Entrepreneurial Investment Dynamics And The Wealth Distribution

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Abstract

This paper studies entrepreneurial investment behavior, and investigates the extent to which entrepreneurial risk taking contributes to wealth inequality. Using representative firm level data on startups, I find evidence that an entrepreneur's capital investment is illiquid. To quantify this, I construct an endogenous occupational choice model, where an entrepreneur holds two assets: liquid bonds and illiquid capital. I find that entrepreneurs lose about 68% to 79% of their investment when selling capital. Relative to a counter-factual economy where capital is fully liquid, high productivity entrepreneurs operate smaller firms while low productivity entrepreneurs operate larger firms; moreover, high productivity entrants delay entry, while low productivity entrepreneurs delay exit. This implies that capital is allocated towards low productivity entrepreneurs, and results in a higher population of wealthy but low productivity entrepreneurs. Consequently, welfare and total factor productivity is substantially lower relative to the frictionless economy; moreover, fiscal policy that addresses capital illiquidity can considerably improve welfare. Finally, I find that the benchmark economy has lower wealth inequality than the counter-factual economy, driven primarily by high productivity entrepreneurs earning a lower ex-post return on their investment. This finding suggests that macroeconomic models of entrepreneurship that ignore illiquidity might over-state the contribution of entrepreneurial risk taking to the dispersion of wealth in the data. (*JEL* D31, E21, E22, L26)

*Duke University. email: eugene.tan@duke.edu. Acknowledgements: I am indebted to my advisors, Andrea Lanteri and Craig Burnside, for their tireless support and advice in writing this paper. I would also like to thank the rest of my committee members Matthias Kehrig, Cosmin Ilut, and Kyle Jurado, as well as the rest of the macroeconomics group at Duke for their helpful comments and advice. I have also benefited substantially from conversations with Sungki Hong, B. Ravikumar, Juan Sanchez, Daniel Xu and Chen Yeh. I would also like to thank the participants at the Fall 2017 Midwest Macro conference, 2017 SEA conference, and the 2018 NASMES conference for helpful discussion and comments. In addition, I would like to thank the St Louis Federal Reserve for hosting me during the writing of this paper. Finally, I would like to thank the Kauffman Foundation for providing sponsorship to access the Kauffamn Firm Survey at the NORC enclave. All errors are my own.

1 Introduction

Ownership of a business is highly correlated with being wealthy - a fact that has been well documented in the earlier literature[1]. However, the returns to entrepreneurship is much less well understood[2]. As such, the question of whether entry into entrepreneurship is the driver of heterogeneity in wealth accumulation, or whether the wealthy simply disproportionately pursue entrepreneurship, remains an open question. The goal of this paper is to (a) provide direct empirical evidence of entrepreneurial investment dynamics, (b) rationalize their behavior using a macroeconomic model of entrepreneurship, and (c) through the lens of the model, discuss whether entrepreneurship is a key determinant in generating the large wealth dispersion in the real economy.

In this paper, I present two key observations of entrepreneurial investment dynamics from the Kauffman Firm Survey (KFS), a representative panel survey of newly formed firms. Firstly, I find that the distribution of the rates of return to capital, as proxied by the (log) average revenue product of capital (henceforth just ARPK), of young entrepreneurial firms is highly left skewed. Secondly, firms also exhibit large autocorrelation in ARPK, with greater persistence in the left tail of the cross-section distribution than the right.

Taken together, this suggests that the returns to entrepreneurship is generally very low; crucially, the downside risk can be much larger than the upside risk, as evinced by the left skewness of the distribution. Moreover, entrepreneurs with low rates of return to capital appear to persistently deliver low rates of return, and at a higher persistence than their high performing counterparts.

The evidence runs counter to the current paradigm of entrepreneurial firm dynamics, which has focused largely on financial constraints driven by collateralized borrowing. In that framework, high productivity entrepreneurs face investment constraints, and thus operate sub-optimally small firms relative to their productivity. This increases the population of entrepreneurs with high returns to capital, generating a right-skewed distribution of returns. Moreover, as financially constrained entrepreneurs are also more likely to be constrained for several periods, this elevated returns to capital will persist for several periods. In contrast, as entrepreneurs do not face downsizing frictions in that paradigm, poor performing entrepreneurs downsize within a period, reverting quickly back to the mean. Therefore, the collateral constraints model predicts that the right tail is more persistent than the left, a counterfactual prediction given my findings[3].

[1] c.f. Quadrini (1999), Cagetti and De Nardi (2006), Kuhn and Rios-Rull (2015)

[2] For instance, there is still a debate regarding whether entrepreneurial undertaking has higher or lower returns than labor work, c.f. Hamilton (2000), Manso (2015), Dillon and Stanton (2017)

[3] The theoretical proof of this effect is provided in the appendix

In order to explain these findings, I construct a model that incorporates partial irreversibility of entrepreneurial capital[4]. I build on the framework of Cagetti and De Nardi (2006), and extend their endogenous occupational choice incomplete markets model by differentiating between liquid assets (bonds) and illiquid assets (entrepreneur's capital). Due to an asymmetry in the purchase and resale price of capital, rather than downsizing or exiting instantly when hit by a bad shock, entrepreneurs adopt a "wait-and-see" attitude. This *illiquidity effect*, originating from a real options value effect, leads them to operate larger than optimal firm sizes[5], generating a naturally left-skewed distribution of log ARPK. Moreover, this also creates persistence in the left tail as entrepreneurs wait out the transient bad shock.

To evaluate the quantitative significance of this friction, I calibrate the model to identifying micro-level moments drawn from the KFS. I find that for every unit of capital sold, the entrepreneur loses 68% of the underlying real value of the capital asset. Upon exit, the entrepreneur faces an additional 31% write down on her assets[6].

A direct consequence of these large downsizing frictions is capital and talent mis-allocation relative to a counter-factual economy where capital is fully liquid. Along the intensive margin, low productivity entrepreneurs operate larger firms while high productivity entrepreneurs operate smaller firms; the latter effect arising due to a fall in the put option value of capital. Along the extensive margin, high productivity entrants delay entry while low productivity incumbents delay exit. As a result, wealthier entrepreneurs have lower average productivity, while poorer entrepreneurs have higher average productivity. This outcome leads to substantial TFP and welfare loses relative to the counter-factual economy: Steady state aggregate TFP is 11% higher, while average welfare, factoring transitional dynamics, is 4.4% higher in consumption equivalent terms.

In addition, capital illiquidity also magnifies the effect of un-insurable capital income risk, leading entrepreneurs to accumulate more liquid assets relative to the counter-factual economy[7]. Consequently, fiscal policy that provides consumption insurance against this illiquidity risk can improve economic efficiency. For instance, I find that a policy that directly increases the resale value of capital, funded by a proportional tax on bond returns,

[4]Partial irreversibility could arise naturally because the resale price of "used" capital is lower than the purchase price of new capital, as discussed in Lanteri (2016)

[5]"Optimal" here refers to the case where the (expected) marginal product of capital is set equal to the sum of the interest rate and the user cost of capital.

[6]This is not the only source of friction in the model that makes entrepreneurial capital illiquid. To match the data, the model also includes a proportional investment fixed cost of about 3.4%. While these frictions might seem large, they are comparable to prior research in both the households (c.f. Diaz and Luengo-Prado (2010), Kaplan and Violante (2014) and Berger and Vavra (2014)) and firm dynamics (c.f. Cooper and Haltiwanger (2006), Bloom (2009), and Gilchrist et al (2014) literature.

[7]This effect is reminiscent of the question studied in Aiyagari (1994), where the author studies whether precautionary savings leads to savings in excess of the complete markets representative agent model.

can improve economic outcomes in terms of welfare and total factor productivity.

The results have important implications for both the applied microeconomics and macroeconomics research on entrepreneurship. On the applied microeconomics front, recent research have exploited variations in asset prices (typically house prices) to test the presence of collateral constraints. A general strategy for this literature is to regress some macroeconomic outcome (such as startup rates) on house price variations[8], and attribute the correlation to collateral constraints. In this paper, the illiquidity effect has qualitatively similar effects to a collateral constraint; for instance, negative shocks to the resale value of capital depresses entry due to a fall in the option value of capital (and hence entrepreneurship). Therefore, to the extent that house or asset prices covary positively with resale capital prices, the results in the preceding literature might be contaminated by the illiquidity effect.

On the macroeconomics front, relative to a standard Aiyagari (1994) model of labor income risk, entrepreneurship in a heterogeneous household model improves the model fit to the top wealth shares of the US economy (the top 1%, 5% and 10%). However, the model wealth distribution is more egalitarian than the real economy (Gini coefficient of about 0.68; the data is about 0.8). This result arises because the illiquidity effect depresses the overall returns to capital: The same effect that allows the model to capture the left skewness of the ARPK distribution also generate substantial numbers of low returns entrepreneurs. As a consequence, wealth dispersion is lower in the benchmark economy. In contrast, a counterfactual “frictionless" economy has substantially higher wealth dispersion and matches the empirical wealth distribution closer; for instance, the Gini is about 0.76. This implies that macroeconomic models of entrepreneurship that ignore illiquidity will overstate the contribution of entrepreneurship to wealth dispersion.

1.1 Related literature

This paper contributes to three main strands of the macroeconomics literature: Household dynamics, entrepreneur dynamics, and firm dynamics.

Household dynamics This paper is most related to the research agenda framed by Quadrini (2000) and Cagetti and De Nardi (2006), which study the contribution of entrepreneurship to wealth inequality. In that framework, the entrepreneur's business wealth and savings wealth are perfect substitutes. In this paper, I generalize their framework to one that distinguishes between business wealth and savings wealth. Business wealth (entrepreneurial capital) is illiquid and subject to transaction costs. Unlike in Cagetti and

[8]See for example, Adelino et al (2015) or Schamlz et al (2017). Schott (2015) studies a similar mechanism, but explores it in a macroeconomic context.

De Nardi (2006), I find that entrepreneurial risk taking alone does not account for the full dispersion of wealth; as capital is illiquid, overall returns to entrepreneurship are heavily depressed, resulting in lower wealth dispersion compared to a frictionless economy.

I also add to that research agenda by contributing direct empirical evidence regarding the investment behavior of entrepreneurs, and calibrating my model directly to data on entrepreneurs. In contrast, Quadrini (2000) and Cagetti and De Nardi (2006) calibrate their model to match the income process of entrepreneurs identified from the Panel Survey of Income Dynamics (PSID) or Survey of Consumer Finances (SCF). In the latter strategy, one is generally unable to identify who constitutes entrepreneurs, as well as what portion of the entrepreneur's income should be considered capital income, and what is consider labor income. Moreover, the book value of capital is not available in household surveys, making it difficult to assess the model's ability in matching actual investment behavior.

In addition, this paper is connected to a growing literature on rates of returns heterogeneity and its effect on the wealth distribution. Recent empirical work has showed that highly persistent and heterogeneous returns to wealth can largely explain the income and wealth inequality in the economy[9], while recent theoretical work by Benhabib et al (2015) provide the theoretical underpinning to explain why this particular mechanism is so successful in matching the wealth distribution. This paper adds to the literature that uses entrepreneurship as a micro-foundation to understand heterogeneity in rates of return among households.

More broadly speaking, this paper also adds to the growing literature on using quantitative heterogeneous agent macroeconomic models to understand the determinants of the empirical wealth distribution. This literature, dating back to Aiyagari (1994), has largely documented that a model of only exogenous labor income risk cannot, in general, match the wealth distribution without resorting to a largely counter-factual labor income process[10]. This paper adds to the research agenda of using entrepreneurship to augment the basic Aiyagari model to understand wealth inequality.

Entrepreneur dynamics This paper is relevant to a broad range of research that studies entrepreneurial investment dynamics and issues associated with it, such as capital taxation, estate taxation, or financial frictions[11]. This paper finds that the entrepreneur's entry/exit

[9] c.f. Saez and Zucman (2014) for tax returns data from the United States, Bach et al (2016) for data on Sweden, and Fagereng et al (2016) for data on Norway.

[10] See in particular Castaneda et al (2003) and Benhabib et al (2016) for discussions on how calibrated labor income risk are typically counterfactual to the real amount of risk households face. Hurst et al (2010) also provides a discussion on the importance of directly modeling business ownership when studying the wealth distribution.

[11] For capital income taxation, see for instance Kitao (2008) and Panousi and Angeletos (2012); for estate taxation, see for example Cagetti and De Nardi (2009); for financial frictions and collateral constraints, see

and investment decision in a two asset framework differs starkly from the one asset model. For instance, I find that the illiquidity effect endogenously induces entrepreneurs to run smaller firms or delay entry. This replicates the effect of collateral constraints, but is a uniquely different channel that cannot be addressed by fiscal policy targeted towards addressing financing constraints. Moreover, this paper raises an identification challenge for the applied microeconomics literature that relates financing constraints, via house price variations, to macroeconomic outcomes through an entrepreneurship channel[12]. Taken together, this paper motivates the use of a liquid-illiquid asset framework for future research that is concerned with understanding entrepreneurial investment dynamics.

Firm dynamics This paper relates to the broad literature that studies the relationship between real and financial frictions and capital mis-allocation[13]. Here, I developed a simple diagnostic to disentangle financial frictions (driven by collateral constraints) and illiquidity effects driven by partial irreversibility. Specifically, I show that the dynamic behavior of the ARPK distribution is critical in helping distinguish between investment or disinvestment frictions. To the best of my knowledge, this is the first paper that has exploited this aspect of the ARPK distribution.

This paper also adds to that literature by providing a bridge between understanding the dispersion of wealth and the dispersion of marginal products of capital. In this paper, I connect the mis-allocation of capital back to the wealth distribution through the lens of a heterogeneous agents model that merges an incomplete markets model with a firm dynamics model.

The rest of this paper is divided as follow. In section 2, I give a summary and description of the data and stylized moments. Following that, in section 3, I describe the model, as well as give a short discussion on why standard investment models are unable to explain this data. In section 4, I discuss in detail the calibration strategy; in particular, I explain how my model can discriminate between illiquidity frictions and collateral constraints. Section 5 discusses the model results of the calibration exercise. Section 6 extends the model to study the impact of illiquidity frictions on fiscal policy instruments and dynamic outcomes, and show why modeling illiquidity in this framework is important. Section 7 concludes the paper.

for instance Evans and Jovanovic (1989), Buera et al (2011), Buera and Shin (2013), Adelino et al (2015), Bassetto et al (2015).

[12]See for instance, Adelino et al (2015) and Schmalz et al (2017).

[13]See for instance, Beura et al (2011), Buera and Shin (2013), Asker et al (2014), Midrigan and Xu (2014).

2 Stylized Facts

In this section, I first briefly describe the data source. I then report the stylized facts regarding entrepreneurial investment, focusing specifically on the distribution and dynamic behavior of the average revenue of product of capital and investment rates. The data construction is relegated to the appendix.

2.1 Data universe

The data is drawn primarily from the restricted version of the Kauffman Firm Survey (KFS). The KFS is a single cohort panel survey, consisting solely of entrepreneurial firms that were formed in the year 2004, and tracked through 2011. The universe of firms considered for survey inclusion was all newly registered firms in 2004 from the Dun and Bradstreet database. The subset of firms in this universe that was included in the survey must then satisfy the following conditions:

1. Business was started as independent business, or by the purchase of an existing business, or by the purchase of a franchise in the 2004 calendar year.

