To illustrate this, let us return to Example 1, with $J = 2$ and $K = 3$. For $k = 0$, the selection mechanism is described by the intersection $E_1(\boldsymbol{Z}) \cap E_2(\boldsymbol{Z})$. Hence, this case corresponds to $N_0 = 1$ and its atom's indicator function is $\Pi_{0,1}(\boldsymbol{Z}) = S_1(\boldsymbol{Z}) S_2(\boldsymbol{Z})$, where the first subscript 0 denotes the treatment value 0. Similarly, for $k = 1$ we have $N_1 = 1$ and $\Pi_{1,1}(\boldsymbol{Z}) = (1 - S_1(\boldsymbol{Z}))(1 - S_2(\boldsymbol{Z}))$. Finally, for $k = 2$ we have $N_2 = 2$ and

$$\begin{aligned}
\Pi_{2,1}(\boldsymbol{Z}) &= S_1(\boldsymbol{Z})(1 - S_2(\boldsymbol{Z})) \\
\Pi_{2,2}(\boldsymbol{Z}) &= (1 - S_1(\boldsymbol{Z})) S_2(\boldsymbol{Z}).
\end{aligned}$$

In this example the indices are $a_0 = a_1 = 1$ and $a_2 = (-1)^1 + (-1)^1 = -2$.

Appendix A.1 gives some results on indices. With $J = 2$ as in Example 1, the only treatments with a zero index are those which only depend on one threshold: e.g. $\mathbb{1}(V_1 < Q_1)$. But for $J > 2$ it is not hard to generate cases in which a treatment value k depends on all J thresholds and still has $a_k = 0$, as shown in Example 3.

Example 3 (Zero Index)**.** Assume that $J = K = 3$ and take treatment 0 such that

$$\begin{aligned}D_0 = &\mathbb{1}(V_1 < Q_1(\boldsymbol{Z}), V_2 < Q_2(\boldsymbol{Z}), V_3 < Q_3(\boldsymbol{Z})) \\ &+ \mathbb{1}(V_1 > Q_1(\boldsymbol{Z}), V_2 > Q_2(\boldsymbol{Z}), V_3 > Q_3(\boldsymbol{Z})).\end{aligned}$$

This has two atoms; the atom on the first line has $p_1 = 0$, and the second one has $p_2 = 3$. The index is $a_0 = 1 - 1 = 0$. Another way to see this is that the indicator function for $\{D_0 = 1\}$ is

$$S_1S_2S_3 + (1 - S_1)(1 - S_2)(1 - S_3) = 1 - S_1 - S_2 - S_3 + S_1S_2 + S_1S_3 + S_2S_3,$$

which has no degree three term. $\square$

When the index is zero as in Example 3, the indicator function of the corresponding treatment k has degree strictly smaller than J. Since Assumption 2.1 rules out the uninteresting cases when treatment k has probability zero or one, its indicator function cannot be constant; and its leading terms have degree $m \geq 1$. We call m the *degree* of treatment k, and we summarize this discussion in a lemma:

Lemma 2.1. *Under Assumption 2.1, for each $k \in \mathcal{K}$ there exists a subset N_k of $\{1, \ldots, 2^J\}$ such that*

$$D_k = \sum_{n \in N_k} \Pi_n(\boldsymbol{Z}),$$

and for each $n \in N_k$,

$$\Pi_n(\boldsymbol{Z}) := \prod_{j \in \boldsymbol{J} - M_n} S_j(\boldsymbol{Z}) \prod_{l \in M_n} (1 - S_l(\boldsymbol{Z})).$$

The leading terms of the multivariate polynomial

$$D_k(\boldsymbol{S}) = \sum_{n \in N_k} \prod_{j \in \boldsymbol{J} - M_n} S_j \prod_{l \in M_n} (1 - S_l)$$

have degree $m \geq 1$, which we also call the degree *of treatment k.*

Define $p_n \equiv |M_n|$, and $a_k \equiv \sum_{n \in N_k} (-1)^{p_n}$ the index *of treatment k. Treatment k has degree $m = J$ if and only if $a_k \neq 0$; and then the leading term of $D_k(\boldsymbol{S})$ is*

$$a_k \prod_{j=1}^{J} S_j.$$

It is useful to think of atoms as alternatives in a discrete choice model. Any of the 2^J atoms can be interpreted as the choice of alternative n, where the binary representation of n has a one for digit j if $F_j = E_j$ and a zero if $F_j = \bar{E}_j$. The assignment of an observation to treatment k, which is a union of atoms, then is formally equivalent to the choice of an alternative whose number matches that of one of these atoms. In essence, we are dealing with discrete choice models with only partially observed choices.

3 Identification Results

In this section we fix $\boldsymbol{x}$ in the support of $\boldsymbol{X}$ and we suppress it from the notation. All the results obtained below are local to this choice of $\boldsymbol{x}$. Global (unconditional) identification results follow immediately if our assumptions hold for almost every $\boldsymbol{x}$ in the support of $\boldsymbol{X}$.

We will treat separately the non-zero index and the zero index cases. We make this explicit in the following assumption.

Assumption 3.1 (Nonzero index). *The index a_k defined in Lemma 2.1 is nonzero.*

We will return to zero-index treatments in Section 3.2.

We require that $\boldsymbol{V}$ have full support:

Assumption 3.2 (Continuously Distributed Unobserved Heterogeneity in the Selection Mechanism). *The joint distribution of $\boldsymbol{V}$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^J$ and its support is $[0,1]^J$.*

Normalization: We normalize the marginal distribution of each $V_j \in \boldsymbol{V}$ to be $U[0,1]$.

Note that when $J = 1$, Assumptions 2.1 and 3.2 define the usual threshold-crossing model that underlies the LATE and LIV approaches. However, our assumptions allow for a much richer class of selection mechanisms when $J > 1$. Our Example 1 illustrates that our "multiple thresholds model" does not impose any multidimensional extension of the monotonicity condition that is implicit with a single threshold model. Even when $K = 2$ so that treatment is binary, J could be larger than one, allowing for flexible treatment assignment: just modify Example 1 to obtain the double hurdle model

$$D = \mathbb{1}\left(V_1 < Q_1(\boldsymbol{Z}) \text{ and } V_2 < Q_2(\boldsymbol{Z})\right).$$

Let $f_{\boldsymbol{V}}(\boldsymbol{v})$ denote the joint density function of $\boldsymbol{V}$ at $\boldsymbol{v} \in [0,1]^J$. Our identification argument relies on continuous instruments that generate enough variation in the thresholds. This motivates the following three assumptions.

For any function ψ of $\boldsymbol{q}$, define "local equicontinuity at $\overline{\boldsymbol{q}}$" by the following property: for any subset $I \subset \boldsymbol{J}$, the family of functions $\boldsymbol{q}_I \mapsto \psi(\boldsymbol{q}_I, \boldsymbol{q}_{-I})$ indexed by $\boldsymbol{q}_{-I} \in [0,1]^{|\boldsymbol{J}-I|}$ is equicontinuous in a neighborhood of $\overline{\boldsymbol{q}}_I$.

Assumption 3.3 (Local equicontinuity at $\boldsymbol{q}$)**.** *The functions $\boldsymbol{v} \mapsto f_{\boldsymbol{V}}(\boldsymbol{v})$ and $\boldsymbol{v} \mapsto E\left(G(Y_k)|\boldsymbol{V} = \boldsymbol{v}\right)$ are locally equicontinuous at $\boldsymbol{v} = \boldsymbol{q}$.*

Assumption 3.3 will allow us to differentiate the relevant expectation terms. It is fairly weak: Lipschitz-continuity for instance implies local equicontinuity.

The next two assumptions apply to the functions $\boldsymbol{Q}(\boldsymbol{Z})$. These are unknown in most cases, and need to be identified; in this part of the paper we assume that they are known. We will return to identification of the $\boldsymbol{Q}$ functions in Section 3.3.

Assumption 3.4 (Open Mapping at $\boldsymbol{q}$)**.** *The function $\boldsymbol{Q}$ is an open map at every point $\boldsymbol{z}$ such that $\boldsymbol{Q}(\boldsymbol{z}) = \boldsymbol{q}$.*

Assumption 3.4 requires that the image by $\boldsymbol{Q}$ of every small neighborhood of $\boldsymbol{z}$ contain a neighborhood of $\boldsymbol{Q}(\boldsymbol{z})$. It ensures that we can generate any small variation in $\boldsymbol{Q}(\boldsymbol{Z})$ by varying the instruments around $\boldsymbol{z}$. This makes the instruments strong enough to deal with multidimensional unobserved heterogeneity $\boldsymbol{V}$. It is crucial to our approach. In Example 1 for instance, Assumption 3.4 would fail if Q_1 and Q_2 were functionally dependent around $\boldsymbol{z}$, with say $Q_1 \equiv Q_2^3$. More generally, Assumption 3.4 ensures both that there are "enough instruments" and that they have enough variation locally[3]. In its absence, we would only get partial identification of the marginal treatment effects.

We also consider a global version of Assumptions 3.3 and 3.4.

Assumption 3.5 (Global Condition)**.** *Assumptions 3.3 and 3.4 hold at all* $\boldsymbol{q} \in (0,1)^J$.

3.1 Identification with a Non-Zero Index

We are now ready to prove identification of $EG(Y_k)$ when treatment k has a non-zero index. In the following theorem, for any real-valued function $\boldsymbol{q} \mapsto h(\boldsymbol{q})$, the notation

$$Th(\boldsymbol{q}) \equiv \frac{\partial^J h}{\prod_{j=1}^J \partial q_j}(\boldsymbol{q})$$

refers to the J-order derivative that obtains by taking derivatives of the function h at $\boldsymbol{q}$ in each direction of $\boldsymbol{J}$ in turn.

Theorem 3.1 (Identification with a non-zero index)**.** *Let Assumptions 2.1, 2.2, 3.1, and 3.2 hold. Fix a value* $\boldsymbol{q}$ *in the support of* $\boldsymbol{Q}(\boldsymbol{Z})$ *and assume that 3.3 and 3.4 hold at* $\boldsymbol{q}$*. Then the density of* $\boldsymbol{V}$ *and conditional expectation of* $G(Y_k)$ *are given by*

$$f_{\boldsymbol{V}}(\boldsymbol{q}) = \frac{1}{a_k} T \Pr(D = k | \boldsymbol{Q}(\boldsymbol{Z}) = \boldsymbol{q})$$

$$E[G(Y_k)|\boldsymbol{V} = \boldsymbol{q}] = \frac{TE\left(G(Y)D_k|\boldsymbol{Q}(\boldsymbol{Z}) = \boldsymbol{q}\right)}{T \Pr(D = k|\boldsymbol{Q}(\boldsymbol{Z}) = \boldsymbol{q})}.$$

If in addition Assumption 3.5 holds, then

$$EG(Y_k) = \frac{1}{a_k} \int_{[0,1]^J} TE\left(G(Y)D_k|\boldsymbol{Q} = \boldsymbol{q}\right) d\boldsymbol{q}. \tag{3.1}$$

Proof of Theorem 3.1. Our proof has three steps. We first write conditional moments as integrals with respect to indicator functions. Then we show that these integrals are

[3]Note that Assumption 3.4 does not require a rank condition at $\boldsymbol{z}$. If $\boldsymbol{Q}$ has a Jacobian, this could have reduced rank at $\boldsymbol{z}$ as long as it has full row rank in small neighborhood of $\boldsymbol{z}$—as it must if $\boldsymbol{Q}$ is an open map at $\boldsymbol{z}$. Since critical points of non-constant maps are typically isolated, this is a much weaker requirement.

differentiable and we compute their multidimensional derivatives. Finally, we impose Assumption 3.1 and we derive the equalities in the theorem.

Step 1:

Under the assumptions imposed in the theorem, for any $\boldsymbol{q}$ in the range of $\boldsymbol{Q}$,

$$\begin{aligned}
&E[G(Y)D_k|\boldsymbol{Q}(\boldsymbol{Z})=\boldsymbol{q}]\\
&=E[G(Y_k)|D=k,\boldsymbol{Q}(\boldsymbol{Z})=\boldsymbol{q}]\Pr(D=k|\boldsymbol{Q}(\boldsymbol{Z})=\boldsymbol{q})\\
&=E[G(Y_k)|g_k(\boldsymbol{V},\boldsymbol{Q}(\boldsymbol{Z}))=0,\boldsymbol{Q}(\boldsymbol{Z})=\boldsymbol{q}]\Pr(g_k(\boldsymbol{V},\boldsymbol{Q}(\boldsymbol{Z}))=0|\boldsymbol{Q}(\boldsymbol{Z})=\boldsymbol{q})\\
&=E[G(Y_k)|g_k(\boldsymbol{V},\boldsymbol{q})=0]\Pr\left(g_k(\boldsymbol{V},\boldsymbol{q})=0\right)\\
&=E[G(Y_k)\mathbb{1}\left(g_k(\boldsymbol{V},\boldsymbol{q})=0\right)]\\
&=E\left(E[G(Y_k)\mathbb{1}\left(g_k(\boldsymbol{V},\boldsymbol{q})=0\right)|\boldsymbol{V}\right)\\
&=E\left(E[G(Y_k)|\boldsymbol{V}]\mathbb{1}\left(g_k(\boldsymbol{V},\boldsymbol{q})=0\right)\right),
\end{aligned}$$

where the third equality follows from Assumption 2.2 and the others are obvious. As a consequence,

$$\begin{aligned}
&E[G(Y)D_k|\boldsymbol{Q}(\boldsymbol{Z})=\boldsymbol{q}]\\
&=\int\mathbb{1}\left(g_k(\boldsymbol{v},\boldsymbol{q})=0\right)E[G(Y_k)|\boldsymbol{V}=\boldsymbol{v}]f_{\boldsymbol{V}}(\boldsymbol{v})d\boldsymbol{v}.
\end{aligned}\tag{3.2}$$

Let $b_k(\boldsymbol{v})\equiv E[G(Y_k)|\boldsymbol{V}=\boldsymbol{v}]f_{\boldsymbol{V}}(\boldsymbol{v})$ and $B_k(\boldsymbol{q})=E[G(Y)D_k|\boldsymbol{Q}(\boldsymbol{Z})=\boldsymbol{q}]$. Then (3.2) takes the form

$$B_k(\boldsymbol{q})=\int\mathbb{1}(g_k(\boldsymbol{v},\boldsymbol{q})=0)b_k(\boldsymbol{v})d\boldsymbol{v}.$$

Now remember from Lemma 2.1 that the indicator function of $D=k$ is a multivariate polynomial of the indicator functions S_j for $j\in\boldsymbol{J}$. Moreover,

$$S_j(\boldsymbol{Z})=\mathbb{1}(V_j<Q_j(\boldsymbol{Z}))=H(Q_j(\boldsymbol{Z})-V_j),$$

where $H(t)=\mathbb{1}(t>0)$ is the one-dimensional Heaviside function. Therefore we can rewrite the selection of treatment k as

$$\mathbb{1}(g_k(\boldsymbol{v},\boldsymbol{q})=0)=\sum_{n\in N_k}\prod_{j\in\boldsymbol{J}-M_n}H(q_j-v_j)\prod_{j\in M_n}\left(1-H(q_j-v_j)\right),\tag{3.3}$$

and it follows that

$$B_k(\boldsymbol{q}) = \sum_{n\in N_k} \int \left(\prod_{j\in \boldsymbol{J}-M_n} H(q_j - v_j) \prod_{j\in M_n} (1 - H(q_j - v_j)) \right) b_k(\boldsymbol{v}) d\boldsymbol{v}.$$

Expanding the products, the right-hand side can be written as a sum

$$B_k(\boldsymbol{q}) = \sum_l c_l \int \prod_{j\in I_l} H(q_j - v_j) b_k(\boldsymbol{v}) d\boldsymbol{v}, \tag{3.4}$$

where for each l, the set $I_l \subset \boldsymbol{J}$ and $c_l = \sum_{n\in N_k} (-1)^{|I_l \cap M_n|}$ is an algebraic integer.

