Robust Predictions in Dynamic Policy Games*

Juan Passadore † **Juan Pablo Xandri** ‡

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Abstract

Dynamic policy games feature a wide range of equilibria. The goal of this paper is to provide a methodology for obtaining robust predictions. We focus on a model of sovereign debt, although our methodology applies to other settings, such as models of monetary policy or capital taxation. Our main result is a characterization of outcomes that are consistent with a subgame perfect equilibrium conditional on the observed history. As an application of our methodology, given a data on observed play, we compute: (a) the set of possible continuation prices of debt and comparative statistics regarding this set (b) the probability of crises (c) bounds on means and variances across all equilibria.

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†Einaudi Institute for Economics and Finance. Email: `juan.passadore@eief.it`

‡Princeton University, Deptartment of Economics. Email: `jxandri@princeton.edu`.

1 Introduction

Following Kydland and Prescott (1977) and Calvo (1978) the literature on optimal government policy without commitment has formalized these situations by employing dynamic game theory, finding interesting applications for capital taxation (e.g. Chari and Kehoe, 1990, Phelan and Stacchetti, 2001, Farhi et al., 2012), monetary policy (e.g. Ireland, 1997, Chang, 1998, Sleet, 2001) and sovereign debt (e.g. Calvo, 1988, Eaton and Gersovitz, 1981, Chari and Kehoe, 1993, Cole and Kehoe, 2000). This research has helped us understand the distortions introduced by the lack of commitment and the extent to which governments can rely on a reputation for credibility to achieve better outcomes.

One of the challenges in applying dynamic policy games is that these settings typically feature a wide range of equilibria with different predictions over outcomes. For example, for "good" equilibria the government may achieve, or come close to achieving, the optimum with commitment, while there are "bad" equilibria where this is far from the case, and the government may be playing the repeated static best response. In studying dynamic policy games, which of these equilibria should we employ? One approach is to impose refinements, such as various renegotiation-proof notions, that select an equilibrium or significantly reduce the set of equilibria. Unfortunately, no consensus has emerged on the appropriate refinements.

Our goal is to overcome the challenge multiplicity raises by providing predictions in dynamic policy games that are not sensitive to any equilibrium selection. The approach we offer involves making predictions for future play that depend on past play. The key idea is that, even when little can be said about the *unconditional* path of play, quite a bit can be said once we *condition* on past observations. To the best of our knowledge, this simple idea has not been exploited as a way of deriving robust implications from the theory. Formally, we introduce and study a concept which we term "equilibrium consistent outcomes": outcomes of the game in a particular period that are consistent with a subgame perfect equilibrium, conditional on the observed history.

Although it will be clear that the notions we propose and results we derive are general and apply to any dynamic policy game, we first develop them for a specific application, using a model of sovereign debt along the lines of Eaton and Gersovitz (1981). This model constitutes a workhorse in international economics. In the model, a small open economy faces a stochastic stream of income. To smooth consumption, a benevolent government can borrow from international debt markets, but lacks commitment to repay. If it defaults on its debt, the only punishment is permanent exclusion from financial markets; it can never borrow again.[1]

Our main result provides a characterization of equilibrium consistent outcomes in any period (debt prices, debt issuance, and default decisions). Aided by this characterization, as a first application, we obtain bounds for equilibrium consistent debt prices that are history dependent. The highest equilibrium consistent price is the best Markov equilibria and, thus, independent of past play. The lowest equilibrium consistent price is strictly positive and depends on past play. In our baseline case, due to the recursive nature of equilibria, only the previous period play matters and acts as a sufficient statistic for the set of equilibrium consistent prices. The fact that the last period is a sufficient statistic may seem surprising. This result is a direct expression of robustness: the expected payoff rationalizing a decision may have been realized for histories that have not occurred. When income is continuous, any particular history has probability zero, so the realized expected payoff rationalizing past behavior can always been expected for those realizations that did not materialize. This intuition was first introduced by Gul and Pearce (1996) to show that Forward induction has much less predictive power as a solution concept if there are correlating devices.

In our sovereign debt application, equilibrium consistent debt prices improve whenever the government avoids default under duress. In particular, if the country just repaid a high amount of debt, or did so under harsh economic conditions, for example, when output was low, the lowest equilibrium consistent price is higher. The choice to repay under these conditions reveals an optimistic outlook for bond prices that narrows down the set of possible equilibria for the continuation game. This result captures the idea that reputation is built for the long run by short-run sacrifices.

The first part of the paper characterized equilibrium outcomes for the model as in Eaton and Gersovitz (1981). This model is usually utilized to study default due to fundamentals. We then turn to study a variation of the model that allows for coordination failures and crisis. For this we introduce a sunspot variable after the government chooses its policy. Our main result characterizes equilibrium consistent distribution over out-

[1]Given that our approach tries to overcome the challenges of multiplicity, as a starting point we first ensure that there is multiplicity in the first place. We show that in the standard Eaton and Gersovitz (1981) model, restrictions on debt, often adopted in the quantitative sovereign-debt literature, imply the existence of multiple equilibria. Our multiplicity relies on the existence of autarky as another Markov equilibrium. This result may be of independent interest, since it implies that rollover crises are possible in this setting. The quantitative literature on sovereign debt following Eaton and Gersovitz (1981) features defaults on the equilibrium path, but to shocks to fundamentals. A recent exception is Stangebye (2014) that studies the role of non fundamental shocks in sovereign crises in a model as in Eaton and Gersovitz (1981). Another strand literature studies self-fulfilling debt crises following the models in Calvo (1988) and Cole and Kehoe (2000). Our results suggest that crises, defined as episodes where the interest rates are very high but not due to fundamentals, are a robust feature in models of sovereign debt.

comes. As in the baseline model, building on these results we perform four applications. First, we study comparative statistics over the set of equilibrium consistent distributions. Second, we apply our results to bound the probability of a rollover debt crises. As we argued above, rollover debt crises may occur on the equilibrium path for any fundamentals. However, the probability of a rollover crisis, after a certain history, may be constrained. We derive these constraints, showing that rollover crises are less likely if the borrower has recently made sacrifices to repay. This result may be contrasted with Cole and Kehoe (2000). In their setting the potential for rollover crises induces the government to lower debt below a threshold that rules rollover crises out. Thus, the government's efforts have no effect in the short run, but payoff in the long run. In our model, an outside observer will witness that rollover crises are less likely immediately after an effort to repay. Third, we study bounds on moments of distributions over outcomes. In particular, we characterize bounds over the expected value of debt prices given a history for any equilibrium. Fourth, and finally, we characterize bounds on variances.

As we argue in the first Sections of the paper, our characterization of equilibrium consistent outcomes extends to other dynamic policy games. In order to show this, in the last Section of the paper, we provide a general model of credible government policies that follows the seminal contribution of Stokey (1991). The key features that the general setup tries to capture are: lack of commitment, a time inconsistency problem, infinite horizon that creates reputation concerns in the sense of trigger-strategy equilibria, and short run players that form an expectation regarding the policies of the government. Most dynamic policy games, such as applications to capital taxation and monetary policy, share these features. After proposing the general model, and showing that widely used frameworks such as Eaton and Gersovitz (1981) and the New Keynesian model as in Woodford (2011), fit in the setup, we prove the main results of the paper for this general model.

Literature Review. Our paper relates to several strands of the literature. First, to the literature on credible government policies. The seminal papers on optimal policy without commitment are Kydland and Prescott (1977) and Calvo (1978).[2] Recent applications range from from capital taxation as in Phelan and Stacchetti, 2001 and Farhi et al., 2012; monetary policy a in Ireland, 1997, Chang, 1998, Sleet, 2001; and sovereign debt Arellano (2008), Aguiar and Gopinath (2006), and Cole and Kehoe (2000). Recent applications are Farhi et al., 2012, Bocola and Dovis (2016), and Waki et al. (2015). We feel that our paper is more closely related to Chari and Kehoe (1990) and Stokey (1991). These two papers Chari

[2] Atkeson (1991) extends the approach to the case with a public state variable. Phelan and Stacchetti (2001) and Chang (1998) extend the approach to study models where individual agents hold stocks (capital and money respectively).

and Kehoe (1990) and Stokey (1991) adapt the techniques developed in Abreu (1988) to dynamic policy games. We contribute this literature in two ways. First, by providing a characterization of equilibrium that focus on outcomes in a particular period instead of the whole sequence of play. Second, we provide a methodology to obtain robust predictions across all equilibria, instead of focusing in one particular equilibrium.

Second, our paper shares the objective with the papers in the literature on robust predictions. The two papers more closely related to our work are Angeletos and Pavan (2013) and Bergemann and Morris (2013). The first paper obtains predictions that hold across every equilibrium in a global game with an endogenous information structure. The second paper obtains restrictions over moments of observable endogenous variables that hold across every possible information structure in a class of coordination games. Our paper relates to them in that we obtain predictions that hold across all equilibria. In a sense, our results are weaker than Angeletos and Pavan (2013) because our predictions are regarding the equilibrium set. However, it is also true that our problem has the additional challenge of being a (repeated) dynamic complete information game. The latter is precisely the root of weaker predictions.

Third, this paper studies robust predictions in a model dynamic policy game where a government borrows from international investors as in Eaton and Gersovitz (1981). This framework has been extensively used to study sovereign borrowing.

One direction, the quantitative literature on sovereign debt, focuses on a model where asset markets are incomplete and there is limited commitment for repayment, following Eaton and Gersovitz (1981), to study the quantitative properties of spreads, debt capacity, and business cycles. The aim of this strand of the literature is to account for the observed behavior of the data. The seminal contributions in this literature are Aguiar and Gopinath (2006) and Arellano (2008) which study economies with short term debt. The quantitative literature of sovereign debt has already been successful in explaining the most salient features in the data.[3] Our paper shares with this literature the focus on a model along the lines of Eaton and Gersovitz (1981) but rather than characterizing a particular equilibrium, it tries to study predictions regarding the set of equilibria.

Another direction of the literature focuses in equilibrium multiplicity, and in particular, in self fulfilling debt crises. The seminal contribution is Calvo (1988). Cole and Kehoe (2000) introduce self-fulfilling debt crises in a full-fledged dynamic model where the equilibrium selection mechanism is a sunspot that is realized simultaneously with

[3]Other examples in this literature are Hatchondo and Martinez (2009), Arellano and Ramanarayanan (2012), Chatterjee and Eyigungor (2012). Yue (2010), Bai and Zhang (2012), Pouzo and Presno (2011), Borri and Verdelhan (2009), D Erasmo (2008), Bianchi et al. (2012).

output. Lorenzoni and Werning (2013) study equilibrium multiplicity in a dynamic version of Calvo (1988). Our paper studies multiplicity but in the Eaton and Gersovitz (1981) setting; the crucial difference between the setting in Calvo (1988) and the one Eaton and Gersovitz (1981) is that in the latter the government issues debt (with commitment) and then the price is realized. This implies that equilibrium multiplicity is coming from the multiplicity of beliefs regarding continuation equilibria. Stangebye (2014) also studies multiplicity in a setting as in Eaton and Gersovitz (1981), but focuses on a Markov equilibrium. Our contribution to this strand of the literature is again to study predictions regarding the set of equilibria. Moreover, by providing sufficient conditions for equilibrium multiplicity in a model as in Arellano (2008), that are novel in the literature. These conditions show that, once we introduce coordination devices, coordination failures are robust feature of models of sovereign borrowing.

Outline. The paper is structured as follows. Section 2 introduces the model. Section E studies equilibrium multiplicity in our model of sovereign borrowing. Section 3 characterizes equilibrium consistent outcomes. Section 4 discusses the characterization of equilibrium consistent outcomes when there are correlating devices available after debt is issued. Section 5 spells out the general model and states the main results of the paper in this setup. Section 6 concludes.

2 A Dynamic Policy Game

Our model of sovereign debt follows Eaton and Gersovitz (1981). Time is discrete and denoted by $t \in \{0, 1, 2, ...\}$. A small open economy receives a stochastic stream of income denoted by y_t. Income follows a Markov process with c.d.f. denoted by $F(y_{t+1} \mid y_t)$. The government is benevolent and seeks to maximize the utility of the households. It does so by selling bonds in the international bond market. The household evaluates consumption streams according to

$$\mathbb{E}\left[\sum_{t=0}^{\infty} \beta^t u(c_t)\right]$$

where $\beta < 1$ and u is increasing and strictly concave. The sovereign issues short term debt at a price q_t. The budget constraint is

$$c_t = y_t - b_t + q_t b_{t+1}.$$

Following Chatterjee and Eyigungor (2012) we assume that the government cannot save

$$b_{t+1} \geq 0.$$

This helps focus our discussion on debt and may implicitly capture political economy constraints that make it difficult for governments to save, as modeled by Amador (2013).

There is limited enforcement of debt. Therefore, the government will repay only if it is more convenient to do so. We assume that the only fallout of default is that the government will remain in autarky forever after. We also do not introduce exogenous costs of default. As we will show below, our assumptions are sufficient for autarky to be an equilibrium. If the government cannot save, and there are no output costs of default, if the government expects a zero bond price for its debt now and in every future period, then it will default its debt. To guarantee multiplicity we need to introduce conditions to guarantee that best Markov equilibrium, the one usually studied in the literature of sovereign debt, has a positive price of debt. In Section E we characterize subgame perfect equilibrium and equilibrium consistent outcomes when the government can save and when defaults do not need to be punished.

Lenders. There is a competitive fringe of risk neutral investors that discount the future at rate $r > 0$. This implies that the price of the bond is given by

$$q_t = \frac{1 - \delta_t}{1 + r}$$

where δ_t if the default probability on bonds b_{t+1} issued at date t.

Timing. The sequence of events within a period is as follows. In period t, the government enters with b_t bonds that it needs to repay. Then income y_t is realized. The government then has the option to default $d_t \in \{0, 1\}$. If the government does not default, the government runs an auction of face value b_{t+1}. Then, the price of the bond q_t is realized. Finally, consumption takes place, and is given by $c_t = y_t - b_t + q_t b_{t+1}$. If the government decides to default, consumption is equal to income, $c_t = y_t$. The same is true if the government has ever defaulted in the past. We adopt the convention that if $d_t = 1$ then $d_{t'} = 1$ for all $t' \geq t$.

Histories and Outcomes. An income history is a vector $y^t = (y_0, y_1, \ldots, y_t)$ of all income realizations up to time t. A history is a vector $h^t = (h_0, h_1, ..., h_{t-1})$, where $h_t = (y_t, d_t, b_{t+1}, q_t)$ is the description of all realized values of income and actions, and $h = h'h''$

is the append operator. A partial history is an initial history h^t concatenated with a part of h_t. For example, $h = \left(h^t, y_t, d_t, b_{t+1}\right)$ is a history where we have observed h^t, output y_t has been realized, the government decisions (d_t, b_{t+1}) have been made, but market price q_t has not yet been realized. We will denote these histories $h = h_-^{t+1}$. The set of all partial histories (initial and partial) is denoted by $\mathcal{H}$, and $\mathcal{H}_g \subset \mathcal{H}$ are those where the government has to make a decision; i.e., $h = \left(h^t, y_t\right)$. Likewise, $\mathcal{H}_m \subset \mathcal{H}$ is the set of partial histories where investors set prices; i.e., $h_-^{t+1} = \left(h^t, y_t, d_t, b_{t+1}\right)$. An outcome path is a sequence of measurable functions[4]

$$x = \left(d_t\left(y^t\right), b_{t+1}\left(y^t\right), q_t\left(y^t\right)\right)_{t\in\mathbb{N}}$$

The set of all outcomes is denoted by $\mathcal{X}$. To make explicit that the default, bond policies and prices are the ones associated with the path x, sometimes we will write

$$\left(d_t^x\left(y^t\right), b_{t+1}^x\left(y^t\right), q_t^x\left(y^t\right)\right)_{t\in\mathbb{N}}.$$

An outcome x_t (the evaluation of a path at a particular period) is a description of the government's policy function and market pricing function at time t where the functions in x_t are $d_t : Y \to \{0,1\}$, $b_{t+1} : Y \to \mathbb{R}_+$, and $q_t : Y \to \mathbb{R}_+$. Our focus will be on a shifted outcome, $x_{t-} \equiv (q_{t-1}, d_t(\cdot), b_{t+1}(\cdot))$. The reason to do this is that the prices in q_{t-1} will only be a function of the next period default decision.

Strategies, Payoffs, Equilibrium. A strategy profile is a complete description of the behavior of both the government and the market, for any possible history. Formally, a strategy profile is defined as a pair of measurable functions $\sigma = \left(\sigma_g, q_m\right)$, where $\sigma_g : \mathcal{H}_g \to \{0,1\} \times \mathbb{R}_+$ and $q_m : \mathcal{H}_m \to \mathbb{R}_+$. The government decision will usually be written as

$$\sigma_g\left(h^t, y_t\right) = \left(d_t^{\sigma_g}\left(h^t, y_t\right), b_{t+1}^{\sigma_g}\left(h^t, y_t\right)\right)$$

so that $d_t^{\sigma_g}(\cdot)$ and $b_{t+1}^{\sigma_g}(\cdot)$ are the default decision and bond issuance decision for strategy σ_g. Σ_g is the set of all strategies for the government, and Σ_m is the set of market pricing strategies. $\Sigma = \Sigma_g \times \Sigma_m$ is the set of all strategy profiles. Given a history h^t, we define the continuation strategy induced by h^t as

$$\sigma_{|h^t}\left(h^s\right) = \sigma\left(h^t h^s\right).$$

[4]For our baseline case, where after default the government is permanently in autarky, the functions have the restriction that bond issues and prices are not defined after a default has been observed: $b_{t+s+1}\left(y^t y^s\right) = q_{t+s}\left(y^t y^s\right) = \emptyset$ for all y^s and y^t such that $d_t\left(y^t\right) = 1$.

Every strategy profile σ generates an outcome path $x := x(\sigma)$.[5] Given a set $S \subseteq \Sigma$ of strategy profiles, we denote $x(S) = \cup_{\sigma \in S} x(\sigma)$ for the set of outcome paths of profiles $\sigma \in S$. For any strategy profile $\sigma \in \Sigma$, we define the continuation at $h^t \in \mathcal{H}_g$

$$V(\sigma \mid h^t) = \mathbb{E}_t \left\{ \sum_{s=t}^{\infty} \beta^s \left[d_s u(y_s - b_s + q_s b_{s+1}) + (1 - d_s) u(y_s) \right] \right\}$$

where (y_s, d_s, b_{s+1}, q_s) are on the path $x = x(\sigma_{|h^t})$.[6] A strategy profile $\sigma = (\sigma_g, q_m)$ constitutes a *Subgame Perfect Equilibrium (SPE)* if and only if, for all partial histories $h^t \in \mathcal{H}_g$

$$V(\sigma \mid h^t) \geq V(\sigma'_g, q_m \mid h^t) \text{ for all } \sigma'_g \in \Sigma_g, \tag{2.1}$$

and for all histories $h_-^{t+1} = (h^t, y_t, d_t, b_{t+1}) \in \mathcal{H}_m$

$$q_m\left(h_-^{t+1}\right) = \frac{1}{1+r} \int (1 - d^{\sigma_g}(h^{t+1}, y_{t+1}) dF(y_{t+1} \mid y_t). \tag{2.2}$$

That is, the strategy of the government is optimal given the pricing strategy of the lenders $q_m(\cdot)$, and likewise $q_m(\cdot)$ is consistent with the default policy generated by σ_g. The set of all subgame perfect equilibria is denoted as $\mathcal{E} \subset \Sigma$.

3 Equilibrium Consistent Outcomes

This section contains the main result of the paper, a characterization of equilibrium consistent outcomes. We work with the baseline case where income is a continuous random variable as in Eaton and Gersovitz (1981). After stating our main result, we apply it to obtain predictions for bond prices across all equilibria.

Equilibrium Prices, Continuation Values. For any history h_-^{t+1} we define the highest and lowest prices equilibrium prices:

$$\overline{q}(h_-^{t+1}) := \max_{\sigma \in \mathcal{E}(h_-^{t+1})} q_m\left(h_-^{t+1}\right)$$

[5] It can be defined recursively as follows: at $t = 0$ jointly define $(d_0(y_0), b_1(y_0), q_1(y_0)) \equiv (d_0^{\sigma_g}(y_0), b_1^{\sigma_g}(y_0), q_m(y_0, b_1^{\sigma_g}(y_0))$ and $h^1 = (y_0, d_0(y_0), b_1(y_0), q_1(y_0))$. For $t > 0$, we define $(d_t(y^t), b_{t+1}(y^t), q_t(y^t)) \equiv (d_0^{\sigma_g}(h^t, y_t), b_1^{\sigma_g}(h^t, y_t), q_m(h^t, y_t))$ and $h^{t+1} = (h^t, y_t, d_t(y^t), b_{t+1}(y^t), q_t(y^t))$

[6] The utility of a strategy profile that specifies negative consumption is $-\infty$.

$$\underline{q}(h_{-}^{t+1}) := \min_{\sigma \in \mathcal{E}(h_{-}^{t+1})} q_m \left(h_{-}^{t+1}\right).$$

In the Online Appendix, Section E, we provide necessary and sufficient conditions for equilibrium multiplicity.[7] We also show that the worst SPE price is zero and the best SPE price is the one of the Markov equilibrium that is characterized in the literature of sovereign debt as in Arellano (2008) and Aguiar and Gopinath (2006) with no role for coordination failures. The lowest price $\underline{q}(h_{-}^{t+1})$ will be attained by a fixed strategy for all histories h_{-}^{t+1}. It will deliver the utility level of autarky for the government. Thus, the lowest price is associated with the worst equilibrium, in terms of welfare. Likewise, the highest price $\overline{q}(h_{-}^{t+1})$ is associated with a, different, fixed strategy for all histories (the maximum is attained by the same σ for all h_{-}^{t+1}) and delivers the highest equilibrium level of utility for the government. Thus, the highest price is associated with the best equilibrium in terms of welfare. The expected autarky continuation is

$$\mathbb{V}^d(y) \equiv \int u(y')dF\left(y' \mid y\right),$$

and the autarky utility (conditional on defaulting) is simply

$$V^d\left(y\right) \equiv u\left(y\right) + \beta \mathbb{V}^d(y). \tag{3.1}$$

The continuation utility (conditional on not defaulting) of a choice b' given bonds (b, y) is

$$\overline{V}^{nd}\left(b, y, b'\right) = u\left(y - b + b'\overline{\mathbf{q}}\left(y, b'\right)b'\right) + \beta \overline{\mathbb{W}}\left(y, b'\right), \tag{3.2}$$

where $\overline{q}\left(b'\right)$ is the bond price schedule under the best continuation equilibrium (the Markov equilibrium that we just characterized), if $y_t = y$ and the bonds to be paid tomorrow are $b_{t+1} = b'$. Finally, the continuation value of the best equilibrium is:

$$\overline{\mathbb{W}}(y, b') = \mathbb{E}_{y'|y}\left[\max\left\{\overline{V}^{nd}(b, y'), V^D(y')\right\}\right].$$

Consistent Histories. We first define the notion of consistent histories. A history h is consistent with (or generated by) an outcome path x if and only if $d_s = d_s^x\left(y^s\right)$, $b_{s+1} = b_{s+1}^x\left(y^s\right)$ and $q_s = q_s^x\left(y^s\right)$ for all $s < l\left(h\right)$ (where $l\left(h\right)$ is the length of the history). If a history h is consistent with an outcome path x we denote it as $h \in \mathcal{H}\left(x\right)$. Intuitively,

[7] Our analysis may be of independent interest, providing conditions under which there are multiple Markov equilibria in a sovereign debt model along the lines of Eaton and Gersovitz (1981). The importance of this result is that it opens up the possibility of confidence crises in models as in Eaton and Gersovitz (1981). Thus, confidence crises are not necessarily a special feature of the timing in Calvo (1988) and Cole and Kehoe (2000) but a robust feature in most models of sovereign debt.

consistency of a history with an outcome means that, given the path of exogenous variables, the endogenous observed variables coincide with the ones that are generated by the outcome. A history h is consistent with strategy profile σ if and only if $h \in \mathcal{H}\left(x\left(\sigma\right)\right)$. If a history h is consistent with a strategy σ we denote it as $h \in \mathcal{H}\left(\sigma\right)$. Intuitively, a history is consistent with a strategy if the history is consistent with the outcome that is generated by the strategy. Given a set $S \subseteq \Sigma$ of strategy profiles, we use $x\left(S\right) = \cup_{\sigma \in S} x\left(\sigma\right)$ to denote the set of outcome paths of profiles $\sigma \in S$. The inverse operator for $\mathcal{H}\left(\cdot\right)$ are respectively $X\left(\cdot\right)$ for the outcomes consistent with history h. We use $\Sigma\left(h\right)$ to denote the strategy profiles consistent with h. For a given set of strategy profiles $S \subseteq \Sigma$, we write $\mathcal{H}\left(S\right) = \bigcup_{\sigma \in S} \mathcal{H}\left(\sigma\right)$ as the set of $S-$consistent histories. When $S = \mathcal{E}$, we call $\mathcal{H}\left(\mathcal{E}\right)$ the set of equilibrium consistent histories. The set of equilibria consistent with history h is defined as $\mathcal{E}_{|h} := \mathcal{E} \cap \Sigma\left(h\right)$.[8] An outcome path $x = \left(d_t\left(\cdot\right), b_{t+1}\left(\cdot\right), q_t\left(\cdot\right)\right)_{t \in \mathbb{N}}$ is consistent with history $h^t \iff \exists\, \sigma \in \mathcal{E} \cap \Sigma\left(h^t\right)$ such that $x = x\left(\sigma\right)$.