2. Business was *not* started as a branch or a subsidiary owned by an existing business, that was inherited, or that was created as a not-for-profit organization in the 2004 calendar year.

3. Business had a valid business legal status (sole proprietorship, limited liability company, subchapter S corporation, C-corporation, general partnership, or limited partnership) in 2004.

4. Business reported at least one of the following activities:

    (a) Acquired employer identification number during the 2004 calendar year

    (b) Organized as sole proprietorships, reporting that 2004 was the first year they used Schedule C or Schedule C-EZ to report business income on a personal income tax return

    (c) Reported that 2004 was the first year they made state unemployment insurance payments

    (d) Reported that 2004 was the first year they made federal insurance contribution act payments

As one can observe, the inclusion criterion is very strict, and relates largely to the common idea of an entrepreneur. For a more complete description of the broad characteristics of the data, especially on the firm's balance sheet, I refer the reader to Robb and Robinson (2014).

A big advantage of utilizing the KFS in this paper is the ability to directly observe the investment dynamics of new and privately-owned firms (via the balance sheet of the firm). As a result, the findings in the KFS can be directly mapped into a model that jointly describes heterogeneous household and firm dynamics. In contrast, the prior literature that has studied entrepreneurial investment dynamics has utilized either household surveys such as the Panel Survey of Income Dynamics (PSID), or datasets comprising large firms (such as Compustat).

Both methods have several disadvantages relative to the KFS for understanding entrepreneurship and its relation to the wealth distribution. In the context of household surveys, important information on the returns on firm assets, such as the marginal product of capital, is not computable as households do not report the book value of the firm's asset[14]. Moreover, the sample has substantial attrition. As a result, it is difficult to draw substantive inference on the dynamics of capital accumulation (or de-cumulation), which requires a long panel survey.

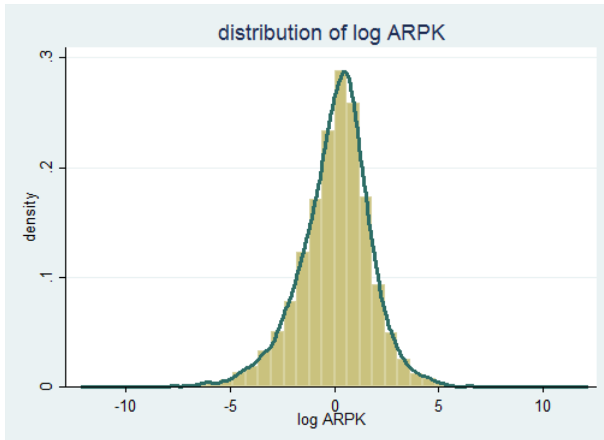
For the literature that uses datasets on large firms, the nature of large firms mean that it is generally possible for the primary owner or manager to fully diversify the idiosyncratic risks of the firm. As a result, observations drawn from this data is not directly applicable to research on new firms. Moreover, this makes data on large firms an inappropriate data source for a incomplete markets household model, which is a necessary model primitive for understanding the wealth distribution.

Finally, there is also a small but growing literature that uses Census micro-data to study new firm behavior [15]. Unfortunately, the census micro-data in general does not provide much information on the balance sheet of private firms. As a result, while it is possible to match a primary owner to her/his firm, we are not able to study the return on firm assets using these data sets.

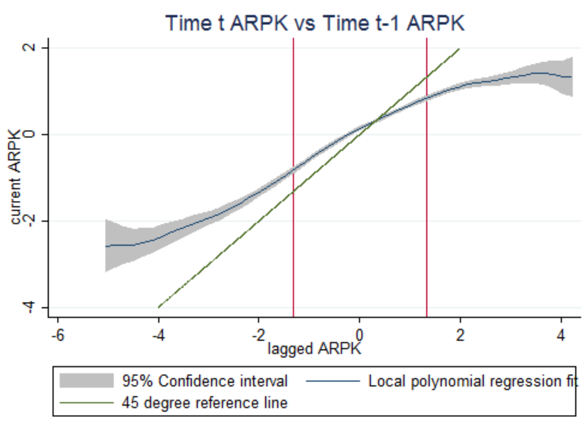
[14]The PSID typically asks the households for their perceived value of their firm. A typical question, such as the 2013 wave, asks respondents the following: "How much is your part of the business worth, that is, how much would it sell for".

[15]for instance, the Longitudinal Business Database (LBD) or the Longitudinal Employer-Household Dynamics (LEHD)

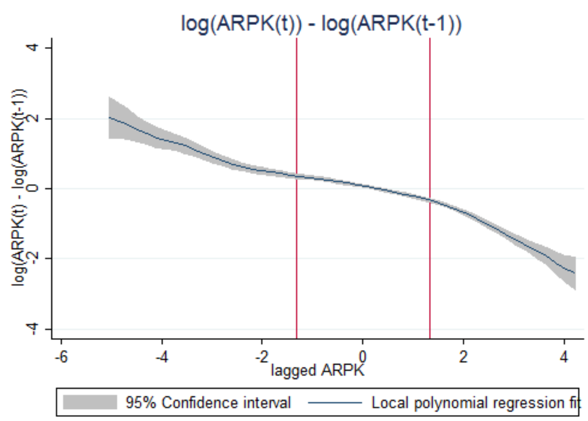
2.2 Distribution and Dynamics of the Average Revenue Product of Capital



(a) Distribution of log ARPK.



(b) Local polynomial fit of log ARPK at time t v.s log ARPK at time $t-1$. The two vertical lines are reference lines for the first and fifth quintile of the cross-sectional distribution.



(c) Local polynomial fit of the first difference of log ARPK at time t vs time log ARPK at time $t-1$. The two vertical lines are reference lines for the first and fifth quintile of the cross-sectional distribution.

Figure 1: Top: The cross-sectional distribution of log ARPK. Bottom: The dynamics of log ARPK.

2.2.1 Cross-sectional moments

Figure 1a shows the cross-sectional distribution of log $ARPK$ in the sample population, while table 1 below documents some key cross-sectional moments of this distribution. Visually, one sees that the distribution is left-skewed. Moreover, just as the prior literature has documented in the context of other industries or countries, log $ARPK$ is very dispersed.

Mean	Standard Deviation	Skewness	Kurtosis
0.00	1.75	-0.39	5.7

Table 1: Selected moments from distribution of $\log ARPK$

2.2.2 Dynamics of $\log ARPK$

Figure 1b plots a local polynomial fit of the current period log $ARPK$ against the last period log $ARPK$. A 45° reference line is overlaid on the plot. Figure 1c plots a local polynomial fit of the first difference of log $ARPK$ (i.e. $\log ARPK_t - \log ARPK_{t-1}$) against last period log $ARPK$.

What do these two graphs tell us about the dynamics of the average revenue product of capital? Note that if $ARPK$ is perfectly persistent, then all the data points should line up on the 45° line in figure 1b; along a similar vein, figure 1c would reflect a flat horizontal line at 0. In contrast, if $ARPK$ was *i.i.d.*. then the local polynomial fit in figure 1b would feature a flat horizontal line at 0. As we can see from the graphs, $ARPK$ is highly persistent in the sample, but there is still some significant churning.

We can also tell from these graphs that this persistence is asymmetric, being greater on the left tail than the right. For instance, looking at figure 1b, a firm that starts out with $\log ARPK = -4$ (i.e. in the left tail) will, on average, end up with $\log ARPK \approx -2.5$ in the next period (equivalent, the difference would be about 1.5, when view from the perspective of figure 1c). In contrast, a firm that starts out with $\log ARPK = 4$ will end up, on average, with $\log ARPK \approx -1.5$ in the next period.

To quantify this asymmetry, I document this fact in two ways: (1) By estimating a transition matrix for $\log ARPK$, and (2) by estimating the conditional autocorrelation of $\log ARPK$.

Transition matrix Here, I estimate the persistence in relative rankings by first binning the firms into quintiles (estimated at the industry level) on a year-by-year basis[16], and then estimating the transition matrix M (across quintiles) for the entire sample. I report the estimated transition matrix below, with standard errors in parenthesis below the estimated value:

[16] I have also estimated the transition matrix by fixing the quintiles to the bins in 2004. The results are very similar. Refer to the appendix for additional robustness checks.

$$\hat{M} = \begin{bmatrix} \underset{(0.017)}{0.58} & \underset{(0.015)}{0.24} & \underset{(0.010)}{0.096} & \underset{(0.007)}{0.046} & \underset{(0.007)}{0.042} \\ \underset{(0.014)}{0.23} & \underset{(0.016)}{0.39} & \underset{(0.014)}{0.22} & \underset{(0.011)}{0.11} & \underset{(0.007)}{0.057} \\ \underset{(0.010)}{0.093} & \underset{(0.014)}{0.21} & \underset{(0.016)}{0.37} & \underset{(0.014)}{0.22} & \underset{(0.008)}{0.010} \\ \underset{(0.008)}{0.058} & \underset{(0.010)}{0.11} & \underset{(0.013)}{0.21} & \underset{(0.016)}{0.37} & \underset{(0.014)}{0.26} \\ \underset{(0.007)}{0.057} & \underset{(0.009)}{0.073} & \underset{(0.011)}{0.12} & \underset{(0.015)}{0.28} & \underset{(0.017)}{0.47} \end{bmatrix}$$

The reader should keep in mind that the columns reflect "today's" relative rankings, and the rows reflect "tomorrow's" relative rankings. As such, entry (i, j) reflect the probability that a firm that was in quantile i transitions to quantile j tomorrow.

To better clarify the import of these results, I direct the reader to look at the values along the diagonal. The values along the diagonal reflect the probability of staying in quantile q, given that you were in quantile q yesterday. As such, a *null* hypothesis of no persistence should reflect that this probability is 0.2. In contrast, looking across all the values along the diagonal, we see that all values are statistically different from (greater than) 0.20, thus reflecting what we already saw in the graph earlier.

In addition, we also see that there is higher conditional persistence at the tails than in the center of the distribution, and that this difference is also statistically significant, indicating that there is greater churning around the center of the distribution than at the tails.

Finally, there is also a significant asymmetry in the persistence of $\log ARPK$ in the left and right tail - the persistence of $\log ARPK$, conditional on being in the first quintile, is statistically and economically significantly larger than the persistence of $\log ARPK$, conditional on being in the last quintile. This again reflects the qualitative description earlier.

Conditional autocorrelation Here, I estimate the conditional autocorrelation of $\log ARPK$. A simple way to study this is to form a regression of the form

$$\log ARPK_{i,t} = \alpha + \sum_{q=1}^{5} \rho_q \log ARPK_{i,t-1} + \varepsilon_{it}$$

where α is the intercept term, and ρ_q is a coefficient that depends on the $\log ARPK$ quantile q that the firm is in currently. As table 2 shows, ρ_1, which is the autocorrelation of $\log ARPK$ when the firm is in quantile 1, is much larger than ρ_5. A Wald test also rejects the null that the two estimates are equivalent, as such supporting the evidence that there is greater persistence in $\log ARPK$ at the bottom quintile relative to the top quintile.

	Dependent variable: $\log ARPK_{i,t}$
a	0.075 (0.037)
ρ_1	0.643 (0.034)
ρ_2	0.897 (0.058)
ρ_3	0.672 (0.109)
ρ_4	0.697 (0.052)
ρ_5	0.443 (0.035)

Table 2: Regression result of $\log ARPK_{i,t} = a + \rho_q \log ARPK_{i,t-1} + \varepsilon_{it}$. $q = 1$ refers to the first quintile, $q = 2$ the second quintile, and so on. Standard errors in parentheses.

2.3 Distribution of investment rates

Figure 2 shows the distribution of investment rates, and Table 3 reports some of the key moments associated with this. The investment distribution presented in Figure 2 is winsorized at the 1st and 95th percentile for clarity of presentation, while the moments in Table 3 are constructed for the distribution winsorized at the 1st and 99th percentile.

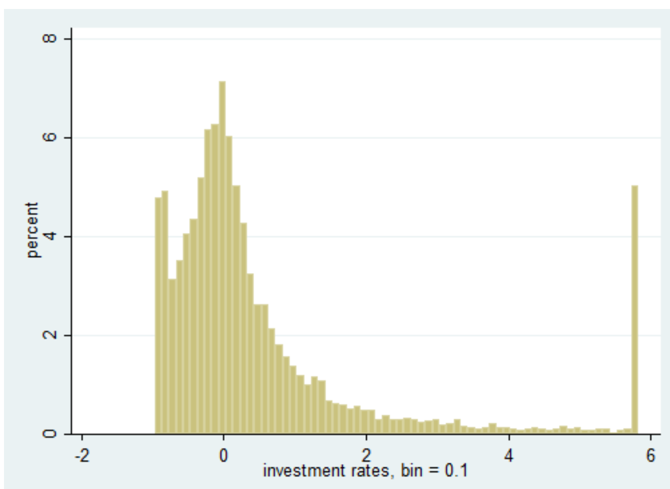


Figure 2: Distribution of investment rates (winsorized at 5th and 95th percentile)

Moments of $\frac{I}{K}$	Value
mean	1.17
standard deviation	4.80
5th percentile	-0.866
95th percentile	5.739
Autocorrelation	-0.072
skewness	2.12
kurtosis	6.95
% of $\frac{I}{K} > 0$	52.4%
% of $\lvert\frac{I}{K}\rvert < \frac{1}{4} \times mean(\frac{I}{K})$	34.9%

Table 3: Selected moments from distribution of investment rates

As Figure 2 shows, the distribution of investment rates is highly right skewed, along with a very long right tail. The 95th percentile of investment rates is around 5.7, that is, at the 95th percentile, firms invest up to 5.7 times their capital stock in the same year; in contrast, the 5th percentile of investment rates is around -0.87, which corresponds to the firm owner essentially selling all his capital stock. Across the entire sample of firm-year observations, 44.9% of firms are downsizing and 55.1% are increasing their capital stock. Broadly speaking, this observation is reflective of the results on investment moments found in the firm dynamics literature, such as the seminal paper by Cooper and Haltiwanger (2006); that is, that investment at the firm level is typically very lumpy.

One might notice that the investment rate distribution features a standard deviation that is much larger than that in the firm dynamics literature (for example, the standard deviation of investment rates in Cooper and Haltiwanger is 0.33). This is in fact unsurprising, given the scale that entrepreneurial firms operate at. The median capital stock is \$23,288 (the mean is \$190,729, reflecting the sizable right-skewness of the firm size distribution). At the 95th percentile, where $\frac{I}{K} \approx 5.7$, this means that the median firm is investing around \$132,741. While sizable for many individuals, in terms of a business investment, this is small. For example, this could correspond to a small moving firm simply buying a couple of new trucks.

2.4 Summary of empirical results

As noted in the firm dynamics literature, the stark dispersion of ARPK implies that there is potentially widespread capital mis-allocation[17], with many low productivity firms operating inefficiently large firms and high productivity firms operating inefficiently small firms. Unlike the preceding literature, I also find that the distribution is left-skewed, implying that there

[17]When compared to a static investment model

might be substantially more firms that are large with low productivity, than small firms with high productivity. Moreover, the asymmetric persistence implies that the former operate at this inefficient level for longer periods of time than the latter, which then naturally gives rise to the left skewed cross-sectional distribution. This suggests that entrepreneurial firms are facing frictions in both capital accumulation and decumulation, but the latter friction appears to be stronger.