Step 2:

By Assumption 3.3, the function $\boldsymbol{b}$ is locally equicontinuous. This implies that all terms in (3.4) are differentiable along all dimensions of $\boldsymbol{q}$. To see this, start with dimension $j = 1$. Any term l in (3.4) such that I_l does not contain 1 is constant in q_1 and obviously differentiable. Take any other term and rewrite it as

$$A_l(q_1) \equiv c_l \int_0^{q_1} \int \prod_{j\in I_l, j\neq 1} H(q_j - v_j) b_k(v_1, \boldsymbol{v}_{-1}) d\boldsymbol{v}_{-1} dv_1,$$

where $\boldsymbol{v}_{-1}$ collects all directions of $\boldsymbol{v}$ in $I_l - \{1\}$.

Then for any $\varepsilon \neq 0$,

$$\begin{aligned} \frac{A_l(q_1 + \varepsilon) - A_l(q_1)}{\varepsilon} &- c_l \int \prod_{j\in I_l, j\neq 1} H(q_j - v_j) b_k(q_1, \boldsymbol{v}_{-1}) d\boldsymbol{v}_{-1} \\ &= \frac{c_l}{\varepsilon} \int_{q_1}^{q_1+\varepsilon} \int \prod_{j\in I_l, j\neq 1} H(q_j - v_j) \left(b_k(v_1, \boldsymbol{v}_{-1}) - b_k(q_1, \boldsymbol{v}_{-1}) \right) d\boldsymbol{v}_{-1} dv_1. \end{aligned}$$

Since the functions $(b_k(\cdot, \boldsymbol{v}_{-1}))$ are locally equicontinuous at q_1, for any $\eta > 0$ we can choose ε such that if $|q_1 - v_1| < \varepsilon$,

$$|b_k(q_1, \boldsymbol{v}_{-1}) - b_k(v_1, \boldsymbol{v}_{-1})| < \eta;$$

and since the Heaviside functions are bounded above by one, we will have

$$\left| \frac{A_l(q_1 + \varepsilon) - A_l(q_1)}{\varepsilon} - c_l \int \prod_{j\in I_l, j\neq 1} H(q_j - v_j) b_k(q_1, \boldsymbol{v}_{-1}) d\boldsymbol{v}_{-1} \right| < |c_l|\, \eta.$$

This proves that A_l is differentiable in q_1 and that its derivative with respect to q_1, which we denote A_l^1, is

$$A_l^1 = c_l \int \prod_{j \in I_l, j \neq 1} H(q_j - v_j)\; b_k(q_1, \boldsymbol{v}_{-1}) d\boldsymbol{v}_{-1}.$$

But this derivative itself has the same form as A_l. Letting $\boldsymbol{v}_{-1,2}$ collect all components of $\boldsymbol{v}$ except (q_1, q_2), the same argument would prove that since the functions $(b_k(\cdot, \boldsymbol{v}_{-1,2}))$ are locally equicontinuous at (q_1, q_2), the function A_l^1 is differentiable with respect to q_2 and its derivative is

$$c_l \int \prod_{j \in I_l, j \neq 1,2} H(q_j - v_j)\; b_k(q_1, q_2, \boldsymbol{v}_{-1,2}) d\boldsymbol{v}_{-1,2}.$$

Continuing this argument finally gives us the cross-derivative with respect to $(\boldsymbol{q}^{I_l})$ as

$$c_l \int b_k(\boldsymbol{q}^{I_l}, \boldsymbol{v}_{-I_l}) d\boldsymbol{v}_{-I_l},$$

where $\boldsymbol{v}_{-I_l}$ collects all components of $\boldsymbol{v}$ whose indices are not in I_l.

Step 3:

Lemma 2.1 and Assumption 3.1 also imply that the full-degree term $I_l = \boldsymbol{J}$ has $c_l = a_k$. To put it differently, the leading term in the H_j's is

$$a_k \prod_{j=1}^{J} H(q_j - p_j).$$

Now take the J-order derivative of $B(\boldsymbol{q})$ with respect to all q_j in turn. By Lemma 2.1, the highest-degree term of B in $\boldsymbol{q}$ is

$$a_k \int \left(\prod_{j=1}^{J} H(q_j - v_j) \right) b_k(\boldsymbol{v}) d\boldsymbol{v}$$

as $a_k \neq 0$ under Assumption 3.1; all other terms have a smaller number of indices j.

This term contributes a cross-derivative

$$a_k b_k(\boldsymbol{q}),$$

and all other terms generate zero-value contributions since each of them is constant in at least one of the directions j.

More formally,

$$TB_k(\boldsymbol{q}) = \frac{\partial^J B_k(\boldsymbol{q})}{\prod_{j \in \boldsymbol{J}} \partial q_j} = a_k b_k(\boldsymbol{q}). \tag{3.5}$$

Given Assumption 3.3, equation (3.5) also applies to the pair of functions

$$\bar{B}_k(\boldsymbol{q}) = \Pr[D = k | \boldsymbol{Q}(\boldsymbol{Z}) = \boldsymbol{q}] \text{ with } \bar{b}_k(\boldsymbol{v}) = f_{\boldsymbol{V}}(\boldsymbol{v}).$$

This gives the first equality in the theorem. To obtain the second equality, we use

$$\tilde{B}_k(\boldsymbol{q}) = E[G(Y)D_k | \boldsymbol{Q}(\boldsymbol{Z}) = \boldsymbol{q}] \text{ and } \tilde{b}_k(\boldsymbol{v}) = E[G(Y_k) | \boldsymbol{V} = \boldsymbol{v}] f_{\boldsymbol{V}}(\boldsymbol{v}),$$

which again is locally equicontinuous by Assumption 3.3.

Under Assumption 3.5, the final conclusion of the theorem follows by using

$$EG(Y_k) = \int E\left(G(Y_k) | \boldsymbol{V} = \boldsymbol{v}\right) f_{\boldsymbol{V}}(\boldsymbol{v}) d\boldsymbol{v}.$$

$\square$

It follows from Theorem 3.1 that if k and k' are two treatments to which all of our assumptions apply, then we can identify the average treatment effect, as well as the marginal treatment effect and the quantile treatment effect of moving between these two treatments.

To identify the average treatment effect, we need the full support condition in Assumption 3.5. This is a stringent assumption that may not hold in many applications. In such cases we can extend Carneiro, Heckman, and Vytlacil (2010) to identify the marginal policy relevant treatment effect (MPRTE) and the average marginal treatment effect (AMTE). The MPRTE is a marginal version of the policy relevant treatment effect (PRTE) of Heckman and Vytlacil (2001), which measures the average effect of moving from a baseline policy to an alternative policy. The AMTE is the average benefit of treatment for people at the margin of indifference between participation in treatment and nonparticipation. We could obtain identification results for a generalized version of the MPRTE by specifying marginal changes for our selection mechanism.

There are applications in which the analyst may not have enough continuous instruments to identify even the MRPTE or AMTE. We consider the case of discrete instruments in Section 5.

3.2 Identification with a Zero Index

Theorem 3.1 required that the index of treatment k be non-zero (Assumption 3.1). Therefore it does not apply to Example 3 for instance. Recall that in that example,

$$D_0 = 1 - S_1 - S_2 - S_3 + S_1 S_2 + S_1 S_3 + S_2 S_3$$

and treatment 0 has degree $m^0 = 2 < J^0 = 3$.

Note, however, that steps 1 and 2 of the proof of Theorem 3.1 apply to zero-index treatments as well; the relevant polynomial of Heaviside functions has leading term

$$H(q_1 - v_1)H(q_2 - v_2) + H(q_1 - v_1)H(q_3 - v_3) + H(q_2 - v_2)H(q_3 - v_3),$$

and we can take the derivative in (q_1, q_2) for instance to obtain an equation that replaces (3.5):

$$\frac{\partial^2}{\partial q_1 \partial q_2} B_0(\boldsymbol{q}) = \int b_0(q_1, q_2, v_3) dv_3.$$

Applying this to $B_0(\boldsymbol{q}) = \Pr[D = 0 | \boldsymbol{Q}(\boldsymbol{Z}) = \boldsymbol{q}]$ and $b_0(\boldsymbol{v}) = f_{\boldsymbol{V}}(\boldsymbol{v})$, and then to $B_0(\boldsymbol{q}) = E[Y D_0 | \boldsymbol{Q}(\boldsymbol{Z}) = \boldsymbol{q}]$ and $b_0(\boldsymbol{v}) = E[G(Y_0) | \boldsymbol{V} = \boldsymbol{v}] f_{\boldsymbol{V}}(\boldsymbol{v})$, identifies

$$\int f_{V_1, V_2, V_3}(q_1, q_2, v_3) dv_3 = f_{V_1, V_2}(v_1, v_2)$$

and

$$\int E[G(Y_0) | V_1 = q_1, V_2 = q_2, V_3 = v_3] f_{V_1, V_2, V_3}(q_1, q_2, v_3) dv_3$$
$$= E[G(Y_0) | V_1 = q_1, V_2 = q_2] f_{V_1, V_2}(v_1, v_2);$$

and dividing through identifies a local counterfactual outcome:

$$E[G(Y_0)|V_1 = q_1, V_2 = q_2].$$

Under assumption 3.5, this also identifies $EG(Y_0)$. Moreover, we can apply the same logic to the pairs (q_1, q_3) and (q_2, q_3) to get further information on the treatment effects.

This argument is quite general. It allows us to state the following theorem:

Theorem 3.2 (Identification with a zero index). *Let Assumptions 2.1, 2.2, and 3.2 hold. Fix a value $\boldsymbol{q}$ in the support of $\boldsymbol{Q}(\boldsymbol{Z})$ and assume that 3.3 and 3.4 hold at $\boldsymbol{q}$. Let m be the degree of treatment k, and $c_l \prod_{i=1,\dots,m} S_{j_i}$ be any of the leading terms of the indicator function of $\{D = k\}$. Denote $I = \{j_1, \dots, j_m\}$, and $\widetilde{T}$ the differential operator*

$$\widetilde{T} = \frac{\partial^m}{\prod_{i=1,\dots,m} \partial_{j_i}}.$$

Then for $\boldsymbol{q} = (\boldsymbol{q}^{\boldsymbol{I}}, \boldsymbol{q}^{\boldsymbol{J}-\boldsymbol{I}})$,

$$f_{\boldsymbol{V}^I}(\boldsymbol{q}^I) = \frac{1}{c_l} \widetilde{T} \ \Pr[D = k | \boldsymbol{Q}(\boldsymbol{Z}) = \boldsymbol{q}]$$

$$E[G(Y_k)|\boldsymbol{V}^I = \boldsymbol{q}^I] = \frac{\widetilde{T}\, E[G(Y)D_k|\boldsymbol{Q}(\boldsymbol{Z}) = \boldsymbol{q}]}{\widetilde{T} \ \Pr[D = k|\boldsymbol{Q}(\boldsymbol{Z}) = \boldsymbol{q}]}.$$

If in addition Assumption 3.5 holds, then

$$EG(Y_k) = \frac{1}{c_l} \int_{[0,1]^J} \widetilde{T}\, E[G(Y)D_k|\boldsymbol{Q}(\boldsymbol{Z}) = \boldsymbol{q})d\boldsymbol{q}.$$

Proof of Theorem 3.2. The proof of Theorem 3.2 is basically the same as that of Theorem 3.1; it is included in Appendix B.1. □

Theorem 3.2 is a generalization of Theorem 3.1 (just take $m = J$). It calls for three remarks. First, we could weaken its hypotheses somewhat. We could for instance replace $(0, 1)^J$ with $(0, 1)^m$ in the statement of Assumption 3.5.

Second, when $m < J$ the treatment effects are overidentified. This is obvious from the equalities in Theorem 3.2, in which the right-hand side depends on $\boldsymbol{q}$ but the left-hand side only depends on $\boldsymbol{q}^I$.

Finally, considering several treatment values can identify even more, since $\boldsymbol{V}$ is assumed to be the same across k. Theorem 3.1 implies for instance that if there is any treatment value k with a nonzero index, then the joint density $f_{\boldsymbol{V}}$ is identified from that treatment value.

3.3 Identification of Q

So far we assumed that the functions $\{\boldsymbol{Q}_j(\boldsymbol{Z}) : j = 1, \ldots, J\}$ were known (see Assumption 2.1). In practice we will often need to identify them from the data before applying Theorems 3.1 or 3.2. The most natural way to do so starts from the generalized propensity scores $\{P_j(\boldsymbol{Z}) : j = 1, \ldots, J\}$, which are identified as the conditional probabilities of treatment[4].

First note that by definition (and by Assumption 2.2),

$$\begin{aligned} P_k(\boldsymbol{z}) &= \Pr(D = k | \boldsymbol{Z} = \boldsymbol{z}) \\ &= \int \mathbb{1}\left(g_k\left(\boldsymbol{v}, \boldsymbol{Q}(\boldsymbol{z})\right) = 0\right) f_{\boldsymbol{V}}(\boldsymbol{v}) d\boldsymbol{v}. \end{aligned}$$

Note that this is a J-index model. Ichimura and Lee (1991) consider identification of multiple index models with the indices are specified parametrically. Matzkin (1993, 2007) obtains nonparametric identification results for discrete choice models[5]; we build on her results in Section 4.4 to obtain the identification of $\boldsymbol{Q}$ for multiple hurdle models. Matzkin's results only apply to a subset of the types of selection mechanisms we consider (discrete choice models when all choices are observed). Section 4 discusses identification of the $\boldsymbol{Q}$'s in some specific models in more detail.

4 Applications

Our framework covers a wide variety of commonly used models. For simplicity, we only illustrate its usefulness on two-threshold selection models in this section.

[4] It would also be possible to seek identification jointly from the generalized propensity scores and from the cross-derivatives that appear in Theorems 3.1 or 3.2, especially when they are over-identified. We do not pursue this here.

[5] See Heckman and Vytlacil (2007, Appendix B) for an application to treatment models.

4.1 Monotone Treatment

We assume in this subsection that there are three treatments, $\mathcal{K} = \{0, 1, 2\}$. Given the results in Vytlacil (2002), the monotonicity assumption is essentially equivalent to the existence of a family of threshold crossing rules.

Example 4 (Monotone Treatment with $K = 3$)**.** For each treatment value $k = 0, 1, 2$, we assign the treatments in the following way:

- $D = 0$ iff $V_1 > Q_1(\boldsymbol{Z})$,
- $D = 1$ iff $V_1 < Q_1(\boldsymbol{Z})$ and $V_2 > Q_2(\boldsymbol{Z})$,
- $D = 2$ iff $V_1 < Q_1(\boldsymbol{Z})$ and $V_2 < Q_2(\boldsymbol{Z})$,

where V_1 and V_2 are independent $U[0, 1]$. This generates a model of treatment that satisfies our Assumption 2.1[6].

Remark 4.1. Note that the traditional ordered choice model only uses a common scalar random variable v, which we can normalize to be $U[0, 1]$: for $k = 0, 1, 2$,

$$D_k = 1 \ \text{ iff } \ F_k(\boldsymbol{Z}) < v < F_{k+1}(\boldsymbol{Z}),$$

with $F_0 \equiv 0$ and $F_2 \equiv 1$. This model of assignment to treatment is observationally equivalent to ours, provided that the probabilities of treatment $P_k = F_k - F_{k-1}$ coincide. We could for instance define the Q_k functions recursively by $1 - Q_1(\boldsymbol{Z}) = F_1(\boldsymbol{Z})$ and $1 - Q_2(\boldsymbol{Z}) = [F_2(\boldsymbol{Z}) - F_1(\boldsymbol{Z})]/[1 - F_1(\boldsymbol{Z})]$. This also shows that our assumption of independence of V_1 and V_2 is not restrictive in this setting. □

Going back to the original nonparametric model in Example 4, the thresholds are easily identified from

$$Q_k(\boldsymbol{Z}) = \Pr(D \geq k | D \geq k - 1, \boldsymbol{Z}).$$

Treatments $k = 1, 2$ consist of a single atom: respectively $E_1 \cap \bar{E}_2$ and $E_1 \cap E_2$. Therefore they have a nonzero index, with $a_1 = -1$ and $a_2 = 1$. Treatment value $k = 0$ comprises two atoms, $\bar{E}_1 \cap E_2$ and $\bar{E}_1 \cap \bar{E}_2$; and it has a zero index, with a leading coefficient $c_l = -1$.