3.1 Main Result

Suppose that we have observed so far $h^t_- = \left(h^{t-1}, y_{t-1}, d_{t-1}, b_t\right)$ an equilibrium consistent history (where price at time t has not yet been realized), and we want to characterize the set of *shifted* outcomes $x_{t-} = \left(q_{t-1}, d_t\left(\cdot\right), b_{t+1}\left(\cdot\right)\right)$ consistent with this history[9]. Proposition 1 provides a full characterization of the set of equilibrium consistent outcomes $x_{t-}\left(\mathcal{E}_{|h^t_-}\right)$, showing that past history only matters through the opportunity cost of not defaulting at $t-1$, $u\left(y_{t-1}\right) - u\left(c_{t-1}\right)$.

Proposition 1 (Equilibrium Consistent Outcomes)**.** *Suppose $h^t_- = \left(h^{t-1}, y_{t-1}, d_{t-1}, b_t\right)$ is an equilibrium consistent history, with no default so far. Then $x_{t-} = \left(q_{t-1}, d_t\left(\cdot\right), b_{t+1}\left(\cdot\right)\right)$ is equilibrium consistent with h^t_- if and only in the following conditions hold:*

a. Price is consistent

$$q_{t-1} = \frac{1}{1+r}\left(1 - \int d_t(y_t) dF(y_t \mid y_{t-1})\right), \tag{3.3}$$

b. IC government

$$(1 - d(y_t)) \left[u(y_t - b_t + \overline{q}(y_t, b_{t+1}) b_{t+1}) + \beta \overline{W}(y_t, b_{t+1})\right] + d(y_t) V^d(y_t) \geq V^d(y_t), \tag{3.4}$$

[8] This notation is useful to precisely formulate questions such as: "Is the observed history the outcome of some subgame perfect equilibria?" In our notation "$h \in \mathcal{H}\left(\mathbf{SPE}\right)$".

[9] An outcome in period t was given by $x_t = \left(d^x_t\left(\cdot\right), b^x_{t+1}\left(\cdot\right), q^x_t\left(\cdot\right)\right)$; the policies and prices of period t. x_{t-} has the policies of period t but the prices of period $t-1$. The focus in x_{t-} as opposed to x_t simplifies the characterization of equilibrium consistent outcomes.

c. Promise keeping

$$\beta\left[\int_{d_t=0}\overline{V}^{nd}\left(b_t,y_t,b_{t+1}(y_t)\right)dF(y_t\mid y_{t-1})+\int_{d_t=1}V^d\left(y_t\right)dF(y_t\mid y_{t-1})\right]\geq$$
$$\left[u\left(y_{t-1}\right)-u\left(y_{t-1}-b_{t-1}+q_{t-1}b_t\right)\right]+\beta\mathbb{V}^d(y_{t-1}). \tag{3.5}$$

Proof. See Appendix A. □

If conditions (a) through (c) hold, we write simply

$$\left(q_{t-1},d_t\left(\cdot\right),b_{t+1}\left(\cdot\right)\right)\in\mathbb{ECO}\left(b_{t-1},y_{t-1},b_t\right),$$

where $\mathbb{ECO}$ stands for "equilibrium consistent outcomes".

First, note that conditions (3.3) and (3.4) in Proposition 1 provide a characterization of the set of SPE outcomes. Condition (3.3) states that the price q_{t-1} needs to be consistent with the default policy $d_t(\cdot)$. Condition (3.4) states that a policy $d_t\left(\cdot\right),b_{t+1}\left(\cdot\right)$ is implementable in an SPE if it is incentive compatible given that following the policy is rewarded with the best equilibrium and a deviation is punished with the worst equilibrium. The argument in the proof follows Abreu (1988).

Second, Equilibrium consistent outcomes are characterized by an additional condition, (3.5), which is the main contribution of this paper. This condition characterizes how past observed history (if *assumed* to be generated by an *equilibrium* strategy profile) introduces restrictions on the set of equilibrium consistent policies. In our setting, condition (3.5) will guarantee that the government's decision at $t-1$ of not defaulting was optimal. That is, on the path of some SPE profile $\hat{\sigma}$, the incentive compatibility constraint from government's utility maximization in $t-1$ is

$$u\left(c_{t-1}\right)+\beta V\left(\hat{\sigma}\mid h^t\right)\geq u\left(y_{t-1}\right)+\beta\mathbb{V}^d(y_{t-1}), \tag{3.6}$$

where $V\left(\hat{\sigma}\mid h^t\right)$ is the continuation value of the equilibrium, as defined before. One interpretation of (3.6) is that the net present value (with respect to autarky) that the government must expect from not defaulting, must be greater (for the choice to have been done optimally) than the opportunity cost of not defaulting: $u\left(y_{t-1}\right)-u\left(c_{t-1}\right)$. This must be true for *any* SPE profile that could have generated h^t_-.

The intuition for why (3.5) is necessary for equilibrium consistency is as follows. Notice that the previous inequality also holds for the case the continuation equilibrium is actually the best continuation equilibrium. Therefore, for any equilibrium consistent policy $(d(\cdot), b'(\cdot))$ it has to be the case that

$$V(\hat{\sigma} \mid h^t) = \int_{y_t : d_t(y_t)=1} V^d(y_t) dF(y_t \mid y_{t-1}) +$$

$$\int_{y_t : d(y_t)=0} \left[u\left(y_t - b_t + b'(y)\, \hat{q}_m\left(h^t, y_t, d_t, b_{t+1}(y_t)\right)\right) + \beta V\left(\hat{\sigma} \mid h^{t+1}\right)\right] dF(y_t \mid y_{t-1})$$

$$\leq \int_{y_t : d_t(y_t)=1} V^d(y_t)\, dF(y_t \mid y_{t-1}) + \int_{y_t : d(y_t)=0} \overline{V}^{nd}(b_t, y_t, b_{t+1})\, dF(y_t \mid y_{t-1}). \tag{3.7}$$

Equations (3.6) and (3.7) imply

$$[u(y_{t-1}) - u(y_{t-1} - b_{t-1} + q_{t-1} b_t)] + \beta \underline{\mathbb{V}}^d(y_{t-1})$$

$$\leq \beta \left[\int_{d_t=0} \overline{V}^{nd}(b_t, y_t, b_{t+1}(y_t))\, dF(y_t \mid y_{t-1}) + \int_{d_t=1} V^d(y_t) dF(y_t \mid y_{t-1}) \right]. \tag{3.8}$$

This is condition (3.5). So if the policies do not satisfy (3.5), they are not part of an SPE that generated the history h^t_-; in other words, there is no SPE consistent with h^t_- with policies $(d_t(\cdot), b_{t+1}(\cdot))$ for period t.

We also show that this condition is sufficient, so if $(d_t(\cdot), b_{t+1}(\cdot))$ satisfy conditions (3.3), (3.4), and (3.5), we can always find at least one SPE profile $\hat{\sigma}$ that would generate x_{t-} on its equilibrium path. Even after a long history the sufficient statistics to forecast the outcome x_{t-} are

$$(b_{t-1}, b_t, y_{t-1}).$$

Thus, effectively

$$\mathbb{ECO}(h^t_-) = \mathbb{ECO}(b_{t-1}, y_{t-1}, b_t).$$

This result may seem surprising, but it is where robustness of the analyst (uncertainty about the equilibrium selection) is expressed. Because income y is a continuous random variable, any promises (in terms of expected utility) that rationalized past choices are "forgotten" each period; the reason is that the outside observer needs to take into account that promises *could* have been be realized in states that did not occur.

Third, finally, notice that even though the outside observer is using just a small fraction of the history, the set of equilibrium consistent outcomes exhibits history dependence beyond that of the set of SPE. The set of equilibrium consistent outcomes is a function variables (b_{t-1}, y_{t-1}, b_t). Thus, there is a role for past actions to signal future behavior. In contrast the set of subgame perfect equilibria after any history only depends on the

Markovian states y_{t-1}, b_t.[10]

3.2 Application: Equilibrium Consistent Prices

we provide answer the following question: The question that we would like to answer now is the following: given an observed history h^t_-, which are the possible continuation prices? Aided with the characterization of equilibrium consistent outcomes in Proposition 1 we will characterize the set of equilibrium debt prices that are consistent with the observed history $h^t_- = \left(h^{t-1}, y_{t-1}, d_{t-1}, b_t\right)$. There are two objects of interest. The highest equilibrium consistent price solves

$$\overline{q}\left(h^t_-\right) = \max_{(\hat{q}, d_t(\cdot), b_{t+1}(\cdot))} \hat{q}$$

subject to

$$\left(\hat{q}, d_t\left(\cdot\right), b_{t+1}\left(\cdot\right)\right) \in \mathbb{ECO}\left(b_{t-1}, y_{t-1}, b_t\right).$$

The lowest equilibrium consistent price solves

$$\underline{q}\left(h^t_-\right) = \min_{(\hat{q}, d_t(\cdot), b_{t+1}(\cdot))} \hat{q} \tag{3.9}$$

subject to

$$\left(\hat{q}, d_t\left(\cdot\right), b_{t+1}\left(\cdot\right)\right) \in \mathbb{ECO}\left(b_{t-1}, y_{t-1}, b_t\right).$$

Now we make these two definition operational.

Highest Equilibrium Consistent Price. The highest equilibrium consistent price is the one of the Markov Equilibrium that we characterized in Section E. Note that the expected value of the incentive compatibility constraint (3.4), is the value of the option to default $\overline{\mathbb{W}}(y_t, b_{t+1})$, for the best equilibrium. The promise-keeping will be generically not binding for the best equilibrium (given that the country did not default). For these two reasons, the best equilibrium consistent price is the one obtained with the default policy and bond policy that maximize the value of the option. Thus,

$$\overline{q}\left(h^t_-\right) = \overline{\mathbf{q}}\left(y_{t-1}, b_t\right). \tag{3.10}$$

[10]Notice that this role contrasts the dependence of the quantitative literature for sovereign debt that follows Eaton and Gersovitz (1981) as in Arellano (2008) and Aguiar and Gopinath (2006) where the fact that a country has just repaid a high quantity of debt, does not affect the future prices that will obtain.

Lowest Equilibrium Consistent Price. Our focus will be on the characterization of the lowest equilibrium consistent price. Note that the lowest SPE price is zero. This follows because default is implementable after any history if we do not take into account the promise keeping constraint (3.5). On the contrary, we will show that lowest equilibrium consistent price is positive, for every equilibrium history. Furthermore, because the set of equilibrium consistent outcomes after history h_{-}^{t} depends only on (b_{t-1}, y_{t-1}, b_t), it holds that

$$\underline{q}\left(h_{-}^{t}\right) = \underline{\mathbf{q}}\left(b_{t-1}, y_{t-1}, b_t\right). \tag{3.11}$$

Proposition 2 establishes the main result of this subsection: a full characterization of $\underline{\mathbf{q}}\left(b, y, b'\right)$ (we drop time subscripts) as a solution to a convex minimization program, which can be reduced to a one equation/one variable problem.

Proposition 2. *Suppose (b, y, b') are such that $\overline{V}^{nd}\left(b, y, b'\right) > V^{d}\left(y\right)$ (i.e., not defaulting was feasible under the best continuation equilibrium). Then there exists a constant $\gamma = \gamma\left(b, y, b'\right) \geq 0$, such that*

$$\underline{\mathbf{q}}\left(b, y, b'\right) = \frac{1 - \int \underline{d}\left(y'\right) dF\left(y' \mid y\right)}{1+r},$$

where

$$\underline{d}\left(y'\right) = 0 \iff \overline{V}^{nd}\left(b', y'\right) \geq V^{d}\left(y'\right) + \gamma \text{ for all } y' \in Y;$$

γ is the minimum solution to the equation:

$$\beta \int_{\Delta^{nd} \geq \gamma} \Delta^{nd} d\hat{F}\left(\Delta^{nd}\right) = u\left(y\right) - u\left(y - b + \frac{1 - \hat{F}\left(\gamma \mid y\right)}{1+r} b'\right) \tag{3.12}$$

where $\Delta^{nd} \equiv \overline{V}^{nd}\left(b', y'\right) - V^{d}\left(y'\right)$ and $\hat{F}\left(\Delta^{nd}\right)$ its conditional cdf. If $dF\left(\cdot\right)$ is absolutely continuous, then γ is the unique solution to equation 3.12.

Proof. See Appendix A. □

The proof is in the appendix. We provide a sketch of the argument. First, note that, by choosing the bond policy of the best equilibrium, all of the constraints imposed by equilibrium consistency are relaxed because the value of no default increases. So, finding the lowest ECO price will amount to finding the default policy that yields the lowest price and is consistent with equilibrium. Second, notice that the promise keeping constraint needs to be binding in the optimum. If not, the minimization problem has as its only constraint the incentive compatibility constraint, and the minimum price is zero (with a policy of default in every state). But, if the price is zero, the promise keeping constraint will not be satisfied. Third, notice that the incentive compatibility constraint will not be binding. Intuitively, imposing default is not costly in terms of incentives, and for the lowest equilibrium consistent price, we want to impose default in as many states as possible.

From these observations, note that the trade-off of the default policy of the lowest price will be: imposing defaults in more states will lower the price at the expense of a tighter promise keeping constraint. This condition pins down the states where the government defaults; as many defaults as possible, but not so many that no default in the previous period was not worth the effort. This, implies that the policy is pinned down by

$$\underline{d}\left(y'\right) = 0 \iff \overline{V}^{nd}\left(b', y'\right) \geq V^{d}\left(y'\right) + \gamma$$

where γ is a constant to be determined. This constant solves a single equation: is the minimum value such that the promise keeping holds with equality, with the optimal bond policy, evaluated at the best continuation

$$\beta \int_{\Delta^{nd} \geq \gamma} \Delta^{nd} d\hat{F}(\Delta^{nd} \mid y) = u(y) - u(y - b + \frac{1 - \hat{F}(\gamma \mid y)}{1+r} b'). \tag{3.13}$$

Note also how policies are tilted in the best and worst continuation equilibrium. For the best equilibrium default policy at t, it holds that $d(y_t) = 0$ if and only if $\overline{V}^{nd}(b_t, y_t) \geq V^d(y_t)$. On the other hand, for the lowest equilibrium consistent price is $\overline{V}^{nd}(b_t, y_t) \geq V^d(y_t) + \gamma$, where γ is the constant that solves (3.13) and depends on (b_{t-1}, y_{t-1}, b_t). The default policy is shifted to create more defaults, to lower the price, but not so many that the promise-keeping was not satisfied (i.e., we cannot rationalize previous choices). Equilibrium consistent outcomes uncovers a novel tension that is not present in SPE. At a particular history h^t_-, implementing default is not costly because it is always as good as the worst equilibrium. However, implementing default today lowers the prices that the government was expecting in the past and makes it harder to rationalize a particular history.

Comparative Statistics. The next Corollary describes how the set of equilibrium consistent prices changes with the history of play.

Corollary 1. *Let $\underline{q}\left(b, y, b'\right)$ be the lowest $\mathbb{ECO}\left(b, y, b'\right)$ price. It holds that: (a) $\underline{q}\left(b, y, b'\right)$* is decreasing in *$b'$; (b) $\underline{q}\left(b, y, b'\right)$ is increasing in b; (c) For every equilibrium $\left(b, y, b'\right)$, $-b + b'\underline{q}\left(b, y, b'\right) \leq 0$; if income is i.i.d., $\underline{q}$ is decreasing in y, and so is the set $Q = \left[\underline{q}\left(b, y, b'\right), \overline{q}\left(y, b'\right)\right]$.*

Proof. See Appendix A. □

First, note that as in the best equilibrium, the lowest equilibrium consistent price is decreasing in the amount of debt issued b'. The intuition is that higher amounts of debt issued imply a more relaxed promise keeping constraint. In other words, the past choices of the government can be rationalized with a lower price. A similar intuition holds for b; if the country just repaid a high amount of debt (i.e., made an effort for repaying), past choices are rationalized by higher prices. Second, note that if there is a positive capital inflow with the lowest equilibrium consistent price, it implies that

$$u(y) - u\left(y - b + b'\underline{q}(b, y, b')\right) < 0.$$

Intuitively, the country is not making any effort in repaying the debt. Therefore, it need not be the case that the country was expecting high prices for debt in the next period. Mathematically, when there is a positive capital outflow with the lowest equilibrium consistent price, γ is infinite. This implies that $\frac{1-\hat{F}(\gamma)}{1+r^*} = \underline{q}(b, y, b') = 0$, which contradicts a positive capital inflow. Finally, because there are no capital inflows with the lowest equilibrium consistent price, repaying debt at this price will become more costly as income is lower; this due to the concavity of the utility function.[11] Mathematically, because of concavity,

$$u(y) - u\left(y - b + b'\underline{q}(b, y, b')\right),$$

is[12] increasing as income decreases, and therefore, the promise keeping constraint tightens as income decreases. Note that, in the non i.i.d. case, this property will not hold, because, even though the burden of repayment is higher, the value of repayment in terms of the continuation value can be increasing.

4 Sunspots

We are now interested in adding a sunspot variable. Adding a sunspot that is realized together with output adds nothing to the analysis. Effectively, output could already acting as a random coordination device. Thus, the interesting question is to add a sunspot variable after the bond issuance, but before the price is determined. As we shall see, conditional on any single realization, the set of equilibrium consistent outcomes then coincides with the set of subgame perfect equilibria. Despite this we can obtain relevant

[11]This observation is used in the literature of sovereign debt. For example, to show that default occurs in bad times, as in Arellano (2008), or to show monotonicity of bond policies with respect to debt, as in Chatterjee and Eyigungor (2012).

[12]The change in this expression will depend on the sign of $u(y) - u\left(y - b + b'\frac{1-\hat{F}(\gamma)}{1+r^*}\right)$, that is positive due to the result of no capital inflows with the lowest equilibrium consistent price.

history dependent predictions. In this section we do three things. First, we characterize what we term are equilibrium consistent distributions. Those are distributions over prices that consistent with a subgame perfect equilibrium given history. Second, aided with this characterization we obtain bounds of the expectation over prices that hold across all equilibrium. This provides a way to obtain set identification of the set of structural parameters in our particular application. Finally, we provide an intuitive application of our results, and we find a bound on the probability of a non-fundamental debt crises; by crisis we mean an event where the realized price falls below a given threshold $\hat{q}$, which we treat as a parameter.

4.1 Main Results: Equilibrium Consistency

Denote the sunspot by ζ_t, realized after the bond issue of the government but before the price q_t; i.e, a sunspot is realized after h^t_-. Without loss of generality[13] we will assume $\zeta_t \sim \text{Uniform}\,[0,1]$ i.i.d. over time. If we assume that the game is on the equilibrium path of some subgame perfect equilibrium, then the government strategy before the realization of the sunspot was optimal; that is

$$\int \left[u\left(y_t - b_t + q_t\left(\zeta_t\right) b_{t+1}\right) + \beta v\left(\zeta_t\right)\right] d\zeta_t \geq V^d\left(y_t\right).$$

The government ex-ante preferred to pay the debt and issue bonds b_{t+1} than to default, where $q\left(\zeta_t\right)$ and $v\left(\zeta_t\right)$ are the market price and continuation equilibrium value conditional on the realization of the sunspot ζ_t. The main difference in the characterization of equilibrium consistent distributions, is that now we cannot rely on the best continuation price, because it might not be realized.

Define the maximum continuation value function $\overline{v}\left(b,q\right)$ given bonds b and price q as

$$\overline{v}\left(b,q\right) = \max_{\sigma \in \mathbf{SPE}(b)} V\left(\sigma \mid b_0 = b\right)$$

subject to

$$\frac{\mathbb{E}(1 - d(y_0))}{1+r} = q$$

This gives the best possible continuation value if we start at bonds b and we restrict prices to be equal to q. In Appendix C we provide a method to compute the function $\overline{v}\left(b,q\right)$ and

[13] This is because of robustness: we will try to map all equilibria that can be contingent on the randomizing device, and hence as long as the random variable remains absolutely continuous, any time dependence in ζ_t can be replicated by time dependence on the equilibrium itself.

show that is non-decreasing and concave in q. is $\overline{v}(b,q)$ non-decreasing in b and concave in q. The fact that the function is non-decreasing in q follows from the fact that better prices are associated with better continuation equilibrium, as well as higher contemporaneous consumption (since $b_{t+1} \geq 0$). This follows from the fact that defaults are punished but when the government does not default, it obtains the best continuation equilibrium (under the strategy associated with value $\overline{v}(b_{t+1}, q_t)$). Concavity, follows from the the fact that $\overline{v}(b,q)$ solves a linear programming problem. We use both properties to obtain sharper characterizations of the set of equilibrium consistent distributions and to obtain testable predictions.

For a given equilibrium σ at history $h = \left(h^t, y_t, d_t, b_{t+1}\right)$ the equilibrium price distribution is defined by

$$\Pr(q \in A) = \Pr\left(\zeta_t : q^{\sigma}(h, \zeta_t) \in A\right)$$

Let $\mathcal{Q}(h)$ be the family of price distributions from history consistent equilibria. The following Proposition provides a characterization of this family.

Proposition 3. *Suppose $h = \left(h^t, y_t\right)$ is equilibrium consistent. Then, a distribution $P \in \Delta(\mathbb{R}_+)$ is an equilibrium consistent price distribution; i.e. $P \in \mathcal{Q}(h)$ if and only if* $\mathbf{Supp}(P) \subseteq [0, \overline{\mathbf{q}}(b_{t+1})]$ *and*

$$\int \left\{u\left(y_t - b_t + q b_{t+1}\right) + \beta \overline{v}\left(b_{t+1}, q\right)\right\} dP(q) \geq V^d(y_t) \tag{4.1}$$

Proof. See Appendix B. □

Condition 4.1 parallels equation (3.5) in Proposition 1. There are some differences. We are now characterizing distribution over prices consistent with a decision of not defaulting $d_t = 0$ and issuing debt b_{t+1}. Note that we are taking an expectation with respect to q: the government does not know what particular price will be realized after it chooses a particular policy. Necessity, again, comes from the fact that we can always implement outcomes promising the best continuation equilibrium. On the other hand, the idea of sufficiency is again coming from the fact that both output and the sunspot are non-atomic. How this condition related to the case without sunspots? Note that the case without sunspots that we analyzed in the previous section, the conditions for equilibrium consistency will be

$$\left\{u\left(y_t - b_t + q b_{t+1}\right) + \beta \overline{v}\left(b_{t+1}, q\right)\right\} dP(q) \geq V^d(y_t).$$

The lowest equilibrium consistent price that we characterized in the previous section will be pinned down by this condition with equality. Note that the set of equilibrium consistent distributions will be given by$\mathcal{Q}(h) = \mathcal{Q}(b_t, y_t, b_{t+1})$.