In addition, the investment distribution reveals that entrepreneurial firm investment is lumpy and infrequent, which suggests that non-convex adjustment costs play a potentially huge role in influencing entrepreneurial firm investment behavior.

Taken as a whole, the results are suggestive that partial irreversibility of capital, which induces an asymmetry in the purchase and resale value of capital, could rationalize the dynamic investment behavior observed here[18]. In particular, partial irreversibility, when modeled as a proportional transaction cost, induces the (s,S) investment dynamics that is crucial in generating lumpy (dis)-investment. The lower resale value of used capital also induces entrepreneurs to adopt a wait-and-see attitude when a bad shock is realized, thus operating firms that are "too large" relative to their productivity. This leads to an elongated left tail of ARPK, generating a left skewness in the distribution of ARPK. In addition, this also leads to greater persistence in the left tail. These heuristics therefore provide the motivation to include partial irreversibility as a key mechanism in the model, which I discuss in the following section.

3 Model

3.1 Households, occupation, and wealth

There is a continuum of mass 0 households in this economy, each indexed by $i \in [0, 1]$. Households are infinitely lived, and time is discrete.

All households are endowed with identical time-separable utility function with constant relative risk aversion (CRRA), and discount future utility at rate β. Households value a consumption bundle $\tilde{c}$, which is the sum of non-durable consumption c and (non-market) home production output $\bar{c}$. $\bar{c}$ is an endowment that is constant across time and households, whereas c depends on the household's endowment of other factors and savings choices.

[18] See for example, Cabellero (1999) for a summary of models with partial irreversibility

Taken together, the household's lifetime expected utility can be written as

$$\begin{aligned} V_0 &\equiv \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U\left(\tilde{c}_{i,t}\right) \\ &= \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \frac{\tilde{c}_{i,t}^{1-\gamma}}{1-\gamma} \\ \tilde{c}_{i,t} &= c_{i,t} + \bar{c} \end{aligned}$$

where γ is the coefficient of relative risk aversion. The household's objective is to maximize expected lifetime utility.

There are two types of occupations available to the households: To become an *entrepreneur*, or to become a *worker*. The key distinction between entrepreneurs and workers lie in their endowments, their way of generating income, and their way of accumulating wealth. The latter sections provide a more detailed description, but I briefly summarize the key differences between *entrepreneurs* and *workers* here:

Endowments Entrepreneurs receive a stochastic endowment of a private business productivity draw $z \in \mathcal{Z}$, while worker's receive a labor efficiency draw $\theta \in \Theta$. This productivity draw is exclusive. Workers also receive a stochastic signal of business prospects ψ^z. ψ^z informs the worker about the potential business prospects that she could engage in if she switches to becoming a business owner. Entrepreneurs receive a stochastic signal of labor prospects ψ^θ. ψ^θ informs the entrepreneur about the potential labor prospects that she could engage in if she switches to becoming a worker.

Wealth Entrepreneurs accumulate two forms of wealth: illiquid physical capital k (which they use in production), and liquid bonds b, which pay off a fixed interest rate. Workers can only save in liquid bonds. Physical capital is illiquid, as adjustment of the capital stock will lead the owner to incur specific adjustment costs (to be detailed later).

Technology All households are endowed with a fixed amount of labor supply $\bar{l}$. Workers can combine their labor supply endowment with their labor productivity to produce effective labor $\theta\bar{l}$. They supply effective labor to a centralized labor market, and earn the corresponding market wage w, giving them an income of $w\theta\bar{l}$. Entrepreneurs have access to a production function $f(z,k,l)$, which allows them to combine illiquid capital k, labor l (including a fraction ι of their labor supply endowment), and business productivity z to produce a consumption good y^e which they sell into a centralized market for income.

3.2 Sources of uncertainty and occupational choice

3.2.1 Sources of uncertainty

There are four sources of idiosyncratic (and uninsurable) uncertainty in this economy. They are:

1. A stochastic Markov process $P_{z|z_{-1}} \equiv Pr\left(z|z_{-1}\right)$ over private business productivity, with support $z \in \mathcal{Z}$ and invariant distribution F_z.

2. A stochastic Markov process $P_{\theta|\theta_{-1}} \equiv Pr\left(\theta|\theta_{-1}\right)$ over labor productivity, with support $\theta \in \Theta$ and invariant distribution F_θ.

3. An IID process of signals regarding business prospects, with support $\psi^z \in \Psi^z$, and distribution F_{ψ^z}. Conditional on drawing a signal ψ^z, next period private business productivity is assumed to be drawn from the conditional probability $P_{z|\psi^z_{-1}} \equiv Pr\left(z|\psi^z_{-1}\right)$.

4. An IID process of signals regarding labor prospects, with support $\psi^\theta \in \Psi^\theta$, and distribution F_{ψ^θ}. Conditional on drawing a signal ψ^θ, next period labor productivity is assumed to be drawn from the conditional probability $P_{\theta|\psi^\theta_{-1}} \equiv Pr\left(\theta|\psi^\theta_{-1}\right)$.

3.2.2 Occupational choice

The occupation of the households determine the types of uncertainty they face. Specifically, as described earlier, only entrepreneurs receive endowments of business productivity and labor prospect signal draws, and only workers receive endowments of labor productivity and business prospect signal draws.

Households are allowed to endogenously choose their occupation, but they have to select their occupation one period in advanced. In other words, a worker today who chooses to stay a worker tomorrow must pursue his chosen occupation for the next period. Occupational choice is exclusive, so a household cannot be both a worker and an entrepreneur simultaneously. This combination of "time-to-build" and exclusivity makes occupational choice itself risky.

Since there is endogenous occupational choice, we can in fact group the households into four types. The exact classification of the household is important, as this determines the sources of uncertainty that they face. Specifically, we have the following:

1. *Entrant entrepreneurs.* Entrants entrepreneurs are entrepreneurs who were workers last period. They are endowed with a productivity z that is drawn from the conditional distribution $P_{z|\psi^z_{-1}}$, where ψ^z_{-1} is the signal that the entrants had received in the

last period (as workers). They also receive a signal ψ^{θ} regarding next period's labor prospects.

2. *Incumbent entrepreneurs.* Incumbents entrepreneurs are entrepreneurs who were entrepreneurs last period. They are endowed with a productivity z that is drawn from the conditional distribution $P_z(z|z_{-1})$, where z_{-1} is last period productivity that they received as entrepreneurs. Like entrant entrepreneurs, they also receive a signal ψ^{θ} regarding next period's labor prospects.

   Upon drawing ψ^{θ}, if the entrepreneur chooses to exercise this option and becomes a worker, her next period labor productivity follows the same Markov process as the worker. Otherwise, she loses the signal and draws a fresh signal next period from the invariant distribution of signals.

3. *Entrant workers.* Entrants workers are workers who were entrepreneurs last period. They are endowed with a productivity θ that is drawn from the conditional distribution $P_{\theta|\psi^{\theta}_{-1}}$. In addition, they receive a signal ψ^{z} regarding next period's business prospects.

4. *Incumbent workers.* Incumbents workers are workers who were workers last period. They are endowed with labor productivity θ that is drawn from the conditional distribution $P_{\theta}(\theta|\theta_{-1})$, where θ_{-1} is last period labor productivity that they received as workers. Like entrant workers, they also receive a signal ψ^{z} regarding next period's business prospects.

   Upon drawing ψ^{z}, if the worker chooses to exercise this option and starts a business, her next period labor productivity follows the same Markov process as the entrepreneurs. Otherwise, she loses the signal and draws a fresh signal next period from the invariant distribution of signals.

3.3 Asset structure

Households have access to 2 types of assets to smooth inter-temporal consumption: Liquid bonds b and illiquid physical capital k.

3.3.1 Entrepreneurial physical capital

Only households who elect to become (or stay) entrepreneurs tomorrow can save in illiquid physical capital k. The primary purpose of the physical capital is as input for entrepreneurial production. However, it also serves a secondary purpose as consumption insurance. In particular, entrepreneurs can use the illiquid asset to smooth consumption by selling off parts of his asset stock in bad times, or to use it as collateral to borrow for consumption.

Capital is illiquid because of frictions associated with adjusting the capital stock. Here, I assume 4 forms of frictions.

The first friction affects investment. Entrepreneurs who choose to increase their capital stock face a proportional fixed adjustment cost $f_s k$. The fixed cost parametrizes the disruptions associated with expansion.

Two frictions influence the entrepreneur's disinvestment decision. If the adjustment is small, the entrepreneur pays a per-unit transaction cost λ, such that she recoups $(1-\lambda)$ of the transacted asset. If the adjustment is large, she has to pay another proportional selling cost ζ on top of the earlier transaction cost. As a result, the net return from selling a unit of capital is $(1-\zeta)(1-\lambda)$. An adjustment is considered small if the volume of disinvestment i_k is less than a fraction $\eta \in [0,1]$ of the entrepreneur's depreciated capital stock (i.e. $i_k \leq \eta(1-\delta)k)$).

This preceding formulation imposes that small adjustments are more costly than large changes, and captures the idea that large scale sales are akin to a "fire sale" of assets. This also simultaneously parameterizes the difficulty of exiting a business and having to conduct a fire sale of all the physical capital stock.[19] In particular, $\eta = 0$ corresponds to a case where only exiting entrepreneurs are affected by this cost, whereas $\eta = 1$ captures a case where all entrepreneurs are affected.

Finally, the last friction involves entry. Specifically, it is assumed that workers who want to enter entrepreneurship must start their business with a minimum level of capital k^w_{min}. Households who are already entrepreneurs do not face this friction. Rather, this assumption reflects the fact that entry typically requires some form of minimum capital investment.

3.3.2 Liquid bonds

All households can trade in bonds, either by buying bonds (i.e. saving), or selling them (i.e. borrowing). The total volume of bonds supplied composed of all the bonds issued by individual households, as well as equity issued by a corporate sector. The cost of a bond that pays off next period is 1 unit of consumption.

All bonds, regardless of whether it was issued by the corporate sector or individuals, pay

[19] It is important to note that this does not necessarily say that every entrepreneur who exits a business has to incur a loss. Rather, it says that when an entrepreneur exists his business, the physical capital stock that he owns is typically worth less than even the actual depreciated value. On the other hand, this paper has nothing to say about the potential profits to be made on intangible capital, such as brand name or R&D capital. I refer the reader to an accompanying paper in Tan (2018), where I extend this model framework to allow entrepreneurs to sell the business itself.

off the same interest rate r. As the corporate sector is large and is in fact issuing claims against its profits, I assume that there is no spread between the cost at which the corporate sector borrows, and the interest rate received by savers. In contrast, individuals issuing bonds are essentially issuing unsecured debt, and hence are "riskier". As such, I assume that the household must pay a per unit intermediation cost ϕ_d when they borrow. As a result, this induces a spread between the interest rate r received by savers, and interest rate r_d paid by borrowers, given by $r_d = r + \phi_d$.

In addition, households who decide to invest in physical capital can also elect to use the liquid value of their physical capital as collateral to borrow in bonds. Specifically, I define the liquid value of capital as $(1-\lambda)(1-\delta)k$, and the collateral constraint is defined as

$$b' \geq -\varphi(1-\lambda)(1-\delta)k' - \underline{b}$$

where $\varphi \in [0, \infty)$, with $\varphi = 0$ representing no collateralized borrowing, and $\varphi \to \infty$ representing no collateral required for borrowing. Households are allowed to borrow up to the liquid value of depreciated capital. $\underline{b}$ represents an ad-hoc borrowing constraint that does not require any collateral. This reflects the nature that some households can acquire unsecured debt. As such, both workers and entrepreneurs can potentially borrow to smooth consumption. The only difference is that entrepreneurs can borrow more.

Default Depending on the exact parametrization of the stochastic productivity process, a subset of entrepreneurs might not be able to fully pay off their stock of debt even after fully liquidating. For this subset, the entrepreneurs are allowed to default on the remaining stock of their debt, but have to exit the entrepreneurial sector. The cost of default D (i.e. all the debt that is unpaid) is transferred in a lump sum manner in equal proportions to the rest of the households[20].

3.4 Sectors of the economy

There are two sectors of production in this economy: a corporate sector and an entrepreneurial sector. Both sector produce a single homogeneous non-durable consumption good.

3.4.1 Corporate sector

I assume that there exist a large corporate sector encompassing all non-entrepreneurial firms. This corporate sector is represented by a representative firm, which owns physical capital K^c

[20] In general, default does not occur in equilibrium in this model under the benchmark calibration. However, when studying unexpected shocks, fully leveraged households might find themselves unexpectedly in default.

and hires labor L^c from a centralized labor market. It has access to a standard Cobb-Douglas production technology of the form

$$Y^c = A\left(K^c\right)^{\alpha}\left(L^c\right)^{1-\alpha}$$

where A is aggregate TFP, and α is the capital share. All households purchase equity in corporate firms, and are paid a dividend each period.

The corporate firm decides independently on how much physical capital to invest, and how much labor to hire at the prevailing wage w. As such, the representative firm solves the standard recursive problem

$$\begin{aligned}
\Pi\left(K\right) = \max_{K'} \pi + \frac{1}{1+r}\Pi\left(K'\right) \\
s.t. \\
\pi = Y^c - \left(K^{c'} - (1-\delta)K^c\right) - wL^c
\end{aligned}$$

where π represents dividends paid out to the firms' investors, and the firm discounts future profits at rate $\frac{1}{1+r}$. Note that because of the size of the corporate sector, I assume here that there are no adjustment costs associated with the corporate sector. Moreover, unlike entrepreneurs, corporate firms are allowed to issue equity, which is in line with the real world observation of larger corporate firms being listed on a stock market. This market arrangement leads to the standard first order condition

$$\begin{aligned}
r + \delta = \alpha A\left(\frac{K^c}{L^c}\right)^{\alpha-1} \\
w = \left(1-\alpha\right)A\left(\frac{K^c}{L^c}\right)^{\alpha}
\end{aligned}$$

3.4.2 Entrepreneurial sector

The entrepreneurial sector is composed of entrepreneurial households. They utilize the production function $y_e = f\left(z,k,l\right) = z\left(k^{\alpha_e}l^{1-\alpha_e}\right)^{\nu}$. As discussed earlier, the inputs to production are physical capital k and labor l. $\nu \in (0,1)$ here denotes the span-of-control, capturing the fact that managerial skills become stretched over larger and larger projects.