[6] Appendix A.2 discusses monotone treatment for any finite K.

To apply Theorems 3.1 and 3.2, we assume the existence of enough continuous instruments $\boldsymbol{Z}$. It follows from Theorem 3.2 and $V_1 \sim U[0,1]$ that

$$E(Y_0|V_1 = v_1) = -\frac{\partial}{\partial q_1} E(YD_0|\boldsymbol{Q}(\boldsymbol{Z}) = \boldsymbol{v}).$$

Theorem 3.1, combined with the assumption that V_1 and V_2 are independent with marginal $U[0,1]$ distributions, gives for $k = 1$:

$$E(Y_1|\boldsymbol{V} = \boldsymbol{v}) = -\frac{\partial^2}{\partial q_1 \partial q_2} E(YD_1|\boldsymbol{Q}(\boldsymbol{Z}) = \boldsymbol{v})$$

and for $k = 2$:

$$E(Y_2|\boldsymbol{V} = \boldsymbol{v}) = \frac{\partial^2}{\partial q_1 \partial q_2} E(YD_2|\boldsymbol{Q}(\boldsymbol{Z}) = \boldsymbol{v}).$$

These formulæ can be used to estimate marginal treatment effects, and to run overidentifying tests.

Now take for instance the unconditional average treatment effect of moving to treatment value $k = 2$ from treatment value $k = 1$. Assume that $\boldsymbol{Z}$ contains at least two continuous instruments that generate full support variation in $\boldsymbol{Q}(\boldsymbol{Z})$. Then by integrating we obtain

$$E(Y_2 - Y_1) = \int_{(0,1)^2} \frac{\partial^2}{\partial q_1 \partial q_2} E\left(Y(D_1 + D_2)|\boldsymbol{Q}(\boldsymbol{Z}) = \boldsymbol{q}\right) d\boldsymbol{q}.$$

It is a natural extension of the binary treatment model ($K = 2$), for which the average treatment effect (ATE) is written simply as

$$E(Y_1 - Y_0) = \int_0^1 \frac{\partial E(Y|Q(\boldsymbol{Z}) = q)}{\partial q} dq$$

since $D_0 + D_1 = 1$. This is the standard formula that derives the ATE from the MTE (Heckman and Vytlacil, 2005).

4.2 Selection into Schooling and Employment

Let $\tilde{S}$ denote a binary schooling decision (say, college education) and $\tilde{E}$ a binary employment decision. We observe the outcome Y (wages) only when an individual is employed (say, $\tilde{E} = 1$). We are interested in the returns to a college education in the form of higher wages. Table 1 summarizes the selection problem in this example. Crossing $\tilde{E}$ and $\tilde{S}$ gives four treatment values $D = 0, 1, 2, 3$. We observe the value of D for each individual, and their wages iff $\tilde{E} = 1$; we denote Y_0 (resp. Y_1) the wages of an employee without (resp. with) a college education, and our parameters of interest are the moments of the college premium $(Y_1 - Y_0)$.

Table 1: Schooling, employment, and wages

	$\tilde{E} = 0$ (non-employed)	$\tilde{E} = 1$ (employed)
$\tilde{S} = 0$ (no college education)	D_0	(D_2, Y_0)
$\tilde{S} = 1$ (college education)	D_1	(D_3, Y_1)

In line with our general model, we assume that both assignments $\tilde{S}$ and $\tilde{E}$ are characterized by a single crossing model based on a one-dimensional unobserved heterogeneity term:

$$\begin{aligned}\tilde{S} &= 1 \text{ iff } V_1 < Q_1(\boldsymbol{Z})\\ \tilde{E} &= 1 \text{ iff } V_2 < Q_2(\boldsymbol{Z}),\end{aligned}$$

where the unobservables V_1 and V_2 are independent of $\boldsymbol{Z}$, marginally distributed as $U[0,1]$; and their codependence structure is unknown. Here Q_1 and Q_2 are identified from the population directly by $Q_1(\boldsymbol{Z}) = \Pr(\tilde{S} = 1|\boldsymbol{Z})$ and $Q_2(\boldsymbol{Z}) = \Pr(\tilde{E} = 1|\boldsymbol{Z})$.

To use the notation of Section 2.2, we have

$$\begin{aligned}D_2 &= S_2(\boldsymbol{Z})(1 - S_1(\boldsymbol{Z}))\\ D_3 &= S_1(\boldsymbol{Z})S_2(\boldsymbol{Z}).\end{aligned}$$

Note that the indices for both treatment values 2 and 3 are non-zero: $a_2 = -1$ and $a_3 = 1$. Therefore Theorem 3.1 applies to $k = 2, 3$, provided in particular that $\boldsymbol{Q}_1(\boldsymbol{Z})$ and $\boldsymbol{Q}_2(\boldsymbol{Z})$ are functionally independent—which is generically true if $\boldsymbol{Z}$ contains two continuous instruments. Under these assumptions,

$$\begin{aligned}E(Y_0|V_1 = q_1, V_2 = q_2) &= \frac{\partial^2 E[YD_2|Q_1(\boldsymbol{Z}) = q_1, Q_2(\boldsymbol{Z}) = q_2]/\partial q_1 \partial q_2}{\partial^2 \Pr[D_2 = 1|Q_1(\boldsymbol{Z}) = q_1, Q_2(\boldsymbol{Z}) = q_2]/\partial q_1 \partial q_2}\\ E(Y_1|V_1 = q_1, V_2 = q_2) &= \frac{\partial^2 E[YD_3|Q_1(\boldsymbol{Z}) = q_1, Q_2(\boldsymbol{Z}) = q_2]/\partial q_1 \partial q_2}{\partial^2 \Pr[D_3 = 1|Q_1(\boldsymbol{Z}) = q_1, Q_2(\boldsymbol{Z}) = q_2]/\partial q_1 \partial q_2}\end{aligned}$$

and the marginal treatment effect obtains by simple difference.

To identify the average treatment effect $E(Y_1 - Y_0)$, we use Theorem 3.1 again under the "full support" Assumption 3.5.[7] Since $a_2 = -1$ and $a_3 = 1$, we obtain

$$EY_0 = -\int_0^1 \int_0^1 \frac{\partial^2 E[YD_2|Q_1(\boldsymbol{Z}) = q_1, Q_2(\boldsymbol{Z}) = q_2]}{\partial q_1 \partial q_2} dq_1 dq_2$$
$$EY_1 = \int_0^1 \int_0^1 \frac{\partial^2 E[YD_3|Q_1(\boldsymbol{Z}) = q_1, Q_2(\boldsymbol{Z}) = q_2]}{\partial q_1 \partial q_2} dq_1 dq_2$$

so that, since $D_2 + D_3 = \tilde{E}$,

$$E(Y_1 - Y_0) = \int_0^1 \int_0^1 \frac{\partial^2 E[YE|Q_1(\boldsymbol{Z}) = q_1, Q_2(\boldsymbol{Z}) = q_2]}{\partial q_1 \partial q_2} dq_1 dq_2.$$

This formula is very intuitive: integrating the right hand side of the equation above gives

$$\begin{aligned} E(Y_1 - Y_0) = {} & E[YE|Q_1(\boldsymbol{Z}) = 1, Q_2(\boldsymbol{Z}) = 1] \\ & - E[YE|Q_1(\boldsymbol{Z}) = 0, Q_2(\boldsymbol{Z}) = 1] \\ & - E[YE|Q_1(\boldsymbol{Z}) = 1, Q_2(\boldsymbol{Z}) = 0] \\ & + E[YE|Q_1(\boldsymbol{Z}) = 0, Q_2(\boldsymbol{Z}) = 0]. \end{aligned}$$

The last two terms are zero since the probability of employment is zero when $Q_2(\boldsymbol{Z}) = 0$; and conversely, the probability of employment is one when $Q_2(\boldsymbol{Z}) = 1$. That leaves us with

$$E(Y_1 - Y_0) = E[Y|Q_1(\boldsymbol{Z}) = 1, Q_2(\boldsymbol{Z}) = 1] - E[Y|Q_1(\boldsymbol{Z}) = 0, Q_2(\boldsymbol{Z}) = 1],$$

the difference in average wages between the surely-employed populations who are surely college-educated or surely not.

Our approach yields much more than this fairly trivial result, since it identifies the whole function $(q_1, q_2) \mapsto E(Y_1 - Y_0|V_1 = q_1, V_2 = q_2)$, as well as the joint density. The

[7] Remember that all of our analysis is conditional on covariates $\boldsymbol{X}$. In practice, it is often impossible to do so nonparametrically. In their study of returns to schooling, Carneiro, Heckman, and Vytlacil (2011) and Carneiro and Lee (2009) circumvent this difficulty by assuming that both the covariates $\boldsymbol{X}$ and instruments $\boldsymbol{Z}$ are independent of the error terms U_k and the scalar V. Then $\boldsymbol{Q}$ can be constructed as a function of both $\boldsymbol{X}$ and $\boldsymbol{Z}$. Such an assumption would allow us to obtain full support even if $\boldsymbol{Z}$ is discrete, by interacting $\boldsymbol{Z}$ with continuous components of $\boldsymbol{X}$.

joint density $f_{V_1,V_2}(q_1, q_2)$ is of interest in itself, as (conditioning on the instruments) it reveals the dependence structure between the likelihood of graduation and the likelihood of employment. Note that $f_{V_1,V_2}(q_1, q_2)$ is over-identified, since it can be obtained from taking cross partial derivatives of $\Pr[D_2 = 1|Q_1(\boldsymbol{Z}) = q_1, Q_2(\boldsymbol{Z}) = q_2]$ or of $\Pr[D_3 = 1|Q_1(\boldsymbol{Z}) = q_1, Q_2(\boldsymbol{Z}) = q_2]$:

$$\begin{aligned} f_{V_1,V_2}(q_1, q_2) &= \frac{\partial^2 \Pr[\tilde{E} = 1, \tilde{S} = 1|Q_1(\boldsymbol{Z}) = q_1, Q_2(\boldsymbol{Z}) = q_2]}{\partial q_1 \partial q_2} \\ &= -\frac{\partial^2 \Pr[\tilde{E} = 1, \tilde{S} = 0|Q_1(\boldsymbol{Z}) = q_1, Q_2(\boldsymbol{Z}) = q_2]}{\partial q_1 \partial q_2}. \end{aligned}$$

Comparing the two resulting estimators provides a specification check.

To conclude this example, note that we could allow for a direct effect of schooling on employment, by adding an argument in Q_2:

$$\tilde{E} = 1 \text{ iff } V_2 < Q_2(\boldsymbol{Z}, \tilde{S}).$$

We could try to rewrite this selection rule as

$$\tilde{E} = 1 \text{ iff } V_2' < Q_2'(\boldsymbol{Z})$$

for a different unobserved heterogeneity term V_2'; but since $\tilde{S}$ is a discontinuous function of V_1, this would violate the continuity requirements that drive Theorem 3.1.

On the other hand, we may still be able to apply our results since we deal with $D = 2$ and $D = 3$ separately. The threshold Q_1 is still directly identified from the probability of graduation. The probability of employment now depends on both $Q_2(\cdot, 0)$ and $Q_2(\cdot, 1)$; we will assume here that their variations are restricted so that they are still identified. With obvious changes in notation, we now have

$$\begin{aligned} D_2 &= (1 - S_1(\boldsymbol{Z}))S_2(\boldsymbol{Z}, 0) \\ D_3 &= S_1(\boldsymbol{Z})S_2(\boldsymbol{Z}, 1); \end{aligned}$$

and the conditional expectations are identified by

$$E(Y_0|V_1 = q_1, V_2 = q_2) = \frac{\partial^2 E[YD_2|Q_1(\boldsymbol{Z}) = q_1, Q_2(\boldsymbol{Z}, 0) = q_2]/\partial q_1 \partial q_2}{\partial^2 \Pr[D_2 = 1|Q_1(\boldsymbol{Z}) = q_1, Q_2(\boldsymbol{Z}, 0) = q_2]/\partial q_1 \partial q_2}$$
$$E(Y_1|V_1 = q_1, V_2 = q_2) = \frac{\partial^2 E[YD_3|Q_1(\boldsymbol{Z}) = q_1, Q_2(\boldsymbol{Z}, 1) = q_2]/\partial q_1 \partial q_2}{\partial^2 \Pr[D_3 = 1|Q_1(\boldsymbol{Z}) = q_1, Q_2(\boldsymbol{Z}, 1) = q_2]/\partial q_1 \partial q_2},$$

from which we can compute marginal and average treatment effects. This shows that interesting models that do not seem to fit our assumptions at first sight can still yield to our approach.

We should mention here a recent paper by Fricke, Frölich, Huber, and Lechner (2015). They consider a model with both treatment endogeneity and non-response bias that has a structure similar to this schooling-employment example. Using a discrete instrument for the binary treatment and a continuous instrument for attrition, they identify the average treatment effect for both the compliers and the total population. In contrast, we identify the marginal treatment effects with two continuous instruments.

* * *

The selection mechanism in Table 4.2 has a single atom for each of the four treatment values[8]. As explained in the introduction, our approach covers all cases in which the analyst only observes limited information on the set of alternatives. We illustrate this by combining the treatment values D_1 and D_2 in Table 4.2 into one common treatment in Section 4.3 and by putting (D_0, D_1, D_2) together in Section 4.4. These models generate different selection patterns; and not surprisingly, the identification conditions demand somewhat stronger instruments as the number of treatment values—the information available to the analyst—decreases.

4.3 Selection with Two-Way Flows

Now return to Example 1, in which

- $D = 0$ iff $V_1 < Q_1(\boldsymbol{Z})$ and $V_2 < Q_2(\boldsymbol{Z})$,
- $D = 1$ iff $V_1 > Q_1(\boldsymbol{Z})$ and $V_2 > Q_2(\boldsymbol{Z})$,

[8] This belongs to the class of "fully partitioned treatment assignments", which we treat in more detail in Appendix A.3.

- $D = 2$ iff $(V_1 - Q_1(\boldsymbol{Z}))$ and $(V_2 - Q_2(\boldsymbol{Z}))$ have opposite signs.

It is useful to start with some exclusion restrictions that help us identify $Q_1(\mathbf{Z})$ and $Q_2(\mathbf{Z})$ separately from the generalized propensity scores given in (2.1). Assume that

Assumption 4.1 (Two Continuous Instruments with Exclusion Restrictions)**.**

1. *The density of* (V_1, V_2) *is continuous on* $[0,1]^2$*, with marginal uniform distributions.*
2. *The instruments* $\mathbf{Z} \equiv (Z_1, Z_2)$ *consists of two scalar random variables whose joint distribution is absolutely continuous with respect to the Lebesgue measure.*
3. $Q_1(\mathbf{Z})$ *does not depend on* Z_2*, and it is continuously differentiable with respect to* Z_1*.*
4. $Q_2(\mathbf{Z})$ *does not depend on* Z_1*, and it is continuously differentiable with respect to* Z_2*.*

The crucial part of Assumption 4.1 is in the exclusion restrictions: Z_1 affects Q_1 but not Q_2, and Z_2 affects Q_2 but not Q_1.