We now delve into four applications of the main result. The first application will study comparative statistics for the set of equilibrium consistent distributions $\mathcal{Q}(b_t, y_t, b_{t+1})$. Second, we characterize the maximum probability of a crisis. Third, we provide bounds over expectation of prices, across all equilibria. Finally, we also provide bound for the variance of prices.

4.2 Application: Comparative Statistics

We start by providing comparative statistics over the set of distributions.

Proposition 4. *The set of equilibrium price distributions $\mathcal{Q}(b_t, y_t, b_{t+1})$ is non-increasing (in set order sense) with respect to b_t; if income is i.i.d, it is non-decreasing in y_t. Suppose that $P \in \mathcal{Q}(b_t, y_t, b_{t+1})$ and $P' \in \Delta([0, \overline{\mathbf{q}}(b_{t+1})])$. If $P' \trianglerighteq P$ (i.e. it first order stochastically dominates P), then $P' \in \mathcal{Q}(b_t, y_t, b_{t+1})$.*

Proof. See Appendix B. □

The intuition of the first part of this comparative statistics is again coming from the revealed preference argument. If the government repaid a higher amount of debt, then the distribution of prices that they could be expecting needs to shift towards higher prices. If the set does not change, then there will be some distribution that will be inconsistent with equilibrium because it will violate the promise keeping constraint. For the second part, if $Q' \trianglerighteq Q$ denote the relationship "Q' first order stochastically dominates Q", the proposition shows that once that a distribution is consistent with equilibrium, any distribution that first order stochastically dominates it will be an equilibrium consistent distribution. Intuitively, higher prices give both higher consumption and higher continuation equilibrium values for the government, since both are weakly increasing in the realizations of debt price q_t.

4.3 Application: Probability of Crises

Our goal in this subsection will be to infer the maximum probability (across equilibria) that the government assigns to the market setting a price $q(\zeta) = \hat{q}$; i.e., a financial crises. Formally,

$$\underline{P}(\hat{q}) \equiv \max_{P \in \mathcal{Q}(b_t, y_t, b_{t+1})} \Pr_P(q \leq \hat{q}) \tag{4.2}$$

where $\Pr_P(q \leq \hat{q}) := \int_0^{\hat{q}} dP(q)$. These bounds are independent of the nature of the sunspots (i.e. the distribution of sunspots, its dimensionality, and so on), in the same way as the set of correlated equilibria does not depend on the actual correlating devices.[14] Furthermore this bound will yield a necessary condition for a distribution to be an element in $\mathcal{Q}(b_t, y_t, b_{t+1})$.

We start by constructing an Upper bound on $\Pr(q=0)$. To construct the maximum equilibrium consistent probability that $q_t = 0$, we make the promise keeping constraint be as relaxed as possible. We do this by considering continuation equilibria with two properties: first, assign the best continuation equilibria if $q \neq 0$ (i.e, under price $\overline{\mathbf{q}}(y_t, b_{t+1})$). Second, note that autarky is the best continuation equilibria feasible with $q = 0$; if the government receives a price of zero, in equilibrium, it will default with probability one in the continuation equilibrium[15]. The IC constraint is now:

$$\underline{P}(\hat{q}=0)\left[u(y_t - b_t + b_{t+1} \times 0) + \beta \mathbb{V}^d(y_t)\right] + (1 - \underline{P}(\hat{q}=0))\left[\overline{V}^{nd}(b_t, y_t, b_{t+1})\right] \geq V^d(y_t).$$

Then

$$\underline{P}(\hat{q}=0) = \frac{\Delta^{nd}(b_t, y_t, b_{t+1})}{\Delta^{nd}(b_t, y_t, b_{t+1}) + u(y_t) - u(y_t - b_t)} < 1,$$

where $\Delta^{nd}(\cdot)$ denotes the maximum utility difference between not defaulting and defaulting (under the best equilibrium)

$$\Delta^{nd}(b_t, y_t, b_{t+1}) \equiv \overline{V}^{nd}(b_t, y_t, b_{t+1}) - V^d(y_t).$$

Thus, the probability of $q = 0$ is bounded away from 1 from an ex-ante perspective (i.e. before the sunspot is realized, but after the government decision). So we obtain a history dependent bound on the probability of a financial crises.

Following this approach, we can generalize it, and characterize an upper bound for general price $\hat{q}$ such that $\hat{q} < \underline{\mathbf{q}}(b_t, y_t, b_{t+1})$. Denote this bound by $\underline{P}(\hat{q})$. Using the same strategy as before, to get the less tight the incentive compatibility constraint for the government we need to: for $\zeta : q(\zeta) > \hat{q}$, we consider equilibria that assign the best continuation equilibria; maximize equilibrium utility for $q : q \leq \hat{q}$. Thus:

$$\underline{P}(\hat{q}) = \frac{\Delta^{nd}(b_t, y_t, b_{t+1})}{V^d(y_t) - [u(y_t - b_t + \hat{q} b_{t+1}) + \beta \overline{v}(b_t, \hat{q})] + \Delta^{nd}(b_t, y_t, b_{t+1})}.$$

[14] As long as out interest is in characterizing all correlated equilibria.

[15] The default decision in equilibrium needs to be consistent with the price: a price of zero is only consistent with default in every state of nature. And we assume that after default the government is in autarky forever.

Note that this is not an innocuous constraint only when the right hand side is less than 1. This happens only when

$$u\left(y_t - b_t + \hat{q} b_{t+1}\right) + \beta \overline{v}\left(b_t, \hat{q}\right) \geq V^d\left(y_t\right).$$

This constraint holds when and this holds if $\hat{q} \geq \underline{\mathbf{q}}\left(b_t, y_t, b_{t+1}\right)$ The last inequality comes from the characterization of $\underline{\mathbf{q}}\left(b_t, y_t, b_{t+1}\right)$. The following Proposition summarizes the results of this section:

Proposition 5. *Take an equilibrium consistent history* $h = \left(h^t, y_t, d_t, b_{t+1}\right)$ *and let* $\Delta^{nd} = \overline{V}^{nd}\left(b_t, y_t, b_{t+1}\right) - V^d\left(y_t\right)$. *For any* $\hat{q} < \underline{\mathbf{q}}\left(b_t, y_t, b_{t+1}\right)$

$$\underline{P}\left(\hat{q}\right) = \frac{\Delta^{nd}}{\Delta^{nd} - \left[u\left(y_t - b_t + \hat{q} b_{t+1}\right) + \beta \overline{v}\left(b_t, \hat{q}\right) - u(y_t) - \beta \mathbb{V}^d\right]} < 1$$

For any $\hat{q} \geq \underline{\mathbf{q}}\left(b_t, y_t, b_{t+1}\right)$, $\underline{P}\left(\hat{q}\right) = 1$.

Proof. See Appendix B. □

In Proposition 5 we find the ex-ante probability (before ζ_t is realized) of observing $q_t = \hat{q}$ is less than $\underline{P}\left(\hat{q}\right) < 1$ for any equilibrium consistent outcome. Note that if the income realization is such that $\overline{V}^{nd}\left(b_t, y_t\right) = V^d\left(y_t\right)$ (i.e. under the best continuation equilibrium, the government was indifferent between defaulting or not, and still did not default), then $\underline{P}\left(\hat{q}\right) = 0$ for any $\hat{q} < \underline{\mathbf{q}}\left(b_t, y_t, b_{t+1}\right) = \overline{\mathbf{q}}\left(y_t, b_{t+1}\right)$, which implies that at such income levels, even with these kind of correlating devices available, only $q = \overline{\mathbf{q}}\left(y_t, b_{t+1}\right)$ is the equilibrium consistent price. We also show that any price $q \in \left[\underline{\mathbf{q}}\left(\cdot\right), \overline{\mathbf{q}}\left(\cdot\right)\right]$ could be observed with probability 1, since they are part of the path of a pure strategy SPE profile. When adding sunspots, any price in $[0, \overline{\mathbf{q}}\left(\cdot\right)]$ can be observed ex-post, and since the econometrician has no information about the realization of the sunspot (or the particular equilibrium selection and use of the correlating device) any price is feasible ex ante. However, before more information is realized, the econometrician can place bounds on how likely different prices are, which can not be 1, so that the government incentive constraint is satisfied.

Aided with the characterization of Proposition 5 we find a restriction satisfied by equilibrium consistent distributions: they stochastically dominate $\underline{P}$, in the first order sense. Note that it is a cumulative distribution function on q: it is a non-increasing, right-continuous function with range $[0, 1]$, hence implicitly defining a probability measure over debt prices. [16]

[16] The distribution $\underline{P}\left(\cdot\right)$ is the maximum lower bound (in the FOSD sense) of the set equilibrium consistent

4.4 Application: Bounding Expectations

One application that is of particular interest is bounding moments across all equilibrium. The importance of this bounds comes from the fact that permit us to obtain restrictions that can be used to recover structural parameters, as in Chernozhukov et al. (2007).. The set of equilibrium consistent expected prices is given by

$$E\left(b_t, y_t, b_{t+1}\right) = \left\{a \in \mathbb{R}_+ :\ a = \mathbb{E}_P\left(q\right) \text{ for some } P \in \mathcal{Q}\left(b_t, y_t, b_{t+1}\right)\right\}$$

where $\mathbb{E}_P\left(q\right) \equiv \int q dP$. The following Proposition shows that in fact, the set of expected values is identical to the set of equilibrium consistent prices when there are no sunspots.

Proposition 6. *Suppose history $h = \left(h^t, y_t, d_t = 0, b_{t+1}\right)$ is equilibrium consistent. Then the set of expected prices is equal to the set of prices without sunspots $Q\left(b_t, y_t, b_{t+1}\right)$; i.e.*

$$E\left(b_t, y_t, b_{t+1}\right) = \left[\underline{\mathbf{q}}\left(b_t, y_t, b_{t+1}\right), \overline{\mathbf{q}}\left(b_{t+1}\right)\right]$$

Moreover, if $b_{t+1} > 0$ then the minimum expected value is achieved uniquely at the Dirac distribution $\hat{P}$ that assigns probability one to $q = \underline{\mathbf{q}}\left(b_t, y_t, b_{t+1}\right)$.

Proof. See Appendix B. □

The result comes from the concavity of the value function $\overline{v}\left(b_{t+1}, \hat{q}\right)$ and the fact that $\underline{\mathbf{q}}\left(\cdot\right)$ is the minimum price that satisfies:

$$u\left(y_t - b_t + \underline{\mathbf{q}} b_{t+1}\right) + \beta \overline{v}\left(b_{t+1}, \underline{\mathbf{q}}\right) = V^d\left(y_t\right) \tag{4.3}$$

The equality at $q = \underline{\mathbf{q}}\left(\cdot\right)$ follows from the strict monotonicity in q of the left hand side expression: if the inequality was strict, then we can find a lower equilibrium consistent price, which contradicts the definition of $\underline{\mathbf{q}}\left(\cdot\right)$. Therefore, the integrand in 4.1 is bigger than $V^d\left(y_t\right)$ only when $q \geq \underline{\mathbf{q}}\left(b_t, y_t, b_{t+1}\right)$. Concavity of $\overline{v}\left(b, q\right)$ and Jensen's inequality then imply that for any distribution $P \in \mathcal{Q}\left(b_t, y_t, b_{t+1}\right)$:

$$\begin{aligned} u\left(y_t - b_t + \mathbb{E}_P\left(q\right) b_{t+1}\right) + \beta \overline{v}\left(b_{t+1}, \mathbb{E}_P\left(q\right)\right) &\geq \int \left\{u\left(y_t - b_t + q b_{t+1}\right) + \beta \overline{v}\left(b_{t+1}, q\right)\right\} dP\left(q\right) \\ &\geq V^d\left(y_t\right) \end{aligned}$$

and therefore $\mathbb{E}_P\left(q_t\right) \geq \underline{\mathbf{q}}\left(b_t, y_t, b_{t+1}\right)$.

distributions; i.e. for every $P \in \mathcal{Q}\left(b_t, y_t, b_{t+1}\right)$ we have $P \trianglerighteq \underline{P}$, and if P' is some other lower bound, then $P' \trianglerighteq \underline{P}$. Moreover, $\underline{P} \notin \mathcal{Q}\left(b_t, y_t, b_{t+1}\right)$

As we just mentioned, the previous Proposition actually provides testable implications for the model. In particular, it yields a necessary and sufficient moment condition for equilibrium consistency at histories $h = \left(h^t, y_t, d_t, b_{t+1}\right)$,

$$\mathbb{E}_{q_t}\left\{u\left(y_t - b_t + b_{t+1}\right) + \beta\overline{v}\left(b_{t+1}, q_t\right) - V^d\left(y_t\right) \mid h\right\} \geq 0 \tag{4.4}$$

The bounds that we just derived yields moment inequalities that are easier to check

$$\mathbb{E}_{q_t}\left\{q_t \mid h\right\} \in \left[\underline{\mathbf{q}}\left(b_t, y_t, b_{t+1}\right), \overline{\mathbf{q}}\left(b_{t+1}\right)\right] \tag{4.5}$$

Aided with these bounds we can perform estimation of the structural set of parameters as in Chernozhukov et al. (2007).

4.5 Application: Bounding Variances

Proposition 7 (Variance restrictions). *Suppose* $V^{nd}\left(b_t, y_t, b_{t+1}\right) \geq V^d\left(y_t\right)$ *and that* $y_t \leq b_t$. *Then*

$$\mathbf{Var}\left(q_t \mid h\right) \leq \overline{\lambda}\left(1 - \overline{\lambda}\right)\left[\overline{\mathbf{q}}\left(b_{t+1}\right)\right]^2$$

where

$$\overline{\lambda} := \min\left\{\frac{1}{2}, \frac{u\left(y_t\right) - u\left(y_t - b_t\right)}{V^{nd}\left(b_t, y_t, b_{t+1}\right) - V^d\left(y_t\right) + u\left(y_t\right) - u\left(y_t - b_t\right)}\right\}$$

5 General Model

We will follow the notation in Stokey et al. (1989). There are two players: an infinitely lived player (the policy maker) and "agents" (price setters) that set expectations according to a particular rule. At each period t, agents play a extensive form stage game, with 4 sub periods $(t, \tau_i)_{i \in \{1,4\}}$. The payoff relevant states will be an exogenous random shock y_{t-1}, and an endogenous state variable x_t. The timeline of the stage game is as follows (all sets are subsets of some euclidean real vector space).

- $\tau = \tau_1$: A publicly observable random variable $y_t \in Y \subseteq \mathbb{R}^l$ is realized, which follows a (controlled) markov process: $y_t \sim f\left(y \mid y_{t-1}, b_t\right)$. Sometimes, we will that say *y includes a sunspot* if $\exists\left\{y_t^*, z_t\right\}$ such that $(\mathbf{1})$ $y_t^* \perp z_t$ for all t, **(2)** y_t^* is a controlled markov process; i.e. $y_t^* \sim g\left(y_t^* \mid y_{t-1}^*, b_t\right)$ and **(3)** $z_t \sim_{i.i.d}$ Uniform $[0,1]$.

- $\tau = \tau_2$: The long lived player chooses a control $d_t \in D \subseteq \mathbb{R}^d$ and a next period state variable $b_{t+1} \in B \subset \mathbb{R}^b$ (where both D and B are compact sets) that are jointly feasible, given (b_t, y_t). This we will write by imposing the constraint $(d_t, b_{t+1}) \in \Gamma(b_t, y_t)$, where $\Gamma : B \times Y \rightrightarrows D \times B$ is a non-empty, compact valued , continuous correspondence.

- $\tau = \tau_3$: Agents set expectations about future play. This is modeled in reduced form, with the market choosing $q_t \in \mathbb{R}^k$ to satisfy:

$$q_t = \mathbb{E}_t \left\{ \sum_{s=t}^{\infty} \delta^{s-t} T\left(b_{s+1}, y_{s+1}, d_{s+1}, b_{s+2}\right) \right\}$$

  where $\delta \in (0, 1)$ and $T : B \times Y \times D \times B \to \mathbb{R}^k$ is a continuous, bounded function. The expectation is taken over future shocks $\{y_{t+s}\}_{s=1}^{\infty}$, knowing the strategy profile of the policy maker.

- $\tau = \tau_4$: Payoffs for the policy maker are realized, given by a continuous utility function $u(b_t, y_t, d_t, b_{t+1}, q_t)$. Lifetime utility is given then by

$$U_0 := \mathbb{E}_0 \left\{ \sum_{t=0}^{\infty} \beta^t u\left(b_t, y_t, d_t, b_{t+1}, q_t\right) \right\}$$

  where $\beta \in (0, 1)$.

Example 1. This model incorporates the sovereign debt model studied above, where y_t is national income, $b_t \geq 0$ is the outstanding public debt to be replayed (we assume no savings in this example), $d \in \{0, 1\}$ is the default decision and $q_t = \mathbb{E}_t \{1 - y_{t+1}/R^*\}$ is the risk neutral price set by lenders in equilibrium. Flow utility is given by $U = (1 - d_t) u(y_t - b_t + q_t b_{t+1}) + d_t u(y_t)$, assuming that when the government defaults on its debt, she gets to consume its income, but cannot issue debt in that period.

Example 2. It also incorporates new Keynesian models, as in Woodford (2003) and more recently Waki et al. (2015), with no endogenous state, and the control is $d_t = \pi_t$ where π_t is inflation. Agents set inflation expectations to match future inflation, as $q_t := \pi_t^e = \mathbb{E}_t(\pi_{t+1})$. Inflation and output are related according to a forward looking Phillips curve $g_t = \pi_t - \beta\pi_t^e + \epsilon_t$, where g_t is the output gap and ϵ_t is a supply shock. Also, let π_t^* be a random variable that gives the optimal natural level of inflation (absent inflation gap). The random shocks are then $y_t = (\epsilon_t, \pi_t^*)$, and the government is assumed to minimize a loss function

$$\mathcal{L}(\pi, \pi^e, \epsilon_t, \pi_t^*) = \frac{1}{2} g_t^2 + \frac{1}{2}\chi\left(\pi_t - \pi_t^*\right)^2 = \frac{1}{2}\left(\pi_t - \beta\pi_t^e + \epsilon_t\right)^2 + \frac{1}{2}\chi\left(\pi_t - \pi_t^*\right)^2$$

5.1 Definition of Equilibrium Consistency

In this section we describe the basic notation for the dynamic game setup, and introduce the main concept of the paper: equilibrium consistency, which is the class of equilibrium paths of the game. A *history* is a vector $h^t = (h_0, h_1, ..., h_{t-1})$, where $h_t = (y_t, d_t, b_{t+1}, q_t)$ is the description of the outcome of the stage game at time t. A partial history is an initial history h^t concatenated with a history of the stage game at period t. For example, $h = (h^t, y_t, d_t, b_{t+1})$ is the typical history after which price setters must choose q_t. The set of all partial histories (initial and partial) is denoted by $\mathcal{H}$, and $\mathcal{H}_p \subset \mathcal{H}$ are those where the policy maker has to choose (d_t, b_{t+1}); i.e., $h = (h^t, y_t)$. Likewise, $\mathcal{H}_m \subset \mathcal{H}$ is the set of partial histories where expectation setters (or "market") ; i.e., $h = (h^t, y_t, d_t, b_{t+1})$.

A policy maker's strategy is a function $\sigma_g (h^t, y_t) = (d_t, b_{t+1})$ for all histories, and a rational expectation strategy is a pricing function $q_m (h^t, y_t, d_t, b_{t+1}) \in \mathbb{R}_k$. For a strategy profile $\sigma = (\sigma_g, q_m)$ we write $V (\sigma \mid h)$ for the continuation expected utility, after history h, of the representative consumer if agents play according to profile . We say a strategy profile $\sigma = (\sigma_g, q_m)$ is a Rational Expectations Equilibrium of the game (REE) if, for all histories $h = (h^t, y_t)$ we have

a. $V (\sigma \mid h^t, y_t) \geq V \left(\sigma'_p, q_m \mid h^t, y_t\right)$ for all (h^t, y_t) , $\sigma'_g \in \Sigma_g$

b. $q_m (h^t, y_t, d_t, b_{t+1}) = \mathbb{E}_t \left\{\sum_{s=t}^{\infty} \delta^{s-t} T (b_{s+1}, y_{s+1}, d_{s+1}, b_{s+2})\right\}$ where $(b_{s+1}, d_{s+1}, b_{s+2})$ are generated by σ

and we write $\sigma \in$ **REE**. The methodology we develop derives statistical predictions for the data generated by the set of rational expectations equilibria. We know introduce the concept of equilibrium consistency. Given a REE profile $\sigma = (\sigma_g, q_m)$, we define its *equilibrium path* $\pi = x (\sigma)$ as a sequence of measurable functions[17]

$$\pi = \left(d_t \left(y^t\right), b_{t+1} \left(y^t\right), q_t \left(z^t\right)\right)_{t \in \mathbb{N}}$$

that are generated by following the profile σ. A history h is *equilibrium consistent* if if is on some equilibrium path $x = x (\sigma)$, for some **REE** σ. Or said differently: a history h is equilibrium consistent if we can find at least some equilibrium σ that explains the data. This definition will be instrumental in finding the defining conditions of equilibrium paths, by providing a recursive representation. A history is part of an equilibrium path if and only

[17]For our baseline case, where after default the government is permanently in autarky, the functions have the restriction that bond issues and prices are not defined after a default has been observed: $b_{t+s+1} (y^t y^s) = q_{t+s} (y^t y^s) = \emptyset$ for all y^s and y^t such that $d_t (y^t) = 1$.

if **(a)** the history up to $t-1$ is part of an equilibrium path and **(b)** the partial history at time t is also consistent with it.

The main questions we answer below are: how do we know if a particular history h^t is equilibrium consistent? If so, what are the equilibria that are consistent with it? And most importantly: what are the forecasting predictions common across the rationalizing equilibrium profiles of history h^t?

5.2 General Model: Main Results

The main object we study is the equilibrium value correspondence, in the spirit of Abreu et al. (1990); Atkeson (1991). Formally,

$$\mathcal{E}\left(y_{-},b\right) := \left\{ (q,v) \in \mathbb{R}^k \times \mathbb{R} : \exists \sigma \in \mathbf{REE}\left(y_{-},b\right) \text{ with } \begin{cases} v = V\left(\sigma \mid h_0 = \left(y_{-},b\right)\right) \\ q = q_m\left[h_0, \left(d_0^{\sigma_g}, b_1^{\sigma_g}\right)\left(h_0\right)\right] \end{cases} \right\}$$

and let $\mathcal{Q}\left(y_{-},b\right) \subseteq \mathbb{R}^k$ be its projection over q. In the Appendix we show how one can characterize $\mathcal{E}\left(y_{-},b\right)$ using the concept of self-generation and enforceability in Abreu (1988); Abreu et al. (1990) and Atkeson (1991), and show it is compact, and convex valued if **(a)** y includes a sunspot (and is therefore non-atomic) and **(b)** u is concave in q (risk aversion of the policy maker), which are both satisfied in the examples given.

We then consider two important functions: the *best value function*, and the *maxmin value*. The best value function gives the maximum equilibrium value for the policy maker, if $q_t = q$ is realized; i.e.

$$\overline{v}\left(y_{-},b,q\right) = \max_{v \in \mathbb{R}} v \text{ subject to } (q,v) \in \mathcal{E}\left(y_{-},b\right)$$

In the Appendix we also show that if $\mathcal{E}\left(y_{-},b\right)$ is convex valued and $u\left(\cdot\right)$ is concave in q, then so is $\overline{v}\left(y_{-},b,q\right)$. The *maxmin value* is the worst possible value that the policy maker can obtain in a Rational Expectations Equilibrium, going forward. Formally,

$$\underline{U}\left(y,b\right) := \max_{(d,b') \in \Gamma(b,y)} \left\{ \min_{(q,v) \in \mathcal{E}(y,b')} u\left(b,y,d,b',q\right) + \beta v \right\}$$

For example, in the sovereign debt model, $\underline{U}\left(y,b\right) = V^d\left(y\right)$, the autarky value. In the online appendix, we show (following Waki et al. (2015)) that $\overline{v}\left(y_{-},b,q\right)$ can be expressed as the unique fixed point of a contraction mapping, given $\underline{U}\left(y,b\right)$ (which is natural, given that this is a game of perfect monitoring).