Physical capital stock is chosen last period, and cannot be altered in the current period. z, as discussed earlier, is stochastic productivity and realized in the same period. Entrepreneurs are endowed with labor supply $\bar{l}$, and they can use a fixed fraction ι to run their own business. If they choose to hire extra workers, they have to pay the prevailing market wage w. As such, the profit of the entrepreneurial firm can be written as

$$\pi\left(z,k\right)=z\left(k^{\alpha_e}l^{1-\alpha_e}\right)^{\nu}-w\left(l-\iota\bar{l}\right)$$

Note that given this setup, the labor choice is a static decision and completely independent of the structure of the rest of the problem. Consequently, we see that optimal labor demand satisfies:

$$l^{*}=\begin{cases}\iota\bar{l} & \text{if } l^{*}\leq\iota\bar{l}\\ \left[\frac{(1-\alpha)\nu}{w}\right]^{\frac{1}{1-(1-\alpha)\nu}}z^{\frac{1}{1-(1-\alpha)\nu}}k^{\frac{\alpha\nu}{1-(1-\alpha)\nu}} & \text{if } l^{*}>\iota\bar{l}\end{cases}$$

and optimal profits is given by

$$\pi^{*}=\begin{cases}z\left(k^{\alpha}\left(\iota\bar{l}\right)^{1-\alpha}\right)^{\nu} & \text{if } l^{*}\leq\iota\bar{l}\\ \left[\mathcal{A}\left(w\right)-w\mathcal{A}\left(w\right)^{\frac{1}{(1-\alpha)\nu}}\right]z^{\Theta_z}k^{\Theta_k}+w\bar{l} & \text{if } l^{*}>\iota\bar{l}\end{cases}$$

where $\mathcal{A}(w)\equiv\left[\frac{(1-\alpha)\nu}{w}\right]^{\frac{(1-\alpha)\nu}{1-(1-\alpha)\nu}}$, $\Theta_z\equiv\frac{1}{1-(1-\alpha)\nu}$ and $\Theta_k\equiv\frac{\alpha\nu}{1-(1-\alpha)\nu}$.

3.5 Recursive formulation of the problem

The preceding problem can be recast compactly into recursive formulation. Denote by V_e and V_w the value functions of entrepreneurs and workers respectively, by $\mathcal{C}\left(k',k,h',h\right)$ the adjustment cost function as described earlier, and h the occupational state (with 1 denoting worker and 0 denoting entrepreneur). Denote by $'$ variables all next-period variables, and unprimed variables current variables. Given this, we have the following problem:

For entrepreneurs,

$$
\begin{aligned}
V_e\left(\psi^{\theta}, z, k, b\right) = & \max_{h',k',b',l} U\left(\tilde{c}\right) \\
& + \left(1-h'\right) \times \beta \int_{\psi'_{\theta}} \int_{z'} V_e\left(\psi^{\theta'}, z', k', b'\right) dP_{z'|z} dF_{\psi^{\theta}} \\
& + h' \times \beta \int_{\psi'_{z}} \int_{\theta'} V_w\left(\psi^{z'}, \theta', b'\right) dP_{\theta'|\psi^{\theta}} dF_{\psi^{z}} \\
& s.t. \\
& \hat{\pi} \equiv zf\left(k, \iota\bar{l}+l\right) - wl + \left(1 + r \times 1_{\{b\geq 0\}} + r_d \times 1_{\{b<0\}}\right) b - \mathcal{C}\left(k', k, h', h\right) - D \\
& c = \max\left\{\hat{\pi}, 0\right\} - k' - b' \geq 0 \\
& k' \begin{cases} > 0 & \text{if } h' = 0 \\ = 0 & \text{if } h' = 1 \end{cases} \\
& h' = 1 \quad (\text{ if } \hat{\pi} < 0) \\
& b' \geq -\varphi\left(1-\lambda\right)\left(1-\delta\right)k' - \underline{b} \\
& \tilde{c} = c + \bar{c}
\end{aligned}
$$

For workers,

$$
\begin{aligned}
V_w\left(\psi^{z}, \theta, b\right) = & \max_{h',k',b'} U\left(\tilde{c}\right) \\
& + \left(1-h'\right) \times \beta \int_{\psi'_{\theta}} \int_{z'} V_e\left(\psi^{\theta'}, z', k', b'\right) dP_{z'|\psi^{z}} dF_{\psi^{\theta}} \\
& + h' \times \beta \int_{\psi'_{z}} \int_{\theta'\in\Theta} V_w\left(\psi^{z'}, \theta', b'\right) dP_{\theta'|\theta} dF_{\psi^{z}} \\
& s.t. \\
& c = \theta w\bar{l} + \left(1 + r \times 1_{\{b\geq 0\}} + r_d \times 1_{\{b<0\}}\right) b - k' - b' - D \\
& k' \begin{cases} > 0 & \text{if } h' = 0 \\ = 0 & \text{if } h' = 1 \end{cases} \\
& b' \geq -\varphi\left(1-\lambda\right)\left(1-\delta\right)k' - \underline{b} \\
& \tilde{c} = c + \bar{c}
\end{aligned}
$$

3.6 Equilibrium definition

The state space of the model can be described by bond holdings $b \in \mathbb{B}$, capital holdings $k \in \mathbb{K}$, occupational choice $h \in \mathbb{H}$, entrepreneurial productivity $z \in \mathbb{Z}$, labor productivity

$\theta \in \Theta$, entrepreneurial productivity signal $\psi^z \in \Psi^z$, and labor productivity signal $\psi^\theta \in \Psi^\theta$. As such, the complete state space $\mathbb{S}$ can be written as $\mathbb{S} = \mathbb{B} \times \mathbb{K} \times \mathbb{H} \times \mathbb{Z} \times \Theta \times \Psi^z \times \Psi^\theta$.

A stationary competitive equilibrium of the model consist of the interest rate r, wage rate w, value functions of households and firms $\{V_e, V_w, \Pi\}$, allocations $\{k', b', l\}$ and distribution of agents Λ over the state space $\mathbb{S}$ such that,

1. Taking r and w as given, the households' and firms' choices are optimal.

2. Markets clear,

   (a) Bonds: $\int b' d\Lambda = K^c$

   (b) Labor: $\int \theta h d\Lambda = \int l d\Lambda + L^c$

3. The distribution Λ is time-invariant, given by

$$\Lambda = \Gamma(\Lambda)$$

   Where Γ is the one-period transition operator on the distribution

The method by which I compute the solution to the individual's problem, and the stationary equilibrium, is documented in the appendix.

3.7 Illiquid capital - Skewness and left tail persistence, and prevailing theory

The goal of this sub-section is to demonstrate that the prevailing theoretical framework of firm dynamics that ignore illiquidity cannot, in general, replicate the left tail persistence and left skewness that I documented. In the calibration section, I will further demonstrate, numerically, how illiquidity is key in generating the left tail persistence.

For reference, I consider a (simplified) canonical model of firm production, operating a production function $Y = zK^\alpha$, where z is firm level TFP, K is capital, Y is output, α is the returns to scale, r is the interest rate and δ is the user cost of capital (depreciation)[21]. In addition, I assume that TFP evolves as an AR(1) process as follows:

$$\log z_{t+1} = \rho \log z_t + \epsilon_{t+1}$$

[21] All the derivations hold if we include labor as an input, but I ignore it in the interest of algebraic clarity. I choose this simple model to illustrate my point, as these models allow me to analytically characterize the skewness and persistence of ARPK. In contrast, a full scale model does not easily admit an analytical expression.

where ϵ_t is any *i.i.d.* innovation to z. Note that ϵ can assume any non-degenerate distribution. I will use this to discuss four common frameworks adopted in this literature:

1. A static model of investment with no frictions (as a reference benchmark)

2. A dynamic investment model with a single period time-to-build

3. The standard static model of investment with collateral constraints resulting from limited commitment[22]

4. A dynamic investment model with collateral constraints; this amounts to (3) with a time-to-build

The full description of the models I consider are relegated to the proofs in the appendix.

Note that in this context, "static model" simply implies that the choice of capital at time t is measurable with respect to time $t+1$ innovations ϵ; it does not necessarily mean that the model is actually static. The key point here is that because firms have advanced information regarding the innovations to production, the choice of capital is both ex-ante and ex-post optimal with respect to the relevant technological constraints (such as financial frictions). In contrast, "dynamic model" refers to the fact that ϵ_{t+1} is not measurable with respect to the time t information set

3.7.1 Static investment model with no frictions

Proposition 1 Consider a static investment model with no frictions. Then this model yields a degenerate distribution for ARPK; therefore, there are no higher order moments associated with ARPK.

Proof The proof is trivial. The canonical firm investment model with static investment choices yield the first order condition for capital as

$$\begin{aligned} \alpha\frac{Y}{K} &\equiv MRPK = r+\delta \\ \log\left(MRPK\right) &= \log\left(r+\delta\right) \\ \implies \log\left(ARPK\right) &= r+\delta-\log\alpha \end{aligned}$$

In this framework, firms always set their (log) ARPK to the (adjusted) user cost of capital. Consequently, ARPK has a degenerate distribution, and has no higher order moments associated with it.

[22]c.f. Cagetti and De Nardi (2006), Kitao (2008), Cagetti and De Nardi (2009), Buera et al (2011), Midrigan and Xu (2014). All these papers utilize the same static investment framework discussed here

While this result is clearly trivial, it is often the starting point for many papers in the recent literature in understanding capital mis-allocation using cross-sectional moments of the distribution of marginal product of capital[23]. Moreover, the vast majority of macro-entrepreneurship papers have utilize this framework in tandem with collateral constraints to explain the dispersion in marginal products. As such, I recount this result here.

3.7.2 Dynamic investment model with no frictions

Proposition 2 Consider a dynamic investment model with no frictions. Then (log) ARPK can be expressed as (denoted by a superscript TTB for "time-to-build"),

$$\log ARPK^{TTB} = \vartheta + \epsilon$$

where ϑ is a collection of parameters. As such, the distribution of log ARPK is simply a mean-shifted distribution of the underlying innovations ϵ. Moreover, this implies that log ARPK has no persistence.

Proof The derivation of the preceding equation is relegated to the appendix. The result that log ARPK has no persistence comes directly from the equation. Since ϵ is an I.I.D innovation, log ARPK will not feature any persistence.

An additional result here is that the *skewness* of the distribution log APRK is equal to the skewness of the distribution of the underlying innovations. In most standard firm (or entrepreneurial) dynamics model, the innovations are typically Gaussian or Pareto; the implication then is that log ARPK has zero or positive skewness, a result that is counter-factual to my empirical findings. As I will show in the next section, the illiquid aspect of capital can naturally generate a left skewed distribution, without having to engineer a left skewed distribution of innovations.

Finally, the results here also relate to a key point raise by Asker et al (2014). There, using numerical examples, the authors show that a substantial portion of the dispersion in log MRPK (log ARPK) observed by Hsieh and Klenow (2009) can be explained using simply time-to-build. My results here are the analytical counterpart to their numerical results.

[23]See, in particular, the seminal paper by Hsieh and Klenow (2009). The authors use this result as their motivation for studying "wedges" that distort a firm's investment decision, leading to dispersion in log ARPK (log MRPK).

3.7.3 Static investment model with collateral constraints

Proposition 3 Consider a static investment model with collateral constraints. Specifically, consider a linear reduced form collateral constraint of the following form

$$k \leq (1+\varphi)\,a$$

where k is working capital and a are net assets of the firm, and φ capture the extent of financial frictions. Then the distribution of (log) ARPK has a left-truncated support, given by (denoted with a superscript FF for financial frictions)

$$\log ARPK^{FF} \in [\log(r+\delta) - \log\alpha, \infty)$$

As such, the distribution of log ARPK is right-skewed regardless of the distribution of the underlying innovations. Moreover, log ARPK is also persistent; the persistence is also higher in the right tail of the distribution than the left tail.

Proof The proof is relegated to the appendix.

The key takeaway here is that this class of models will *always* generate a right skewed distribution with higher right tailed persistence, and as such cannot replicate the findings in the KFS.

3.7.4 Dynamic investment model with collateral constraints

Proposition 4 Consider a dynamic investment model with collateral constraints. Specifically, consider a linear reduced form collateral constraint of the following form

$$b' \geq -\varphi(1-\delta)k'$$

where b' is next period debt (or savings, if $b' > 0$), and k' is next period capital, and φ captures the strength of the financial frictions. Then log ARPK in the collateral constraints model is related to the frictionless time-to-build model through the following relationship (denoted by a superscript TTB, FF for "time-to-build with financial frictions")

$$\log ARPK^{TTB,FF} = \log ARPK^{TTB} + \xi_{-1}$$

where ξ_{-1} is a random variable that has a right-skewed distribution (and is pre-determined last period). As such, if the skewness of $\log ARPK^{TTB}$ is lower than some threshold that is strictly bigger than 0, the distribution of $\log ARPK^{TTB,FF}$ will be more right-skewed than the distribution of the underlying innovations. If the skewness of $\log ARPK^{TTB}$ if larger than the threshold, than the distribution of $\log ARPK^{TTB,FF}$ is less right skewed than the distribution of the underlying innovations, but it will always be right-skewed. Moreover, the right tail of the distribution will be more persistent than the left tail.

Proof The proof is relegated to the appendix.

The key point here is that, like in the static model with collateral constraints, the dynamic model will likewise feature a right skewed distribution of ARPK. As such, this class of models will not be able to replicate the results I presented in the earlier section.

3.8 Illiquid capital - Additional Evidence

As a final additional piece of evidence supporting capital illiquidity as driving the left tail persistence, I estimate the impact of changes in the capital portfolio mix on the probability of switching ranks. Specifically, I fit the following regression:

$$Y_{i,j,t} = \sum_s \beta_s \frac{k_{i,s,t}}{K_{i,t}} + \gamma X_{i,j,t} + \xi_j + \delta_t + \epsilon_{i,t} \tag{1}$$

where,

- s indicates capital type: Equipment, Vehicles, Land and buildings, Inventory, Accounts receivables.
- ξ and δ are industry (j) and time (t) fixed effects
- X is vector of individual (i) level controls, including gender, legal form, capital stock, revenue
- $Y_{i,j,t} = 1$ if individual stays in current ARPK rank tomorrow, $= 0$ if individual switches out

The coefficient of interest here is β_s, which can be interpreted as follows: For every 1 percentage point increase in the share of capital asset s as a fraction of total assets, the probability that a firm stays in it's current ranking along the ARPK distribution increases by β_s percentage points.

How does this statistical model provide support for my theoretical model? In my proposed model to explain the high left tail persistence, the key model mechanism is capital illiquidity which impedes downsizing. However, capital illiquidity has a much weaker impact on the right tail, since it does not directly impede a firm from investing. As such, to the extent that the capital types on the firm's balance sheet have differential extent of liquidity, we can test whether a firm that has more illiquid assets is also more likely to persist in a low ARPK state; likewise, we can test whether a firm with more illiquid assets is more likely to persist is a high ARPK state.

Table 4 reports the regression results for firms that started out the period in rank 1 (i.e. firms in the bottom 20%) and rank 5 (i.e. firms in the top 20%). From column (1), we see that firms that own a larger share of equipment, product inventories or land or buildings are (statistically and economically) significantly more likely to persist in a low ARPK state when they start the year with low ARPK. In contrast, firms that own a large share of vehicles and accounts receivables are not statistically more likely to persist in their current state.

Notice that equipment and product inventory are highly firm specific, which makes resale of these capital assets very difficult. Consequently, these assets can be considered highly illiquid. Along the same vein, land and buildings are much less liquid due to the search costs associated with selling them. As such, the results are suggestive that firms with greater share of their assets invested in illiquid assets are also more likely to persist in a low ARPK state.

In contrast, firms that have more of their assets invested in vehicles or accounts receivables are not statistically more likely to persist in a low ARPK state. Unlike equipment or product inventory, vehicles are generally not firm specific. Moreover, there is a large market for resale vehicles. Accounts receivables are also relatively more liquid; unlike the previous assets which are physical assets, these are debt owed to the firms.

In column (2), we see that firms that start out with high ARPK (i.e rank 5) are not affected by their portfolio mix: none of the asset shares show any statistical significance with respect to their effect on the probability that a high ARPK firm stays in the top 20%. This is reflective of the theory that capital illiquidity plays a very small role in driving the right tail persistence.