It follows from (2.1) on page 6 that

$$Q_1(\boldsymbol{Z}) + Q_2(\boldsymbol{Z}) = 2P_0(\boldsymbol{Z}) + P_2(\boldsymbol{Z}). \tag{4.1}$$

The right hand side of (4.1) is identified directly from the data. Suppose that $\tilde{Q}_1(\boldsymbol{Z})$ and $\tilde{Q}_2(\boldsymbol{Z})$ also satisfy $\tilde{Q}_1(\boldsymbol{Z}) + \tilde{Q}_2(\boldsymbol{Z}) = 2P_0(\boldsymbol{Z}) + P_2(\boldsymbol{Z})$, as well as Assumption 4.1. Then writing $\Delta_j(\boldsymbol{Z}) = Q_j(\boldsymbol{Z}) - \tilde{Q}_j(\boldsymbol{Z})$ $(j = 1, 2)$ gives $\Delta_1(\boldsymbol{Z}) = -\Delta_2(\boldsymbol{Z})$. But by Assumption 4.1, Δ_1 does not depend on Z_2, and Δ_2 does not depend on Z_1. Therefore we must have $\tilde{Q}_1(Z_1) = Q_1(Z_1) + C$ and $\tilde{Q}_2(Z_2) = Q_2(Z_2) - C$, where C is a constant. This proves that Q_1 and Q_2 are identified up to an additive constant.

To get rid of the constant, we use (2.1) again: if F is the joint distribution of (V_1, V_2), we must have

$$P_0(\boldsymbol{Z}) = F(Q_1(\boldsymbol{Z}), Q_2(\boldsymbol{Z})). \tag{4.2}$$

This implies that $(\tilde{Q}_1, \tilde{Q}_2)$ must be associated with a joint distribution

$$\tilde{F}(q_1, q_2) \equiv F(q_1 - C, q_2 + C).$$

But since $V_1 \sim U[0,1]$, we must have $\tilde{F}(q_1, 1) \equiv q_1$. Now $\tilde{F}(q_1, 1) = F(q_1 - C, 1 + C)$; and if $C > 0$,

$$F(q_1 - C, 1 + C) = F(q_1 - C, 1) = q_1 - C < q_1.$$

Therfore $C \leq 0$. Similarly, $\tilde{F}(1, q_2) \equiv q_2$ gives $C \geq 0$. Hence $C = 0$ and both (Q_1, Q_2) and F are point-identified.

Using the identified Q_1 and Q_2, since the indices are $a_0 = a_1 = 1$ and $a_2 = -2$ we apply Theorem 3.1 to identify the joint density by

$$f_{V_1,V_2}(q_1, q_2) = \frac{1}{a_k} \frac{\partial^2 \Pr[D = k | Q_1(\boldsymbol{Z}) = q_1, Q_2(\boldsymbol{Z}) = q_2]}{\partial q_1 \partial q_2}, \tag{4.3}$$

where $k = 0, 1, 2$. Note that $f_{V_1,V_2}(q_1, q_2)$ is overidentified; checking equality between the right hand sides of (4.3) provides a specification test[9]. Similar remarks apply to the conditional expectations $E(Y_k | V_1 = q_1, V_2 = q_2)$; and as

$$E(Y_k | V_1 = q_1, V_2 = q_2) = \frac{\partial^2 E[Y D_k | Q_1(\boldsymbol{Z}) = q_1, Q_2(\boldsymbol{Z}) = q_2] / \partial q_1 \partial q_2}{\partial^2 \Pr[D = k | Q_1(\boldsymbol{Z}) = q_1, Q_2(\boldsymbol{Z}) = q_2] / \partial q_1 \partial q_2}$$

for each $k = 0, 1, 2$, the identification of the marginal and average treatment effects follows immediately.

4.4 Double Hurdle Model

Let us now return to the double hurdle model of the introduction, where treatment is binary and the selection mechanism is governed by

$$D = 1 \text{ iff } V_1 < Q_1(\boldsymbol{Z}) \text{ and } V_2 < Q_2(\boldsymbol{Z}), \tag{4.4}$$

and $D = 0$ otherwise.

Both treatment values have non-zero indices: $a_1 = 1$ and $a_0 = -1$.

Identification of Q_1 and Q_2, which is a requisite to applying Theorem 3.1, is not as straightforward as in the schooling/employment model of Section 4.2. In fact, this

[9] Since probabilities add up to one, only one of these equalities generates a specification test.

case is more demanding than the selection model with two-way flows in Section 4.3 since we only have two treatment values. We observe

$$\Pr(D=1|\boldsymbol{Z}) = F_{V_1,V_2}\left(Q_1(\boldsymbol{Z}), Q_2(\boldsymbol{Z})\right), \tag{4.5}$$

which is a nonparametric double index model in which both the link function F_{V_1,V_2} and the indices Q_1 and Q_2 are unknown. This is clearly underidentified without stronger restrictions. Matzkin (1993, 2007) considers nonparametric identification and estimation of polychotomous choice models. Our multiple hurdle model has a similar but not identical structure. We build on Lewbel (2000) and on Matzkin's results to identify $\boldsymbol{Q}$. To do so, we assume that the thresholds have the following structure:

$$\begin{aligned} Q_1(\boldsymbol{Z}) &= G_1\left(Z_1 + q_2(\boldsymbol{Z}_2)\right) \\ Q_2(\boldsymbol{Z}) &= G_3\left(Z_3 + q_4(\boldsymbol{Z}_4)\right), \end{aligned} \tag{4.6}$$

where G_1, G_3, q_2 and q_4 are unknown functions; we also allow for $q_2 = q_4 = 0$. We impose that

Assumption 4.2 (Identifying the Thresholds)**.** *The density of (V_1, V_2) is continuous on $[0,1]^2$, with marginal uniform distributions. Furthermore,*

1. *G_1 and G_3 are strictly increasing C^1 functions from possibly unbounded intervals $[a_1, b_1]$ and $[a_3, b_3]$ onto $[0,1]$;*
2. *there exists a point $(\bar{\boldsymbol{z}}_2, \bar{\boldsymbol{z}}_4)$ in the support of $(\boldsymbol{Z}_2, \boldsymbol{Z}_4)$ such that*
   (a) *the support of (Z_1, Z_3) conditional on $\boldsymbol{Z}_2 = \bar{\boldsymbol{z}}_2, \boldsymbol{Z}_4 = \bar{\boldsymbol{z}}_4$ is the rectangle $R_{13} = [a_1, b_1] \times [a_3, b_3]$;*
   (b) *the support of $\boldsymbol{Z}_2$ conditional on $\boldsymbol{Z}_4 = \bar{\boldsymbol{z}}_4$ equals its unconditional support;*
   (c) *the support of $\boldsymbol{Z}_4$ conditional on $\boldsymbol{Z}_2 = \bar{\boldsymbol{z}}_2$ equals its unconditional support.*
3. *if q_2 and/or q_4 are known to be zero, drop the corresponding conditioning statements in 2.*

Theorem 4.1 (Identification in the double-hurdle model)**.** *Under Assumption 4.2, the functions $F_{\boldsymbol{V}}, G_1, G_3$ and (if nonzero) q_2 and q_4 are identified from the propensity score* $\Pr(D=1|\boldsymbol{Z})$.

Proof of Theorem 4.1. The proof is in Appendix B.2. □

While Theorem 4.1 requires at least four continuous instruments when q_2 and q_4 are nonzero, various additional restrictions would relax this requirement. If for instance the functional forms of q_2 and q_4 were known, then $\boldsymbol{Z}_4$ could coincide with $\boldsymbol{Z}_2$. And if q_2 and q_4 were linear, we would be back to the linear multiple index model of Ichimura and Lee (1991).

Once $Q_1(\boldsymbol{Z})$ and $Q_2(\boldsymbol{Z})$ are identified, then under our assumptions we identify the joint density by

$$f_{V_1,V_2}(q_1, q_2) = \frac{\partial^2 \Pr[D = 1|Q_1(\boldsymbol{Z}) = q_1, Q_2(\boldsymbol{Z}) = q_2]}{\partial q_1 \partial q_2}. \tag{4.7}$$

Note that under Assumption 4.2, F_{V_1,V_2} is already identified; so that we overidentify f_{V_1,V_2}. The marginal treatment effect is given by

$$E(Y_1 - Y_0|V_1 = q_1, V_2 = q_2) f_{V_1,V_2}(q_1, q_2) = \frac{\partial^2 E[Y|Q_1(\boldsymbol{Z}) = q_1, Q_2(\boldsymbol{Z}) = q_2]}{\partial q_1 \partial q_2}. \tag{4.8}$$

Under Assumption 4.2, both $Q_1(\boldsymbol{Z})$ and $Q_2(\boldsymbol{Z})$ have full support, and the average treatment effect is identified by

$$E[Y_1 - Y_0] = \int_0^1 \int_0^1 \frac{\partial^2 E[Y|Q_1(\boldsymbol{Z}) = p_1, Q_2(\boldsymbol{Z}) = p_2]}{\partial p_1 \partial p_2}\Big|_{(p_1,p_2)=(q_1,q_2)} dq_1 dq_2. \tag{4.9}$$

Example 5. As another illustration, consider the following model of employment, adapted from Laroque and Salanié (2002). An employee ($D = 1$) must be employable, in the sense that her unobserved productivity ρ must be above the minimum wage $\underline{Y}$. Specify productivity as

$$\rho = R_1(\boldsymbol{Z}) - v_1,$$

where v_1 is independent of $\boldsymbol{Z}$. This gives a first hurdle $v_1 < R_1(\boldsymbol{Z}) - \underline{Y}$; and transforming both sides by the cdf of v_1 gives $V_1 < Q_1(\boldsymbol{Z})$.

In addition, employees must be willing to work at the offered wage. Assume that each employee receives her full productivity. Then with a disutility of work specified as

$$d = R_2(\boldsymbol{Z}) - v_2,$$

with again v_2 independent of $\boldsymbol{Z}$, the second hurdle $\rho > d$ translates to $v_1 + v_2 < R_1(\boldsymbol{Z}) - R_2(\boldsymbol{Z})$. Again, this can be transformed into $V_2 < Q_2(\boldsymbol{Z})$ using the cdf $F_{v_1+v_2}$.

The impact of employment on outcomes Y can then be assessed using (4.7), (4.8) and (4.9). Note that this particular structure naturally suggests ways of identifying Q_1 and Q_2, as Q_1 only depends on R_1 and Q_2 depends on both R_1 and R_2. $\square$

Example 6. Finally, consider a parental choice problem: the choice of a school for a child, given nontransferable utility. The child will go to a private school ($D = 1$) if both parents agree that she should: $V_1 < Q_1(\boldsymbol{Z})$ and $V_2 < Q_2(\boldsymbol{Z})$. Otherwise the child will attend a public school ($D = 0$). If Y is any child outcome, then the effect of attending a private school can be identified from (4.7), (4.8) and (4.9). $\square$

5 Discrete Instruments

Continuous instruments are a luxury that may not be available to the analyst. While our method seems to be extremely dependent on them, it is sometimes possible to use it with discrete-valued instruments, in the same way that LATE is an integrated version of the MTE. To see this, take the nonzero index case. Theorem 3.1 gave us equalities of the general form: for some functions $b(\boldsymbol{q})$ and $F(Y, D_k)$ (which will be different in different uses)

$$b(\boldsymbol{q}) = \frac{1}{a_k} TE\left(F(Y, D_k) | \boldsymbol{Q}(\boldsymbol{Z}) = \boldsymbol{q}\right), \tag{5.1}$$

where T is the linear differential operator

$$TH = \frac{\partial^J H}{\prod_{j=1}^J \partial q_j}.$$

With discrete-valued instruments, we cannot make sense of the right-hand side of (5.1); on the other hand, we can invert the operator T to obtain

$$a_k b(\boldsymbol{q}) = E\left(F(Y, D_k) | \boldsymbol{Q}(\boldsymbol{Z}) = \boldsymbol{q}\right) + F_0(\boldsymbol{q}), \tag{5.2}$$

where F_0 is any function with $TF_0 = 0$; that is,

$$F_0(\boldsymbol{q}) = \sum_{j=1}^{J} F_{0j}(\boldsymbol{q}_{-j}),$$

where each term in the sum excludes one of the components of $\boldsymbol{q}$.

Given discrete-valued instruments, we can apply the finite-difference $\bar{T}$ version of T to (5.2). The terms F_{0j} generate null finite differences, and we point-identify finite differences of b. In many models this will allow us to identify the average effect of a treatment on a family of observations that comprises several groups of "compliers."

If moreover the instruments vary in the "right" way, it is in fact easy to identify some local average treatment effects. Consider a model woih J thresholds. Assume that some of the values the instruments $\boldsymbol{Z}$ take in the data generate threshold values $\boldsymbol{Q}(\boldsymbol{Z})$ that form a hyperrectangle in $(Q_1, \ldots, Q_J)$ space. The analyst may know this because (s)he has identified the functions Q_j, or simply from prior knowledge on the role the instruments play in selection to treatment. We will return to the identification of $\boldsymbol{Q}$ with discrete instruments later in this section. For now, note that this obviously requires that the instruments take 2^J different values—and a specific alignment of the 2^J threshold values they generate.

Given this "rectangular case", pick two opposite summits of the hyperrectangle[10], and call them $\boldsymbol{q}^-$ and $\boldsymbol{q}^+$. Index all summits by $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_J) \in \Sigma_J = \{0, 1\}^J$, where summit $\boldsymbol{\sigma}$ has coordinates

$$q_j^{\boldsymbol{\sigma}} = (1 - \sigma_j) q_j^- + \sigma_j q_j^+;$$

and for each summit $\boldsymbol{\sigma}$, denote $n_{\boldsymbol{\sigma}}$ the number of indices j such that $\sigma_j = 1$. For instance, when $J = 1$, the hyperrectangle reduces to the interval between q^- and q^+; and $n_{\boldsymbol{\sigma}} = 0$ for q^- and $n_{\boldsymbol{\sigma}} = 1$ for q^+. When $J = 2$, it consists of four points in the two-dimensional plane: $(q_1^-, q_2^-), (q_1^-, q_2^+), (q_1^+, q_2^-), (q_1^+, q_2^+)$ and the corresponding $n_{\boldsymbol{\sigma}}$'s are respectively 0, 1, 1, and 2.

We claim that

Theorem 5.1 (The rectangular case)**.**

[10] They could be for instance the summits with the smallest and the largest coordinates; but any pair of opposite summits will do.

1. *If any treatment value k has a non-zero index a_k, then the probability that $\boldsymbol{V}$ belongs to the hyperrectangle is identified*[11] *by*

$$\Pr(q_j^- \leq V_j \leq q_j^+ \ \forall j = 1, \ldots, J) = \frac{1}{a_k} \sum_{\boldsymbol{\sigma} \in \Sigma_J} (-1)^{J+n_{\boldsymbol{\sigma}}} \Pr\left(D = k | \boldsymbol{Q}(\boldsymbol{Z}) = \boldsymbol{q}^{\boldsymbol{\sigma}}\right).$$

2. *If treatment value k has a non-zero index a_k, then the density-weighted average value of $E(Y_k|\boldsymbol{V} = \boldsymbol{q})$ in the hyperrectangle is identified by*

$$E_w E(Y_k|\boldsymbol{V} = \boldsymbol{q}) = \frac{\sum_{\boldsymbol{\sigma} \in \Sigma_J} (-1)^{J+n_{\boldsymbol{\sigma}}} E\left(Y D_k | \boldsymbol{Q}(\boldsymbol{Z}) = \boldsymbol{q}^{\boldsymbol{\sigma}}\right)}{\sum_{\boldsymbol{\sigma} \in \Sigma_J} (-1)^{J+n_{\boldsymbol{\sigma}}} \Pr\left(D = k | \boldsymbol{Q}(\boldsymbol{Z}) = \boldsymbol{q}^{\boldsymbol{\sigma}}\right)},$$

where $w(\boldsymbol{q})$ is $f_{\boldsymbol{V}}(\boldsymbol{q})$ normalized to integrate to one within the hyperrectangle.