The main proposition of this section is to characterize what period t outcomes $h_t = (z_t, y_t, x_{t+1}, q_t)$ are equilibrium consistent.

Proposition 8 (Equilibrium Consistency)**.** *Suppose y is non-atomic, and that h^t is an equilibrium consistent history. Then, an outcome $h_t = (y_t, d_t, b_{t+1}, q_t)$ is equilibrium consistent if and only if*

a. Prices are equilibrium consistent, given government decision:

$$q_t \in \mathcal{Q}\left(y_t, b_{t+1}\right)$$

b. Incentive compatibility for policy maker:

$$u\left(b_t, y_t, d_t, b_{t+1}, q_t\right) + \beta \overline{v}\left(y_t, b_{t+1}, q_t\right) \geq \underline{U}\left(y_t, b_t\right) \tag{5.1}$$

Proof. See Appendix D. □

This gives an if and only if condition to check if data is equilibrium consistent. It replicates the same intuition we had on the sovereign debt model: if the policy maker's choice (d_t, b_{t+1}) could be rationalized as optimal under some equilibrium σ that predicted the price to be $q = q_t$, then it should also be rationalizable under an equilibrium that gives the highest equilibrium value consistent with prices being q_t if she chooses precisely (d_t, b_{t+1}), and gives the harshest possible punishment if the policy maker deviates (giving $\underline{U}\left(y_t, b_t\right)$ in lifetime utility in the continuation game). Sufficiency of this condition comes again from the fact that y_t is non-atomic, and hence, any particular realization of y_t has no marginal effect on expected lifetime utilities from previous periods; i.e. the promise keeping constraints can always be satisfied if we change the realization of the continuation value on a single y_t.

Proposition 8 also implies that for equilibrium consistency, we only need to know whether $h = h^t$ is equilibrium consistent or not. However, no information in h^t is relevant to decide whether $h^{t+1} = \left(h^t, h_t\right)$ is equilibrium consistent or not, besides the fact that there exist *some* equilibrium consistent with it. Therefore, we can characterize all equilibrium consistent histories recursively: start with the null history $h^0 = (y_-, b_0)$ (the starting state) and, h^{t+1} is equilibrum consistent if and only if h^t is equilibrium consistent and $h_t = (y_t, d_t, b_{t+1}, q_t)$ satisfies conditions **(1)** and **(2)** of Proposition 8.

Resembling what we did for the model of sovereign debt, how can we use the previous Proposition to obtain Robust Predictions on prices? Condition (2) defines, given y_t and the policy maker's choice (d_t, b_{t+1}), a set of equilibrium consistent prices:

$$Q(b_t, y_t, d_{t,}, b_{t+1}) := \{q_t \in \mathcal{Q}(y_t, b_{t+1}) : u(b_t, y_t, d_t, b_{t+1}, q_t) + \beta\overline{v}(y_t, b_{t+1}, q_t) \geq \underline{U}(y_t, b_t)\} \tag{5.2}$$

If $\overline{v}(y_t, b_{t+1}, q_t)$ is concave in q_t (which happens if $\mathcal{E}$ is convex valued and u concave in q), then the set of equilibrium consistent prices $Q(b_t, y_t, b_{t+1})$ will be a convex set as well. In the case of $k = 1$, this implies that Q is a compact interval; $Q(b_t, y_t, d_t, b_{t+1}) = \left[\underline{q}(b_t, y_t, d_t, b_{t+1}), \overline{q}(b_t, y_t, d_t, b_{t+1})\right]$ as in the sovereign debt model.

5.3 Case with sunspots

The random variable y is allowed to be multidimensional. Therefore, the above results include the case with sunspots, by adding an extra absolutely continuous random variable ζ_t variable (without loss of generality, we can assume $\zeta_t \sim U[0,1]$) and making the shock $\hat{y} = (y, \zeta_t)$. The different case is when the sunspot is realized in between the long lived agent's decision and the myopic agents expectations setting:

a. At $\tau = 1$ shock y_t is realized

b. At $\tau = 2$ long lived agent chooses (d_t, b_{t+1})

c. At $\tau = 3$ sunspot ζ_t is realized

d. At $\tau = 4$ myopic players choose $q_t(y_t, d_t, b_{t+1}, \zeta_t)$

Let $\mathcal{E}^s(y_-, b)$ be the equilibrium value correspondence, with sunspots. Note that in general, $\mathcal{E}^s(y_-, b) \supseteq \mathcal{E}(y_-, b)$. As we studied in the sovereign debt model, an outcome given history $h = (h^t, y_t)$ will be a triple $x_t = (d_t, b_{t+1}, P_t) \in D \times B \times \Delta(\mathbb{R}^k)$, predicting the long lived player decision, and the distribution of q conditional on (d_t, b_{t+1}) and the realization of the sunspot ζ_t. The main result of this section relies on the assumption that $\mathcal{E}(y_-, b)$ (the equilibrium value set without sunspots) is convex-valued. If that is the case, then $\mathcal{E}^s(y_-, b) = \mathcal{E}(y_-, b)$ and if $u(\cdot)$ is concave in q, then the maximum continuation value function $\overline{v}(y_-, b, q)$ coincides with the case without sunspots. Moreover, if $\mathcal{E}$ is convex valued and u is concave in q, then so is $\overline{v}(y, b, q)$ (the proof is left to the Appendix D).

We can now present the main result of this section, which generalizes the result of equilibrium consistency with sunspots in the sovereign debt model (its proof is also relegated to Appendix D).

Proposition 9. *Suppose y_t is non-atomic, $\mathcal{E}(y_-, b)$ is convex valued and u is concave in q . If h^t is an equilibrium consistent history, then an outcome $x_t = (d_t, b_{t+1}, P_t) \in D \times B \times \Delta\left(\mathbb{R}^k\right)$ is equilibrium consistent if and only if*

a. *Prices with positive probability are equilibrium consistent, given the long-lived player decision.*

$$\textbf{Supp}\,(P_t) \subseteq \mathcal{Q}(y_t, b_{t+1}) \tag{5.3}$$

b. *Incentive compatibility for policy maker:*

$$\int \left[u(b_t, y_t, d_t, b_{t+1}, \hat{q}) + \beta\overline{v}(y_t, b_{t+1}, \hat{q})\right] dP_t(\hat{q}) \geq \underline{U}(y_t, b_t) \tag{5.4}$$

Proof. See Appendix D. □

Clearly Proposition 9 generalizes Proposition 8 when $\mathcal{E}$ is convex valued and u is concave. When this is not the case, the proposition remains valid, only that now we define the functions $\overline{v}$ and $\underline{U}$ over the correspondence $\mathcal{E}^s(y_-, b)$ instead; i.e. $\overline{v}^s(y, b, q) = \max\{v : (q, v) \in \mathcal{E}^s(y, b)\}$ and $\underline{U}^s(y, b) := \max_{(d,b') \in \Gamma(b,y)} \min_{(q,v) \in \mathcal{E}^s(y,b)} u(b, y, d, b', q) + \beta v$.

As in the case without sunspots, conditions (5.3) and (5.4) define now a set of equilibrium consistent price distributions

$$Q^s(b_t, y_t, d_t, b_{t+1}) = \left\{P \in \Delta\left(\mathbb{R}^k\right) : x_t = (d_t, b_{t+1}, P) \text{ is eqm. consistent}\right\}$$

and see that this is a convex set of measures (since condition (2) is a linear inequality on measures P_t). Under the assumptions of Proposition 9 is easy to see that the function $g(\hat{q} \mid h_t) := u(b_t, y_t, d_t, b_{t+1}, \hat{q}) + \beta\overline{v}(y_t, b_{t+1}, \hat{q}) - \underline{U}(y_t, b_t)$ is concave in $\hat{q}$ as well. Therefore, as in the sovereign debt model, we have that the set of expected prices $E(b_t, y_t, b_{t+1}) := \left\{q \in \mathbb{R}^k : q = \int \hat{q}\, dP(\hat{q}) \text{ for some } P \in Q^s(b_t, y_t, d_t, b_{t+1})\right\}$ equals the set of equilibrium consistent prices without sunspots; i.e. $E(b_t, y_t, d_t, b_{t+1}) = Q(b_t, y_t, d_t, b_{t+1})$.

6 Conclusion and Discussion

Dynamic policy games have been extensively studied in macroeconomic theory to increase our understanding on how the outcomes that a government can achieve are restricted by its lack of commitment. One of the challenges in studying dynamic policy games is equilibrium multiplicity. Our paper acknowledges equilibrium multiplicity, and for this reason focuses on obtaining predictions that hold across all equilibria. To do this, we conceptually introduced and characterized equilibrium consistent outcomes. We did so under different settings, and we found that the assumption that a history was generated by the path of a subgame perfect equilibrium puts restrictions on current policies, and therefore on observables. In addition, we found intuitive conditions under which past decisions place restrictions on future policies; if the past decision occurred far away in time or in a history where the current history had low probability of occurrence, then it is less likely that a particular past decision influences current policies. In the extreme case that every particular history has probability zero, the restrictions of past decisions in current outcomes die out after one period. At first glance, this is surprising; but as we showed in the paper, this a direct consequence of robustness.

As we discussed in the text, equilibrium consistency is a general principle. Even though we focus on a model of sovereign debt that follows Eaton and Gersovitz (1981), our results generalize to other dynamic policy games. An example is the model of capital taxation as in Chari and Kehoe (1990). In that model, the entrepreneur invests and supplies labor, then the government taxes capital, and finally, the entrepreneur receives a payoff. The worst subgame perfect equilibrium is one where the government taxes all the capital. Note that, if the government has been consistently abstaining from taxing capital, then as outside observers we can rule out that the government will tax all capital. Past behavior, and the sole assumption of equilibrium, is giving information to the outside observer about future outcomes.

We think equilibrium consistency might have applications beyond policy games. The reason is that the sole assumption of equilibrium yields testable predictions. For example, the literature of risk sharing studies barriers to insurance and tries to test among different economics environments. Two environments that have received a lot of attention are Limited Commitment and Hidden Income. To test these two environments, a property of the efficient allocation with limited commitment is exploited: lagged consumption is a sufficient statistic of current consumption. If this hypothesis is rejected, then hidden income is favored in the data. However, the test is rejecting two hypotheses at the same time: efficiency and limited commitment. Our approach could, in principle, be suitable for a test that is tractable and robust to equilibrium multiplicity.

Over the course of the paper, we have been silent with respect to optimal policy. An avenue of future research is to relate equilibrium consistent outcomes and forward reasoning in dynamic games. Our conjecture is that, the set of equilibrium consistent outcomes will be intimately related with the set of outcomes if there is common knowledge of strong certainty of rationality. The reason is that, in the model of sovereign debt that we studied, the outside observer and the lenders have the same information set. Even in the motivating example, equilibrium consistent outcomes and outcomes when the solution concept is strong certainty of rationality are the same. In that case, our results have a different interpretation: the government is choosing the history to manage the expectations of the public.

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A Proofs Main Results

Proof. (Proposition 1). (**Necessity,** $\Longrightarrow$) If $(d(\cdot), b'(\cdot))$ is SPE - consistent, there exists an SPE profile $\hat{\sigma}$ such that $h^t \in \mathcal{H}(\hat{\sigma})$ and

$$d(y_t) = d_t^{\hat{\sigma}}\left(h^t, y_t\right) \text{ and } b'(y) = b_{t+1}^{\hat{\sigma}}\left(h^t, y_t, d = 0\right)$$

That is, there exists a SPE that generated the history h_-^t, specifies the contingent policy $d(\cdot), b'(\cdot)$ in period t, and satisfies conditions (3.3) to (3.5). Because $\hat{\sigma}$ is an SPE, using the results of Abreu et al. (1990) we know that if $d(y) = 0$ at $h^t = \left(h_-^t, q_{t-1}\right)$ then

$$u\left(y_t - b_t + b'(y_t)\, q_m^{\hat{\sigma}}\left(h^t, d_t = 0, b'(y_t)\right)\right) + \beta W\left(\hat{\sigma} \mid h^{t+1}\right) \geq u(y_t) + \beta \mathbb{V}^d(y_t) \qquad \text{(A.1)}$$

By definition of best continuation values and prices

$$W\left(\hat{\sigma} \mid h^{t+1}\right) \leq \overline{\mathbb{W}}\left(y_t, b'(y_t)\right) \text{ and } q_m^{\hat{\sigma}}\left(h^t, d_t = 0, b'(y_t)\right) \leq \overline{q}\left(y_t, b'(y_t)\right) \qquad \text{(A.2)}$$

Because $b'(y_t) \geq 0$ (no savings assumption), and $u(\cdot)$ is strictly increasing, we can plug in (A.2) into (A.1) to conclude that

$$u\left(y_t - b + b'(y_t)\, \overline{q}\left(y_t, b'(y_t)\right)\right) + \beta \overline{\mathbb{W}}\left(y_t, b'(y_t)\right) \geq$$

$$u\left(y - b_t + b'(y_t)\, q_m^{\hat{\sigma}}\left(h^t, d_t = 0, b'(y_t)\right)\right) + \beta W\left(\hat{\sigma} \mid h^{t+1}\right)$$

Proving condition (3.4). Further, since $\hat{\sigma}$ generated the observed history, past prices must be consistent with policy $(d(\cdot), b'(\cdot))$. Formally:

$$q_{t-1} = q_m^{\hat{\sigma}}\left(h^{t-1}, y_{t-1}, d_{t-1}, b_t\right) = \frac{1}{1+r^*}\left(1 - \int_{y_t \in Y} d^{\hat{\sigma}}\left(h^t, y_t\right) dF(y_t \mid y_{t-1})\right)$$

$$= \frac{1}{1+r}\left(1 - \int_{y_t \in Y} d(y_t) dF(y_t \mid y_{t-1})\right)$$

proving also condition (3.3). Condition (3.5) is the same as condition (3.4) but at $t-1$, using the usual promise keeping accounting. Formally, if $\hat{\sigma}$ is SPE and $h^t \in \mathcal{H}(\hat{\sigma})$ then the government's default and bond issue decision at $t-1$ was optimal given the observed expected prices

$$u(\underbrace{y_{t-1} - b_{t-1} + b_t q_{t-1}}_{=c_{t-1}}) + \beta W\left(\hat{\sigma} \mid h^t\right) \geq u(y_{t-1}) + \beta \mathbb{V}^d(y_{t-1})$$

Using the recursive formulation of $W(\cdot)$ we get the following inequality:

$$
\begin{aligned}
W\left(\hat{\sigma} \mid h^t\right) &= \int_{y_t:d(y_t)=0} \left[u(y_t - b_t + b'(y_t) q_m^{\hat{\sigma}}(h^t, y_t, d_t = 0, b'(y_t))) + W\left(\hat{\sigma} \mid h^{t+1}\right)\right] dF(y_t \mid y_{t-1}) \\
&\qquad + \int_{y_t:d(y_t)=1} \left[u\left(y_t\right) + \beta \mathbb{V}^d(y_t)\right] dF(y_t \mid y_{t-1}) \\
&\leq \int_{y_t:d(y_t)=0} \left[u\left(y_t - b_t + b'(y_t)\overline{q}(b'(y_t))\right) + \overline{\mathbb{W}}(b'(y_t))\right] dF(y_t \mid y_{t-1}) \\
&\qquad + \int_{y_t:d(y_t)=1} \left[u\left(y_t\right) + \beta \mathbb{V}^d(y_t)\right] dF(y_t \mid y_{t-1})
\end{aligned}
$$

From the previous two inequalities, we show (3.5).

(**Sufficiency,** $\Longleftarrow$) We need to construct a strategy profile $\sigma \in SPE$ such that $h_-^t \in \mathcal{H}(\sigma)$ and $d(\cdot) = d_t^{\sigma}\left(h^t, \cdot\right)$ and $b'(\cdot) = b_{t+1}^{\sigma}\left(h^t, \cdot\right)$. Given that $h_-^t \in \mathcal{H}(SPE)$, we know there exists some SPE profile $\hat{\sigma} = \left(\hat{\sigma}_g, \hat{q}_m\right)$ that generated h_-^t. Let $\overline{\sigma}(b, y)$ be the best continuation SPE (associated with the best price $\overline{q}(\cdot)$) when $y_t = y$ and $b_{t+1} = b$. Let σ^{aut} be the strategy profile for autarky (associated with $q_m = 0$ for all continuation histories). Also, let $h^{t+1}(y_t) = \left(h^t, y_t, d(y_t), b'(y_t), \overline{q}(y_t, b'(y_t))\right)$ be the continuation history at $y_t = y$ and the policy $(d(\cdot), b'(\cdot))$ if the government faces the best possible prices. Define $(h^s, y_s) \prec h^t$ as the histories that precede h^t and are not equal to h^t. That is, if we truncate h^t to period s, we obtain h^s. Denote $(h^s, y_s) \not\prec h^t$ as the histories that do not precede h^t. The symbol $\preceq$ denotes, histories that precede and can be equal. Construct the following strategy profile $\sigma = \left(\sigma_g, q_m\right)$:

$$
\sigma_g\left(h^s, y_s\right) = \begin{cases}
\hat{\sigma}_g\left(h^s, y_s\right) & \text{for all } (h^s, y_s) \prec h^t \\
\sigma^{aut}\left(y_s\right) & \text{for all } s < t \text{and } (h^s, y_s) \not\prec h^t \\
d_t\left(h^t, y_t\right) = d\left(y_t\right) \text{ and } b_{t+1}\left(h^t, y_t\right) = b'\left(y_t\right) & \text{for } \left(h^t, y_t\right) \text{ for all } y_t \\
\overline{\sigma}_g\left(b_{s+1}, y_s\right)\left(h^s, y_s\right) & \text{for all } h^s \succeq h^{t+1}\left(y_t\right) \\
\sigma^{aut}\left(y_s\right) & \text{for all } s > t,\ h^s \not\succeq h^{t+1}\left(y_t\right)
\end{cases}
$$

and

$$
q_m\left(h^s, y_s, d_s, b_{s+1}\right) = \begin{cases}
\hat{q}_m\left(h^s, y_s, d_s, b_{s+1}\right) & \text{for all } (h^s, y_s) \prec h^t \\
0 & \text{for all } s < t \text{ and } \left(h^s, y_s\right) \not\prec h^t \\
\overline{q}\left(y_s, b'\left(y_s\right)\right) & \text{for all } h^s \succeq \left(h^t, y_t, d\left(y_t\right), b'\left(y_t\right)\right) \\
0 & \text{for all } h^s / \succ \left(h^t, y_t, d\left(y_t\right), b'\left(y_t\right)\right)
\end{cases}
$$

By construction $h^t_- \in \mathcal{H}(\sigma)$. This is because, $\sigma = \hat{\sigma}_g$ for histories $(h^s, y_s) \preceq h^t$. Also, the strategy σ, prescribes the policy $(d(\cdot), b'(\cdot))$ on the equilibrium path. Now we need to show that the constructed strategy profile is indeed an SPE. For this, we will use the one deviation principle. See that for all histories with $s > t$ the continuation profile is an SPE (by construction); it prescribes the best continuation equilibrium, that is a SPE by definition. Now, we need to show that at h^t this is indeed an equilibrium. This comes from the second constraint, the incentive compatibility constraint

$$(1 - d(y_t)) \left[u(y_t - b_t + \overline{q}(y_t, b_{t+1}((y_t))) b_{t+1}(y_t)) + \beta \overline{\mathrm{W}}(y_t, b_{t+1}((y_t))) \right]$$

$$+ d(y_t) V^d(y_t) \geq V^d(y_t)$$

Note also that the default policy at $t-1$ was consistent with σ (and is an equilibrium) and that q_{t-1} is consistent with the policy $(d(\cdot), b'(\cdot))$. The promise keeping constraint (3.5) translates into the exact incentive compatibility constraint for profile σ, showing that the default decision at $t-1$ was indeed optimal given profile σ. The "price keeping" (3.3) constraint also implies that q_{t-1} was consistent with policy $(d(\cdot), b'(\cdot))$. The final step in sufficiency is to show that, $s < t-1$ (that is $h^s \prec h^t$). Note that, because y is absolutely continuous, the particular y that is realized, has zero probability. So, the expected value of this new strategy is the same

$$W(\hat{\sigma} \mid h^s) = W(\sigma \mid h^t)$$

for all $h^s \prec h^t$ with $s < t-1$; the probability of the realization of h^t, is zero. All this together implies that σ is indeed an SPE and generates history h^t_- on the equilibrium path, proving the desired result. □

Proof. (**Proposition** 2) By Proposition 1, we can rewrite program (3.9) as,

$$\underline{q}(b, y, b') = \min_{q, d(\cdot) \in \{0,1\}^Y, b''(\cdot)} q$$

subject to

$$q = \frac{1 - \int d(y') dF(y' \mid y)}{1 + r} \tag{A.3}$$

$$\left(1 - d\left(y'\right)\right) \left(\overline{V}^{nd}\left(b', y', b''\left(y'\right)\right) - V^d\left(y'\right)\right) \geq 0 \tag{A.4}$$

and

$$\beta \int \left[d\left(y'\right) V^{d}\left(y'\right) + \left(1 - d\left(y'\right)\right) \overline{V}^{nd}\left(b', y', b''\left(y'\right)\right) \right] dF\left(y' \mid y\right) - \beta \mathbb{V}^{d}(y) \geq u\left(y\right) - u\left(y - b + b'q\right) \tag{A.5}$$

First, note that we can relax the constraint (A.4) and (A.5) by choosing

$$b''\left(y'\right) = \underset{\hat{b} \geq 0}{\operatorname{argmax}}\ \overline{V}^{nd}\left(b', y', \hat{b}\right)$$

Second, define the set $R\left(b'\right) = \left\{y' \in Y : \overline{V}^{nd}\left(b', y'\right) \geq V^{d}\left(y'\right)\right\}$ to be the set of income levels for which the government does not default, under the best continuation equilibrium. Note that, if $y' \notin R\left(b'\right)$, it implies that no default is not equilibrium feasible for any continuation equilibrium (it comes from the fact that (A.4) is a necessary condition for no default). The minimization problem can now be written as

$$\underline{q}\left(b, y, b'\right) = \min_{q, d(\cdot) \in \{0,1\}^{Y}} q$$

subject to

$$q = \frac{1 - \int d\left(y'\right) dF\left(y' \mid y\right)}{1 + r}$$

$$\left(1 - d\left(y'\right)\right) \left[\overline{V}^{nd}\left(b', y'\right) - V^{d}\left(y'\right)\right] \geq 0 \text{ for all } y' \in R\left(b'\right) \tag{A.6}$$

$$d\left(y'\right) = 1 \text{ for all } y' \notin R\left(b'\right) \tag{A.7}$$

$$\beta \int \left[d\left(y'\right) V^{d}\left(y'\right) + \left(1 - d\left(y'\right)\right) \overline{V}^{nd}\left(b', y'\right) \right] dF\left(y'\right) - \beta \mathbb{V}^{d}(y) \geq u\left(y\right) - u\left(y - b + b'q\right)$$

As a preliminary step, we need to show that this problem has a non-empty feasible set. For that, choose the default rule that makes all constraints be less binding: i.e. $d\left(y'\right) = 0 \iff \overline{V}^{nd}\left(b', y'\right) \geq V^{d}\left(y'\right)$. This corresponds to the best equilibrium policy. If this policy is not feasible, then the feasible set is empty. Under this default policy, the one of the best equilibrium, the price q is equal to the best equilibrium price $q = \overline{\mathbf{q}}\left(y, b'\right)$. The feasible set is non-empty if and only if

$$\beta \int \left[d\left(y'\right) V^{d}\left(y'\right) + \left(1 - d\left(y'\right)\right) \overline{V}^{nd}\left(b', y'\right) \right] dF\left(y' \mid y\right) - \beta \mathbb{V}^{d}(y) \geq u\left(y\right) - u\left(y - b + b'\overline{\mathbf{q}}\left(y, b'\right)\right)$$

$$u\left(y - b + b'\overline{\mathbf{q}}\left(y, b'\right)\right) + \beta \mathbb{W}\left(b'\right) \geq u\left(y\right) + \beta \mathbb{V}^{d}(y) \iff$$

$$\overline{V}^{nd}\left(b, y, b'\right) \geq V^{d}\left(y\right)$$

where $\mathbb{W}\left(y, b'\right)$ is the value of the option of defaulting b' bonds; this is the initial assumption of this proposition. Also, note that

$$\mathbb{V}^d(y) = \int \left[d\left(y'\right) V^d\left(y'\right) + (1 - d\left(y'\right))V^d\left(y'\right)\right] dF\left(y' \mid y\right)$$