COEFFICIENT	Pr(stay in rank 1)	Pr(stay in rank 5)
share of equipment	0.409**	0.400
	(0.171)	(0.247)
share of inventory	0.701***	0.287
	(0.154)	(0.291)
share of vehicles	0.240	0.268
	(0.177)	(0.271)
share of land / buildings	0.651***	-0.271
	(0.188)	(0.345)
share of accounts receivables	0.0365	0.305
	(0.171)	(0.277)
log(value added)	-0.0500***	0.0563**
	(0.0162)	(0.0223)
log(real capital stock)	0.0368**	-0.0577***
	(0.0182)	(0.0199)
Observations	1055	875
R-squared	0.644	0.535
Standard errors in parentheses		
*** p<0.01, ** p<0.05, * p<0.1		

Table 4: Effect of portfolio share of each asset mix on probability that entrepreneur stays in current rank. Other controls (Year + Industry FE, gender, legal form) are included.

3.9 Parametric forms for productivity

Before moving forward to discussing the calibration strategy in the next section, I first close out the model section by assigning functional forms to the productivity processes.

Business productivity Business productivity is assumed to follow an AR(1) process of the form $\log z' = (1-\rho_z)\,\mu_z + \rho_z \log z + \sigma_z \epsilon_z'$, with $\epsilon_z' \sim iid, N(0,1)$. The process is discretized with a 15 point Markov transition matrix using the Tauchen (1986) method.

Business productivity signal The signal is assumed to be drawn from a distorted invariant distribution of the actual invariant distribution of business productivity. Note that business productivity has the invariant distribution $N(\mu_z, \sqrt{\frac{1}{1-\rho^2}}\sigma_z)$. In the case of the signal, I assume worker households draw a signal from the "twisted" signal distribution $N(\tilde{\mu_z}, \sqrt{\frac{1}{1-\rho_z^2}}\sigma_z)$, where the $\tilde{\mu_z}$ parameter distorts the household's perception of the true mean of the distribution of productivity. The primary effect of this is to vary the entry rate into entrepreneurship, and thus steady-state population of entrepreneurs.

Labor productivity Labor productivity is assumed to follow an AR(1) process of the form $\log\theta' = (1-\rho_\theta)\,\mu_\theta + \rho_\theta \log\theta + \sigma_\theta\epsilon'_\theta$, with $\epsilon'_\theta \sim iid, N(0,1)$. The process is discretized with a 15 point Markov transition matrix using the Tauchen (1986) method.

Labor productivity signal The signal is assumed to be drawn from a distorted invariant distribution of the actual invariant distribution of labor productivity. Similar to the case of business productivity signals, labor productivity has the invariant distribution $N(\mu_\theta, \sqrt{\frac{1}{1-\rho_\theta^2}}\sigma_\theta)$; I assume then that entrepreneurial households draw a signal from the "twisted" distribution $N(\tilde{\mu_\theta}, \sqrt{\frac{1}{1-\rho_\theta^2}}\sigma_\theta)$, where $\tilde{\mu_\theta}$ distorts the entrepreneur's perception of the true mean of the distribution of labor productivity. The primary effect of this is to vary the exit rate.

4 Calibration

The model frequency is annual, which corresponds to the frequency in the KFS. As in the literature, many standard parameters (such as the corporate sector's capital share, depreciation rate, labor income process) are taken from the preceding literature. The exact numbers used and their rationale are reported under "Group A" in table 5.

For "Group A" parameters, two parameters are set in an ad-hoc fashion: The value of home production $\bar{c}$ and the fraction of disinvestment that is considered small "η". Here, I assume $\bar{c} = 0$ and $\eta = 1$; therefore, I assume that home production has no value, and that any volume of transaction short of a complete exit is considered "small". I make this assumption as there are no known references to which I can set these numbers to. Unfortunately, there is also no clear strategy to discriminate the value of these parameters using the micro-data. However, as a further form of robustness checks, I experimented with different values of $\bar{c}$ and η, and found that the results are qualitatively very similar.

Three parameters are inferred directly from the data: The depreciation rate of the entrepreneur's capital stock δ_k, the capital intensity α_k, and the returns to scale ν. The depreciation rate for the entrepreneur's capital stock differs from the corporate sector, as the composition of the aggregated "capital" stock is different for an entrepreneur as compared to that of a corporate firm. Consequently, I construct the depreciation rate for the entrepreneur's capital as a weighted average of the individual depreciation rates of the components of the types of capital that make up an entrepreneur's stock of capital. The capital intensity and returns to scale are estimated using a hybrid cost shares approach and production function regression. The exact method from which these two parameters are constructed is relegated to the appendix. The estimated values used in the model is reported below under “Group B" in table 5.

Parameter	Value	Description	Identifying moment
Group A parameters			
γ	Risk aversion	2	Standard. See for instance, Storesletten et al (2004)
μ_θ	Unconditional mean of labor productivity	0	Normalization
ρ_θ	Persistence of labor productivity	0.94	Storesletten et al (2004)
σ_θ	Conditional variance of labor productivity	0.20	Storesletten et al (2004)
μ_z	Mean of business productivity process process	0	Normalization
α	Capital share of corporate sector	0.33	Fixed to value in Cagetti and De Nardi (2006)
A	Corporate sector TFP	1	Normalization
δ	Depreciation rate of corporate sector capital	0.10	Standard
η	Parameter for “small" adjustment	1	See text
$\bar{c}$	Consumption value of home production	0	See text
Group B parameters			
δ_k	0.15	Depreciation rate of entrepreneur’s capital	From data (See appendix)
α_e	0.42	Capital intensity of entrepreneurial production function	From data (See appendix)
ν	0.79	Returns to scale of entrepreneurial production function	From data (See appendix)

Table 5: Fixed and estimated parameters

The rest of the ten parameters are inferred indirectly from the data by jointly calibrating them to identifying moments from the data. A brief description of these parameters, as well as the mapping of data moments to parameter, is summarized under “Group C" in table 6; this is done with the understanding that variation in any parameter will inadvertently trigger changes in other moments. The idea here is that these moments are most strongly associated with their corresponding parameters.

Parameter	Value	Description	Identifying moment
Group C parameters			
λ	0.68	Resale transaction cost	Probability that firm stays in quintile 1 of ARPK distribution
ζ	0.31	Exit friction	Skewness of ARPK
φ	0.93	Collateral constraint	Skewness / Probability that firm stays in quintile 5 of ARPK distribution
f_s	0.035	Investment fixed cost	Rate of positive investment reported
ρ_z	0.81	Autocorrelation of productivity shock	Autocorrelation of investment rates
σ_z	0.35	Volatility of productivity shock	Coefficent of variation of investment, firm size distribution
$\bar{l}$	0.23	Entrepreneur's endowed labor	% of firms that are employers
$\tilde{\mu_z}$	0.51	Mean of entrepreneurial prospects signal shock	Fraction of households that are entrepreneurs in steady-state
$\tilde{\mu_\theta}$	0.75	Mean of labor prospects signal shock	Exit rate
β	0.9265	Discount factor	Interest rate of 3.5%

Table 6: Calibrated parameters

In table 7 below, I report the model fit by comparing the data moments that I target, against the model implied moments. Broadly speaking the model does a good job at both matching the micro-level moments (i.e. from the KFS) and the macro-level moments (taken from the PSID).

The calibration of λ, ζ, and φ, which is one of the key contributions of this paper, is further discussed in the rest of this section. In particular, I will demonstrate how the skewness of the ARPK distribution, as well as it's persistence in the two tails of the distribution, can be an informative device in discriminating between downsizing frictions (in this context, partial irreversibility) and investment frictions (such as collateral constraints). The calibration of the other parameters are relegated to the appendix.

Moments	Data	Model
Pr(1,1)	0.58	0.57
Pr(5,5)	0.47	0.44
Skew log(Y/K)	-0.39	-0.29
% +ve investment	54%	50%
CV of I/K	4.0	4.0
% Employer firms	53%	58%
KFS exit rate	10%	30%
% of households that are entrepreneurs	(8%,20%)	8.9%
% Exit rate	(20%,40%)	32%
% Startup rate	(3%,10%)	3.2
Interest rate	3.5%	3.5

Table 7: Model fit: Data and corresponding model moments

4.1 Downsizing frictions and collateral constraints: Skewness and asymmetric persistence

How important are the downsizing frictions in helping the model to match the skewness and asymmetric persistence noted in the data? Table 8 below reports the model implied moments for the skewness and the tail persistence when the resale frictions are removed (i.e. $\lambda = \zeta = 0$).

Moment	Data	Benchmark calibration	Counter-factual: No resale frictions
		Targeted	
$skew\left(log\frac{Y}{K}\right)$	-0.39	-0.29	0.18
$Pr(1 \rightarrow 1)$	0.58	0.57	0.37
$Pr(5 \rightarrow 5)$	0.47	0.44	0.47
		Not targeted	
ρ_1	0.71	0.84	0.31
ρ_5	0.45	0.50	0.63

Table 8: Effect of illiquidity on skewness and symmetric persistence - Comparison with counter-factual "frictionless" model. ρ_1 and ρ_5 refer to the autocorrelation of ARPK in quintile 1 and 5 respectively. These two moments were not targeted.

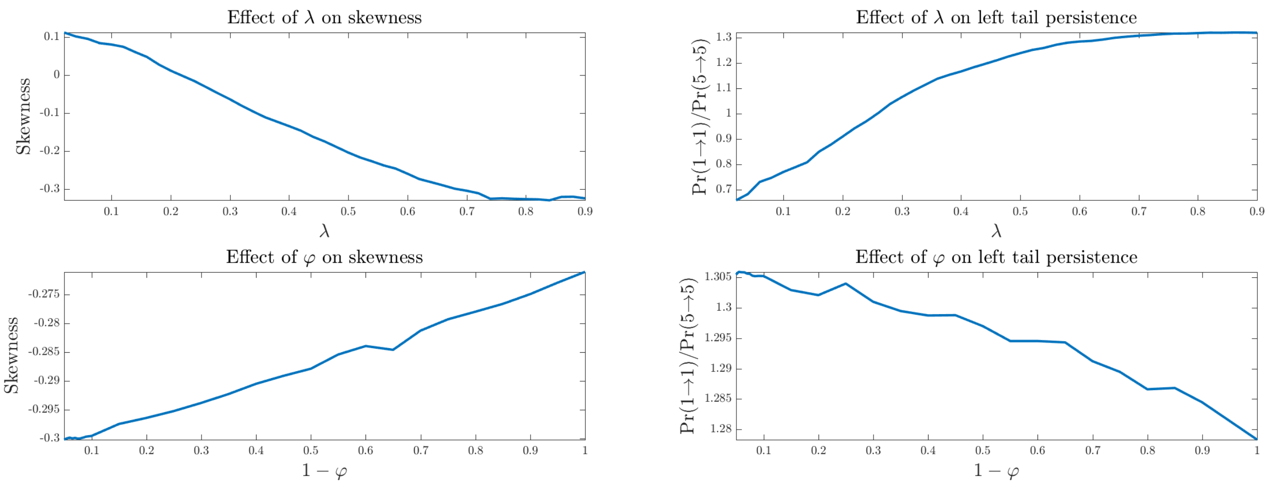
Notice that once the two resale frictions are removed, the overall skewness rises to 0.18; that is, the distribution is now *right-skewed*, as predicted in the earlier section. Similarly, the left-tail persistence falls to 0.37, whereas the right-tail persistence rises to 0.47. In other words, the model now exhibits greater persistence in the right tail than the left tail. As a form of external validation, I also report the effect on the autocorrelation of ARPK in the bottom and top quintiles; these two moments are not explicitly targeted. Similar to the finding with the persistence of relative rankings, the benchmark calibration generates greater autocorrelation in the left than right tail. In contrast, a model without resale frictions flips the asymmetry.

Why is modeling resale frictions so crucial in generating the left skewness and higher left tail persistence?

First, recall that in a fully frictionless economy, all firms always target the same (expected) ARPK. Ex-post, the distribution of ARPK then simply reflects the distribution of all the individuals' expectational errors; in the case of i.i.d log-normal innovations, for example, the ARPK distribution is also log-normal and has no persistence.

Next, when the same model is augmented with collateral constraints, firm that are financially unconstrained behave just as the firms in the frictionless economy. As such, the conditional distribution (of unconstrained firms) simply reflects the expectational errors. Again, when innovations are i.i.d log-normal as assumed in the benchmark calibration, this conditional distribution will have no skewness or persistence. Financially constrained firms, on the other hand, must necessarily operate firm sizes that are "too small" relative to their optimal sizes - that is, their ARPK must be *higher* than the optimal ARPK (in expectation). Consequently, the ex-post conditional distribution of constrained firms will feature a right skew, and the combined distribution of constrained and unconstrained firms is also right-skewed. Moreover, the tighter the collateral constraint, the more right-skewed the distribution will be, as one can see in the bottom panel of figure 3a.

In contrast, downsizing frictions tend to extend the left tail of the distribution. When hit by a bad shock, the options value of capital induced by the asymmetry in purchase and resale price of capital lead downsizing entrepreneurs to target a lower ARPK (in expectation) than the unconstrained ARPK. As a result, the left tail of the ARPK distribution becomes extended, making the distribution more left-skewed. Moreover, resale frictions also lead poor performing entrepreneurs to stay in business for longer periods of time. The combined effect generates greater left skewness when these frictions are added to the model. This result is reflected in the first top panel of figure 3a.



(a) Skewness

(b) Relative persistence of tails

Figure 3: Effect of λ and φ on skewness and relative persistence of ARPK. Figure (b) reports the ratio of the probability of staying in quintile 1 to the probability of staying in quintile 5 (i.e. $Pr(1 \to 1)/Pr(5 \to 5)$). Note that increasing φ represents loosening collateral constraints; therefore, for clarity, I plot the relevant moments against $1 - \varphi$.

A similar effect plays out in driving the asymmetry in the persistence of the tails of the distribution. Tighter collateral constraints mean that firms that are "too small" (i.e. high ARPK) remain too small for longer periods of time, thus generating stronger persistence in the right tail. In contrast, λ induces a "wait-and-see" attitude only for incumbent firms who want to downsize; in contrast, entrepreneurs looking to expand do not face the same constraint. As a result, increases in λ will result in stronger persistence in the left tail. This opposing effect is reflected in figure 3b.

4.1.1 Using skewness and relative persistence to discriminate amongst sources of friction

The results here have important considerations for understanding the underlying frictions that impede efficient capital allocation amongst entrepreneurs, as well as for the broader economy. Specifically, we see that frictions that distort downsizing (λ) or exit (ζ) produces log ARPK distributions that are increasing left skewed. Moreover, it also leads to greater persistence in the left tail of the distribution, and can lead to greater persistence in the left tail than right tail. In contrast, frictions that impede investment, such as financial frictions, lead to greater right skewness in the log ARPK distributions. As such, this paper provides a way to discriminate between these two types of frictions.