Proof of Theorem 5.1. The proof is in Appendix B.3, where we also present analog results for treatment values with a zero index. □

It is important to be clear on the interpretation of the theorem. If treatment values k and l both have a non-zero index, then part 2 of the theorem shows that we identity a marginal treatment effect $E(Y_k - Y_l|\boldsymbol{V} = \boldsymbol{q})$ averaged with known weights (given part 1) over a hyperrectangle. The observations in the hyperrectangle may constitute an interesting population over which to compute this average in some applications. What can be said about this population is that by construction, they had $S_1 = \ldots = S_J = 0$ and they switched to $S_1 = \ldots = S_J = 1$. We may call this group "supercompliers" since the change from $\boldsymbol{q}^-$ to $\boldsymbol{q}^+$ made them jump above all J thresholds; but this group may or may not be of interest as regards the treatment effect of moving from k to l.[12]

Rather than going through a tedious enumeration of all the examples from Section 4, we illustrate Theorem 5.1 and the interpretation of the supercompliers by focusing here on the double hurdle model of Section 4.4 and the two-way flow model of Section 4.3. Remember that both models have two thresholds, so that

$$TH(q_1, q_2) = \frac{\partial^2 H}{\partial q_1 \partial q_2}(q_1, q_2) \ \text{ and } \ \bar{T}H(\boldsymbol{q}, \boldsymbol{q}') = \frac{H(\boldsymbol{q}) + H(\boldsymbol{q}') - H(q_1, q_2') - H(q_1', q_2)}{(q_1' - q_1)(q_2' - q_2)}.$$

[11] As always, if several treatment values have a non-zero index then applying the formula to each of them may overidentify the probability of the hyperrectangle.

[12] As pointed out in footnote 10, the choice of $\boldsymbol{q}^-$ and $\boldsymbol{q}^+$ can be tweaked and that could make this supercomplier group more relevant; this depends on the application of interest.

5.1 The Double Hurdle Model

Let us start with the double hurdle model of Section 4.4. Assume that the values the instruments take in the data contain two vectors $\boldsymbol{Z}^i$ and $\boldsymbol{Z}^l$ such that

$$q_1^i \equiv Q_1(\boldsymbol{Z}^i) < Q_1(\boldsymbol{Z}^l) \equiv q_1^l \text{ and } q_2^i \equiv Q_2(\boldsymbol{Z}^i) < Q_2(\boldsymbol{Z}^l) \equiv q_2^l.$$

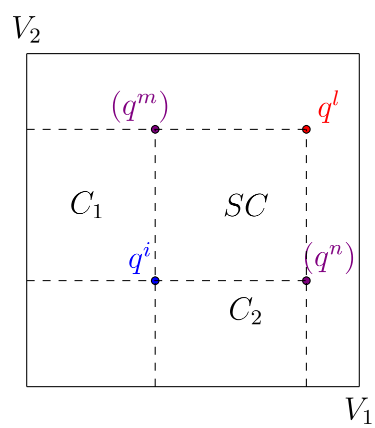
Here, both thresholds are higher under $\boldsymbol{q}^l \equiv (q_1^l, q_2^l)$ than under $\boldsymbol{q}^i \equiv (q_1^i, q_2^i)$. The analyst may know this because (s)he has identified the functions Q_1 and Q_2, or simply from prior knowledge on the role the instruments play in selection to treatment. For now, only ordinal knowledge that $q_1^i < q_1^l$ and $q_2^i < q_2^l$ is assumed—we will return to the identification of Q_1 and Q_2 with discrete instruments at the end of this subsection.

Since both thresholds increase, no observation moves from $D = 1$ to $D = 0$; and three groups move from $D = 0$ to $D = 1$:

1. (C_1): those with $V_1 < q_1^i$ and $q_2^i < V_2 < q_2^l$
2. (C_2): those with $q_1^i < V_1 < q_1^l$ and $V_2 < q_2^i$
3. (SC): those with $q_1^i < V_1 < q_1^l$ and $q_2^i < V_2 < q_2^l$.

To borrow from the language of the LATE literature, there are three different groups of compliers and no defiers, as shown in Figure 3.

Figure 3: Discrete instruments in the double hurdle model



The relative weights of these groups cannot be estimated from the data without further assumptions. If we form the Wald estimator

$$\frac{E(Y|\boldsymbol{Q}(\boldsymbol{Z})=\boldsymbol{q}^l)-E(Y|\boldsymbol{Q}(\boldsymbol{Z})=\boldsymbol{q}^i)}{\Pr(D=1|\boldsymbol{Q}(\boldsymbol{Z})=\boldsymbol{q}^l)-\Pr(D=1|\boldsymbol{Q}(\boldsymbol{Z})=\boldsymbol{q}^i)},$$

we only identify a weighted treatment effect for all three groups combined together. This illustrates the limitations of discrete instruments, and the difficulty of interpreting Wald estimands or their extensions when the selection mechanism is more complex than in the usual single-threshold model.

Let us now move to a more favorable case: we assume that the sample contains not only $\boldsymbol{Z}^i$ and $\boldsymbol{Z}^l$, but also values $\boldsymbol{Z}^m$ and $\boldsymbol{Z}^n$ such that the four vectors of thresholds $(\boldsymbol{q}^i,\boldsymbol{q}^l,\boldsymbol{q}^m,\boldsymbol{q}^n)$ form a rectangle in (q_1,q_2) space. This could arise if the thresholds are varied independently and then the variations are combined. Of course, it requires that the vector of instruments take at least four values.

We form

$$\begin{aligned}\Delta_1 &= \Pr(D=1|\boldsymbol{Q}(\boldsymbol{Z})=\boldsymbol{q}^i)+\Pr(D=1|\boldsymbol{Q}(\boldsymbol{Z})=\boldsymbol{q}^l)\\ &\quad-\Pr(D=1|\boldsymbol{Q}(\boldsymbol{Z})=\boldsymbol{q}^m)-\Pr(D=1|\boldsymbol{Q}(\boldsymbol{Z})=\boldsymbol{q}^n)\\ &= F_{V_1,V_2}(\boldsymbol{q}^i)+F_{V_1,V_2}(\boldsymbol{q}^l)-F_{V_1,V_2}(\boldsymbol{q}^m)-F_{V_1,V_2}(\boldsymbol{q}^n),\end{aligned} \tag{5.3}$$

which identifies the last term. Note in passing that the quantity

$$\frac{\Delta_1}{(q_1^i-q_1^l)(q_2^i-q_2^l)} \tag{5.4}$$

is the value of the density $f_{\boldsymbol{V}}$ at some point between[13] $\boldsymbol{q}^i$ and $\boldsymbol{q}^l$; it is identified if the values of the thresholds are.

Let us turn to the identified quantity

$$\begin{aligned}\Delta_2 &= E(Y|\boldsymbol{Q}(\boldsymbol{Z})=\boldsymbol{q}^i)+E(Y|\boldsymbol{Q}(\boldsymbol{Z})=\boldsymbol{q}^l)-E(Y|\boldsymbol{Q}(\boldsymbol{Z})=\boldsymbol{q}^m)-E(Y|\boldsymbol{Q}(\boldsymbol{Z})=\boldsymbol{q}^n)\\ &= \int_{q_1^i}^{q_1^l}\int_{q_2^i}^{q_2^l}\frac{\partial^2}{\partial q_1\partial q_2}E(Y|V_1=q_1,V_2=q_2)dq_1dq_2.\end{aligned}$$

[13]More precisely, at some point on each arc that links these two points.

Now using (4.8), we get

$$\Delta_2 = \int_{q_1^i}^{q_1^l} \int_{q_2^i}^{q_2^l} E(Y_1 - Y_0 | V_1 = q_1, V_2 = q_2) f_{V_1,V_2}(q_1, q_2) dq_1 dq_2. \tag{5.5}$$

Again,

$$\frac{\Delta_2}{(q_1^i - q_1^l)(q_2^i - q_2^l)} = E(Y_1 - Y_0 | V_1 = t_1, V_2 = t_2) f_{V_1,V_2}(t_1, t_2)$$

for some point $\boldsymbol{q}$ between $\boldsymbol{q}^i$ and $\boldsymbol{q}^l$.

If the rectangle is small enough, it will be a good first approximation to say that Δ_2/Δ_1 identifies the MTE locally. If it is not, then we identify

$$\frac{\Delta_2}{\Delta_1} = \int_{q_1^i}^{q_1^l} \int_{q_2^i}^{q_2^l} E(Y_1 - Y_0 | V_1 = q_1, V_2 = q_2) w(q_1, q_2) dq_1 dq_2,$$

where the function

$$w(q_1, q_2) = \frac{f_{V_1,V_2}(q_1, q_2)}{\Delta_1}$$

defines unknown positive weights[14] that integrate to one. These weights are simply the density of $\boldsymbol{V}$ truncated to the rectangle.

This is an integrated MTE, just like LATE. Note that Δ_1 corresponds to the size of group 3 (SC); in fact the ratio Δ_2/Δ_1 is a density-weighted average of the effect of the treatment for group 3 (SC). If we are lucky enough to observe such a "rectangular" variation in the thresholds, then we can estimate the effect of treatment on this group of "super-compliers", who failed both criteria and now pass both.

We could also construct other Wald estimators in the rectangular case. For instance, assume that $q_1^n > q_1^i$ (so that $q_2^n = q_2^i$), and consider the identified ratio

$$\frac{E(Y|\boldsymbol{Q}(\boldsymbol{Z}) = \boldsymbol{q}^n) - E(Y|\boldsymbol{Q}(\boldsymbol{Z}) = \boldsymbol{q}^i)}{\Pr(D = 1|\boldsymbol{Q}(\boldsymbol{Z}) = \boldsymbol{q}^n) - \Pr(D = 1|\boldsymbol{Q}(\boldsymbol{Z}) = \boldsymbol{q}^i)}.$$

The denominator equals

$$\int_0^1 \int_0^1 \left(\mathbb{1}(v_1 < q_1^n, v_2 < q_2^n) - \mathbb{1}(v_1 < q_1^i, v_2 < q_2^i) \right) f_{V_1,V_2}(v_1, v_2) dv_1 dv_2;$$

[14] Δ_1 is positive given our ordering of i, l, m, n.

and since $q_2^n = q_2^i$, this can be rewritten as

$$\int_0^1 \int_0^1 \mathbb{1}(v_2 < q_2^n)\mathbb{1}(q_1^i < v_1 < q_1^n) f_{V_1,V_2}(v_1, v_2) dv_1 dv_2,$$

which is the size of group (C_1) of compliers. It is easy to see that this new Wald estimator estimates the treatment effect on this group. We could similarly define

$$\frac{E(Y|\boldsymbol{Q} = \boldsymbol{q}^m) - E(Y|\boldsymbol{Q} = \boldsymbol{q}^i)}{\Pr(D = 1|\boldsymbol{Q} = \boldsymbol{q}^m) - \Pr(D = 1|\boldsymbol{Q} = \boldsymbol{q}^i)}$$

and identify the average effect of treatment on group (C_2).

The rectangular case therefore identifies the sizes of the three groups of compliers, as well as the average effect of treatment for each group.

The identification of the values of the thresholds $\boldsymbol{q}$ is more difficult with discrete instruments than it was in Section 4.4; and it requires stronger assumptions.

To illustrate this, suppose that V_1 and V_2 are independent $U[0, 1]$, and that the thresholds are increasing functions of different instruments:

$$q_1 = Q_1(Z_1) \text{ and } q_2 = Q_2(Z_2).$$

Then with $\boldsymbol{Z} = (Z_1, Z_2)$, (4.5) becomes

$$\Pr(D = 1|\boldsymbol{Z}) = Q_1(Z_1)Q_2(Z_2), \tag{5.6}$$

Suppose that we observe four values of the vector of instruments $\boldsymbol{Z} = (Z_1, Z_2)$ of the form

$$\boldsymbol{Z}^i = (z_1^i, z_2^i), \boldsymbol{Z}^l = (z_1^l, z_2^l), \boldsymbol{Z}^m = (z_1^l, z_2^i), \boldsymbol{Z}^n = (z_1^i, z_2^l),$$

Then the vectors of thresholds are ordered in a rectangle as on Figure 3. If moreover $0 < \Pr(D = 1|\boldsymbol{Z}^j) < 1$ for $j = i, n, l, m$, then none of the values of the thresholds

$$q_1^i = Q_1(z_1^i), q_1^l = Q_1(z_1^l), q_2^i = Q_2(z_2^i), q_2^l = Q_2(z_2^l)$$

can be zero or one. The values of the generalized propensity scores at the four $\boldsymbol{Z}^j$ points are

$$P_i = q_1^i q_2^i \ ; \ P_l = q_1^l q_2^l \ ; \ P_m = q_1^l q_2^i \ ; \ P_n = q_1^i q_2^l.$$

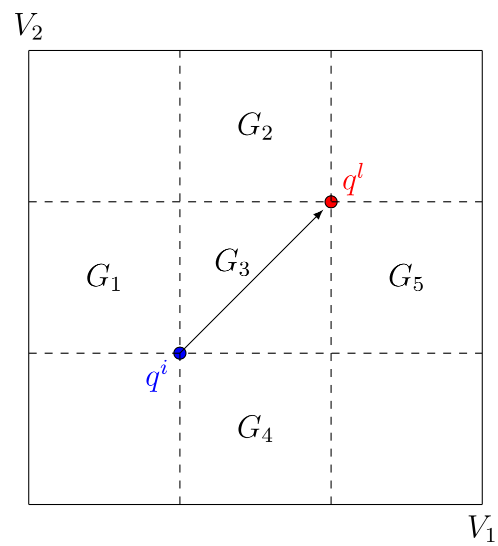
These four equations are linked by $P_i P_l = P_m P_n$, which is an implication of independence (and obviously a testable one). As a consequence, we cannot identify all four q_k^j values. This is natural, given the built-in symmetry: one can pick a value for say q_1^i, which fixes all other three values. Such a choice corresponds to fixing the relative scales of the two axes[15] while conserving the probability.

5.2 Selection with Two-way Flows

Example 1 presents different possibilities. Start again by assuming ordinal knowledge of the thresholds. Figure 4 illustrates the effect of a change in instruments that increases both thresholds, just as in Figure 3. Five groups are involved this time:

- (G_1): moves from $D = 2$ to $D = 0$
- (G_2): moves from $D = 1$ to $D = 2$
- (G_3): moves from $D = 1$ to $D = 0$
- (G_4): moves from $D = 2$ to $D = 0$
- (G_5): moves from $D = 1$ to $D = 2$.

Figure 4: Discrete instruments with two-way flows

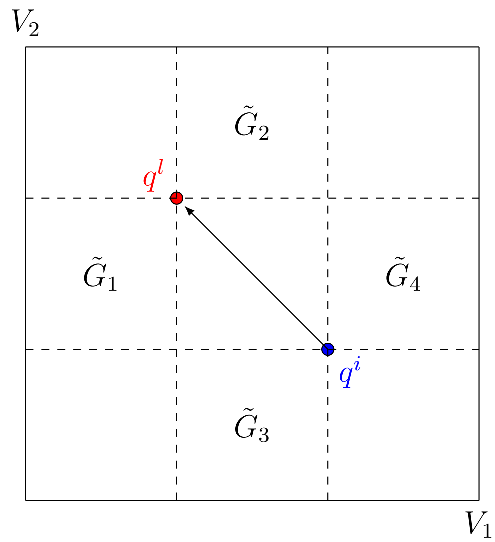


[15]The choice of q_1^i is only constrained by the need for all thresholds to be smaller than one. It is easy to check that q_1^i can be chosen freely between $\max(P_i, P_n)$ and P_l/P_m.

With three treatment values, we identify two changes in probabilities and three changes in expected values; but they are combinations of the weights of these five groups and of no fewer than $5 \times 2 = 10$ expected values like $E(Y\mathbb{1}(D=2) \mid (G_1))$.