So, we can rewrite the promise keeping constraint as

$$\beta \int \left(1 - d\left(y'\right)\right) \left[\overline{V}^{nd}\left(b', y'\right) - V^d\left(y'\right)\right] dF\left(y'\right) \geq u\left(y\right) - u\left(y - b + b'q\right) \tag{A.8}$$

We focus on a relaxed version of the problem. We will allow the default rule to be $d\left(y'\right) \in [0,1]$ for all y'. Given the state variables $\left(b, y, b'\right)$ the relaxed problem is a convex minimization program in the space $\left(q, d\left(\cdot\right)\right) \in \left[0, \frac{1}{1+r}\right] \times \mathbb{D}\left(Y\right)$, where

$$\mathbb{D}\left(Y\right) \equiv \left\{d : Y \rightarrow [0,1] \text{ such that } d\left(y'\right) = 1 \text{ for all } y' \notin R\left(b'\right)\right\}$$

is a convex set of default functions. Also, include the constraint for prices

$$q \geq \frac{1 - \int d\left(y'\right) dF\left(y' \mid y\right)}{1+r}$$

The intuition for this last constraint is that $d\left(y'\right) = 1$ has to be feasible in the relaxed problem. The Lagrangian

$$\mathcal{L}\left(q, \delta\left(\cdot\right)\right) = q + \mu\left(-q + \frac{1 - \int d\left(y'\right) dF\left(y' \mid y\right)}{1+r}\right) +$$

$$\lambda\left(u\left(y\right) - u\left(y - b + b'q\right) - \beta\int \left(1 - d\left(y'\right)\right)\left[\overline{V}^{nd}\left(b', y'\right) - V^d\left(y'\right)\right] dF\left(y' \mid y\right)\right)$$

The optimal default rule $d\left(\cdot\right)$ must minimize the Lagrangian $\mathcal{L}$ given the multipliers $\left(\mu, \lambda\right)$ (where $\mu, \lambda \geq 0$). Notice that for $y' \in R\left(b'\right)$ any $d \in [0,1]$ is incentive constraint feasible, and

$$\frac{\partial \mathcal{L}}{\partial d\left(y'\right)} = \left(-\frac{\mu}{1+r} + \lambda\beta\left[\overline{V}^{nd}\left(b', y'\right) - V^d\left(y'\right)\right]\right) dF\left(y' \mid y\right)$$

So, because it is a linear programming program, the solution is in the corners (and if it is not in the corners, it has the same value in the interior), then the values of y' such that the country does not default are given by

$$d\left(y'\right) = 0 \iff \lambda\Delta^{nd} > \frac{\mu}{\beta\left(1+r\right)} \tag{A.9}$$

Note that $\lambda > 0$ in the optimum. Suppose not; then $d(y') = 1$ for all $y' \in Y$ satisfies the IC and the price constraint. Then, the minimum price is

$$q \geq \frac{1-1}{1+r}$$

So, the minimizer will be zero, $q = 0$. But, this will not meet the promise keeping constraint. Formally,

$$\beta \int V^d(y')\, dF(y' \mid y) - \beta \mathbb{V}^d(y) - u(y) + u(y-b) =$$

$$= \beta\left(\mathbb{V}^d(y) - \mathbb{V}^d(y)\right) + u(y-b) - u(y) = u(y-b) - u(y) < 0$$

This implies $\lambda > 0$. Note that, $\lambda > 0$ implies that $\underline{\mathbf{q}}(b, y, b') > 0$. Define

$$\gamma \equiv \frac{\mu}{\lambda \beta (1+r)}$$

From (A.9)

$$d(y') = 0 \iff \Delta^{nd} \geq \gamma \iff \overline{V}^{nd}(b', y') \geq V^d(y') + \gamma$$

as we wanted to show. Aided with this characterization, from the promise keeping constraint we have an equation for γ as a function of the states

$$\beta \int_{\overline{V}^{nd}(b',y') \geq V^d(y') + \gamma} \left[\overline{V}^{nd}(b', y') - V^d(y')\right] dF(y' \mid y) = u(y) - u(y - b + b'q) \quad \text{(A.10)}$$

where

$$q = \frac{\Pr\left(\overline{V}^{nd}(b', y') \geq V^d(y') + \gamma\right)}{1+r} \quad \text{(A.11)}$$

Define

$$\Delta^{nd}(y') := \overline{V}^{nd}(b', y') - V^d(y')$$

So,

$$q = \frac{\hat{F}\left(\Delta^{nd}(y') \geq \gamma\right)}{1+r}$$

where $\hat{F}$ is the probability distribution of $\Delta^{nd}(y')$. The **last step in the proof** involves showing that the solution is well defined. Define the function

$$G(\gamma) = \beta \int_{\Delta^{nd} \geq \gamma} \Delta^{nd} d\hat{F}\left(\Delta^{nd} \mid y\right) - u(y) + u\left(y - b + b' \frac{1 - \hat{F}(\gamma \mid y)}{1+r}\right)$$

First, note that G is weakly decreasing in γ, that $G(0) > 0$ (from the assumption $\overline{V}^{nd}(b', y') - V^d(y') > 0$) and $\lim_{\gamma \to \infty} G(\gamma) = u(y - b) - u(y) < 0$. Second, note that G is right continuous in γ. These two observations imply that we can find a minimum $\gamma : G(\gamma) \geq 0$. If income is an absolutely continuous random variable, then $G(\cdot)$ is strictly decreasing and continuous, implying the existence of a unique γ such that $G(\gamma) = 0$. This determines the solution to the price minimization problem.

□

B Sunspot Proofs

Proof of Proposition 3. Necessity ($\Longrightarrow$): Suppose there is an equilibrium strategy σ such that $h \in \mathcal{H}(\sigma)$. This implies that the government decided optimally not to default at period t; i.e.

$$\int_0^1 \left[u\left(y_t - b_t + q^\sigma\left(h,\zeta\right)b_{t+1}\right) + \beta V^\sigma\left(h,\zeta\right)\right]d\zeta \geq u\left(y_t\right) + \beta\underline{\mathbb{V}}^d \tag{B.1}$$

Since σ is a SPE, we have that for all sunspot realizations $\zeta \in [0,1]$ we must have

$$\left(V^\sigma\left(h,\zeta\right), q^\sigma\left(h,\zeta\right)\right) \in \mathcal{E}\left(b_{t+1}\right)$$

using the self-generation characterization of $\mathcal{E}(b)$. This further implies two things:

a. $q^\sigma\left(h,\zeta\right) \in \left[0, \overline{\mathbf{q}}\left(b_{t+1}\right)\right]$ (i.e. it delivers equilibrium prices)

b. $V^\sigma\left(h,\zeta\right) \leq \overline{v}\left(b_{t+1}, q^\sigma\left(h,\zeta\right)\right)$ (because $\overline{v}$ is the maximum possible continuation value with price $q = q^\sigma\left(h,\zeta\right)$)

The price distribution given by σ can be defined by a measure P over measurable sets $A \subseteq \mathbb{R}_+$ as

$$P\left(A\right) = \int_0^1 \mathbf{1}\left\{q^\sigma\left(h,\zeta\right) \in A\right\}d\zeta = \Pr\left\{\zeta : q^\sigma\left(h,\zeta\right) \in A\right\}$$

Note that numeral (1) shows that $\mathbf{Supp}\left(P\right) \subseteq \left[0, \overline{\mathbf{q}}\left(b_{t+1}\right)\right]$. To show **??**, we change integration variables in B.1 and using the definitions above and properties (1) and (2), we get

$$\int \left[u\left(y_t - b_t + \hat{q}b_{t+1}\right) + \beta\overline{v}\left(b_{t+1},\hat{q}\right)\right]dP\left(\hat{q}\right) \geq$$

$$\int_0^1 \left[u\left(y_t - b_t + q^\sigma\left(h,\zeta\right)b_{t+1}\right) + \beta V^\sigma\left(h,\zeta\right)\right]d\zeta \geq u\left(y_t\right) + \beta\underline{\mathbb{V}}^d$$

Proving the desired result.

Sufficiency ($\Longleftarrow$). Suppose that P is an equilibrium consistent distribution with cdf F_P. Let

$$\sigma^*\left(b,q\right) \in \underset{\sigma \in SPE\left(b_{t+1}\right)}{\operatorname{argmax}} V^\sigma\left(h^0\right) \text{ s.t. } q_0^\sigma \leq q$$

i.e. it is a strategy that achieves the continuation value $\overline{v}(b,q)$. As we showed before, the constraint in this problem will be binding. Because h^t is equilibrium consistent, we know there exist an equilibrium profile $\hat{\sigma}$ such that $h \in \mathcal{H}(\hat{\sigma})$. For histories h' successors of histories $h^{t+1} = \left(h^t, d_t, \hat{b}_{t+1}, \zeta_t, \hat{q}_t\right)$ we define the profile σ as

$$\sigma(h') = \begin{cases} \sigma^d(h') & \text{if } d_t = 1,\ \hat{b}_{t+1} \neq b_{t+1} \text{ or } \hat{q}_t \notin [0, \overline{\mathbf{q}}(b_{t+1})] \\ \sigma^*(b_{t+1}, \hat{q}_t)\left(h' \sim h^{t+1}\right) & \text{otherwise} \end{cases}$$

and for histories $h' = \left(h^t, d_t = 0, b_{t+1}, \zeta_t\right)$ let

$$q^\sigma\left(h^t, d_t, b_{t+1}, \zeta_t\right) = F_P^{-1}(\zeta_t)$$

where $F_P(q) = P(\hat{q})$ is the cumulative distribution function of distribution P and $F_P^{-1}(\zeta) = \inf\{x \in \mathbb{R} : F_P(q) \geq \zeta\}$ its inverse. It will be optimal to not default at t (if we follow strategy σ for all successor nodes) if

$$\int_0^1 \left[u\left(y_t - b_t + F_P^{-1}(\zeta)\, b_{t+1}\right) + \beta V^\sigma(b_{t+1}, \zeta)\right] d\zeta \geq u(y_t) + \beta \underline{\mathbb{V}}^d \underbrace{\iff}_{(\mathbf{a})}$$

$$\int \left[u(y_t - b_t + \hat{q} b_{t+1}) + \beta \overline{v}(b_{t+1}, \hat{q})\right] dP(\hat{q}) \geq u(y_t) + \beta \underline{\mathbb{V}}^d \tag{B.2}$$

using the classical result that $F_P^{-1}(\zeta) =_d P$ if $\zeta \sim \text{Uniform}[0,1]$ and the fact that $V^\sigma(h') = V(\sigma^*(h')) = \overline{v}(b_{t+1}, q_t)$ from the definition of σ. Conditions B.1 is satisfied, and $\mathbf{Supp}(P) \subseteq [0, \overline{\mathbf{q}}(b_{t+1})]$ imply that, if the government follows profile σ, then h is also on the path of σ, and σ is indeed a Nash equilibrium at such histories (because both σ^d and $\sigma^*(b_{t+1}, \hat{q})$ are subgame perfect profiles). Finally, for histories $h' \not\succ h^t$ define $\sigma(h') = \hat{\sigma}(h')$. Therefore, $\sigma(h')$ is itself a subgame perfect equilibrium profile (since it is a Nash equilibrium at every possible history) and generates $h = \left(h^t, d_t = 0, b_{t+1}\right)$ on its path. □

Proof of Proposition4. This comes from the fact that the function

$$U(P) = \int \left\{u(y_t - b_t + \hat{q} b_{t+1}) + \beta \overline{v}(b_{t+1}, \hat{q})\right\} dP(q)$$

is strictly increasing in y_t and strictly decreasing in b_t, and the set can be rewritten as

$$\mathcal{Q}(b_t, y_t, b_{t+1}) = \left\{P \in \Delta([0, \overline{\mathbf{q}}]) : U(P) \geq V^d(y_t)\right\}$$

The function $H(q) := u(y_t - b_t + q b_{t+1}) + \beta \overline{v}(b_{t+1}, q)$ is strictly increasing in q. Therefore, if $P' \trianglerighteq P$ and $P \in \mathcal{Q}(b_t, y_t, b_{t+1})$ then $\int H(q)\, dP' \geq \int H(q)\, dP \geq V^d(y_t)$. Using Proposition **??** together with assumption (1) gives the result. □

It also has a greatest element,

$$\overline{P}(q \in A) = \begin{cases} 1 & \text{if } \overline{\mathbf{q}}(b_{t+1}) \in A \\ 0 & \text{otherwise} \end{cases}$$

i.e. $\overline{P}$ is the Dirac measure over the best price $q = \overline{\mathbf{q}}(b_{t+1})$. It also has an infimum, with respect to the first order stochastic dominance, given by the Lebesgue-stjeljes measure associated with the cdf $\underline{P}(\cdot)$ we characterize in section 3 below. However, this infimum distribution is not an equilibrium distribution.

Proof of Proposition 5. Upper bound for general $\hat{q} < \underline{\mathbf{q}}(b_t, y_t, b_{t+1})$Here we replicate the same strategy: let $p = \Pr(\zeta : q(\zeta) \leq \hat{q})$. Using the same strategy as before, to get the less binding incentive compatibility constraint for the government we need to maximize equilibrium utility for $\zeta : q(\zeta) \leq \hat{q}$ for $\zeta : q(\zeta) > \hat{q}$, we consider equilibria that assign the best continuation equilibria (to make the incentive constraint of the government as flexible as possible).

For (2) we just follow the same thing we did for the case where $\hat{q} = 0$ and consider the continuation equilibria where $q(\zeta) = \overline{\mathbf{q}}(b_{t+1})$ and $v(\zeta) = \overline{\mathbb{V}}(b_{t+1})$ (the fact that this corresponds to an actual equilibria is easy to check). For (1), we see that focusing on equilibria that have support $q(\zeta) \in \{\hat{q}, \overline{\mathbf{q}}(b_{t+1})\}$ make the government incentive constraint as flexible as possible, since utility of the government is increasing in $\hat{q}$ and moreover, $\overline{v}(b, \hat{q})$ (the biggest continuation utility consistent with $q \leq \hat{q}$) is also increasing in $\hat{q}$ as we saw before. Therefore, if p is the maximum such probability, we must have

$$p\left[u(y_t - b_t + \hat{q}b_{t+1}) + \beta\overline{v}(b_t, \hat{q})\right] + (1-p)\, V^{nd}(b_t, y_t, b_{t+1}) \geq V^d(y_t) \iff$$

$$p \leq \frac{\Delta^{nd}(b_t, y_t, b_{t+1})}{V^d(y_t) - \left[u(y_t - b_t + \hat{q}b_{t+1}) + \beta\overline{v}(b_t, \hat{q})\right] + \Delta^{nd}(b_t, y_t, b_{t+1})}$$

See that this is not an innocuous constraint only when the right hand side is less than 1. This happens only when

$$u(y_t - b_t + \hat{q}b_{t+1}) + \beta\overline{v}(b_t, \hat{q}) \geq V^d(y_t)$$

As we argued

$$\hat{q} \geq \underline{\mathbf{q}}\left(b_t, y_t, b_{t+1}\right)$$

where the last inequality comes from the characterization of $\underline{\mathbf{q}}\left(b_t, y_t, b_{t+1}\right)$. $\square$

Proof of Proposition 6. We already know that $\max E\left(b_t, y_t, b_{t+1}\right) = \overline{\mathbf{q}}\left(b_{t+1}\right)$ since the Dirac distribution $\overline{P}$ over $q = \overline{\mathbf{q}}\left(b_{t+1}\right)$ is equilibrium feasible. In the same way, we also know that the Dirac distribution $\hat{P}$ that puts probability 1 to $q = \underline{\mathbf{q}}\left(b_t, y_t, b_{t+1}\right)$ is also equilibrium consistent; it corresponds to a case where both investors and the government ignore the realization of the correlated device, and the characterization of $\underline{\mathbf{q}}\left(\cdot\right)$ is exactly the only price that satisfies

$$u\left(y_t - b_t + \underline{\mathbf{q}}\left(b_t, y_t, b_{t+1}\right)\ b_{t+1}\right) + \beta\overline{v}\left(b_{t+1}, \underline{\mathbf{q}}\left(b_t, y_t, b_{t+1}\right)\right) = V^d\left(y_t\right).$$

Lemma 3 shows that $\overline{v}\left(b, q\right)$ is a concave function in q, which together with the fact that u is strictly concave and $b' > 0$ implies that the function

$$H\left(q\right) := u\left(y_t - b_t + q b_{t+1}\right) + \beta\overline{v}\left(b_{t+1}, q\right)$$

is strictly concave in q. For any distribution $P \in \mathcal{Q}\left(b_t, y_t, b_{t+1}\right)$, let $\mathbb{E}_P\left(q\right) = \int \hat{q} dP\left(\hat{q}\right)$. Jensen's inequality then implies that

$$u\left(y_t - b_t + \mathbb{E}_P\left(q\right) b_{t+1}\right) + \beta\overline{v}\left(b_{t+1}, \mathbb{E}_P\left(q\right)\right) \underbrace{\geq}_{(1)} \int \left[u\left(y_t - b_t + \hat{q} b_{t+1}\right) + \beta\overline{v}\left(b_{t+1}, \hat{q}\right)\right] dP\left(\hat{q}\right) \geq$$

$$\underbrace{\geq}_{(2)} V^d\left(y_t\right)$$

with strict inequality in (1) if P is not a Dirac distribution. Then, the definition of $\underline{\mathbf{q}}(b_t, y_t, b_{t+1})$ implies that for any distribution $P \in \mathcal{Q}\left(b_t, y_t, b_{t+1}\right)$ we have

$$\mathbb{E}_P\left(q\right) \geq \underline{\mathbf{q}}\left(b_t, y_t, b_{t+1}\right)$$

and therefore the minimum expected value is exactly $\underline{\mathbf{q}}\left(b_t, y_t, b_{t+1}\right)$, which is achieved uniquely at the Dirac distribution $\hat{P}$ (because of strict concavity of $u\left(\cdot\right)$). Finally, knowing that E is naturally a convex set, we then get that

$$\begin{aligned} E\left(b_t, y_t, b_{t+1}\right) &= \left[\min_{P\in\mathcal{Q}(b_t,y_t,b_{t+1})} \int \hat{q}dP\left(\hat{q}\right), \max_{P\in\mathcal{Q}(b_t,y_t,b_{t+1})} \int \hat{q}dP\left(\hat{q}\right)\right] \\ &= \left[\underline{\mathbf{q}}\left(b_t, y_t, b_{t+1}\right), \overline{\mathbf{q}}\left(b_t, y_t, b_{t+1}\right)\right] \end{aligned}$$

as we wanted to show. □

Proof of Corollary . $\underline{P}$ as defined in equation 4.2 cannot be an equilibrium consistent price: this implies that the Lebesgue-stjeljes measure associated with $\overline{P}(\cdot)$ has the property that **Supp** $(\underline{P}) = \left[0, \underline{\mathbf{q}}\left(b_t, y_t, b_{t+1}\right)\right]$ and $\underline{P}\left(q=0\right) = p_0 > 0$, which implies that

$$\begin{aligned} \int \left\{u\left(y_t - b_t + \hat{q}b_{t+1}\right) + \beta\overline{v}\left(b_{t+1}, \hat{q}\right)\right\} d\underline{P}\left(\hat{q}\right) &< u\left(y_t - b_t + \underline{\mathbf{q}}\left(\cdot\right) b_{t+1}\right) + \beta\overline{v}\left(b_{t+1}, \underline{\mathbf{q}}\left(\cdot\right)\right) \\ &= V^d\left(y_t\right) \end{aligned}$$

where the last equation comes from the definition of $\underline{\mathbf{q}}\left(\cdot\right)$ and the function $H\left(\hat{q}\right) \equiv u\left(y_t - b_t + \hat{q}b_{t+1}\right) + \beta\overline{v}\left(b_{t+1}, \hat{q}\right)$ is strictly increasing in $\hat{q}$. □

Online Appendix to "Robust Predictions in Dynamic Policy Games"

Juan Passadore and Juan Xandri

C Characterization of $\overline{v}(b,q)$

C.1 Characterizing the Equilibrium Set

Define the equilibrium value correspondence as

$$\mathcal{E}(b) = \left\{ (v,q) \in \mathbb{R}_2 : \exists \sigma \in SPE(b) : \begin{cases} v = \mathbb{E}\left\{ \sum_{t=1}^{\infty} u\left(c^{\sigma}\left(h^t\right)\right)\right\} \\ q = \frac{1}{1+r}\left(1 - \int d^{\sigma}(y_0)\, dF(y_0)\right) \end{cases} \right\}$$

The set $\mathcal{E}(b)$ has the values and prices that can be obtained in a subgame perfect equilibrium. We need to find a policy that keeps the promise for prices, for one period.

Enforceability

Take a bounded, compact valued correspondence $W : \mathbb{R}_+ \rightrightarrows \mathbb{R}^2$. We will drop the dependence on d, and we will bear in mind that after default the government is not in the market.

Definition 1. Given $b \geq 0$, a government strategy $(d(\cdot), b'(\cdot))$ is enforceable in $W(b)$ if we can find a pair of functions $v(y)$ and $q(y)$ such that

a. $(v(y), q(y)) \in W(b'(y))$ for all $y \in Y$

b. For all $y \in Y$, the policy $(d(y), b'(y))$ solves the problem

$$V^{v(\cdot),q(\cdot)}(b,y) = \max_{\hat{d}\in\{0,1\},\hat{b}\geq 0} \left(1-\hat{d}\right)\left\{u\left[y - b + q(y)\hat{b}\right] + \beta v(y)\right\} + \hat{d}\left\{u(y) + \beta \underline{\mathbb{V}}^d\right\}$$

We will refer to the pair $(v(\cdot), q(\cdot))$ as the enforcing values of policy $(d(y), b'(y))$ and we will write $(d(\cdot), b'(\cdot)) \in \mathbf{E}(W)(b)$.

Definition 2. Given a correspondence $W : \mathbb{R}_+ \rightrightarrows \mathbb{R}^2$, we define the generating correspondence $B(W) : \mathbb{R}_+ \rightrightarrows \mathbb{R}^2$ as

$$B(W)(b) = \left\{ (v, q) \in \mathbb{R}^2 : \exists \left(d(\cdot), b'(\cdot) \right) \in \mathbf{E}(W)(b) : \begin{cases} v = \mathbb{E}\left\{ V^{v(\cdot),q(\cdot)}(b, y) \right\} \\ q = \frac{1}{1+r}\left(1 - \int d(y)\right) \end{cases} \right\}$$

Definition 3. A correspondence $W(\cdot)$ is self-generating if for all $b \geq 0$ we have $W(b) \subseteq B(W)(b)$

Proposition 10. *Any bounded, self-generating correspondence gives equilibrium values: i.e. if $W(b) \subseteq B(W)(b)$ for all $b \geq 0$, then $W(b) \subseteq \mathcal{E}(b)$*

Proof. The proof follows Abreu et al. (1990) and is constructive; we provide a sketch of the argument. Take any pair $(v_{-1}, q_{-1}) \in W(b)$. We need to construct a subgame perfect equilibrium strategy profile $\sigma \in \mathbf{SPE}(b)$. Since $W(b) \subseteq B(W)(b)$ we know we can find functions $(d_0(y_0), b_1(y_0))$ and values $(v_0(y_0), q_0(y_0)) \in W(b)$ for any $b \geq 0$ such that $(d_0(y_0), b_1(y_0))$ is in the argmax of $V^{v_0(\cdot),q_0(\cdot)}(\cdot)$ and

$$v_{-1} = \mathbb{E}_0\left\{ V^{v_0(\cdot),q_0(\cdot)}(y, b) \right\}$$

and

$$q_{-1} = \frac{1}{1+r}\left\{ 1 - \int d_0(y_0)\, dF(y_0) \right\}$$

Define

$$\sigma_g\left(h^0\right) = \left(d_0(y_0), b_1(y_0)\right)$$

and

$$\sigma_m\left(h_-^0\right) = q_0$$

where $h_-^0 = (b_0, q_{-1})$. Because $(v_0(y_0), q_0(y_0)) \in W(b_1(y_0))$ and W is self-generating, we know that for any realization of y_0, we can find policy functions $(d_1(y_1), b_2(y_1))$ and values $(v_1(y_1), q_1(y_1, b_2(y_1))) \in B(W)(b_2(y_1))$ such that $(d_1(y_1), b_2(y_1))$ is in the argmax of $V^{v_1(\cdot),q_1(\cdot)}(\cdot)$ and

$$v_0(y_0) = \mathbb{E}\left(V^{v_1(\cdot),q_1(\cdot)}(\cdot) \right),$$

$$\sigma_m\left(h_-^1\right) = q_1(y_1, b_2) = \frac{1}{1+r}\left(1 - \int d_1(y_1) \right)$$

Also define

$$\sigma_g\left(h_-^2\right) = \left(d_1(y_1), b_2(y_1)\right)$$

is clear to see that strategy profiles σ_m and σ_g defined for all histories of type h_0^1 and h_0^2 satisfy the first constraints of being a subgame perfect equilibrium. Doing it recursively for all finite t, we can then prove by induction (same as APS original proof) that this profile forms a SPE with initial values (v_0, q_0) as we stated. The finiteness of the value function is guaranteed because the set W is bounded. There are no one shot deviations by construction. □

Proposition 11. *The correspondence $\mathcal{E}(b)$ is the biggest correspondence (in the set order) that is a fixed point of B. That is, $\mathcal{V}(\cdot)$ satisfies:*

$$B(\mathcal{E})(b) = \mathcal{E}(b) \tag{C.1}$$

for all $b \geq 0$, and if another operator $W(\cdot)$ also satisfies condition C.1, then $W(b) \subseteq \mathcal{E}(b)$ for all $b \geq 0$.