In contrast, prior papers that focus on misallocation of capital, such as Hsieh and Klenow (2009), Asker et al (2014), and Midrigan and Xu (2014), have focused primarily on studying the *dispersion* of log MRPK (and equivalently, log ARPK). Asker et al (2014), for instance, focuses on capital adjustment costs (equivalent to the f_s and λ parameters in this model) while ignoring financing constraints, while Midrigan and Xu (2014) focuses on financing constraints (equivalent to φ in this model) but largely abstracting away from adjustment costs. In both cases, frictions can lead to observationally equivalent outcomes in the dispersion of log MRPK (or ARPK). This paper therefore merges these two frameworks by incorporating *both* adjustment and financial frictions, and providing a simple framework to discriminate between the two (i.e. using the skewness and relative persistence).

4.1.2 Are entrepreneurs financially constrained?

In the benchmark calibration, $\varphi = 0.93$. φ in this model refers to the limited commitment problem in the Kiyotaki-Moore framework; in this context, this means that entrepreneurs are able to collateralize up to 93% of the face value of their capital assets.

What does this mean? At face value, this suggests that early stage entrepreneurs are in fact less encumbered by financial frictions than one suspects. In prior research such as that in Evans and Jovanovic (1989) or Cagetti and De Nardi (2006), financial friction play a large role in determining the dispersion of wealth and firm sizes[24]. The inability to fully borrow and fund their investment leads to large distortions in the economy, primarily by deterring high productivity potential entrants from entering, and forcing poorer entrepreneurs to run sub-optimally small firms. In contrast, my results appear to be more in line with that reported in Hurst and Lusardi (2004), Nanda (2011), and Robb and Robinson (2014), which find no evidence that regular entrepreneurs face severe financing constraints. In particular, Robb and Robinson (2014), using the same data set as this paper, find that most entrepreneurs are able to finance their investment by simply going to the bank.

Does this mean that entrepreneurs don't face borrowing constraints? On the contrary, the calibration indicates that entrepreneurs face tight collateral constraints similar to that in the preceding literature. Note that while φ determines the *financial* frictions that arise from the limited commitment problem, the true collateral constraint is given by $\tilde{\varphi} \equiv (1 - \lambda) \times \varphi$, i.e. the entrepreneur can only collateralize the net depreciated *resale value* of entrepreneurial capital. When this effect is taken into account, the "net collateral constraint" parameter $\tilde{\varphi}$ is about 0.30, i.e. 30% of the real value of entrepreneurial capital is collateralizable. Given that

[24] Evans and Jovanovic (1989) report an equivalent point estimate of 0.44, Cagetti and De Nardi (2006) uses a benchmark calibration that amounts to 0.25 in this model. Other similar papers such as Kitao (2008) and Beura and Shin (2013) generally assume an equivalent collateral constraint of 0.33.

my calibration targets moments that are independent of the debt-to-asset ratio (a standard target of the macroeconomic literature for the collateral constraint parameter), my results in fact provide an additional independent support for the assumptions of the prior literature.

The main implication here then, is that rather than entrepreneurs simply facing tight financial constraints resulting from limited commitment (i.e small φ), a large portion of the observed "financial friction" can simply be attributed to the illiquid nature of an entrepreneur's capital (i.e large λ). Banks and other financial intermediaries recognize that used entrepreneurial capital have a low resale value, and hence adjust their lending according to this. The finding here is important. Unlike issues of limited commitment, which might be intrinsically hard to address, the friction arising from illiquid entrepreneurial capital can be addressed by a policy as straightforward as simply having a government policy that purchases an entrepreneur's capital at a higher price, or supporting leasing markets for capital. These policy options are further explored in section 6.

5 Discussion

I begin this section with a discussion of the model mechanisms that operate at the individual level. Following that, I show the aggregate outcomes of the micro-level interactions, and continue into a discussion of how entrepreneurship relates to the wealth distribution. I also connect the wealth distribution to the distribution of productivity, showing how high illiquidity can result in a fall in overall productivity. In particular, I emphasize a new channel of mis-allocation - where wealthy entrepreneurs operate low productivity firms due to capital illiquidity. Subsequently, I discuss the implications for welfare and aggregate total factor productivity relative to an economy where these frictions did not exist.

5.1 Model mechanism

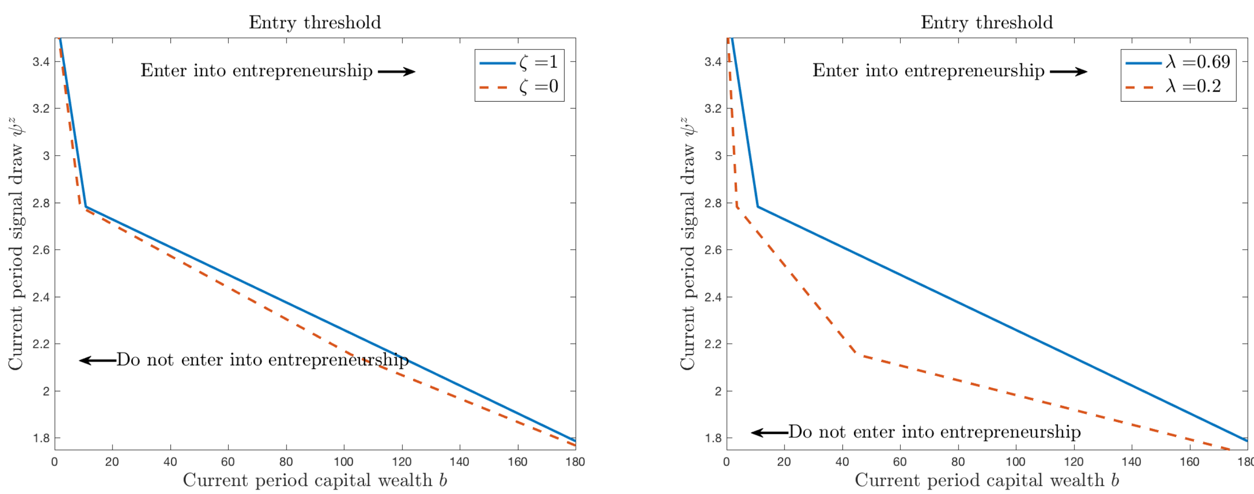
There are two main channels through which capital illiquidity operates: An *options value effect* channel, and a *collateral constraint* channel.

5.1.1 Options value effect

Entry margin

When capital is more illiquid (an increase in ζ or λ), the value of owning capital and the value of running a business falls from the perspective of potential entrants (holding all else constant). This happens because of a rise in illiquidity risk. Potential entrants compare the value of staying a worker, which guarantee a rate of return of r on their asset, against investing in a business where they run the risk of losing $(1-\lambda)\zeta$ of their investment if they have to exit the next period. When ζ or λ increases, the losses incurred by entrepreneurs become larger upon a bad shock. The value of entrepreneurship falls from the perspective of workers, thus raising the bar for entry into entrepreneurship. Given any wealth level, workers have to receive higher signal shocks (ψ^z) to be convinced to switch into entrepreneurship. Stated equivalently, potential entrants have to be richer (in terms of liquid wealth) in order to enter into entrepreneurship, in order to sufficiently self-insure against the illiquidity risk.

Consequently, many marginal potential entrants now choose to stay as workers. This generates a distortion along the entry margin, where the economy loses out on many potential entrants who should be entrepreneurs, but now choose to be workers instead. This effect plays out in figure 4, where I plot the entry threshold in (ψ^z, b)-space for different values of ζ and λ[25]. The space right of the threshold is the space where workers will choose to enter into entrepreneurship; conversely the space left of the threshold is where workers choose to stay as workers.



(a) ζ

(b) λ

Figure 4: Entry policy in (ψ^z, b) space for different values of ζ and λ.

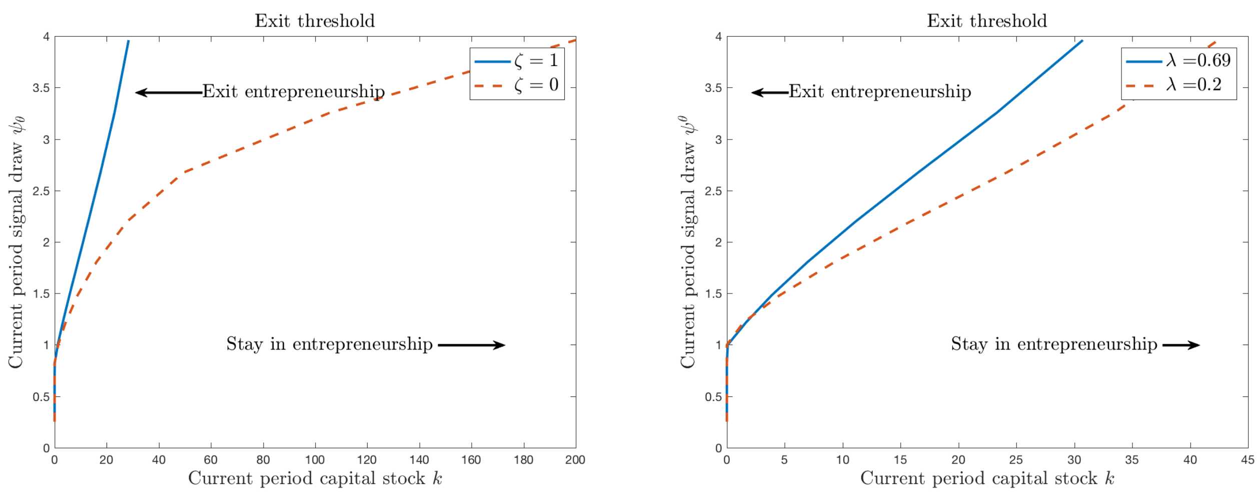
As we can see, the threshold shifts right when capital is more illiquid, meaning that for any given wealth level, the signal draw must be higher for a worker to switch into entrepreneurship.

Exit margin

[25]Figures 4 to 10 are drawn such that the net collateral constraint, $\varphi(1-\lambda)$ is held constant. This avoids the confounding secondary effect where changing λ also affects the collateral constraint. This latter effect is differed to the next subsection.

The choice to exit depends on two competing options: The value of exit and the value of entrepreneurship (i.e. staying in business).

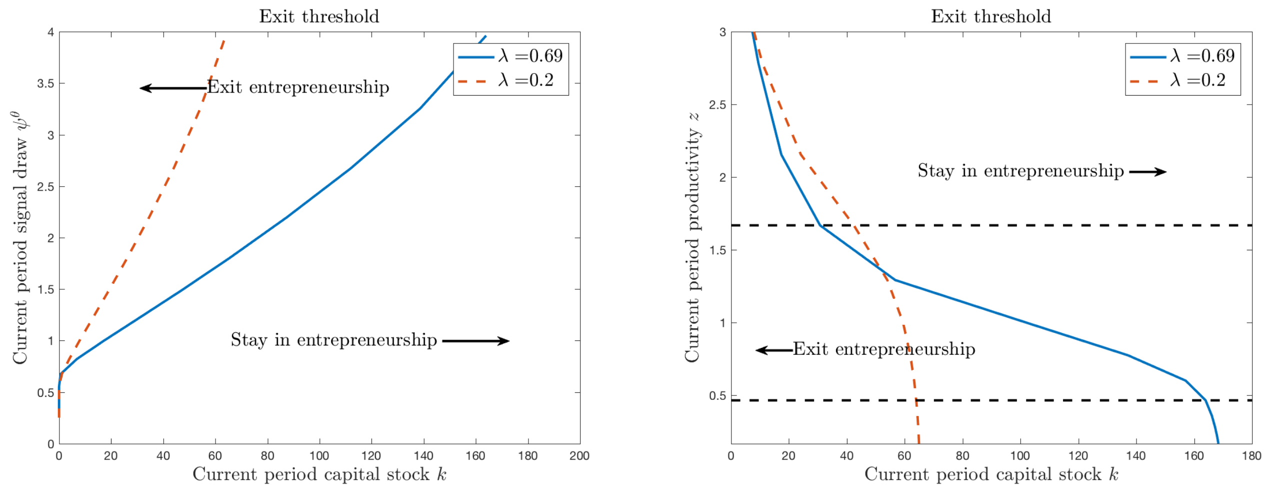
Higher transaction costs always decreases the value of the exit option, because the entrepreneur loses more of her capital when the transaction cost is higher. As such, fixing the continuation value of entrepreneurship, an entrepreneur must receive a better signal if she is to sell her business. This effect is reflected in figure 6 below. Here, I plot the exit threshold in (ψ^θ, k)-space for different values of ζ and λ. The space left of the threshold is the space where entrepreneurs choose to exit; conversely, the space to the right of the threshold is where entrepreneurs will stay in entrepreneurship. In both figures, we see clearly that the exit threshold shifts left when capital becomes more illiquid, meaning that for any firm size (in terms of capital stock), the signal required to trigger exit must be higher.



(a) ζ (b) λ

Figure 5: Exit policy in (ψ^θ, k) space for different values of ζ and λ (low = solid line; high = dashed line), holding b and z fixed. $z = 1.67$.

However, this propensity to exit is not strictly decreasing in the transaction costs. As discussed earlier, increases in transaction costs also decreases the value of entrepreneurship. As a result, the choice to exit is a two horse race between the value of entrepreneurship and the value of becoming a worker (i.e. exit): When the value of entrepreneurship falls less than the value of exit, the exit propensity of the entrepreneur will fall; in contrast, the exit propensity rises when the reverse happens. In figure 5b, we see a case when the former situation arises; in figure 6a below, we see the latter situation happening. As a result, in figure 6a, the threshold shifts *right* when λ increases.



(a) Exit policy in (ψ^θ, k) space for $z = 0.46$. (b same as in figure 5b)

(b) Exit policy in (z, k) space, holding ψ^θ fixed. The two black dashed lines are reference lines corresponding to the z values used in figures 5b (top line) and 6a (bottom line).

Figure 6: Exit policies in (ψ^θ, k)-space (left) and (z, k)-space (right) for different values of λ (low = solid line; high = dashed line).

Figure 6b provides another perspective on this effect, by showing the threshold in (z, k)-space for a single signal value ψ^θ, for two values of λ. There, notice that for *high* values of z, the exit threshold shifts left when λ increasing, indicating a decrease propensity to exit. In contrast, low values of z sees a rightward shift of the threshold, indicating an increase in the exit propensity. Higher z values corresponds to higher continuation value for entrepreneurs, and this shift in the threshold reflects the effect where entrepreneurs with high continuation values are more likely to stay in business when the transaction costs increase.

Finally, this two-horse effect predominantly shows up in variations in λ, as λ has both a direct impact on the value of entrepreneurship and the value of exit. In contrast, variations in ζ only has an indirect impact on the value of entrepreneurship - it indirectly lowers the value of entrepreneurship by lowering the value of exit. Consequently, increases in ζ generally decreases the exit propensity, as the fall in the value of exit due to ζ (a first order effect) is always greater than the fall in the value of entrepreneurship (a second order effect). We can see this in figure 7 below, which replicates figure 6b for two values of ζ.

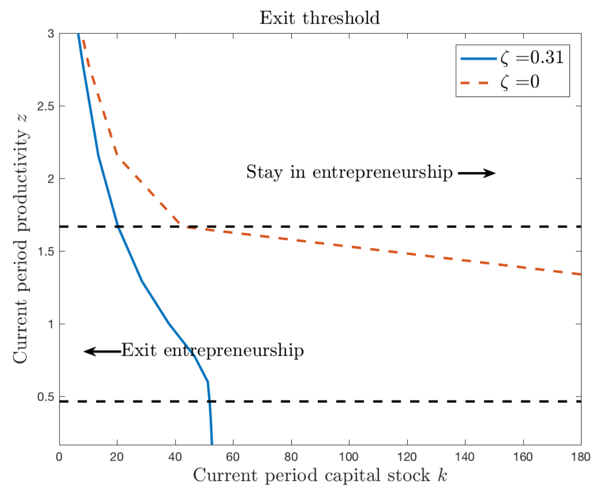


Figure 7: Exit policies in (ψ^{θ}, k)-space (left) and (z, k)-space (right) for different values of ζ. The two black dashed lines are reference lines corresponding to the z values in figures 5b and 6a .