In the "rectangular case" we would identify the effect of moving from $D=0$ to $D=1$ for group (G_3) by moving around the rectangle. By moving up from q^i we would combine groups $(G_1), (G_3)$, and (G_5); by moving to the right from q^i we would combine groups $(G_2), (G_3)$, and (G_4); etc.

Figure 5: Discrete instruments with two-way flows—bis



Note also that a change in instruments that moves the thresholds in opposite directions would only involve four groups, as shown in Figure 5. The groups are:

- $(\tilde{G}_1)$: moves from $D=2$ to $D=0$
- $(\tilde{G}_2)$: moves from $D=2$ to $D=1$
- $(\tilde{G}_3)$: moves from $D=0$ to $D=2$
- $(\tilde{G}_4)$: moves from $D=1$ to $D=2$.

Moving up from q^i would involve groups $(\tilde{G}_1)$ and $(\tilde{G}_4)$; and moving to the left from q^i would involve groups $(\tilde{G}_2)$ and $(\tilde{G}_3)$.

Identification of the thresholds $\boldsymbol{Q}$ is simpler, since with three treatment values we have more information (two generalized propensity score functions). As in the previous subsection, assume here that V_1 and V_2 are independent. Then (4.1) and (4.2) imply that

$$Q_1(\boldsymbol{Z}) + Q_2(\boldsymbol{Z}) = 2P_0(\boldsymbol{Z}) + P_2(\boldsymbol{Z}) \text{ and } Q_1(\boldsymbol{Z})Q_2(\boldsymbol{Z}) = P_0(\boldsymbol{Z}),$$

This defines a quadratic equation for $Q_1(\boldsymbol{Z})$ and $Q_2(\boldsymbol{Z})$. Solving the equation gives two solutions between 0 and 1 if the model is well-specified.[16] Given the symmetry between $Q_1(\boldsymbol{Z})$ and $Q_2(\boldsymbol{Z})$ in the selection mechanism of Example 1, we can only achieve identification of $(Q_1(\boldsymbol{Z}), Q_2(\boldsymbol{Z}))$ up to labeling.

Note that this discussion does not rely on exclusion restrictions as we did before. It shows again that independence is a powerful alternative identifying assumption.

6 Relation to the Existing Literature

Several papers have analyzed multivalued treatments under the unconfoundedness assumption. Imbens (2000) and Hirano and Imbens (2004) develop generalizations of the propensity score to discrete treatments and to continuous treatments, respectively. Cattaneo (2010) show that the semiparametric efficiency bound can be achieved in discrete treatment models by first estimating the generalized propensity score, then applying an inverse probability weighted estimator.

Since we do not assume conditional independence between potential outcomes and unobservables governing the selection mechanism, the rest of this section discusses selection on unobservables in models with multivalued treatment. The most popular approaches rely on instruments, like ours.

6.1 Ordered Treatments with Discrete Instruments

Angrist and Imbens (1995) consider two-stage least-squares estimation of a model in which the ordered treatment takes a finite number of values, and a discrete-valued instrument is available. Let $z = 0, \dots, M-1$ be the possible values of the instrument, ordered so that $E(D|Z = z)$ increases with z; and $D = 0, \dots, K-1$. Angrist and Imbens show that the TSLS estimator obtained by regressing outcome Y on a preestimated $E(D|Z)$ converges to $\beta_{TSLS} \equiv \sum_{m=1}^{M-1} \mu_m \beta_m$, where β_m's are called the

[16] One in the limit case when they are equal.

average causal responses, defined by

$$\beta_m \equiv \frac{E(Y|Z=m) - E(Y|Z=m-1)}{E(D|Z=m) - E(D|Z=m-1)}$$

for $m = 1, \dots, M-1$, and the family of weights $\{\mu_m\}_{m=1}^{M-1}$ is given by the joint distribution of D and Z.

The average causal response β_m itself can only be interpreted as causal under a stronger monotonicity assumption. Denote D_z the counterfactual treatment for $Z = z$, and assume that $D_m \geq D_{m-1}$ with probability one. Angrist and Imbens (1995) prove that under these assumptions, β_m is a weighted average of the effects of treatment on the various groups of compliers:

$$\beta_m = \sum_{k=1}^{K-1} \omega_k E\left(Y_k - Y_{k-1} | D_m \geq k > D_{m-1}\right).$$

The weights (ω_k) are given by the joint distribution of D_{m-1} and D_m, and they can be estimated under the monotonicity assumption. On the other hand, the individual terms

$$E\left(Y_k - Y_{k-1} | D_m \geq k > D_{m-1}\right)$$

cannot be identified; only their weighted average β_{TSLS} is.

Heckman, Urzua, and Vytlacil (2006, 2008) go beyond Angrist and Imbens (1995) by showing how the TSLS estimate can be reinterpreted in more transparent ways in the MTE framework. They also analyze a family of discrete choice models, to which we now turn.

6.2 Discrete Choice Models

Heckman, Urzua, and Vytlacil (2008, see also Heckman and Vytlacil (2007)) consider a multinomial discrete choice model of treatment. They posit

$$D = k \iff R_k(\boldsymbol{Z}) - U_k > R_l(\boldsymbol{Z}) - U_l \text{ for } l = 0, \dots, K-1 \text{ such that } l \neq k,$$

where the U's are continuously distributed and independent of $\boldsymbol{Z}$.

Define

$$\boldsymbol{R}(\boldsymbol{Z}) = \left(R_k(\boldsymbol{Z}) - R_l(\boldsymbol{Z})\right)_{l \neq k} \quad \text{and} \quad \boldsymbol{U} = \left(U_k - U_l\right)_{l \neq k}.$$

Then $D_k = \mathbb{1}(\boldsymbol{R}(\boldsymbol{Z}) > \boldsymbol{U})$; and defining $Q_l(\boldsymbol{Z}) = \Pr[\boldsymbol{U}_l < \boldsymbol{R}_l(\boldsymbol{Z})|\boldsymbol{Z}]$ allows us to write the treatment model as

$$D = k \text{ iff } \boldsymbol{V} < \boldsymbol{Q}(\boldsymbol{Z}), \tag{6.1}$$

where each V_l is distributed as $U[0,1]$.

Heckman, Urzua, and Vytlacil (2008) then study the identification of marginal and local average treatment effects under assumptions that are similar to ours: continuous instruments that generate enough dimensions of variation in the thresholds.

As they note, the discrete choice model with an additive structure implicitly imposes monotonicity, in the following form: if the instruments $\boldsymbol{Z}$ change in a way that increases $R_k(\boldsymbol{Z})$ relative to all other $R_l(\boldsymbol{Z})$, then no observation with treatment value k will be assigned to a different treatment. In our notation, D_k is an increasing function of $\boldsymbol{Q}(\boldsymbol{Z})$. We make no such assumption, as Example 1 and Figure 1 illustrate. Our results extend those of Heckman, Urzua, and Vytlacil (2008) to any model with identified thresholds.

Example 7 (Discrete Choice Model with Three Alternatives)**.** We consider a special case of (6.1). Suppose that $\mathcal{K} = \{0,1,2\}$ with $K = 3$. Let $\tilde{R}_{0,1}(\boldsymbol{Z}) = R_0(\boldsymbol{Z}) - R_1(\boldsymbol{Z})$, $\tilde{R}_{0,2}(\boldsymbol{Z}) = R_0(\boldsymbol{Z}) - R_2(\boldsymbol{Z})$ and $\tilde{R}_{1,2}(\boldsymbol{Z}) = R_1(\boldsymbol{Z}) - R_2(\boldsymbol{Z})$. Similarly, let $\tilde{U}_{0,1} = U_0 - U_1$, $\tilde{U}_{0,2} = U_0 - U_2$ and $\tilde{U}_{1,2} = U_1 - U_2$. Let $V_{0,1} = F_{\tilde{U}_{0,1}}(\tilde{U}_{0,1})$ and $Q_{0,1}(\boldsymbol{Z}) = F_{\tilde{U}_{0,1}}(\tilde{R}_{0,1}(\boldsymbol{Z}))$. Define $V_{0,2}$, $V_{1,2}$, $Q_{0,2}(\boldsymbol{Z})$ and $Q_{1,2}(\boldsymbol{Z})$ similarly. Then the selection mechanism in (6.1) can be rewritten as

- $D = 0$ iff $V_{0,1} < Q_{0,1}(\boldsymbol{Z})$ and $V_{0,2} < Q_{0,2}(\boldsymbol{Z})$
- $D = 1$ iff $V_{0,1} > Q_{0,1}(\boldsymbol{Z})$ and $V_{1,2} < Q_{1,2}(\boldsymbol{Z})$
- $D = 2$ iff $V_{0,2} > Q_{0,2}(\boldsymbol{Z})$ and $V_{1,2} > Q_{1,2}(\boldsymbol{Z})$.

Our general result in Section 3 applies immediately once the $Q_{j,k}$'s are identified. This can be done along the lines of Theorem 4.1, or by applying the results of Matzkin (1993, 2007).

6.3 Unordered Monotonicity

In an important recent paper, Heckman and Pinto (2015) introduce a new concept of monotonicity. Their "unordered monotonicity" assumption can be rephrased in our notation in the following way. Take two values $\boldsymbol{z}$ and $\boldsymbol{z}'$ of the instruments $\boldsymbol{Z}$. We want to study the treatment effect of moving from k to k' by exploiting the change of instruments from $\boldsymbol{z}$ to $\boldsymbol{z}'$.

Assumption 6.1 (Unordered Monotonicity)**.** *Denote $D_{\boldsymbol{z}}$ and $D_{\boldsymbol{z}'}$ the counterfactual treatments. Then for $l = k, k'$, there cannot be two-way flows in and out of treatment value l as the instruments change. More succinctly,*

$$\Pr(D_{\boldsymbol{z}} = l \ \ \textit{and} \ \ D_{\boldsymbol{z}'} \neq l) \times \Pr(D_{\boldsymbol{z}} \neq l \ \ \textit{and} \ \ D_{\boldsymbol{z}'} = l) = 0.$$

Unordered monotonicity for treatment value l requires that if some observations move out of (resp. into) treatment value l when instruments change value from $\boldsymbol{z}$ to $\boldsymbol{z}'$, then no observation can move into (resp. out of) treatment value l. For binary treatments, unordered monotonicity is equivalent to the usual monotonicity assumption: there cannot be both compliers and defiers. When $K > 2$, it is much weaker, and also weaker than ordered choice.

Heckman and Pinto (2015) show that unordered monotonicity (for well-chosen changes in instruments) is equivalent to a treatment model based on rules that are additively nonseparable in the unobserved variables. That is,

$$D_k = \mathbb{1}\left(\phi_k(\boldsymbol{V}) \leq \psi_k(\boldsymbol{Z})\right)$$

for some functions ϕ_k and ψ_k that assign all observations to a unique treatment value. This is almost, but not quite, equivalent to a discrete choice model with additively separable utilities. In this interpretation, changes in instruments that increase the mean utility of an alternative relative to all others are unordered montonicity for that alternative, for instance. We refer the reader to Section 6 of Heckman and Pinto (2015) for a more rigorous discussion, and to Pinto (2015) for an application to the Moving to Opportunity program.

Unlike us, Heckman and Pinto (2015) do not require continuous instruments; all of their analysis is framed in terms of discrete-valued instruments and treatments. Beyond this (important) difference, unordered monotonicity clearly obeys our assumptions—just redefine $\phi_k(\boldsymbol{V})$ and $\psi_k(\boldsymbol{Z})$ above so that the unobserved variable is distributed as $U[0, 1]$. On the other hand, we allow for much more general models of treatment. It would be impossible, for instance, to rewrite our Examples 1, 2 and 3 so that they obey unordered monotonicity: to use the terminology of Heckman and Pinto (2015), they are both unordered and non-monotonic. We illustrate this point using Example 1 below.

Example 1 (continued)**.** In Example 1, we have that $D = 2$ iff $(V_1 - Q_1(\boldsymbol{Z}))$ and $(V_2 - Q_2(\boldsymbol{Z}))$ have opposite signs. Note that there are two unobserved categories within $D = 2$:

$$\begin{aligned} D = 2a \ \text{ iff } \ V_1 < Q_1 \ \text{ and } \ V_2 > Q_2, \\ D = 2b \ \text{ iff } \ V_1 > Q_1 \ \text{ and } \ V_2 < Q_2. \end{aligned}$$

Each one is unordered monotonic; but because we only observe their union, $D = 2$ is not unordered monotonic—increasing Q_1 brings more people into $2a$ but moves some out of $2b$, so that in the end we have two-way flows, contradicting unordered monotonicity. To put it differently, the selection mechanism in Example 1 becomes a discrete choice model when each of four alternatives $d = 0, 1, 2a, 2b$ is observed; however, we only observe whether alternative $d = 0$, $d = 1$ or $d = 2$ is chosen in Example 1. This amounts to "filtering" unordered monotonic treatment through a coarser information partition; this coarsening destroys unordered monotonicity. $\square$

Our formalism allows us to derive a new characterization of the unordered monotonicity property defined by Heckman and Pinto (2015). Take any treatment value k. In our model, a change in instruments $\boldsymbol{Z}$ acts on the treatmemt assigned to an observation with unobserved characteristics $\boldsymbol{V}$ through the indicator functions $S_j = \mathbb{1}(V_j < Q_j(\boldsymbol{Z}))$, which depend on the thresholds $\boldsymbol{Q}$.

Unordered monotonicity requires that there exist changes in thresholds $\Delta\boldsymbol{Q}$ such that for $\boldsymbol{Q}' = \boldsymbol{Q} + \boldsymbol{\Delta}\boldsymbol{Q}$,

$$\Pr\left\{D_k(\boldsymbol{V},\boldsymbol{Q}) = 0 \text{ and } D_k(\boldsymbol{V},\boldsymbol{Q}') = 1\right\} \times \Pr\left\{D_k(\boldsymbol{V},\boldsymbol{Q}) = 1 \text{ and } D_k(\boldsymbol{V},\boldsymbol{Q}') = 0\right\} = 0,$$

where the probabilities are computed over the joint distribution of $\boldsymbol{V}$.

In our framework, several thresholds are typically relevant for each treatment value. This makes the analysis of unordered monotonicity complex in general. To understand why, we start from

$$D_k = \sum_{n \in N_k} \Pi_n(\boldsymbol{S})$$

with $N_k \subset \{1, \dots, 2^J\}$ the atoms included in treatment value k and

$$\boldsymbol{S} = (S_1, \dots, S_J) \text{ for } S_j(\boldsymbol{V}, \boldsymbol{Q}) = \mathbb{1}(V_j < Q_j).$$

Here

$$\Pi_n = \prod_{j \in J - M_n} S_j \prod_{j \in M_n} (1 - S_j).$$

Consider D_k and Π_n as multivariate polynomials of $\Delta \boldsymbol{S}$. For any change in thresholds $\Delta \boldsymbol{Q}$ that induces changes in the indicators $\Delta \boldsymbol{S}$,

$$\Delta D_k = \sum_{m=1}^{J} \frac{1}{m!} \sum_{j_1 \neq \ldots \neq j_m} \frac{\partial^m D_k}{\partial S_{j_1} \ldots \partial S_{j_m}} \prod_{l=1}^{m} \Delta S_{j_l}. \tag{6.2}$$

Note that this is an exact expansion since D_k is a polynomial. Moreover, note that given a change in thresholds ΔQ_j,

$$\Delta S_j(\boldsymbol{V}) = \mathbb{1}(|V_j - Q_j| < |\Delta Q_j|) \times \delta_j, \tag{6.3}$$

where $\delta_j = \pm 1$ as $\Delta Q_j \gtrless 0$, and $q_j = 0$ otherwise.