Proof. Is sufficient to show that $\mathcal{E}(b)$ is itself self-generating. As in APS, we start with any strategy profile $\sigma = (\sigma_g, \sigma_m)$ and the values associated with it (v_0, q_0) with initial debt b. From the definition of SPE, we know that the policy $d_1(y_1) = d^{\sigma_g}(h^1, y_1)$ and $b'(y_1) = b_2^{\sigma_g}(h^1, y_1)$ is implementable with functions $q\left(y_1, \hat{b}\right) = q_m^{\sigma}(y_1, d(y_1), b'(y_1))$ and $v\left(y_1, \hat{b}\right) = V\left(\sigma \mid h^2\left(y_1, \hat{b}\right)\right)$, where $h^2\left(y_1, \hat{b}\right) \equiv \left(h^1, y_1, d_1(y_1), b'(y_1), q\left(y_1, \hat{b}\right)\right)$. Moreover, because σ is an SPE strategy profile, it means it also is a subgame perfect equilibria for the continuation game starting with initial bonds $b = \hat{b}$, and hence

$$\left(v\left(y_1, \hat{b}\right), q\left(y_1, \hat{b}\right)\right) \in \mathcal{V}\left(\hat{b}\right).$$

This then means that $(v_0, q_0) \in B(\mathcal{V})(b)$, and hence $\mathcal{V}(\cdot)$ is a self-generating correspondence. □

Bang Bang Property

Now we are going to relate the APS characterization with the characterization in the main text .First, notice that the singleton set $\{(v,q)\} = \left\{\left(\underline{\mathbb{V}}^{aut}, 0\right)\right\}$ (corresponding to the autarky subgame perfect equilibria) is itself self-generating, and hence an equilibrium value. Let $(v,q) = \left(\overline{\mathbb{V}}(b), \overline{\mathbf{q}}(b)\right)$ denote the expected utility and debt price associated with the best equilibrium.

Proposition 12. *Let $(d(\cdot) .b'(\cdot))$ be an enforceable policy on $\mathcal{V}(b)$ (i.e. they are part of a subgame* *perfect equilibrium). Then, it can be enforced by the following continuation value functions:*

$$v\left(y,\hat{d}\right)=\begin{cases}\overline{\mathbb{V}}\left(b'\left(y\right)\right) & \textit{if } d\left(y\right)=0 \textit{ and } \hat{d}=b'\left(y\right)\\ \underline{\mathbb{V}}^{d} & \textit{otherwise}\end{cases} \tag{C.2}$$

and

$$q\left(y,\hat{d}\right)=\begin{cases}\overline{\mathbf{q}}\left(b'\left(y\right)\right) & \textit{if } d\left(y\right)=1 \textit{ and } \hat{d}=b'\left(y\right)\\ 0 & \textit{otherwise}\end{cases} \tag{C.3}$$

Proof. Notice that the functions $v\left(\cdot\right),q\left(\cdot\right)$ satisfy the restriction $\left(v\left(y,\hat{d}\right),q\left(y,\hat{d}\right)\right)\in\mathcal{E}\left(\hat{d}\right)$ for all $\hat{b}$. Since $\left(d\left(\cdot\right),b'\left(\cdot\right)\right)$ are enforceable, there exist functions $\left(\hat{v}\left(\cdot\right),\hat{q}\left(\cdot\right)\right)$ such that for all $y:d\left(y\right)=0$ we have

$$u\left[y-b+\hat{q}\left(y,b'\left(y\right)\right)b'\left(y\right)\right]+\beta\hat{v}\left(y,b'\left(y\right)\right)\geq u\left[y-b+\hat{q}\left(y,\hat{b}\right)\hat{b}\right]+\beta\hat{v}\left(y,\hat{b}\right) \tag{C.4}$$

for all $\hat{b}\geq 0$. Now, because the left hand side argument is an equilibrium value (since it is generated by an equilibrium policy), its value must be less than the best equilibrium value for the government, characterized by $q=\overline{\mathbf{q}}\left(b'\left(y\right)\right)$ and $v=\overline{\mathbb{V}}^{nd}\left(b'\left(y\right)\right)$ (that is, the best equilibrium from tomorrow on, starting at a debt value of $\hat{b}=b'\left(y\right)$. This means that

$$V^{nd}\left(b,y,b'\left(y\right)\right)\equiv u\left[y-b+\overline{\mathbf{q}}\left(y,b'\left(y\right)\right)b'\left(y\right)\right]+\beta\overline{\mathbb{V}}\left(b'\left(y\right)\right)\geq$$

$$\geq u\left[y-b+\hat{q}\left(y,b'\left(y\right)\right)b'\left(y\right)\right]+\beta\hat{v}\left(y,b'\left(y\right)\right) \tag{C.5}$$

On the other side, we also have that autarky is the worst equilibrium value (since it coincides with the min-max payoff) which implies

$$u\left[y-b+\hat{q}\left(y,\hat{b}\right)\hat{b}\right]+\beta\hat{v}\left(y,\hat{b}\right)\geq u\left(y\right)+\beta\underline{\mathbb{V}}^{d}\text{ for all }\hat{b}\geq 0 \tag{C.6}$$

Combining C.4 with the inequalities given in C.5 and C.6 we get

$$u\left[y-b+\overline{\mathbf{q}}\left(y,b'\left(y\right)\right)b'\left(y\right)\right]+\beta\overline{\mathbb{V}}\left(b'\left(y\right)\right)\geq u\left(y\right)+\beta\underline{\mathbb{V}}^{d} \tag{C.7}$$

which is the enforceability constraint (conditional on not defaulting) of the proposed functions $\left(v,q\right)$ in equations C.2 and C.3. To finish the proof, we need to show that if it is indeed optimal to choose $d\left(y\right)=0$ under the functions $\left(\hat{v}\left(\cdot\right),\hat{q}\left(\cdot\right)\right)$, then it will also be so under functions $\left(v\left(\cdot\right),q\left(\cdot\right)\right)$. This is readily given by condition C.7, since punishment of defaulting coincides with the value of deviating from bond issue rule $\hat{b} = b'(y)$. Hence, $(v(\cdot), q(\cdot))$ also enforce $(d(\cdot), b'(\cdot))$. $\square$

This proposition greatly simplifies the characterization of implementable policies. Remember the definitions of the objects

$$V^{nd}(b, y, b') \equiv u\left(y - b + \overline{\mathbf{q}}(b') b'\right) + \beta \overline{\mathbb{V}}(b')$$

as the expected lifetime utility under the best continuation equilibrium for any choice of debt b', and

$$V^{d}(y) \equiv u(y) + \beta \underline{\mathbb{V}}^{d}$$

as the expected lifetime utility of autarky.

Corollary 2. *A policy $(d(\cdot), b'(\cdot))$ is enforceable on $\mathcal{E}(b)$ if and only if $d(y) = 0$ implies*

$$V^{nd}\left(b, y, b'(y)\right) \geq V^{d}(y)$$

C.2 Computing $\overline{v}(b, q)$

The function $\overline{v}(b, q)$ gives the highest expected utility that a government can obtain if they raised debt at price q and issued b bonds[18]. This is the Pareto frontier in the set of equilibrium values. We now discuss how we compute $\overline{v}(b, q)$, which can be redefined using the equilibrium correspondence:

$$\overline{v}(b, q) := \max\{v : \exists \hat{q} \geq 0 \text{ such that } (v, \hat{q}) \in \mathcal{E}(b) \text{ and } \hat{q} \leq q\} \tag{C.8}$$

Note that we focus in a relaxes version, where we replace the equality $\hat{q} = q$ by the inequality $\hat{q} \leq q$. We will show a result that will enable us to compute $\overline{v}(b, q)$.

Proposition 13. *For all $q \in [0, \overline{\mathbf{q}}(b)]$ the maximum continuation value $\overline{v}(b, q)$* solves

$$\overline{v}(b, q) = \max_{\delta(\cdot) \in [0,1]^{Y}} \int \left\{\delta(y) V^{d}(y) + [1 - \delta(y)] \overline{V}^{nd}(b, y)\right\} dF(y)$$

subject to

$$q = \frac{1}{1 + r}\left(1 - \int \delta(y) \, dF(y)\right) \tag{C.9}$$

Furthermore, is $\overline{v}(b, q)$ non-decreasing and concave in q.

[18] Because this is the best equilibrium given a price $\hat{q}$ it does not depend on the amount of debt repaid; we are not characterizing equilibrium consistent outcomes.

The proof of Proposition **??** follows from the next three Lemmas.

Lemma 1 (Characterization of $\overline{v}$). *For all $q \in [0, \overline{\mathbf{q}}(b))$ the maximum continuation value $\overline{v}(b,q)$ solves*

$$\overline{v}(b,q) = \max_{\delta(\cdot)\in[0,1]^{Y}} \int \left\{ \delta(y) V^{d}(y) + [1-\delta(y)] \overline{V}^{nd}(b,y) \right\} dF(y) \tag{C.10}$$

subject to

$$q \geq \frac{1}{1+r}\left(1 - \int \delta(y)\, dF(y)\right) \tag{C.11}$$

where the constraint C.11 is always binding for all $q > 0$.

Proof. Take an enforceable policy $(\delta(\cdot), b'(\cdot))$ such that $\frac{1}{1+r}(1 - \int \delta(y)\, dF(y)) = q$. By definition, there must exist functions $(\hat{v}(y,b'), \hat{q}(y,b')) \in \mathcal{E}(b')$ such that for all y

$$\left(\delta(y), b'(y)\right) \in \underset{(\delta,b')}{\operatorname{argmax}}\, \delta V^{d}(y) + (1-\delta)\left\{u\left[y - b + \hat{q}\left(y,b'\right) b'\right] + \beta \hat{v}\left(y,b'\right)\right\}$$

with the right hand side value (at the optimum) being the ex ante value of the policy. We show in Proposition 12 that (1) any enforceable policy can also be enforced by the "bang-bang values"

$$\hat{v}\left(y,b'\right) = \begin{cases} \overline{\mathbb{V}}\left(b'(y)\right) & \text{if } b' = b'(y) \\ \underline{\mathbb{V}}^{d} & \text{otherwise} \end{cases} \text{ and } \hat{q}\left(y,b'\right) = \begin{cases} \overline{\mathbf{q}}\left(b'(y)\right) & \text{if } b' = b'(y) \\ 0 & \text{otherwise} \end{cases}$$

and (2) the continuation value is maximized at this values, since

$$\delta(y) V^{d}(y) + [1-\delta(y)]\left\{u\left[y - b + \hat{q}\left(y, b'(y)\right) b'(y)\right] + \beta \hat{v}\left(y, b'(y)\right)\right\} \leq$$

$$\delta(y) V^{d}(y) + [1-\delta(y)]\left\{u\left[y - b + \overline{\mathbf{q}}\left(b'(y)\right) b'(y)\right] + \beta \overline{\mathbb{V}}\left[b'(y)\right]\right\} =$$

$$\underbrace{=}_{\text{by def.}} \delta(y) V^{d}(y) + [1-\delta(y)] V^{nd}\left(b, y, b'(y)\right) \tag{C.12}$$

Therefore, an enforceable policy $(\delta(\cdot), b'(\cdot))$ policy can generate (conditional on y) a value given by equation C.12. Therefore, we can write the problem of finding the biggest continuation value consistent with a default price less than q as

$$\overline{v}(b,q) = \max_{(\delta(\cdot),b'(\cdot))} \int \left\{ \delta(y) V^{d}(y) + [1-\delta(y)] V^{nd}\left(b, y, b'(y)\right) \right\} dF(y)$$

subject to the incentive constraint:

$$V^{nd}\left(b, y, b'\left(y\right)\right) \geq V^{d}\left(y\right) \text{ for all } y : \delta\left(y\right) = 0$$

and that its associated price is less than q:

$$\frac{1}{1+r}\left(1 - \int \delta\left(y\right) dF\left(y\right)\right) \leq q$$

Finally, notice that $b'\left(y\right)$ only enters the problem through the term $V^{nd}\left(b, y, b'\left(y\right)\right)$, and that making this object as large as possible makes both (1) the objective function bigger and (2) the constraints less binding (since it only enters through the incentive compatibility constraint). Therefore, we choose $b'\left(y\right)$ to solve

$$\overline{V}^{nd}\left(b, y\right) = \max_{b' \geq 0} V^{nd}\left(b, y, b'\left(y\right)\right)$$

showing then the desired result. Finally, note that $\overline{v}\left(b, q\right)$ is weakly increasing in q, and that if we remove the price constraint, then the agent would choose the default rule to get price $\overline{\mathbf{q}}\left(b\right)$ (the one associated with the best equilibrium), so for $q < \overline{\mathbf{q}}\left(b\right)$ this constraint must be binding. □

Remark 1. See that this is a linear programming problem in $\delta\left(\cdot\right)$, which we will see is easy to solve. If tractable, this Lemma will help us mapping the boundaries of the equilibrium correspondence $\mathcal{E}\left(b\right)$ for any given q.

The following proposition solves the programming problem shown in Lemma 1, reducing it to solving a problem in one equation in one unknown.

Lemma 2. *Given* (b, q) *there exist a constant* $\gamma = \gamma\left(b, q\right)$ *such that*

$$\overline{v}\left(b, q\right) = \int \left[\hat{\delta}\left(y\right) V^{d}\left(y\right) + \left(1 - \hat{\delta}\left(y\right)\right) \overline{V}^{nd}\left(b, y\right)\right] dF\left(y\right)$$

where

$$\hat{\delta}\left(y\right) = 0 \iff \overline{V}^{nd}\left(b, y\right) \geq V^{d}\left(y\right) + \gamma \textit{ for all } y \in Y$$

and γ *is the (maximum) solution to the single variable equation:*

$$\frac{1}{1+r} \Pr\left\{y \,:\, \overline{V}^{nd}\left(b, y\right) \geq V^{d}\left(y\right) + \gamma\right\} = q$$

Moreover, γ *is also the Lagrange multiplier of constraint C.11 in program C.12, so that* $\frac{\partial \overline{v}(b,q)}{\partial q} =$

$\gamma(b,q)$.

Proof. Using the Lagrangian in the relaxed program of letting $\delta(y) \in [0,1]$ for all output levels for which no-default is feasible; i.e. for all $y \in D(b) \equiv \left\{ y : \overline{V}^{nd}(b,y) \geq V^d(y) \right\}$. The Lagrangian (without the corner conditions for δ) is

$$\mathcal{L} = \int \left[\delta(y) V^d(y) + (1-\delta(y)) \overline{V}^{nd}(b,y) \right] dF(y) +$$

$$+ \int \mu(y) [1-\delta(y)] \left[\overline{V}^{nd}(b,y) - V^d(y) \right] dF(y) +$$

$$+\lambda \left(q(1+r) - 1 + \int \delta(y) dF(y) \right)$$

so that at a $y : \overline{V}^{nd}(y) > V^d(y)$

$$\frac{\partial \mathcal{L}}{\partial [\delta(y)]} = \left[-\overline{V}^{nd}(b,y) + V^d(y) + \lambda \right] dF(y) \Longrightarrow \hat{\delta}(y) = \begin{cases} 0 & \text{if } \overline{V}^{nd}(b,y) \geq V^d(y) + \lambda \\ 1 & \text{otherwise} \end{cases}$$

Defining $\gamma \equiv \lambda$ we get the desired result, using the binding property of constraint for prices. □

Lemma 3 (Concavity of $\overline{v}$). *The function $\overline{v}(b,q) = \max\{v : \exists \hat{q} \leq q$ such that $(v,\hat{q}) \in \mathcal{E}(b)\}$ is concave in q.*

Proof. From Lemma 1 we know that the feasible set of the program in that Lemma is convex, having a linear objective function and an affine restriction. Take $q_0, q_1 \in [0, \overline{\mathbf{q}}(b)]$ and $\lambda \in [0,1]$. We need to show that

$$\overline{v}(b, \lambda q_0 + (1-\lambda) q_1) \geq \lambda \overline{v}(b, q_0) + (1-\lambda) \overline{v}(b, q_1)$$

Let $G[\delta(\cdot)] = \int \left[\delta(y) V^d(y) + (1-\delta(y)) \overline{V}^{nd}(b,y) \right] dF(y)$ be the objective function of the maximization in C.10. Let $\delta_0(y)$ be one of the solutions for the program when $q = q_0$, and likewise $\delta_1(y)$ be one of the solutions of the relaxed program when $q = q_1$. Define

$$\delta_\lambda(y) = \lambda \delta_0(y) + (1-\lambda) \delta_1(y)$$

Clearly this is not a feasible default policy as it is, since δ_λ may be in $(0,1)$, but it is feasible in the relaxed program of Lemma 1. Note that it is feasible when $q = q_\lambda :=$

$\lambda q_0 + (1-\lambda) q_1$, since

$$\frac{1}{1+r}\left(1-\int \delta_\lambda(y)\, dF(y)\right) = \lambda \frac{1}{1+r}\left(1-\int \delta_0(y)\, dF(y)\right) + ..$$

$$+ (1-\lambda)\frac{1}{1+r}\left(1-\int \delta_0(y)\, dF(y)\right) \leq \lambda q_0 + (1-\lambda) q_1 = q_\lambda$$

Therefore, the optimal continuation value at $q = q_\lambda$ must be greater than the objective function evaluated at δ_λ. The reason is that the optimum will be at a corner even in the relaxed problem. Then

$$\overline{v}(b, q_\lambda) \geq G[\delta_\lambda(\cdot)] \underbrace{=}_{(a)} \lambda G[\delta_0(\cdot)] + (1-\lambda) G[\delta_1(\cdot)] \underbrace{=}_{(b)} \lambda \overline{v}(b, q_0) + (1-\lambda)\overline{v}(b, q_1)$$

using in (a) the fact that $G[\delta(\cdot)]$ is an affine function in $\delta(\cdot)$ and in (b) the fact that both $\delta_0(\cdot)$ and $\delta_1(\cdot)$ are the optimizers at q_0 and q_1 respectively. This concludes the proof. $\square$

D Proofs General Model

D.1 Equilibrium Consistency without Sunspots

Enforceability And Decomposability

The main object we study is the equilibrium value correspondence, in the spirit of Abreu et al. (1990); Atkeson (1991). Formally,

$$\mathcal{E}(y_-, b) := \left\{ (q, v) \in \mathbb{R}^k \times \mathbb{R} : \exists \sigma \in \mathbf{REE}(y_-, b) \text{ with } \begin{cases} v = V(\sigma \mid h_0 = (y_-, b)) \\ q = q_m \left[h_0, \left(d_0^{\sigma_g}, b_1^{\sigma_g} \right)(h_0) \right] \end{cases} \right\}$$

We will follow the steps in Abreu, Pearce and Stacchetti (1990) and Atkeson (1991) and define $\mathcal{E}$ as a fixed point of a set operator. The notation $\mathcal{E}(z_-, x)$ serves to remind the reader that y is not random, but the conditional past value of y that conditions the distribution of the next draw of y. We will see that we cannot apply the results in their papers directly, and need to adapt their theory for our setup. A pair of functions $d : Y \to D$ and $b' : Y \to B$ satisfying $(d(y), b'(y)) \in \Gamma(b, y)$ is defined as a policy function. To simplify notation, define the set $C(b) = \{(y, d, b') \in Y \times D \times B : (d, b') \in \Gamma(b, y)\}$.

Definition 4 (Enforceability). Given a compact valued correspondence $W : Y \times X \rightrightarrows \mathbb{R}^k \times \mathbb{R}$, we say that a policy function $(d(\cdot), b'(\cdot))$ is enforceable in W at (y_-, b) if we can find functions $v : C(b) \to \mathbb{R}$ and $q : C(b) \to \mathbb{R}^k$ such that

$$u\left[b, y, d(y), b'(y), q(y, d(y), b'(y))\right] + \beta v(y, d(y), b'(y)) \geq$$
$$\geq u\left[b, y, \hat{d}, \hat{b}, q\left(y, \hat{d}, \hat{b}\right)\right] + \beta v\left(y, \hat{d}, \hat{b}\right) \text{ for all } \left(y, \hat{d}, \hat{b}\right) \in C(b) \tag{D.1}$$

and

$$\left(q(y, d, b'), v(y, d, b')\right) \in W(y, b') \tag{D.2}$$

for all $(y, d, b') \in C(b)$.

We call the functions v and q the **enforcing functions** of policy $(d(\cdot), b'(\cdot))$ over W. Note that the definition of enforceability is actually only function of the state b: this is because the feasible set it is only a function of initial state (set up in the previous period) and current y_t.

The next step is to define decomposable values, with respect to a given correspondence. This will allow us to define self-generating correspondences later.

Definition 5 (Decomposability). A pair $(q, v) \in \mathbb{R}^k \times \mathbb{R}$ is decomposable over $W(\cdot)$ at state (y_-, b) if there exist an enforceable policy $(d(y), b'(y))$ with enforcing functions $(q, v) : C(s) \to \mathbb{R}^k \times \mathbb{R}$ such that

$$v = \int \left\{ u\left[b, y, d(y), b'(y), q\left(y, d(y), b'(y)\right)\right] + \beta v\left(y, d(y), b'(y)\right) \right\} dF(y \mid y_-, b) \tag{D.3}$$

and

$$q = \int T\left(b, y, d(y), b'(y)\right) dF(y \mid y_-, b) \tag{D.4}$$

Let $\mathcal{W}$ be the class of compact-valued correspondences from $S \times Y$ to $\mathbb{R}^{k+1}$. Define $B : \mathcal{W} \to \mathcal{W}$ as

$$B(W)(y_-, b) = \left\{ (q, v) \in \mathbb{R}^{k+1} : \ (q, v) \text{ is decomposable over } W(\cdot) \text{ at } (y_-, b) \right\}$$

We will write $W \preceq W'$ if $W(y_-, b) \subseteq W'(y_-, b)$ for all $(y_-, b) \in Y \times B$. This order defines a complete lattice on $\mathcal{W}$. Finally, we give the main definition of this section: Self-Generating correspondences

Definition 6. Self Generation. We say a correspondence $W(\cdot) \in \mathcal{W}$ is *self-generating* if, for all (y_-, b), we have $W(y_-, b) \subseteq B(W)(y_-, b)$

In the following Proposition, we prove an analogous result to the main result in Abreu et al. (1990); Atkeson (1991): that self-generating and bounded correspondences are included in the Equilibrium correspondence.