Investment and disinvestment margin

Investment and disinvestment behavior also changes when capital becomes more illiquid. The put option value of capital falls when capital becomes more costly to sell, while the call option value of capital rises. Consequently, incumbents looking to invest will target smaller firm sizes, holding all else constant. In contrast, incumbents who are looking to downsize will operate larger firms. This is reflected in figure 8 below.

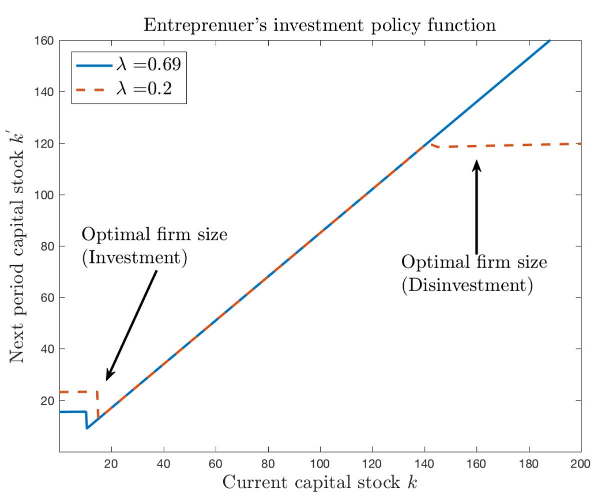
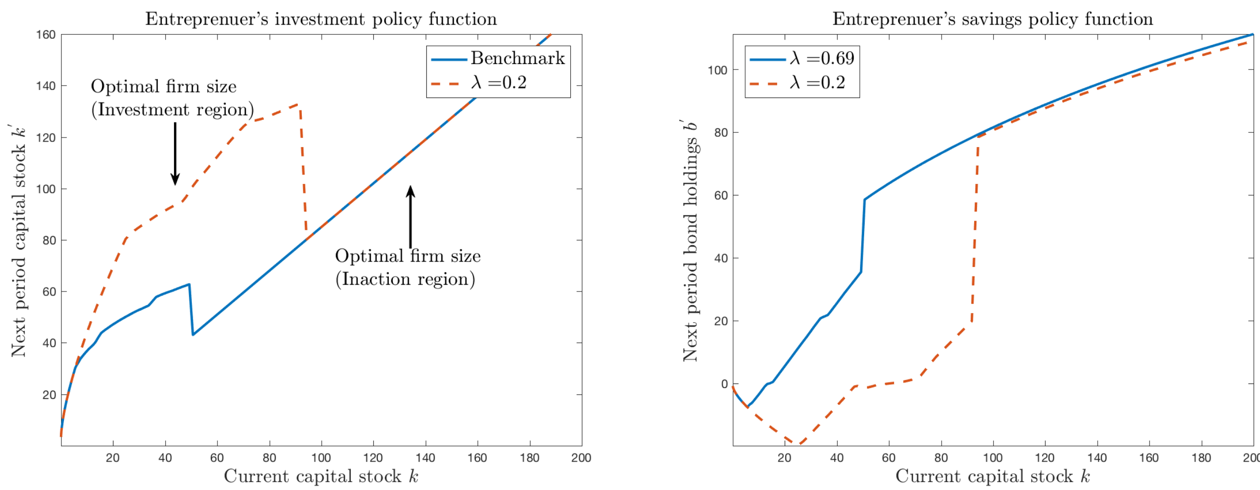


Figure 8: Next period capital holdings policy for different values of λ (low = solid line; high = dashed line)

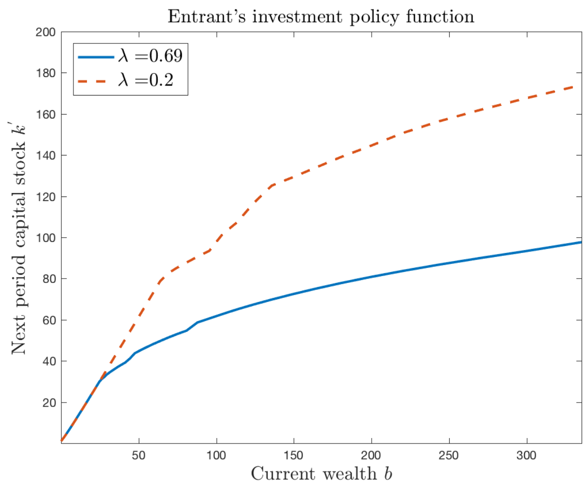
Moreover, the illiquid nature of capital means that households are less able to use their physical capital stock as a means to smooth consumption. In the case of a model where bonds and capital are perfect substitutes, the household does not have to worry about losing their investment in the event of a bad shock. In the worst case scenario where $z \to 0$, the return to investment is $-\delta$ (i.e. the user cost of capital). In contrast, with $\lambda > 0$ and/or $\zeta > 0$, the household also loses more capital due to the transaction costs. As such, for entrepreneurs in an economy where capital is more illiquid, entrepreneurial households will demand more bond holdings to insure themselves against illiquidity risk. Taken together, entrepreneurial households who are expanding their business will, in general, tend to operate smaller firms, and hold more liquid assets, as we can see in figure 9 below.



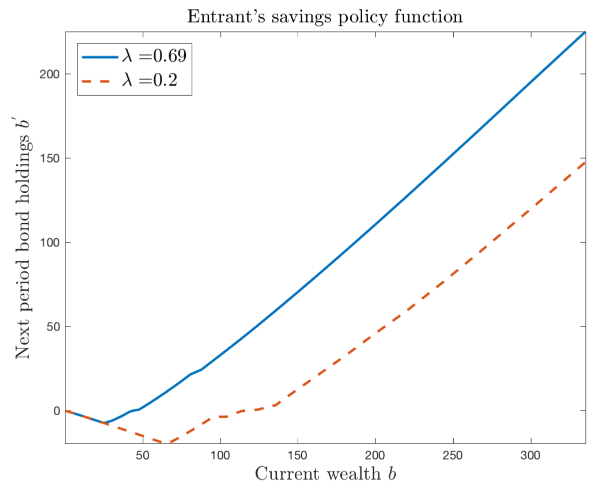
(a) Next period capital holdings vs current capital stock (b) Next period bond holdings vs current capital stock

Figure 9: Entrepreneurs: Incumbent's investment (left) and savings (right) for different values of λ (low = solid line; high = dashed line)

This effect doesn't only affect current incumbents. Workers that are potential entrants, who are not directly affect by λ, also invest in smaller firms and save more, in anticipation of the illiquidity risk that they have to bear upon entry, as we see in figure 10 below.



(a) Next period capital holdings vs current capital stock



(b) Next period bond holdings vs current capital stock

Figure 10: Worker: Entrant's Investment (left) and savings (right) for different values of λ (low = solid line; high = dashed line)

5.1.2 Collateral constraint effect

In addition to the options value effect, which is the primary distortion to the economy, the illiquid nature of capital also reduces the collateral value of capital. Recall that households can only borrow up to $b' \geq -\varphi(1-\lambda)(1-\delta)k'$. Similar to the collateral constraint effect resulting from financial frictions (represented by φ in this model), households now need to accumulate more wealth in order to enter into entrepreneurship when $\lambda > 0$. However, unlike financial frictions, this mechanism is not a result of financial contracting issues that φ represents. Instead, this financing constraint simply reflects the fact that the value of capital has been lowered by $1 - \lambda$.

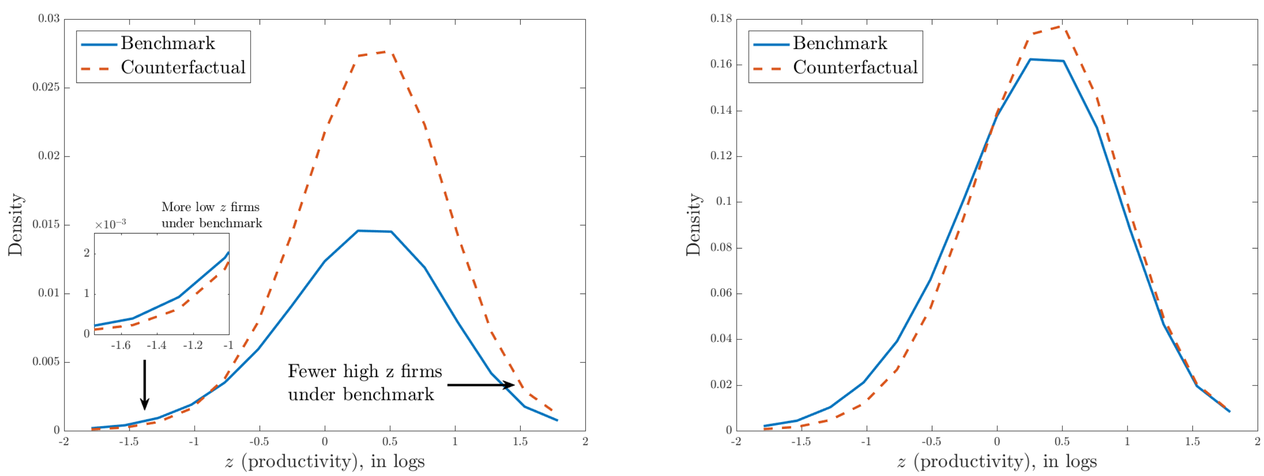
What is the difference between φ and λ with regards to their impact on financial constraints? Note that while both constraints make it harder for households to enter into entrepreneurship, φ does not distort the financially unconstrained optimal firm size; the primary channel of φ is to force households to operate businesses that are "too small" for extended periods of time while they try to accumulate enough capital. In contrast, λ (and ζ) decreases the financially unconstrained optimal firm size, due to the effect of illiquidity risk. That is, even for households who are financially unconstrained, they will choose to operate smaller businesses. For financially constrained households, higher λ has a double whammy effect where it makes them even more constrained while simultaneously lowering their business income (since they operate smaller optimal firm sizes).

5.2 Aggregate Outcomes: Illiquidity, Occupational and Portfolio Choice, and the Allocation of Capital

In this section, I discuss the aggregate impact of illiquidity by contrasting it with a counter-factual economy where capital was fully liquid (i.e $\lambda = 0$ and $\zeta = 0$). Here, I discuss the results in the context of the allocation of capital along the intensive margin, and the allocation of entrepreneurial talent along the extensive margin. The positive analysis done here will then become relevant for a discussion on welfare analysis in the later half of this section. Note that the counter-factual economy presented in this section is solved under partial equilibrium, fixing the interest rate and wages to the benchmark calibration. Moreover, the collateral constraint parameter is re-calibrated such that the net borrowing constraint stays constant. This ensures that the discussion here isolates the impact solely to illiquidity, instead of changes due to price or financial conditions.

5.2.1 Extensive margin

Figure 11 below plots the distributions of entrepreneurial productivities. I first refer the reader to figure 11a, which plots the un-normalized distribution.



(a) Un-normalized distribution

(b) Normalized distribution

Figure 11: Distribution of productivity (z) under the benchmark and counterfactual calibrations. Entrepreneurs, respectively, account for about 9 and 15 percent of the population under the benchmark and counter-factual calibration.

There are two distinct effects arising from illiquidity. Firstly, we can see that there is a clear selection towards low skilled entrepreneurs and away from high skilled entrepreneurs. In the left tail of the distribution, we see that there are more low skilled entrepreneurs under the benchmark calibration; in the right tail, we see that there are less high skilled entrepreneurs under the benchmark calibration. This happens because of the options value effect discussed earlier. For low skilled entrepreneurs, the wait-and-see effect generated by the exit costs lead them to persist longer, therefore giving rise to more low skilled entrepreneurs. For high skilled entrepreneurs, the distortion along the entry margin means that even conditional on the same signal, potential entrants need to be wealthier in order to enter into entrepreneurship. As a result, there are less high skilled entrepreneurs under the benchmark calibration. Notice that this result is not simply a consequence of the lack of normalization. In figure 11b, we see that the effect persists; in fact, it becomes even starker on the left tail.

Secondly, illiquidity also leads to a fall in the size of the population of entrepreneurs. We can see this from the un-normalized distribution, where the area under the density function of the counter-factual economy is visually bigger than that of the benchmark calibration. The fact that this population falls is not an obvious result, but rather the interaction between the exit and entry rates along the transition path towards the steady state.

To understand this point, first recall from the earlier discussion that decreases in the resale value of capital will always lead to a fall in the propensity of entry, but have an ambiguous impact on the propensity to exit. However, in steady-state, as the exit rate must always equate the entry rate, the net result of falling resale values means that the overall rate of reallocation at the extensive margin (i.e. the equilibrium rate at which households switch occupations) must fall. Indeed, this is reflected in figure 12, which plots the number of households who are entering entrepreneurship in steady-state, as a function of ζ (bottom figure) and λ (top figure).

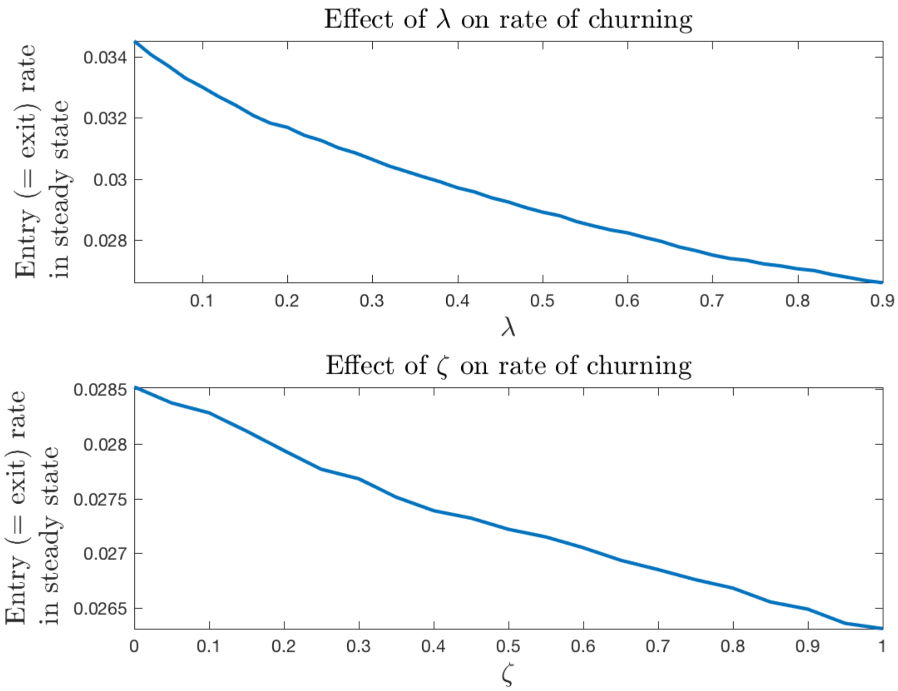


Figure 12: Effect of ζ and λ on steady state entry and exit rates

However, unlike their impact on the entry and exit rates, the impact of capital illiquidity on the fraction of households that are entrepreneurs in steady state can be ambiguous for two reasons. Firstly, the rate of job switching simply gives us an idea on how active reallocation is being pursued, but does not give us any information on the number of entrepreneurs in steady state. Secondly, since the exit and entry rate both simultaneously decrease in steady-state, the steady state number of entrepreneurs depends on which of the two forces are stronger along the transition path towards the steady state. If the exit rate falls at a slower rate than the entry rate, the number of entrepreneurs will decline along the transition path towards the new steady state; on the other hand, if the exit rate falls at a faster rate than the entry rate, the number of entrepreneurs will rise as the economy transitions to the new steady state.