The changes ΔS_j can only take the values 0 or ± 1. In general higher-order terms in the above expansion may be nonzero. However, if the changes in thresholds $\Delta \boldsymbol{Q}$ are small then we can neglect the higher order terms since the set of values of $\boldsymbol{V}$ for which several ΔS_j are nonzero will have very small probability. To make this more precise, we use the following definition:

Definition 1 (Two-Way Flows)**.** A change in thresholds $\Delta \boldsymbol{Q}$ generates two-way flows for treatment value k if and only if

$$\lim_{\varepsilon \to 0} \frac{\Pr\left(D_k(0) = 0 \text{ and } D_k(\varepsilon) = 1\right)}{\varepsilon} \times \frac{\Pr\left(D_k(0) = 1 \text{ and } D_k(\varepsilon) = 0\right)}{\varepsilon} > 0$$

for $D_k(\varepsilon) \equiv D_k(\boldsymbol{V}, \boldsymbol{Q} + \varepsilon \Delta \boldsymbol{Q})$.

We now provide new characterizations of unordered monotonicity.

Theorem 6.1 (Characterizing Unordered Monotonicity in the Small)**.** *Assume that $J > 1$ thresholds are relevant for treatment value k. Fix a value $\boldsymbol{Q}$ of the thresholds. Denote*

$$\boldsymbol{D}'_k(\boldsymbol{S}) = \frac{\partial D_k}{\partial \boldsymbol{S}}(\boldsymbol{S}).$$

1. *If each component of $\boldsymbol{D}'_k(\boldsymbol{S})$ has a constant sign when $\boldsymbol{S}$ varies over $\{0,1\}^{|J|}$, then some changes in thresholds will not generate two-way flows, and others will.*

2. *If the sign of a component j of $\boldsymbol{D}'_k(\boldsymbol{S})$ changes in a neighborhood of the hyperplane $S_j = 0$, then any change in thresholds will generate two-way flows.*

(In these two statements, we take 0 to have the same sign as both -1 and $+1$.)

Proof of Theorem 6.1. The proof is given in Appendix B.4. □

To illustrate the theorem, first consider the double hurdle model, for which $\boldsymbol{D}'_1(\boldsymbol{S}) = (S_2, S_1) \geq 0$. Changes such that ΔQ_1 and ΔQ_2 have the same sign do not generate two-way flows, but changes that generate $\Delta Q_1 \Delta Q_2 < 0$ do. Now turn to the model of Example 1, where $\boldsymbol{D}'_2(\boldsymbol{S}) = (1-2S_2, 1-2S_1)$. To get one way flows only, we would need to choose $q_1, q_2 = \pm 1$ so that

$$0, (1-2S_2)q_1, (1-2S_1)q_2 \text{ and } (1-2S_2)q_1 + (1-2S_1)q_2$$

have the same sign, and that sign is the same for all $\boldsymbol{V}$. But with $(1-2S_1)$ (resp. $(1-2S_2)$) changing sign near $\{S_1 = 0\}$ (resp. $\{S_2 = 0\}$), that is clearly impossible since as $\boldsymbol{V}$ varies the sum takes values $0, q_1, q_2, q_1+q_2, q_1-q_2, q_2-q_1$ and $-q_1-q_2$. Hence any change in instruments creates two-way flows.

6.4 Models with Continuous Treatment

Chesher (2003) develops conditions to identify derivatives of structural functions in nonseparable models by functionals of quantile regression functions. His framework includes the case of a continuous treatment and is based on a control function approach. In addition, Florens, Heckman, Meghir, and Vytlacil (2008) consider a potential outcome model with a continuous treatment. They assume a stochastic polynomial restriction such that the counterfactual outcome Y_d corresponding to the continuous treatment value d has the form:

$$Y_d = EY_d + \sum_{j=0}^{J} d^j \varepsilon_j,$$

where the order of the polynomial, $J < \infty$, is known. They show that the average treatment effect can be identified if a control function $\tilde{V}$ can be found such that

$$E(\varepsilon_j|D, Z) = E(\varepsilon_j|\tilde{V}) \equiv r_j(\tilde{V}),\ j = 0, \dots, J,$$

where D is the realized treatment and Z is a vector of instruments.

Imbens and Newey (2009) also consider selection on unobservables with a continuous treatment. They assume that the treatment (more generally in their paper, an endogenous variable) is given by $D = g(Z, V)$, with g increasing in a scalar unobserved V. They use a control function approach based on the identification of V as $F_{D|Z}(D|Z)$ to identify the average structural function

$$EY_d = EE(Y_d|V) = EE(Y|D = d, V),$$

as well as quantile, average, and policy effects.

Other more recent identification results along this line can be found in Torgovitsky (2015) and D'Haultfœuille and Février (2015) among others. One key restriction in this group of papers is the monotonicity in the scalar V in the selection equation. We do not rely on this type of restriction, but we only focus on the case of multivalued treatments. Hence, our approach and those of the papers cited in this subsection are complementary.

Finally, our approach shares some similarities with Hoderlein and Mammen (2007). They consider the identification of marginal effects in nonseparable models without monotonicity:

$$Y = \phi(X, Z, U),$$

where Z is continuous multivariate and $U \perp\!\!\!\perp X|Z$. They show that

$$E\left(\frac{\partial \phi}{\partial x}(x, z, U)|X = x, Z = z, Y = q_\alpha(x, z)\right) = \frac{\partial q_\alpha}{\partial x}(Y|X = x, Z = z).$$

In this equation, $q_\alpha(Y|X, Z)$ represents the α-quantile of the distribution of Y conditional on X and Z; and the left-hand side is a local average structural derivative. Since the quantiles are clearly identified from the data, so is the left-hand side. Their approach based on differentials is reminiscent of our method of taking derivatives. The parameters of interest they study are quite different, however; and their selection mechanism is not as explicit as ours.

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Online Appendices to "Identifying Effects of Multi-valued Treatments"

A Additional Results

A.1 Some Results on Indices

Assume that the model of treatment has J thresholds. This generates 2^J atoms. A treatment value is defined by the union of any number of atoms, that is by a subset of $\{1,\dots,2^J\}$. There are no fewer than $(2^{2^J}-2)$ possible selection rules (excluding the two trivial cases). The number of treatment models with t treatment values is the number of partitions of the set $\{1,\dots,2^J\}$ into t non-empty sets, which is an exponentially increasing number.

For $m=0,\dots,J$, the number of atoms with m terms $\bar{E}_j$ is $\binom{J}{m}$; and such atoms have an index $(-1)^m$. Since $\sum_{m=0}^{J}\binom{J}{m}(-1)^m=(1-1)^m=0$, the sum of the indices of all atoms is zero; and so is that of the indices of all treatment values since each atom belongs to one treatment and to one only. Moreover, for every atom with index 1 there is one with index -1, and vice versa (just take complements of the E_j and $\bar{E}_j$ sets). It follows that there are 2^{J-1} atoms with index 1 and 2^{J-1} with index -1.

To create a treatment value with all J thresholds relevant and a zero index, we need to combine (at least) an atom with index 1 and one with index -1. Take any such pair of atoms. They must differ on an odd number of threshold-crossing rules. They can differ on *only one* threshold j: but then their union would combine E_j or $\bar{E}_j$, and threshold j would not be relevant any more. It follows that the two-threshold case is very special: for $J=2$ no treatment value that responds to both thresholds can have zero index.

On the other hand, with $J=3$ thresholds one can simply take the complement of the three E_j or $\bar{E}_j$ in any atom; combining the resulting two atoms creates a zero-index treatment value, as in Example 3. And for $J>3$, we can leave all other threshold crossings unchanged.

A.2 Monotone Treatment (General Case)

This section generalizes Section 4.1 for any finite $K \geq 2$. To do so, take a family of thresholds $(Q_1(\boldsymbol{Z}), \ldots, Q_{K-1}(\boldsymbol{Z}))$ and unobserved mutually independent random variables $(V_1, \ldots, V_{K-1})$ whose marginal distributions are $U[0,1]$. First, assign $D = 0$ when $V_1 > Q_1(\boldsymbol{Z})$; and for every $k = 1, \ldots, K-1$ and given that $D \geq k-1$, let the model assign $D \geq k$ if and only if

$$V_k < Q_k(\boldsymbol{Z}).$$

This generates a model of treatment that satisfies our Assumption 2.1. It has $J = K-1$ and a very specific structure:

$$D = \arg\min\{k = 0, \ldots, K-2 \mid V_{k+1} > Q_{k+1}(\boldsymbol{Z})\},$$

with $D = K-1$ if $V_k < Q_k(\boldsymbol{Z})$ for all $k = 1, \ldots, K-1$.

Note that the thresholds are easily identified from

$$Q_k(\boldsymbol{Z}) = \Pr(D \geq k | D \geq k-1, \boldsymbol{Z}).$$

Each treatment value k is defined by k atoms E_j (for $j = 1, \ldots, k-1$) and one event $\bar{E}_{k+1}$, with the exceptions of $k = 0$ which only has atom $\bar{E}_1$ and $k = K-1$ which has atoms E_j for $j = 1, \ldots, K-1$. Therefore only treatment values $(K-2)$ and $(K-1)$ have a nonzero index, with $a_{K-2} = -1$ and $a_{K-1} = 1$. Treatment values $k = 0, \ldots, K-2$ (if $K > 2$) have $a_k = 0$ and their leading coefficient is $c_l = -1$.

To apply Theorems 3.1 and 3.2, we assume the existence of enough continuous instruments $\boldsymbol{Z}$. Using the generic notation $\boldsymbol{x}^n = (x_1, \ldots, x_n)$, we then obtain a series of formulæ for $k = 0, \ldots, K-2$ and all $\boldsymbol{v} \in (0,1)^{K-1}$:

$$E(Y_k | \boldsymbol{V}^{k+1} = \boldsymbol{v}^{k+1}) = -\frac{\partial^{k+1}}{\partial q_1 \ldots \partial q_{k+1}} E(Y D_k | \boldsymbol{Q}(\boldsymbol{Z}) = \boldsymbol{v})$$

along with two slightly different formulæ for $k = K-1$:

$$E(Y_{K-1} | \boldsymbol{V} = \boldsymbol{v}) = \frac{\partial^{K-1}}{\partial q_1 \ldots \partial q_{K-1}} E(Y D_{K-1} | \boldsymbol{Q}(\boldsymbol{Z}) = \boldsymbol{v}).$$

These formulæ can be used to estimate marginal treatment effects, and to run overidentifying tests.

Now take for instance the unconditional average treatment effect of moving to treatment value $(K-1)$ from treatment value $(K-2)$. Assume that $\boldsymbol{Z}$ contains at least $(K-1)$ continuous instruments that generate full support variation in $\boldsymbol{Q}(\boldsymbol{Z})$. Then by integrating we obtain

$$E(Y_{K-1} - Y_{K-2}) = \int_{(0,1)^{K-1}} \frac{\partial^{K-1}}{\partial q_1 \dots \partial q_{K-1}} E\left(Y(D_{K-1} + D_{K-2}) | \boldsymbol{Q}(\boldsymbol{Z}) = \boldsymbol{q}\right) d\boldsymbol{q}.$$

A.3 Fully Partitioned Treatment Assignment

Sometimes the combination of J criteria determines 2^J different treatments[17], according to the value of the binary vector $(V_1 < Q_1(\boldsymbol{Z}), \dots, V_J < Q_J(\boldsymbol{Z}))$. Each of these treatment values is what we called earlier an atom, with an index ± 1. Identification of $(Q_1, \dots, Q_J)$ is straightforward; if for instance the first 2^{J-1} treatment values have $V_1 < Q_1$ and the last 2^{J-1} have $V_1 > Q_1$, then $Q_1(\boldsymbol{Z}) = \Pr(D \leq 2^{J-1} | \boldsymbol{Z})$.

To identify the treatment effects and joint density, we need J continuous instruments in $\boldsymbol{Z}$. To illustrate, order treatment values in the standard binary order, coding $V_j < Q_j$ as a 1. The joint density is multiply overidentified: for each treatment value $d = 0, \dots, 2^J - 1$ of index a_d,

$$f_{V_1, \dots, V_J}(q_1, \dots, q_J) = \frac{1}{a_d} \frac{\partial^J}{\partial q_1 \dots \partial q_J} \Pr(D = d | Q_1(\boldsymbol{Z}) = q_1, \dots, Q_J(\boldsymbol{Z}) = q_J).$$

Say that $J \geq 4$ and we want to identify the treatment effect of moving from treatment value $d = 1011$ (with index $a_d = -1$) to treatment value $d' = 0101$ (which has index $a_d = 1$). The marginal treatment effect is given by

$$\begin{aligned} &E(Y_d - Y_{d'} | V_1 = q_1, \dots, V_J = q_J) f_{V_1, \dots, V_J}(q_1, \dots, q_J) \\ &= \frac{\partial^J}{\partial q_1 \dots \partial q_J} E(Y(D_d + D_{d'}) | Q_1(\boldsymbol{Z}) = q_1, \dots, Q_J(\boldsymbol{Z}) = q_J). \end{aligned}$$

[17] We thank Rodrigo Pinto for suggesting this example to us.

B Proofs

B.1 Proof of Theorem 3.2

As explained in the text, steps 1 and 2 of the proof of Theorem 3.1 do not rely on any assumption about indices. They show that if we define

$$W_l(\boldsymbol{q}) = \int \prod_{j \in I_l} H(q_j - v_j) b_k(\boldsymbol{v}) d\boldsymbol{v}$$

where the set $I_l \subset \boldsymbol{J}$, then its cross-derivative with respect to $(\boldsymbol{p}^{I_l})$ is

$$\int b_k(\boldsymbol{q}^{I_l}, \boldsymbol{v}_{-I_l}) d\boldsymbol{v}_{-I_l},$$

where $\boldsymbol{v}_{-I_l}$ collects all components of $\boldsymbol{v}$ whose indices are not in I_l.

Now let m be the degree of treatment k. In the sum (3.4), take any term l such that $|I_l| = m$. Recall that $\widetilde{T}$ denotes the differential operator

$$\widetilde{T} = \frac{\partial^m}{\prod_{i=1,\dots,m} \partial_{j_i}}.$$

By the formula above, applying $\widetilde{T}$ to term l gives

$$c_l \int b_k(\boldsymbol{q}^{I_l}, \boldsymbol{v}_{-I_l}) d\boldsymbol{v}_{-I_l}.$$

Moreover, applying $\widetilde{T}$ to any other term l' obviously gives zero if term l' has degree lower than m. Turning to terms l' of degree m, any such term must have a $I_{l'} \neq I_l$, or it would be collected in term l. But then $\widetilde{T}$ takes at least one derivative along a direction that is not in l', and that term contributes zero too.

This proves that

$$\widetilde{T} B_k(\boldsymbol{q}) = c_l \int b_k(\boldsymbol{q}^{I_l}, \boldsymbol{v}_{-I_l}) d\boldsymbol{v}_{-I_l};$$

note that it also implies that $\widetilde{T} B_k(\boldsymbol{q})$ only depends on $\boldsymbol{q}^{I_l}$.