Proposition 14 (Self Generation implies Equilibrium). *Suppose $W(\cdot) \in \mathcal{W}$ is self-generating. Then $W(y_-, b) \subseteq \mathcal{E}(y_-, b)$ for all $(y_-, b) \in Y \times B$ (or, simply, $W \preceq \mathcal{E}$). Moreover, $\mathcal{E}$ is itself a self-generating correspondence, implying that $\mathcal{E}$ is the biggest fix point of B : i.e. $B(\mathcal{E}) = \mathcal{E}$ and if $B(W) = W \Longrightarrow W \preceq \mathcal{E}$.*

Proof. The proof follows and is constructive; we provide a sketch of the argument. Take any pair $(q, v) \in W(y_-, b)$. We need to construct an equilibrium strategy profile $\sigma \in$ **REE** (y_-, b). Since $W(y_-, b) \subseteq B(W)(y_-, b)$ we know we can a policy function $\left(\hat{d}_0(y_0), \hat{b}_1(y_0)\right)$ and enforcing functions $(\hat{q}_0, \hat{v}_0) : C(b) \to W(y_-, b)$ such that $\left(\hat{d}(y_0), \hat{b}_1(y_0)\right)$ maximizes the policy maker utility given the schedule $\hat{q}_0 = \hat{q}_0(y, d, b')$ and the continuation value function $\hat{v}_0 = \hat{v}_0(y_0, d, b')$, and moreover

$$v = \mathbb{E}_{y_0} \left\{ u\left(b, y_0, \hat{d}_0(y_0), \hat{b}_1(y_0), q_0^*(y_0)\right) + \beta v^*(y_0) \right\}$$

where we write $q_0^*(y_0) := \hat{q}\left(y_0, \hat{d}_0(y_0), \hat{b}_1(y_0)\right)$ and $v_0^*(y_0) := \hat{v}\left(y_0, \hat{d}_0(y_0), \hat{b}_1(y_0)\right)$ as the expectations and continuation values on the equilibrium path of the equilibrium we are constructing. Moreover

$$q = \mathbb{E}_{y_0}\left\{T\left[b, y_0, \hat{d}_0(y_0), \hat{b}_1(y_0)\right] \mid y_-, b\right\}$$

We will now start defining a strategy profile (σ_g, q_m) recursively, for all possible histories of length t. At $t = 0$, we define

$$\sigma_g\left(h^0, y_0\right) := \left(\hat{d}_0(y_0), \hat{b}_1(y_0)\right)$$

and

$$q_m\left(h^0, y_0, d_0, b_1\right) = \hat{q}_0(y_0, d_0, b_1)$$

for all possible realizations of d_0 and b_1 (not just the ones prescribed by strategy σ_g). Because W is self-generating, we have that $(\hat{q}_0(y_0, d_0, b_1), \hat{v}_0(y_0, d_0, b_1)) \in W(y_0, b_1) \subseteq B(W)(y_0, b_1)$ for all $(y_0, d_0, b_1) \in C(b)$, and hence we can find a policy function $\left(\hat{d}_1(y_1), \hat{b}_2(y_1)\right)$ enforceable over W at (y_0, b_1) with enforcing functions $(\hat{q}_1(y_1, d_1, b_2), \hat{v}_1(y_1, d_1, b_2))$ such that $\left(\hat{d}_1(y_1), \hat{b}_2(y_1)\right)$ maximizes utility given schedule $q = \hat{q}_1$ and continuation value function $v = \hat{v}_1$, and moreover

$$\hat{v}_0(y_0, d_0, b_1) = \mathbb{E}_{y_1}\left\{u\left(b_1, y_1, \hat{d}_1(y_1), \hat{b}_2(y_1), q^*(y_1)\right) + \beta v_1^*(y_1) \mid y_0, b_1\right\} \tag{D.5}$$

and

$$\hat{q}_0(y_0, d_0, b_1) = \mathbb{E}_{y_1}\left\{T\left[b_1, y_1, \hat{d}_1(y_1), \hat{b}_2(y_1)\right] \mid y_0, b_1\right\} \tag{D.6}$$

where $q^*(y_1) = \hat{q}\left(y_1, \hat{d}_1(y_1), \hat{b}_2(y_1)\right)$ and $v_1^*(y_1) = \hat{v}_1\left(y_1, \hat{d}_1(y_1), \hat{b}_2(y_1)\right)$ are the equilibrium path expectations and continuation values. Also note from equation D.6 that $q_m(h^0, y_0, d_0, b_1)$ satisfies rational expectations. Hence, we define the strategy profile for histories of length $t = 1$ as

$$\sigma_g\left(h^1, y_1\right) := \left(\hat{d}_1(y_1), \hat{b}_2(y_1)\right)$$

and

$$q_m\left(h^1, y_1, d_1, b_2\right) = \hat{q}_1(y_1, d_1, b_2)$$

Following in the same fashion for $t > 1$, we construct the strategies σ_g and q_m for all histories. Moreover, once we define them for all histories, the continuation value functions $\hat{v}_t(\cdot)$ give, in fact, subgame perfect values (because they are the values of following

σ_g if the myopic players follow q_m, which itself satisfies the rational expectations condition at all histories). Therefore, to conclude that $\left(\sigma_g, q_m\right) \in \mathbf{REE}\left(y_-, s\right)$, we need to show that they in fact achieve lifetime value of v for the policy maker. This implied by the set $W\left(y_-, b\right)$ being bounded (since W is compact valued). Formally, doing the iterations T steps, we get that

$$v = \mathbb{E}\left\{\sum_{t=0}^{t=T-1} \beta^t u\left(b_t, y_t, d_t, b_{t+1}, q_t\right) + \beta^T v_T^*\left(y_T\right)\right\} = \mathbb{E}\left\{\sum_{t=0}^{\infty} \beta^t u\left(b_t, y_t, d_t, b_{t+1}, q_t\right)\right\}$$

since $v^*\left(y_T\right)$ is bounded.

Moreover, the strategy profile satisfies the single deviation principle (by construction). Therefore, $W\left(y_-, b\right) \subseteq \mathcal{E}\left(y_-, b\right)$. Since this construction is true for any initial conditions, we conclude that $W \preceq \mathcal{E}$.

To finish the proof, we now show that $\mathcal{E}$ is a self-generating correspondence. For this, we only need to show that $\mathcal{E}\left(y_-, b\right) \subseteq B\left(\mathcal{E}\right)\left(y_-, b\right)$ for all $\left(y_-, b\right)$. As in APS, we start with any strategy profile $\sigma = \left(\sigma_g, q_m\right)$ and the values associated with it $\left(q, v\right)$ with initial debt states $\left(y_-, b\right)$. From the definition of REE, we know that the policy $\left(\hat{d}_0\left(y_0\right), \hat{b}_1\left(y_0\right)\right) = \sigma_g\left(h^0, y_0\right)$ is implementable with functions $\hat{q}\left(y_0, d_0, b_1\right) = q_m\left(h^0, y_0, d_0, b_1\right)$ and $\hat{v}\left(y_0, d_0, b_1\right) = V\left(\left(\sigma_g, q_m\right) \mid h = \left(h^0, y_0, d_0, b_1, \hat{q}\left(\cdot\right)\right)\right)$. Moreover, because σ is a REE strategy profile, it means it also is a REE for the continuation game starting with initial state $\left(y, b\right) = \left(y_0, b_1\right)$, and hence

$$\left(\hat{q}\left(y_0, d_0, b_1\right), \hat{v}\left(y_0, d_0, b_1\right)\right) = \left(q_m\left(h^0, y_0, d_0, b_1\right), V\left(\sigma \mid h\right)\right) \in \mathcal{E}\left(y_0, b_1\right)$$

This means that $\left(q, v\right) \in B\left(\mathcal{E}\right)\left(y_-, b\right)$, and hence $\mathcal{E}\left(y_-, b\right) \subseteq B\left(\mathcal{E}\right)\left(y_-, b\right)$ is a self-generating correspondence.

To show $\mathcal{E}$ is a fixed point of B, we use monotonicity of B to get that $B\left(\mathcal{E}\right) \preceq B\left(B\left(\mathcal{E}\right)\right)$, so the correspondence $W = B\left(\mathcal{E}\right)$ is also self-generating. But the first part of the Proposition then implies that $B\left(\mathcal{E}\right) \preceq \mathcal{E}$ and since $\mathcal{E}$ is self-generating, we have $\mathcal{E} \preceq B\left(\mathcal{E}\right)$. Since $\succeq$ is a linear partial order, this implies $B\left(\mathcal{E}\right) = \mathcal{E}$, as we wanted to show. The fact that $\mathcal{E}$ is a fixed point of B follows from the fact that, if $\mathcal{E}$ is self-generating, we know $B\left(\mathcal{E}\right)\left(y_-, s\right) \subseteq \mathcal{E}$ □

Finally, how do we calculate $\mathcal{E}$, and show it is compact? Following Abreu et al. (1990); Atkeson (1991) we only need a big enough correspondence $\mathcal{F}$ such that **(1)** $\mathcal{E} \preceq \mathcal{F}$ **(2)** $\mathcal{F}$ is compact-valued and **(3)** $B\left(\mathcal{F}\right) \preceq \mathcal{F}$ (so it is NOT self generating). We do so in the Online Appendix, also replicating the result that B preserves compactness. If so, then we know that the sequence of correspondences $W_1 = \mathcal{F}$ and $W_n = B(W_{n-1})$ for $n > 1$ is a decreasing sequence of compact valued correspondences (being all uniformly bounded by $\mathcal{F}$), so the limit $W_\infty(y_-, s) = \bigcap_{n=1}^\infty W_n(y_-, s)$ is compact valued and non-empty (Cantor's Intersection Theorem). We then consider the family of correspondences

$$\mathcal{W} = \left\{ W : Y \times S \rightrightarrows \mathbb{R}^{k+1} \text{ such that } W \text{ is compact valued, and } W \preceq \mathcal{F} \right\}$$

This is a complete lattice, and $B : \mathcal{W} \to \mathcal{W}$ is isotone and order continuous. Then, the Knaster-Tarski Theorem implies that there exist a largest (and smallest) fixed point W^* of B, and in particular, that the sequence $W_n = B^n(\mathcal{F})$ has the property that $\{W_n\}$ is decreasing, and $W_\infty = \bigcap_{n=1}^\infty W_n = W^*$. But since we already showed that $\mathcal{E}$ is the largest fixed point, we know $W^* = \mathcal{E}$ and hence $W_\infty = \mathcal{E}$. Moreover, since W_∞ is non-empty and compact valued, and so is $\mathcal{E}$

Necessary and Sufficient conditions for implementability. Equilibrium Consistency

We know are endowed with the technology to find the equilibrium correspondence $\mathcal{E}(y_-, b)$, so for this subsection we will assume to be known. When a policy $(d(y), b'(y))$ is enforceable with $W = \mathcal{E}$ at (y_-, b), with enforcing function $(\hat{q}(y, d, b'), \hat{v}(y, d, b')) \in \mathcal{E}(y, b')$, we will say simply that the *outcome* $(d(y), b'(y), q(y))$ is **implementable**, where $q(y) = \hat{q}(y, d(y), b'(y))$. Intuitively, an *implementable outcome* is the realization of the equilibrium path of a Rational Expectations Equilibrium starting at $h_0 = (y_-, b)$. We are interested in characterizing the equilibrium paths of all equilibria, and hence we do not focus on characterizing behavior off equilibrium; i.e. knowing $q(y, d, b')$ when $(d, b') \neq (d(y), b'(y))$.

Proposition 15 (Implementable outcomes)**.** *Take a state $(y_-, b) \in Y \times B$. An outcome $(d(y), b'(y), q(y))$ is implementable at (y_-, b) if and only if*

a. Equilibrium feasibility:

$$q(y) \in \mathcal{Q}\left(y, b'(y)\right) \text{ for all } y \in Y \tag{D.7}$$

b. Incentive compatibility for policy maker: for all $y \in Y$

$$u\left[b, y, d(y), b'(y), q(y)\right] + \beta \overline{v}\left(y, b'(y), q(y)\right) \geq \underline{U}(b, y) \tag{D.8}$$

Moreover, if $\mathcal{E}(y_-, b)$ is convex-valued, and $u(\cdot)$ is concave in q, then given $d = d(y)$ and

$b' = b'(y)$, $q = q(y)$ satisfies D.8 if and only if $q \in \mathbb{Q}(b, y, d, b')$, a convex-valued and compact valued correspondence.

This proposition can also be stated with respect to a given value correspondence $W(y_-, b)$. That is, we can adapt Proposition 15 to show if an outcome $(d(y), b'(y), q(y))$ is implementable in W at (y_-, b); being $(d(y), b'(y))$ an enforceable policy in W and q a decomposable value in W. Proposition 15 generalizes in this case, with the alternative functions

$$\overline{v}_W(y_-, b, q) := \max\{v : \ (q, v) \in W(y_-, b)\} \tag{D.9}$$

and

$$\underline{U}_W(b, y) := \max_{(d,b') \in \Gamma(b,y)} \left\{ \min_{(q,v) \in W(y,b)} u(b, y, d, b', q) + \beta v \right\} \tag{D.10}$$

To prove Proposition 15, we first need to show two easy Lemmas

Lemma 4. *If $\mathcal{E}(y_-, b)$ is convex $\Longrightarrow \mathcal{Q}(y_-, b)$ is a convex set and $\overline{v}$ is concave in q*

Proof. First, see that $\mathcal{Q}(y_-, b) = P_q(\mathcal{E}(y_-, b))$ the orthogonal projection of a convex set, hence convex. Take $q_0, q_1 \in \mathcal{Q}(y_-, b)$ and $\lambda \in [0, 1]$. If the maximum is attained, then there exist v_0 and v_1 such that

$$\overline{v}(y_-, b, q_0) = v_0 \text{ and } \overline{v}(y_-, b, q_1) = v_1$$

We know then that $(q_0, v_0) \in \mathcal{E}(y_-, b)$ and $(q_1, v_1) \in \mathcal{E}(y_-, b)$, and convexity of the equilibrium value correspondence implies that $(q_\lambda, v_\lambda) \in \mathcal{E}(y_-, b)$ where $q_\lambda = \lambda q_0 + (1-\lambda) q_1$ and $v_\lambda = \lambda v_0 + (1-\lambda) v_1$. This makes v_λ feasible for the maximization problem:

$$\overline{v}(y_-, b, \lambda q_0 + (1-\lambda) q_1) = \overline{v}(y_-, b, q_\lambda) \geq v_\lambda = \lambda v_0 + (1-\lambda) v_1 = \lambda \overline{v}(y_-, b, q_0) + (1-\lambda) \overline{v}(y_-, b, q_1)$$

and hence $\overline{v}$ is concave in q □

Lemma 5. *Take a state $(y_-, b) \in Y \times B$. An outcome $(d(y), b'(y), q(y))$ is implementable at (y_-, b) if and only if we can find a function $v : Y \to \mathbb{R}$ such that*

a. For all $y \in Y$ and all $(\tilde{d}, \tilde{b}) \in \Gamma(b, y)$ there exist $(\tilde{q}, \tilde{v}) \in \mathcal{E}(y, \tilde{b})$ such that

$$u\left[b, y, d(y), b'(y), q(y)\right] + \beta v(y) \geq u\left(b, y, \tilde{d}, \tilde{b}, \tilde{q}\right) + \beta \tilde{v} \tag{D.11}$$

b. $(q(y), v(y)) \in \mathcal{E}(y, b'(y))$ for all $y \in Y$

Proof. Suppose $(d(y), b'(y), q(y))$ satisfy (1) and (2). Then, we can define price and continuation value schedules (on and off equilibrium) as

$$(\hat{q}, \hat{v})\left(y, d, b'\right) = \begin{cases} (q(y), v(y)) & \text{if } (d, b') = (d(y), b'(y)) \\ (\tilde{q}, \tilde{v}) & \text{if } (d, b') = \left(\tilde{d}, \tilde{b}\right) \neq (d(y), b'(y)) \end{cases}$$

which is a well defined function, since the existence of $(\tilde{q}, \tilde{v})$ is known given $\left(y, \tilde{d}, \tilde{b}\right)$. Since $(\tilde{q}, \tilde{v}) \in \mathcal{E}\left(y, \tilde{b}\right)$ and $(q(y), v(y)) \in \mathcal{E}(y, b'(y))$ we have that $(d(y), b'(y))$ is decomposable in $\mathcal{E}$ at (y_-, b) and $q(y) = \hat{q}(y, d(y), b'(y))$ by definition. The other direction is trivial (simply choose $(q(y), v(y)) = (\hat{q}, \hat{v})(y, d(y), b'(y))$ and take $(\tilde{q}, \tilde{v}) = (\hat{q}, \hat{v})\left(y, \tilde{d}, \tilde{b}\right)$. □

This Lemma is useful because it lets us forget about finding specific schedules of expected values and continuation values, since we only need to find some values that satisfy these constraints. What we aim now is to obtain a result akin to the Bang-Bang property, where in order to check implementability of an outcome, the choices of the schedules and the equilibrium continuation value are easy.

To prove Proposition 15 we use Lemma 5: take an outcome $(d(y), b'(y), q(y))$ and we will prove it satisfies D.8 and D.7. Let us first bound the left hand side of the IC constraint: see that by definition, $(q(y), \overline{v}(y, b'(y), q(y))) \in \mathcal{E}(y, b'(y))$ as well, and hence $\overline{v}(y, s'(y), q(y)) \geq v(y)$ (being $v(y)$ feasible in the maximization problem defining $\overline{v}$). Therefore

$$u\left(b, y, d(y), b'(y), q(y)\right) + \beta\overline{v}\left(y, b'(y), q(y)\right) \geq u\left(b, y, d(y), b'(y), q(y)\right) + \beta v(y) \tag{D.12}$$

We turn our attention to the right hand side of D.12. Given $\left(b, y, \tilde{d}, \tilde{b}\right)$, condition D.11 implies there exist a pair $(\tilde{q}, \tilde{v}) \in \mathcal{E}\left(y, \tilde{b}\right)$ satisfying that constraint. Therefore

$$u\left(b, y, \tilde{d}, \tilde{b}, \tilde{q}\right) + \beta\tilde{v} \geq G\left(b, y, \tilde{d}, \tilde{b}\right) := \min_{(q,v)\in\mathcal{E}\left(y,\tilde{b}\right)} u\left(b, y, \tilde{d}, \tilde{b}, q\right) + \beta v \tag{D.13}$$

Conditions D.12 and D.13 imply that, for all $y \in Y$ and all $\tilde{d}, \tilde{b} \in \Gamma(b, y)$:

$$u\left(b, y, d(y), b'(y), q(y)\right) + \beta\overline{v}\left(y, b'(y), q(y)\right) \geq G\left(b, y, \tilde{d}, \tilde{b}\right)$$

and see that the left hand side expression is independent of $\left(\tilde{d}, \tilde{b}\right)$. Therefore, this condition holds for all $\left(\tilde{d},\tilde{b}\right) \in \Gamma\left(b,y\right)$ if and only if

$$u\left(b,y,d\left(y\right),b'\left(y\right),q\left(y\right)\right) + \beta\overline{v}\left(y,b'\left(y\right),q\left(y\right)\right) \geq \max_{\left(\tilde{d},\tilde{b}\right)\in\Gamma(b,y)} G\left(b,y,\tilde{d},\tilde{b}\right)$$

$$= \max_{\left(\tilde{d},\tilde{b}\right)\in\Gamma(b,y)} \min_{(q,v)\in\mathcal{E}\left(y,\tilde{b}\right)} u\left(b,y,\tilde{d},\tilde{b},q\right) + \beta v := \underline{U}\left(b,y\right)$$

proving D.8. For the reciprocal, see that if we choose $v\left(y\right) = \overline{v}\left(y,b'\left(y\right),q\left(y\right)\right)$ and given $\left(y,\tilde{d},\tilde{b}\right)$ we choose $\left(\tilde{q},\tilde{v}\right) \in \underset{(q,v)\in\mathcal{E}\left(y,\tilde{b}\right)}{\operatorname{argmin}} u\left(b,y,\tilde{d},\tilde{b},q\right) + \beta v,$ this choice satisfies the conditions of Lemma 5.

For the second result, if $\mathcal{E}$ is convex valued, then the function $\overline{v}$ is concave in q. If $u\left(\cdot\right)$ is also concave at q, then the function $g\left(q\right) := u\left(b,y,d,b',q\right) + \beta\overline{v}\left(y,b'\left(y\right),q\right) - \underline{U}(b,y)$ is concave in q, and then

$$q\left(y\right) \in \mathbb{Q}\left(b,y,d\left(y\right),b'\left(y\right)\right) \iff \Delta\left[q\left(y\right)\right] \geq 0 \text{ and } q\left(y\right) \in \mathcal{Q}\left(y,s'\right)$$

which are both convex sets in q. Therefore $\mathbb{Q}$ is a convex set, as we wanted to show. Compactness of $\mathbb{Q}$ comes from $\mathcal{Q}$ being a compact set (since it is the projection of $\mathcal{E}\left(y,b'\right)$, a compact set) and u, $\overline{v}$ and $\underline{U}$ being continuous functions (using Berge's Theorem of the maximum).

Equilibrium Consistency - Proof of Proposition 8

Proposition 15 is the main result of this section, characterizing all the possible equilibrium outcomes, on and off path. As a corollary, it then implies that since it holds for all histories, it clearly does for histories on some equilibrium path.

Formally, consider a history h^t which is equilibrium consistent. This implies that there exist some equilibrium profile σ such that $h^t \in \Sigma\left(\sigma\right)$. Therefore, σ also generates an equilibrium outcome $x_t = \left(d_t,b_{t+1},q_t\right) = \left(d\left(y_t\right),b'\left(y_t\right),q\left(y_t\right)\right)$ for some implementable outcome $\left(d\left(\cdot\right),b'\left(\cdot\right),q\left(\cdot\right)\right)$ satisfying conditions (1) and (2) of Proposition 8. This then means that if h^{t+1} is equilibrium consistent, we must have

$$q_t \in \mathcal{Q}\left(y_t,b'\left(y_t\right)\right) = \mathcal{Q}\left(y_t,b_{t+1}\right)$$

and

$$u\left(b_t,y_t,d_t,b_{t+1},q_t\right) + \beta\overline{v}\left(y_t,b_{t+1},q_t\right) \geq \underline{U}\left(b_t,y_t\right)$$

See that since we have not used the assumption of non-atomicity of y_t, Proposition

15 will always provide necessary conditions for equilibrium consistency. To finish the proof of Proposition 8, we need to establish sufficiency. For this, we replicate almost identically the argument in the proof of <<FIXME: SUFFICIENCY IN THE SOVEREIGN DEBT MODEL>>.

D.2 Equilibrium Consistency with Sunspots

The definition of equilibrium value correspondence is the same. However, we need to adapt the definition of enforceability with sunspots

Definition 7 (Enforceability with sunspots). Given a correspondence $W : Y \times B \rightrightarrows \mathbb{R}^k \times \mathbb{R}$, we say that a policy function $(d(y), b'(y)) \in \Gamma(b, y)$ for all y is ***enforceable with sunspots*** in W at (y_-, b) if we can find functions $\hat{v} : C(b) \times [0,1] \to \mathbb{R}$ and $\hat{q} : C(b) \times [0,1] \to \mathbb{R}^k$ such that

$$\int_0^1 \left\{u\left[b, y, d(y), b'(y), \hat{q}\left(y, d(y), b'(y)\right)\right] + \beta v\left(y, d(y), b'(y), \zeta\right)\right\} d\zeta \geq$$

$$\geq \int_0^1 \left\{u\left[b, y, \hat{d}, \hat{b}', \hat{q}\left(y, \hat{d}, \hat{b}, \zeta\right)\right] + \beta v\left(y, \hat{d}, \hat{b}, \zeta\right)\right\} d\zeta \text{ for all } \left(y, \hat{d}, \hat{b}\right) \in C(b) \tag{D.14}$$

and

$$\left(\hat{q}\left(y, d, b', \zeta\right), v\left(y, d, b', \zeta\right)\right) \in W\left(y, b'\right) \tag{D.15}$$

for all $(y, d, b', \zeta) \in C(s) \times [0,1]$.

Clearly, enforceability implies enforceability with sunspots (it is just a putting constant enforcing functions over ζ). We also need to adapt the definition of decomposability.

Definition 8 (Decomposability with Sunspots). A pair $(q, v) \in \mathbb{R}^k \times \mathbb{R}$ is ***decomposable with sunspots*** over $W(\cdot)$ at state (y_-, b) if there exist an **enforceable with sunspots** policy $(d(y), b'(y))$ with enforcing functions $(\hat{q}, \hat{v}) : C(b) \times [0,1] \to \mathbb{R}^k \times \mathbb{R}$ such that

$$v = \int_Y \int_0^1 \left\{u\left[b, y, d(y), b'(y), \hat{q}\left(y, d(y), b'(y), \zeta\right)\right] + \beta v\left(y, d(y), b'(y), \zeta\right)\right\} d\zeta dF\left(y \mid y_-, b\right) \tag{D.16}$$

and

$$q = \int_Y \int_0^1 T\left(b, y, d(y), b'(y)\right) d\zeta dF\left(y \mid y_-, b\right) \tag{D.17}$$

Let $\mathcal{W}$ be the class of compact valued correspondences from $Y \times B$ to $\mathbb{R}^{k+1}$. Define $B^s : \mathcal{W} \to \mathcal{W}$ as

$$B^s(W)(y_-, b) = \left\{(q, v) \in \mathbb{R}^{k+1} : \ (q, v) \text{ is decomposable with sunspots over } W(\cdot) \text{ at } (y_-, b)\right\}$$

We clearly have that for all correspondences W, we have $B(W) \preceq B^s(W)$ because decomposability implies decomposability with sunspots). The question we will analyze is, under what conditions over the enviroment and the correpondence W we have that actually, $B(W) = B^s(W)$.

Proposition 16. *Suppose W is a non-empty, compact and convex-valued correspondence, and u is concave in q. Then, any enforceable with sunspots policy $(d(y), b'(y))$ over W at (y_-, b) is also enforceable (without sunspots) and $\mathbf{P}_q(B(W)(y_-, b)) = \mathbf{P}_q(B^s(W)(y_-, b))$ (i.e. have same set of q's). Moreover, $B^s(W)(y_-, b) \subseteq ch(B(W)(y_-, b))$, where $ch(\cdot)$ denotes the convex hull of a set.*

Proof. Suppose $(d(y), b'(y))$ is enforceable with enforcing function $(q, v)(y, d, b', \zeta) \in W(y, b')$. Define the enforcing function $(\hat{q}, \hat{v})(y, d, b') = \int_0^1 (q, v)(y, d, b', \zeta) d\zeta$ (i.e. it is the expected value over the realization of the sunspot). Since W is convex valued, we know $(\hat{q}, \hat{v})(y, d, b') \in W(y, b')$ as well.

We will first use the fact that $(d(y), b'(y))$ is enforceable with sunspots if and only if

$$u\left[b, y, d(y), b'(y), q\left(y, d(y), b'(y), \zeta\right)\right] + \beta v\left(y, d(y), b'(y), \zeta\right) \geq$$

$$u\left[b, y, \hat{d}, \hat{b}, \hat{q}\left(y, \hat{d}, \hat{b}, \zeta\right)\right] + \beta v\left(y, \hat{d}, \hat{b}, \zeta\right) \text{ for all } \left(y, \hat{d}, \hat{b}\right) \in C(b), \zeta \in [0, 1]$$

and therefore

$$u\left[b, y, d(y), b'(y), q\left(y, d(y), b'(y), \zeta\right)\right] + \beta v\left(y, d(y), b'(y), \zeta\right) \geq \underline{U}_W(s, y) \text{ for all } y \in Y$$

following the same argument as before, where the function $\underline{U}_W$ as in D.10 well defined because W is compact valued and non-empty. Moreover, using the concavity of u in q, we know that

$$u\left[b, y, d(y), b'(y), \hat{q}\left(y, d(y), b'(y)\right)\right] + \beta \hat{v}\left(y, d(y), b'(y)\right) =$$

$$\int_0^1 u\left[b, y, \hat{d}, \hat{b}', \hat{q}\left(y, \hat{d}, \hat{b}, \zeta\right)\right] d\zeta + \beta \int_0^1 v\left(y, \hat{d}, \hat{b}, \zeta\right) d\zeta \geq \int_0^1 \underline{U}_W(b, y) d\zeta = \underline{U}_W(b, y)$$

so $(d(y), b'(y))$ is enforceable over W without sunspots. This also implies that if $(q, v) \in B^s(W)(y_-, b)$ then we can use the enforcing functions to get that

$$\hat{v} = \int_{y \in Y} \left\{u\left[b, y, d(y), b'(y), \hat{q}\left(y, d(y), b'(y)\right)\right] + \beta \hat{v}\left(y, d(y), b'(y)\right)\right\} dF(y) \geq$$

$$\int_{y \in Y} \left\{\int_0^1 u\left[b, y, \hat{d}, \hat{b}, \hat{q}\left(y, \hat{d}, \hat{b}, \zeta\right)\right] d\zeta + \beta \int_0^1 v\left(y, \hat{d}, \hat{b}, \zeta\right) d\zeta\right\} dF(y) = v$$

and

$$\hat{q} = \int_{y \in Y} T\left(b, y, d\left(y\right), b'\left(y\right)\right) dF\left(y\right) = q$$

so, for any $(q, v) \in B^s(W) \implies \exists \hat{v} \geq v : (q, \hat{v}) \in B(W)(y_-, b)$. This in particular means that $\mathcal{Q}(y_-, s) = \mathcal{Q}^s(y_-, s)$ (the set of equilibrium values take $(q, v) \in B^s(W)(y_-, b)$. The above result implies that then there exist $\hat{v} \geq v$ such that $(q, \hat{v}) \in B(W)$. Moreover, see that the extremal point of $B^s(W)(y_-, b)$ at q is the point $(q, \underline{v})$, where $\underline{v} := \min_{(q,v) \in B^s(W)(y_-,b)} v$ is well defined (from the compactness of $W(y_-, b)$). If $(q, \underline{v}) \in B(W)$ we are done: if $B(W)$ is convex valued, we know that for any $\lambda \in [0, 1]$ the element

$$(q_\lambda, v_\lambda) = \lambda(q, \hat{v}) + (1 - \lambda)(q, \underline{v}) = (q, \lambda(\hat{v} - \underline{v}) + \underline{v}) \in \text{ch}\left(B(W)(y_-, b)\right)$$

and hence, by choosing $\lambda = (v - \underline{v}) / (\hat{v} - v)$ we have $(q_\lambda, v_\lambda) = (q, v) \in B(W)(y_-, b)$.

But see that $\underline{v}$, by construction of $\underline{U}_W$ is simply $\underline{v} = \int_{y \in Y} \underline{U}_W(b, y)\, dF(y)$, which is itself also in $B(W)(y_-, b)$. This therefore means that $B^s(W)(y_-, s) \subseteq \text{ch}\left[B(W)(y_-, s)\right]$. □

Corollary 3. *Suppose $\mathcal{E}$ is convex valued and u is concave in q, then $\mathcal{E}^s = \mathcal{E}$*

Proof. We know $\mathcal{E}^s \succeq \mathcal{E}$ and that $\mathcal{E}^s$ is compact and convex valued (because $\mathcal{E}$ is convex), and that $\mathcal{E}^s = B^s(\mathcal{E}^s)$. If $B(\mathcal{E}^s)$ is convex valued, Proposition 16 then implies that $\mathcal{E}^s(y_-, b) = B(\mathcal{E}^s)(y_-, b) \subseteq \text{ch}\left(B(\mathcal{E}^s)(y_-, b)\right) = B(\mathcal{E}^s)(y_-, b)$ for all (y_-, b). This implies $\mathcal{E}^s$ is a fixed point (and in particular, a self-generating correspondence) of B, which then implies that $\mathcal{E}^s \preceq \mathcal{E}$, and hence $\mathcal{E} = \mathcal{E}^s$

Proposition 16 its corollary are the key ingredients now to show the main result of this subsection. We define an outcome with sunspots as a triple $x = (d(y), b'(y), P \mid y)$, where $(d(y), b'(y)) \in \Gamma(b, y)$ for all y, and $P \mid y \in \Delta(\mathbb{R}^k)$; i.e. $P \mid y$ is a conditional probability over q, given y. We say that the outcome x is *implementable* if $(d(y), b'(y))$ is enforceable with sunspots on $\mathcal{E}^s(y_-, b)$, and

$$(P \mid y)(A) = \Pr(q \in A \mid y) = \Pr\left(\{\zeta \in [0, 1] : q(y, \zeta) \in A\}\right) \tag{D.18}$$

for all borel sets $A \subseteq \mathbb{R}^k$ and some decomposable price function $q(y, \zeta)$ on $\mathcal{E}^s(y_-, b)$. While one could just work with an outcome being $q(y, \zeta)$, it is easier to work with the induced distribution over prices, given y. □

Proposition 17. *Suppose $\mathcal{E}(y_-, s)$ is convex valued and $u(\cdot)$ is concave in . A triple $(x(y), s'(y), P \mid y)$ is implementable with sunspots at (y_-, b) if and only if* *a. Equilibrium feasibility:*

$$\mathbf{supp}\left(P \mid y\right) \subseteq \mathcal{Q}\left(y, b'\left(y\right)\right)$$

for all y

b. Incentive compatibility for policy maker: for all $y \in Y$

$$\int \left[u\left(b, y, d\left(y\right), b'\left(y\right), \hat{q}\right) + \beta \overline{v}\left(y, b'\left(y\right), \hat{q}\right)\right] dP\left(\hat{q} \mid y\right) \geq \underline{U}\left(b, y\right)$$

Idea of Proof: Replicate the proof of Proposition 15, with outcome $\left(d\left(y\right), b'\left(y\right), q\left(y, \zeta\right)\right)$. One gets the simplified IC constraint

$$\int_0^1 \left[u\left(b, y, d\left(y\right), b'\left(y\right), q\left(y, \zeta\right)\right) + \overline{v}\left(y, b'\left(y\right), q\left(y, \zeta\right)\right)\right] d\zeta \geq \underline{U}\left(b, y\right)$$

using in this proposition, the fact that $\mathcal{E} = \mathcal{E}^s$ and hence so are the best continuation function $\overline{v}^s = \overline{v}$ and $\underline{U}^s = \underline{U}$. We then do a change of variable in the integration, since ζ only enters through $q\left(y, \zeta\right)$. This implicitely defines a measure over prices, according to 17, showing the desired result.

E Equilibrium

This Appendix characterizes the best and worst equilibrium prices in the Eaton and Gersovitz (1981) model and discusses the scope for multiplicity of equilibria and provides sufficient conditions for equilibrium multiplicity.

Preliminaries. For any history h_-^{t+1} we consider the highest and lowest prices

$$\overline{q}(h_-^{t+1}) := \max_{\sigma \in \mathcal{E}} q_m\left(h_-^{t+1}\right)$$

$$\underline{q}(h_-^{t+1}) := \min_{\sigma \in \mathcal{E}} q_m\left(h_-^{t+1}\right).$$

The best and worst equilibria turn out to be Markov equilibria and we find conditions for multiplicity. The worst SPE price is zero and the best SPE price is the one of the Markov equilibrium that is characterized in the literature of sovereign debt as in Arellano (2008) and Aguiar and Gopinath (2006). Thus, our analysis may be of independent interest, providing conditions under which there are multiple Markov equilibria in a sovereign debt model along the lines of Eaton and Gersovitz (1981).[19] The importance of this result is that it opens up the possibility of confidence crises in models as in Eaton and Gersovitz (1981). Thus, confidence crises are not necessarily a special feature of the timing in Calvo (1988) and Cole and Kehoe (2000) but a robust feature in most models of sovereign debt. The lowest price $\underline{q}(h_{-}^{t+1})$ will be attained by a fixed strategy for all histories h_{-}^{t+1}. It will deliver the utility level of autarky for the government. Thus, the lowest price is associated with the worst equilibrium, in terms of welfare. Likewise, the highest price $\overline{q}(h_{-}^{t+1})$ is associated with a, different, fixed strategy for all histories (the maximum is attained by the same σ for all h_{-}^{t+1}) and delivers the highest equilibrium level of utility for the government. Thus, the highest price is associated with the best equilibrium in terms of welfare.

E.1 Lowest Equilibrium Price and Worst Equilibrium

We start by showing that, after any history h_{-}^{t+1}, the lowest subgame perfect equilibrium price is equal to zero.

Proposition 18. *Denote by* $\mathbf{B}$ *the set of assets for the government. Under our assumption of* $\mathbf{B} \geq 0$*, the lowest SPE price is equal to zero*

$$\underline{q}(h_{-}^{t+1}) = \underline{q}(y_t, b_{t+1}) = 0$$

and associated with a Markov equilibrium that achieves the worst level of welfare.

Whenever the government confronts a price of zero for its bonds in the present period and expects to face the same in all future periods, it is best to default. There is no benefit from repaying. The proof is simple. We need to show that defaulting after every history is a subgame perfect equilibrium. Because the game is continuous at infinity, we need to show that there are no profitable one shot deviations when the government plays that strategy. Note first that, if the government is playing a strategy of always defaulting, it is effectively in autarky. In a history h_{-}^{t+1} with income y_t and debt b_t, the payoff of such a strategy is

$$u(y_t) + \frac{\beta}{1-\beta}\mathbb{E}_{y'|y_t}u(y').$$

[19]Our result complements the results in Auclert and Rognlie (2014); their paper shows uniqueness in the Eaton and Gersovitz (1981) when the government can save.

Note also that, a one shot deviation involving repayment today has associated utility of

$$u(y_t - b_t) + \frac{\beta}{1-\beta}\mathbb{E}_{y'|y_t}u(y').$$

Thus, as long as b_{t+1} is non-negative, a one shot deviation of repayment is not profitable. So, autarky is an SPE with an associated price of debt equal to zero.

Discussion

The equilibrium does not require conditioning on the past history, i.e. it is a Markov equilibrium. Notice, as well, that we have not yet introduced sunspots. Thus, multiplicity does not require sunspots. Sunspots may act as a coordinating device to select a particular continuation equilibrium. We introduce sunspots in Section 4.

Things are different when the government is allowed to save before default and the punishment is autarky, including exclusion from saving. Under this combination of assumptions, the government might want to repay small amounts of debt to maintain the option to save in the future. As a result, autarky is no longer an equilibrium and a unique Markov equilibrium prevails, as shown by Auclert and Rognlie (2014).

A similar result holds when there are output costs of default. The sufficient condition for multiplicity will be that for the government is dominant to default on any amount of debt that it is allowed to hold, for all $b \in \mathbf{B}$. With default costs, the value of defaulting is lower. Thus, we need to increase the static gain of defaulting for any history. A sufficient condition would then be that $\mathbf{B} > 0$. The lower bound on debt will be increasing in the magnitude of the output costs of default.

E.2 Highest Equilibrium Price and Best Equilibrium

We now characterize the best subgame perfect equilibrium and show that it is the Markov equilibrium studied by the literature of sovereign debt. To find the worst equilibrium price, it was sufficient to use the definition of equilibrium and the one shot deviation principle. To find the best equilibrium price it will be necessary to find a characterization of equilibrium prices. Denote by $\overline{\mathrm{W}}(y_t, b_{t+1})$ the highest expected equilibrium payoff if the government enter period $t+1$ with bonds b_{t+1} and income in t was y_t. The next lemma provides a characterization of equilibrium outcomes.

Lemma 6. $x_{t-} = (q_{t-1}, d_t(\cdot), b_{t+1}(\cdot))$ *is a subgame perfect equilibrium outcome at history* h^t_- *if and only in the following conditions hold:*

a. Price is consistent

$$q_{t-1} = \frac{1}{1+r^*}(1 - \int d_t(y_t) dF(y_t \mid y_{t-1})), \tag{E.1}$$

b. IC government

$$(1 - d(y_t)) \left[u(y_t - b_t + \overline{q}(y_t, b_{t+1}) b_{t+1}) + \beta \overline{\mathbb{W}}(y_t, b_{t+1}) \right] + d(y_t) V^d(y_t) \geq V^d(y_t). \tag{E.2}$$

The proof is omitted; it is a particular case of the main result for the model without sunspots. Condition (E.1) states that the price q_{t-1} needs to be consistent with the default policy $d_t(\cdot)$. Condition (E.2) states that a policy $d_t(\cdot), b_{t+1}(\cdot)$ is implementable in an SPE if it is incentive compatible given that following the policy is rewarded with the best equilibrium and a deviation is punished with the worst equilibrium. The argument in the proof follows Abreu (1988). These two conditions are necessary and sufficient for an outcome to be part of an SPE.[20]

Markov Equilibrium. We now characterize the Markov equilibrium that is usually studied in the literature of sovereign debt. The value of a government that has the option to default is given by

$$\overline{\mathbb{W}}(y_-, b) = \mathbb{E}_{y|y_-} \left[\max \left\{ \overline{V}^{nd}(b, y), V^D(y) \right\} \right]. \tag{E.3}$$

This is the expected value of the maximum between not defaulting $\overline{V}^{nd}(b, y)$ and the value of defaulting $V^D(y)$. The value of not defaulting is given by

$$\overline{V}^{nd}(b, y) = \max_{b' \geq 0} u(y - b + q(y, b')b') + \beta \overline{\mathbb{W}}(y, b'). \tag{E.4}$$

That is, the government repays debt, obtains a capital inflow (outflow), and from the budget constraint consumption is given by $y - b + q(y, b')b'$; next period has the option to default b' bonds. The value of defaulting is

$$V^d(y) = u(y) + \beta \frac{\mathbb{E}_{y'|y} u(y')}{1 - \beta}, \tag{E.5}$$

[20]Note that at any history (even on those *inconsistent* with equilibria) SPE policies are a function of only one state: the debt that the government has to pay at time t (b_t). There are two reasons for this. First, the stock of debt summarizes the physical environment. Second, the value of the worst equilibrium only depends on the realized income.

and is just the value of consuming income forever. These value functions define a default set

$$D(b) = \left\{ y \in Y : \overline{V}^{nd}(b,y) < V^d(y) \right\}. \tag{E.6}$$

A Markov Equilibrium (with state b, y) is a: set of policy functions $(c(y,b), d(y,b), b'(y,b))$, a bond price function $q(b')$ and a default set $D(b)$ such that: $c(y,b)$ satisfies the resource constraint; taking as given $q(y,b')$ the government bond policy maximizes $\overline{V}^{nd}$; the bond price $q(y,b')$ is consistent with the default set

$$q(y,b') = \frac{1 - \int_{D(b')} dF(y' \mid y)}{1+r}. \tag{E.7}$$

The next proposition states that the best Markov equilibrium is the best subgame perfect equilibrium.

Proposition 19. *The best subgame perfect equilibrium is the best Markov equilibrium.*

Proof. From lemma 6, the value of the best equilibrium is the expectation with respect to y_t, given y_{t-1}, and is given by

$$\max_{d_t, b_{t+1}} (1 - d_t) \left[u(y_t - b_t + \overline{q}(y_t, b_{t+1}) b_{t+1}) + \beta \overline{\mathbb{W}}(y_t, b_{t+1}) \right] + d_t V^d(y_t).$$

Note that this is equal to the left hand side of (E.3). The key assumption for the best subgame perfect equilibrium to be the best Markov equilibrium is that the government is punished with permanent autarky after a default. □

E.3 Multiplicity

Given that the worst equilibrium is autarky, a sufficient condition for multiplicity of Markov equilibria will be any condition that guarantees that the best Markov equilibria has positive debt capacity, a standard situation in quantitative sovereign debt models. In general some debt can be sustained as long as there is enough of a desire to smooth consumption. This will motivate the government to avoid default, at least for small debt levels. The following proposition provides a simple sufficient condition for this to be the case. Define $\mathcal{V}^{nd}(b, y; B, \frac{1}{1+r})$ as the value function when the government faces the risk free interest rate $q = \frac{1}{1+r}$ and some borrowing limit B as in a standard Bewley incomplete market model. The government has the option to default. This value is not an upper bound on the possible values of the borrower because default introduces state contingency and might be valuable. Our next proposition, however, establishes conditions under which default does not take place.

Proposition 20. *Suppose that for all $b \in [0, B]$ and all $y \in \mathbf{Y}$, the*

$$\mathcal{V}^{nd}(b, y; B, \frac{1}{1+r}) \geq u(y) + \beta \mathbb{E}_{y'|y} V^d(y'). \tag{E.8}$$

Then there exist multiple Markov equilibria.

Proof. If the government is confronted with $q = \frac{1}{1+r}$ for $b \leq B$ condition (E.8) ensures that it will not want to default after any history. This justifies the risk free rate for $b \leq B$. A SPE can implicitly enforce the borrowing limit $b \leq B$ by triggering to autarky and setting $q_t = 0$ if ever $b_{t+1} > B$. Since the debt issuance policy is optimal given the risk free rate, we have constructed an equilibrium. This proves there is at least one SPE sustaining strictly positive debt and prices. The best equilibrium dominates this one and is Markov, as shown earlier, so it follows that there exists at least one strictly positive Markov equilibrium. Finally, note that we only require checking this condition (E.8) for small values of B. However, the existence result then extends an SPE over the entire $\mathbf{B} = [0, \infty)$. Indeed, it is useful to consider small B and take the limit, this then requires checking only a local condition. The following example illustrates this condition. □

Example. Suppose there are two income shocks y_L and y_H that follow a Markov chain (a special case is the i.i.d. case). Denote by λ_i the probability of transitioning from state i to state $j \neq i$. We will construct an equilibrium where debt is risk free, the government goes into debt B and stays there as long as income is low, and repays debt and remains debt free when income is high. Conditional on not defaulting, this bang bang solution is optimal for small enough B. To investigate whether default is avoided, we must compute the values

$$\begin{aligned}
v_{BL} &= u(y_L + (R-1)B) + \beta\left(\lambda_L v_{BH} + (1-\lambda_L) v_{BL}\right) \\
v_{BH} &= u(y_H - RB) + \beta\left(\lambda_H v_{0L} + (1-\lambda_H) v_{0H}\right) \\
v_{0L} &= u(y_L + B) + \beta\left(\lambda_L v_{BH} + (1-\lambda_L) v_{BL}\right) \\
v_{0H} &= u(y_H) + \beta\left(\lambda_H v_{0L} + (1-\lambda_H) v_{0H}\right)
\end{aligned}$$

where $R = 1 + r$. Write the solution to this system as a function of B. To guarantee that the government does not default in any state, we need to check that $v_{BL}(B) \geq v^{aut}$, $v_{BH}(B) \geq v^{aut}$, $v_{0L}(B) \geq v_L^{aut}$ and $v_{0H}(B) \geq v_H^{aut}$ (some of these conditions can be shown to be redundant).

Lemma 7. *A sufficient condition for* $v_{BL} \geq v^{aut}, v_{BH} \geq v^{aut}, v_{0L} \geq v_L^{aut}, v_{0H} \geq v_H^{aut}$ *to hold for some* $B > 0$ *is* $v'_{BL}(0) > 0, v'_{BH}(0) > 0$. *When* $\lambda_H = \lambda_L = 1$ *this simplifies to* $\beta u'(y_L) > Ru'(y_H)$.

Note that the simple condition with $\lambda_H = \lambda_L = 1$ is met whenever u is sufficiently concave or if β is sufficiently close to 1. These conditions ensure that the value from consumption smoothing is high enough to sustain debt.

Proof. (Lemma 7) Note that we can rewrite the system of Bellman equations as

$$A.v(B) = u(B)$$

Thus, a condition in primitives is

$$v'(0) = A^{-1}u'(0) \geq 0$$

For the special case where $\lambda = 1$, note that

$$\begin{aligned} v_{BH} &= \frac{1}{1-\beta^2}\left(u(y_H - RB) + \beta u(y_L + B)\right) \\ v_{0L} &= u(y_L + B) + \beta v_{BH} \end{aligned}$$

Then, $v'_{BH}(0) > 0$ implies that $v'_{0L}(0) > 0$. A sufficient condition is $\beta u'(y_L) > Ru'(y_H)$. The intuition is that, the government is credit constrained in the low state, with no debt, and is willing to tradeoff and have lower consumption in the high state. □