Applying this first to $b_k(\boldsymbol{v}) = f_{\boldsymbol{V}}(\boldsymbol{v})$ and $B_k(\boldsymbol{q}) = \Pr(D = k | \boldsymbol{Q}(\boldsymbol{Z}) = \boldsymbol{q})$, then to $b_k(\boldsymbol{v}) = E[G(Y_k) | \boldsymbol{V} = \boldsymbol{v}] f_{\boldsymbol{V}}(\boldsymbol{v})$ and $B_k(\boldsymbol{q}) = E[G(Y) D_k | \boldsymbol{Q}(\boldsymbol{Z}) = \boldsymbol{q}]$ exactly as in the proof of Theorem 3.1, we get

$$\int f_{\boldsymbol{V}}(\boldsymbol{q}^{I_l}, \boldsymbol{v}_{-I_l}) d\boldsymbol{v}_{-I_l} = \frac{1}{c_l}\widetilde{T}\Pr(D = k|\boldsymbol{Q}(\boldsymbol{Z}) = \boldsymbol{q})$$
$$\int E[G(Y_k)|\boldsymbol{V} = (\boldsymbol{q}^{I_l}, \boldsymbol{v}_{-I_l})] f_{\boldsymbol{V}}(\boldsymbol{q}^{I_l}, \boldsymbol{v}_{-I_l}) d\boldsymbol{v}_{-I_l} = \frac{1}{c_l}\widetilde{T} E(G(Y)D_k|\boldsymbol{Q}(\boldsymbol{Z}) = \boldsymbol{q}).$$

Since the left-hand sides are simply $f_{\boldsymbol{V}^{I_l}}(\boldsymbol{v}^{I_l})$ and $E[G(Y_k)|\boldsymbol{V}^{I_l} = \boldsymbol{q}^{I_l}] f_{\boldsymbol{V}^{I_l}}(\boldsymbol{v}^{I_l})$, the conclusion of the theorem follows immediately. □

B.2 Proof of Theorem 4.1

Without loss of generality[18], we normalize $q_2(\bar{\boldsymbol{z}}_2) = q_4(\bar{\boldsymbol{z}}_4) = 0$. Define H by

$$H(z_1, z_3) = \Pr\left(D = 1|Z_1 = z_1, Z_3 = z_3, \boldsymbol{Z}_2 = \bar{\boldsymbol{z}}_2, \boldsymbol{Z}_4 = \bar{\boldsymbol{z}}_4\right)$$

for any $(z_1, z_3) \in \mathbb{R}^2$.

Let $f_{\boldsymbol{V}}(v_1, v_2)$ denote the density of $\boldsymbol{V}$. By construction,

$$\text{(B.1)} \qquad H(z_1, z_3) = F_{\boldsymbol{V}}(G_1(z_1), G_3(z_3)) = \int_0^{G_1(z_1)} \int_0^{G_3(z_3)} f_{\boldsymbol{V}}(v_1, v_2) dv_1 dv_2.$$

Differentiating both sides of (B.1) with respect to z_1 gives

$$\text{(B.2)} \qquad \frac{\partial H}{\partial z_1}(z_1, z_3) = G_1'(z_1) \int_0^{G_3(z_3)} f_{\boldsymbol{V}}(G_1(z_1), v_2) dv_2.$$

Now letting $z_3 \to b_3$ on the both sides of (B.2) yields

$$\text{(B.3)} \qquad \lim_{z_3 \to b_3} \frac{\partial H}{\partial z_1}(z_1, z_3) = G_1'(z_1) \left[\lim_{z_3 \to b_3} \int_0^{G_3(z_3)} f_{\boldsymbol{V}}(G_1(z_1), v_2) dv_2\right].$$

Note that the expression inside the brackets on the right side side of (B.3) is 1 since $\lim_{z_3 \to b_3} G_3(z_3) = 1$ and the marginal distribution of V_2 is $U[0, 1]$. Therefore we identify G_1 by

$$\text{(B.4)} \qquad G_1(z_1) = \int_{a_1}^{z_1} \lim_{t_3 \to b_3} \frac{\partial H}{\partial z_1}(t_1, t_3) dt_1.$$

[18] We can always adjust G_1 and G_3 to compensate.

Analogously, we identify G_3 by

$$G_3(z_3) = \int_{a_3}^{z_3} \lim_{t_1 \to b_1} \frac{\partial H}{\partial z_3}(t_1, t_3) dt_3. \tag{B.5}$$

Returning to (B.1), since G_1 and G_3 are strictly increasing we also identify $F_{\boldsymbol{V}}$ by

$$F_{\boldsymbol{V}}(v_1, v_2) = H(G_1^{-1}(v_1), G_3^{-1}(v_2)).$$

Once $F_{\boldsymbol{V}}, G_1$ and G_3 are identified, we fix any point $(\bar{z}_1, \bar{z}_3)$ and we identify $q_2(\boldsymbol{z}_2)$ by choosing $Z_1 = \bar{z}_1; Z_3 = \bar{z}_3; Z_2 = \boldsymbol{z}_2$; and $Z_4 = \bar{\boldsymbol{z}}_4$. This gives

$$\Pr\left(D = 1 | Z_1 = \bar{z}_1; Z_3 = \bar{z}_3; Z_2 = \boldsymbol{z}_2, Z_4 = \bar{\boldsymbol{z}}_4\right) = F_{\boldsymbol{V}}\left(G_1(\bar{z}_1 + q_2(\boldsymbol{z}_2)), G_3(\bar{z}_3)\right)$$

which inverts to give the value of $q_2(\boldsymbol{z}_2)$. We proceed in the same way for $q_4(\boldsymbol{z}_4)$. $\square$

B.3 Proof of Theorem 5.1 and Theorem B.1

First define the discrete form $\bar{T}$ of the differential operator

$$Th(\boldsymbol{q}) = \frac{\partial^J h}{\partial q_1 \dots \partial q_J}(\boldsymbol{q})$$

by averaging $Th(\boldsymbol{q})$ over the hyperrectangle:

$$\bar{T}h(\boldsymbol{q}^-, \boldsymbol{q}^+) \equiv \frac{\int_{q_1^-}^{q_1^+} \dots \int_{q_J^-}^{q_J^+} Th(\boldsymbol{q}) d\boldsymbol{q}}{\prod_{j=1}^J (q_j^+ - q_j^-)}.$$

We claim that

$$\bar{T}h(\boldsymbol{q}^-, \boldsymbol{q}^+) = \frac{\sum_{\boldsymbol{\sigma} \in \Sigma_J} (-1)^{J + n_{\boldsymbol{\sigma}}} h(\boldsymbol{q}^{\boldsymbol{\sigma}})}{\prod_{j=1}^J (q_j^+ - q_j^-)}. \tag{B.6}$$

This is easily seen by a recursive argument. It is obviously true for $J = 1$, with

$$\bar{T}h(q^-, q^+) = \frac{1}{q^+ - q^-} \int_{q^-}^{q^+} h(q) dq.$$

Assume that (B.6) holds for $J = p$. For $J = p+1$ we write, integrating over the last coordinate,

$$\begin{aligned}
\bar{T}h(\boldsymbol{q}^-, \boldsymbol{q}^+) &= \frac{\int_{q_1^-}^{q_1^+} \cdots \int_{q_{p+1}^-}^{q_{p+1}^+} \frac{\partial^J h}{\partial q_1 \ldots \partial q_{p+1}} h(\boldsymbol{q}) d\boldsymbol{q}}{\prod_{j=1}^{p+1}(q_j^+ - q_j^-)} \\
&= \frac{1}{q_{p+1}^+ - q_{p+1}^-} \frac{\int_{q_1^-}^{q_1^+} \cdots \int_{q_p^-}^{q_p^+} \frac{\partial^p}{\partial q_1 \ldots \partial q_p} h(q_1, \ldots, q_p, q_{p+1}^+) dq_1 \ldots dq_p}{\prod_{j=1}^{p}(q_j^+ - q_j^-)} \\
&- \frac{1}{q_{p+1}^+ - q_{p+1}^-} \frac{\int_{q_1^-}^{q_1^+} \cdots \int_{q_p^-}^{q_p^+} \frac{\partial^p}{\partial q_1 \ldots \partial q_p} h(q_1, \ldots, q_p, q_{p+1}^-) dq_1 \ldots dq_p}{\prod_{j=1}^{p}(q_j^+ - q_j^-)} \\
&= \frac{1}{\prod_{j=1}^{p+1}(q_j^+ - q_j^-)} \sum_{\boldsymbol{\sigma} \in \Sigma_p} (-1)^{p+n_{\boldsymbol{\sigma}}} \left(h(\boldsymbol{q}^{\boldsymbol{\sigma}}, q_{p+1}^+) - h(\boldsymbol{q}^{\boldsymbol{\sigma}}, q_{p+1}^-) \right),
\end{aligned}$$

using $\boldsymbol{q}^{\boldsymbol{\sigma}}$ to denote summits of the p-th dimensional hyperrectangle. Noting that

$$(p+1) + (n_{\boldsymbol{\sigma}} + 1) \equiv p + n_{\boldsymbol{\sigma}} \pmod 2$$

completes the proof of (B.6).

The proof of Theorem 5.1 is now straightforward. First take $h(\boldsymbol{q}) = \Pr(D = k | \boldsymbol{Q}(\boldsymbol{Z}) = \boldsymbol{q})$. We know from Theorem 3.1 that

$$Th(\boldsymbol{q}) = a_k f_{\boldsymbol{V}}(\boldsymbol{q});$$

it follows that $\bar{T}h(\boldsymbol{q}^-, \boldsymbol{q}^+)$ is a_k times the average of the density of $\boldsymbol{V}$ over the hyperrectangle:

$$\frac{\sum_{\boldsymbol{\sigma} \in \Sigma_J} (-1)^{J+n_{\boldsymbol{\sigma}}} \Pr(D = k | \boldsymbol{Q}(\boldsymbol{Z}) = \boldsymbol{q}^{\boldsymbol{\sigma}})}{\prod_{j=1}^{J}(q_j^+ - q_j^-)} = a_k \frac{\int_{q_1^-}^{q_1^+} \cdots \int_{q_J^-}^{q_J^+} f_{\boldsymbol{V}}(\boldsymbol{q}) d\boldsymbol{q}}{\prod_{j=1}^{p+1}(q_j^+ - q_j^-)}$$

which proves part 1 of the theorem. Using $h(\boldsymbol{q}) = E(YD_k | \boldsymbol{Q}(\boldsymbol{Z}) = \boldsymbol{q})$ gives

$$\frac{\sum_{\boldsymbol{\sigma} \in \Sigma_J} (-1)^{J+n_{\boldsymbol{\sigma}}} E(YD_k | \boldsymbol{Q}(\boldsymbol{Z}) = \boldsymbol{q}^{\boldsymbol{\sigma}})}{\prod_{j=1}^{J}(q_j^+ - q_j^-)} = a_k \frac{\int_{q_1^-}^{q_1^+} \cdots \int_{q_J^-}^{q_J^+} f_{\boldsymbol{V}}(\boldsymbol{q}) E(Y_k | \boldsymbol{Q}(\boldsymbol{Z}) = \boldsymbol{q}) d\boldsymbol{q}}{\prod_{j=1}^{p+1}(q_j^+ - q_j^-)}$$

and dividing through gives part 2. $\square$

It is easy to adapt the proof of Theorem 5.1 to obtain the following result for treatment values with a zero index:

Theorem B.1 (The rectangular case with zero index)**.** *If no treatment value has a non-zero index but treatment k has degree $m < J$, renumber threshold conditions so that one of the highest-degree terms in $D_k(\boldsymbol{S})$ is $c \times S_1 \times \cdots \times S_m$. Then if we can construct a hyperrectangle in m-dimensional $(Q_1, \ldots, Q_m)$ space with fixed $(q_{m+1}, \ldots, q_J)$, adapting the $\boldsymbol{q}^{\boldsymbol{\sigma}}$ in the natural way,*

1. *the probability that $(V_1, \ldots, V_m)$ belongs to the hyperrectangle is identified by*

$$\Pr(q_j^- \leq V_j \leq q_j^+ \ \forall j = 1, \ldots, m) = \frac{1}{c} \sum_{\boldsymbol{\sigma} \in \Sigma_m} (-1)^{m+n_{\boldsymbol{\sigma}}} \Pr\left(D = k | \boldsymbol{Q}(\boldsymbol{Z}) = \boldsymbol{q}^{\boldsymbol{\sigma}}\right);$$

2. *the density-weighted average value of $E(Y_k | \boldsymbol{V} = \boldsymbol{q})$ in the hyperrectangle is identified by*

$$E_w E(Y_k | \boldsymbol{V} = \boldsymbol{q}) = \frac{\sum_{\boldsymbol{\sigma} \in \Sigma_m} (-1)^{m+n_{\boldsymbol{\sigma}}} E\left(Y D_k | \boldsymbol{Q}(\boldsymbol{Z}) = \boldsymbol{q}^{\boldsymbol{\sigma}}\right)}{\sum_{\boldsymbol{\sigma} \in \Sigma_m} (-1)^{m+n_{\boldsymbol{\sigma}}} \Pr\left(D = k | \boldsymbol{Q}(\boldsymbol{Z}) = \boldsymbol{q}^{\boldsymbol{\sigma}}\right)},$$

   where $w(\boldsymbol{q})$ is $f_{\boldsymbol{V}}(\boldsymbol{q})$ normalized to integrate to one within the hyperrectangle.

The changes in the proof are very minor: they only require setting $q_j^+ = q_j^-$ for $j > m$, limiting the σ_j's to be 0 for $j > m$, and integrating over the first m dimensions only.

B.4 Proof of Theorem 6.1

Remember that given a change in thresholds $\varepsilon \Delta Q_j$,

$$\Delta S_j(\boldsymbol{V}) = \mathbb{1}(|V_j - Q_j| < \varepsilon \, |\Delta Q_j|) \times \delta_j,$$

where $\delta_j = \pm 1$ as $\Delta Q_j \gtrless 0$, and $\delta_j = 0$ otherwise.

Therefore under our assumptions on the distribution of $\boldsymbol{V}$ the probability that $\Delta S_j \neq 0$ is of order ε; the probability that $\Delta S_j \Delta S_l \neq 0$ is of order ε^2, etc. Given Definition 1, we only need to work on the first-order terms in expansion (6.2) since the other terms generate vanishingly small corrections.

To prove part 1 of the theorem, assume that the sign of each derivative of D_k with respect to the S_j has a constant sign, independently of $\boldsymbol{S} \in \{0,1\}^{|J|}$. Then it is easy to find changes $\Delta \boldsymbol{Q}$ that only generate one-way flows: take each ΔQ_j to have the sign of $\frac{\partial D_k}{\partial S_j}$ (or take all opposite signs). It is equally easy to find changes in instruments that generate two-way flows. Take two indices $j \neq l$ such that $\frac{\partial D_k}{\partial S_j}$ and $\frac{\partial D_k}{\partial S_l}$ are not identically zero. (Such a pair exists since the number of k-relevant thresholds $J > 1$.) Under our assumptions we can take $\Delta Q_m = 0$ for $m \neq j, l$. Choose some $\Delta Q_j, \Delta Q_l \neq 0$ such that

$$\frac{\partial D_k}{\partial S_j}(\boldsymbol{S})\delta_j \text{ and } \frac{\partial D_k}{\partial S_l}(\boldsymbol{S})\delta_l$$

have opposite signs (which do not vary with S by assumption). Then the first-order terms in the expansion in (6.2) give

$$\Delta D_k \simeq \frac{\partial D_k}{\partial S_j}(\boldsymbol{S})\delta_j \mathbb{1}(|V_j - Q_j| < \varepsilon\,|\Delta Q_j|) + \frac{\partial D_k}{\partial S_l}(\boldsymbol{S})\delta_l \mathbb{1}(|V_l - Q_l| < \varepsilon\,|\Delta Q_l|).$$

Take $|V_j - Q_j|$ small and $|V_l - Q_l|$ not small; then this expression has the sign of $\frac{\partial D_k}{\partial S_j}(\boldsymbol{S})\delta_j$. Permuting j and l generates the opposite sign; therefore such a change in thresholds generates two-way flows.

To prove part 2 of the theorem, take j such that $\frac{\partial D_k}{\partial S_j}$ changes sign with $\boldsymbol{S}$ near the hyperplane $\{S_j = 0\} = \{V_j = Q_j\}$. Make $\Delta Q_j > 0$ and $\Delta Q_m = 0$ for all $m \neq j$. Then the first order term in (6.2) gives

$$\Delta D_k \simeq \frac{\partial D_k}{\partial S_j}(\boldsymbol{S})\delta_j \mathbb{1}(|V_j - Q_j| < \varepsilon\,|\Delta Q_j|),$$

which takes opposite values as V_j varies in a neighborhood of $S_j = 0$. $\square$