This point can be best illustrated by studying the transition path of the benchmark economy to the counter-factual economy where capital is fully liquid. Here, I consider a perfect foresight equilibrium where the transaction costs are suddenly removed, and in figure 13, trace out the equilibrium responses of the aggregate entry and exit rates, as well as the size of the entrepreneur population[26]. We see that along the transition path, the entry rate rises faster than the exit rate; as a consequence, the steady state population of entrepreneurs

[26] For a full discussion of the model, I refer the reader to the appendix

is larger under the counter-factual calibration than the benchmark calibration.

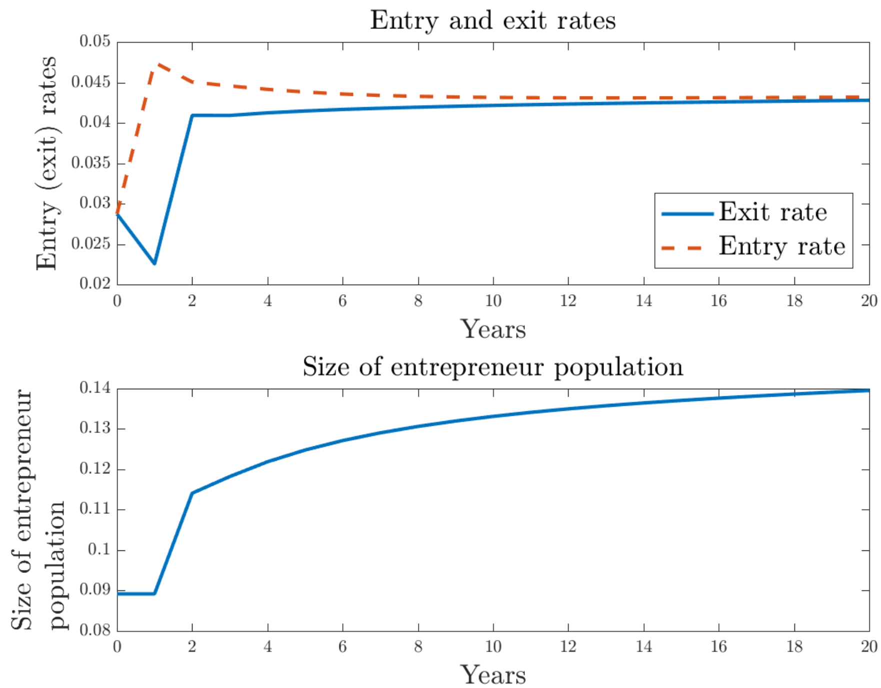
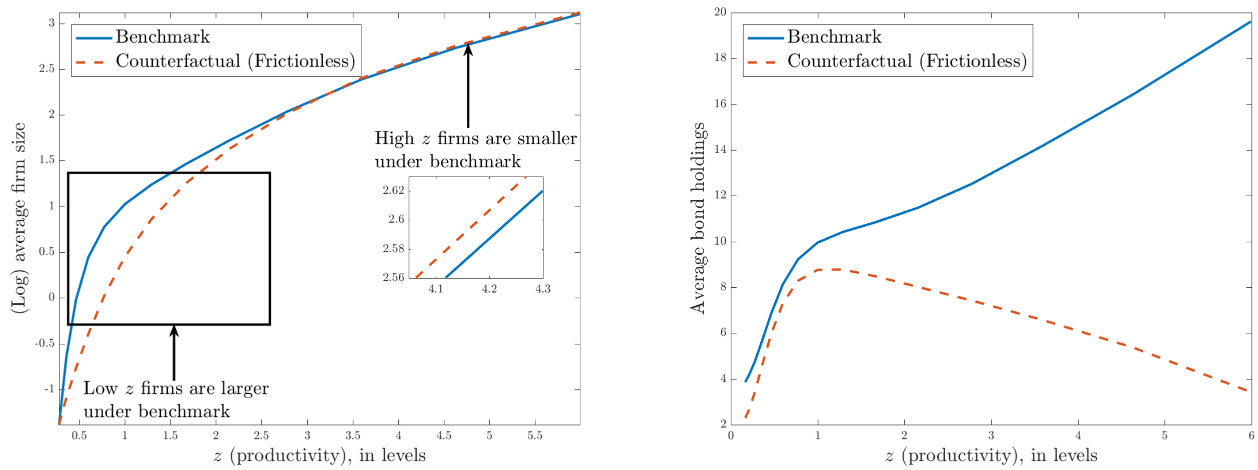


Figure 13: Response of the entrepreneur population size, and entry and exit rates, to a sudden removal of all illiquidity frictions.

5.2.2 Intensive margin

Now, I direct the reader to figure 14a, where I plot the average firm size for each level of productivity (z) under the benchmark calibration, and a counter-factual economy where all resale frictions are removed. Here, we see that under the benchmark calibration, low productivity entrepreneurs operate larger firms on average; moreover, high productivity entrepreneurs operate smaller firms on average.

This result stems from the asymmetric purchase and resale price of capital, which leads incumbents to become more cautious in investing and dis-investing. Entrepreneurs who receive a good productivity shock will invest less than the unconstrained optimum, as they want to avoid being trapped with too much capital in the event of a downturn in the following periods; in contrast, entrepreneurs who receive a bad shock will disinvest less than the unconstrained optimum. In this latter case, entrepreneurs are hedging against the event of an upturn in the following periods. Since newly purchased capital is more expensive than used capital, entrepreneurs would like to reduce their expenditure in the case of an upturn. This leads poorly performing entrepreneurs to hold on larger capital stocks.



(a) Average firm size (capital, log scale) vs Productivity (z).

(b) Average bond holdings vs Productivity (z).

Figure 14: Average firm sizes and bond holdings as a function of productivity.

In addition, capital illiquidity means that households are less able to use their physical capital stock as a means to smooth consumption. In the case of transitory bad shocks, entrepreneurs are unwilling to disinvest much of their capital due to the effect discussed earlier. Consequently, physical capital serves as a poor instrument to insure households. Holding all else constant, this drives households to accumulate relatively more bonds, thus leading to a portfolio with greater liquid assets. We see this in figure 14b below, which plots the average bond holdings of entrepreneurs against their productivity levels. Here, we see that across all productivity levels, entrepreneurs in the benchmark calibration have higher average bond holdings than their frictionless counterpart. Moreover, the average bond holdings is increasing under the benchmark calibration, whereas it has a "hump shape" for the counter-factual. This aggregate outcome reflects the results reported in figure 5.1.1. In the counter-factual economy, entrepreneurs are willing to take up more debt to finance their investment; as a result, high productivity entrepreneurs tend to be more indebted and hence have lower average bond holdings. In contrast, the fear of illiquidity risk drives entrepreneurs under the benchmark calibration to hold more liquid assets, thus leading the average bond holdings to increase.

In fact, taken together, this effect looks very similar to a financial constraint, such as that in Cagetti and De Nardi (2006): (High productivity) entrepreneurs operate firms that are "too small", and tend to accumulate more bonds. However, unlike the literature on financial constraints, this effect is through a different channel. Where the small firm size in the Cagetti and De Nardi (2006) framework is an indication that the entrepreneur would like to expand if she could, the smaller firm sizes here is a result of households simply having no desire to expand to the unconstrained optimum. Likewise, where wealth accumulation in the face of financial frictions results from households trying to save out of their constraints, higher (liquid) wealth accumulation here is simply a result of households trying to self-insure against illiquidity risk.

5.3 The Wealth Distribution

Table 9 below reports the wealth distribution in the US (row 5), and contrasts the model predicted wealth distribution under the benchmark calibration (row 1), and a counter-factual economy where there are no resale frictions. For the model with no resale frictions, I reset the collateral constraint parameter to $\tilde{\varphi}$, such that $\tilde{\varphi} = (1 - \lambda)\varphi$, thus keeping the net collateral constraint constant. The no resale friction model is solved under partial equilibrium (row 2), where I keep the interest rate r at the benchmark calibration, as well as under general equilibrium (row 3). It also reports the Gini coefficient as a single summary statistic for the extent of wealth inequality predicted by the model. I also report the wealth distribution for an equivalent model where entrepreneurship is not modeled (i.e. the plain Aiyagari model).

Concentration of wealth in:	Quintile range					Top 10%			Gini
Model/Data	1st	2nd	3rd	4th	5th	90-95	95-99	99-100	
(1) Benchmark	1.5	4.2	7.9	15	72	11	23	23	0.68
(2) Frictionless (PE)	1.1	3	5.7	11	83	14	28	25	0.75
(3) Frictionless (GE)	1.1	3	5.5	11	80	9.6	29	31	0.76
(4) No entrepreneurs (GE)	1.2	9.1	24.6	28.2	37.0	14.6	16.3	6.1	0.57
(5) Data (USA)	-0.39	1.7	5.7	13	79	13	24	30	0.80

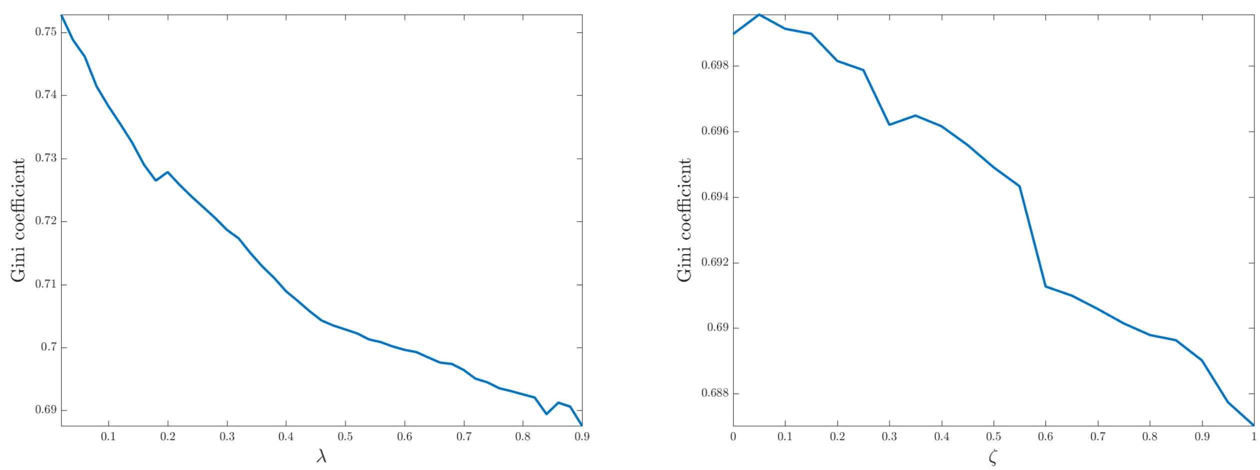
Table 9: Wealth distribution

First, referring to row (1), one notices right away that the model with entrepreneurs is strictly closer to the empirical wealth distribution than a plain vanilla model without entrepreneurs (row 4), in terms of its ability to match the right tail of the wealth distribution. This observation has been made by the preceding literature, and is replicated here. Just as Benhabib et al (2015) note in their paper, the un-insurable, persistent and highly volatile returns to capital leads households to accumulate more wealth, thus extending the right tail of the wealth distribution. Entrepreneurial returns, which is a form of capital income risk, bears this feature, and thus leads to a model economy that features a more unequal distribution of wealth.

However, the wealth distribution under the benchmark calibration is more egalitarian than the actual US wealth distribution. For instance, the top 1% holds about 23% of the economy's wealth; in contrast, the top 1% holds about 30% of the economy's wealth in the data. Likewise, if we look at the Gini coefficient as a measure of dispersion, the benchmark model has a lower Gini coefficient than the true economy. In contrast, prior research such as that by Cagetti and De Nardi (2006) have utilized entrepreneurship to successfully match the wealth distribution.

5.3.1 Revisiting Entrepreneurship and Wealth Inequality

Why are my results different from the prior literature? The key reason can be attributed to the illiquidity of capital[27], which reduces the overall returns to capital, and thus reduces wealth dispersion in the economy. Referring now to rows (2) and (3) of table 9, one sees that a removal of all illiquidity frictions will lead to an increase in the extent of wealth inequality, regardless of general equilibrium effects. In fact, the model wealth distribution in the counter-factual economy almost approaches that of the empirical wealth distribution.



(a) λ (b) ζ

Figure 15: The effect of illiquidity on wealth inequality. The Gini coefficient for the model economy is computed for different values of λ and ζ. The Gini is computed for a fixed interest rate to avoid conflating general equilibrium effects with the pure illiquidity effect.

[27] There is a secondary reason arising from a relatively low returns to scale. I discuss this in the appendix.

How does capital illiquidity affect returns to investment?

Firstly, the transaction costs directly impact investment returns by simply increasing the user cost of capital. Since every entrepreneur has a non-zero probability of exit, this implies that the user cost of capital is larger than the depreciation rate. Consequently, the net return to investment decreases.

Secondly, recall that in section 5.1.1, I showed that the higher transaction costs lead entrepreneur to operate smaller than optimal firm sizes (if they are investing); moreover, they devote a larger portion of their portfolio to buying bonds. This diminishes the ability of the entrepreneur to accumulate wealth. Smaller firm sizes mean that the entrepreneur is making less business income than she could have; while a larger liquid asset stock means that the entrepreneur could strictly be better off moving some bonds into his firm. The gross return on asset, which is $zf(k,l) + rb$, is thus lower.

Finally, section 5.1.1 also showed that many entrepreneurs are operating larger than optimal firm sizes. Just as when firms are too small, firm sizes that are too larger also reduce the gross return on asset. In this case, the entrepreneur could be strictly better off transferring her firm's assets into bonds.

The combined effect leads entrepreneurs to earn lower lifetime incomes when capital is more illiquid, due to the depressed average returns to capital. As a result, the overall income distribution becomes more compressed, and overall wealth dispersion falls. This result is reflected in figure 15, where we see that the Gini coefficient increases as capital becomes more liquid (i.e. as λ or ζ decreases).

The results here show that, while entrepreneurship does improve the model fit of an otherwise standard incomplete markets model to the wealth distribution, it does not fully generate the stark wealth dispersion observed in the data. The finding stands in contrast to the prior literature, which often report exceedingly good model fit to the wealth distribution when entrepreneurship is modeled in a Bewley-Huggett-Aiyagari framework. One of the key differences between my model, and the preceding literature, is that none of the preceding research has considered ***illiquidity*** as an element affecting entrepreneurial investment dynamics. The findings in this paper therefore suggest that macroeconomic models of entrepreneurship that ignore illiquidity might be in fact over-stating the contribution of entrepreneurial investment to the dispersion of wealth in the data.

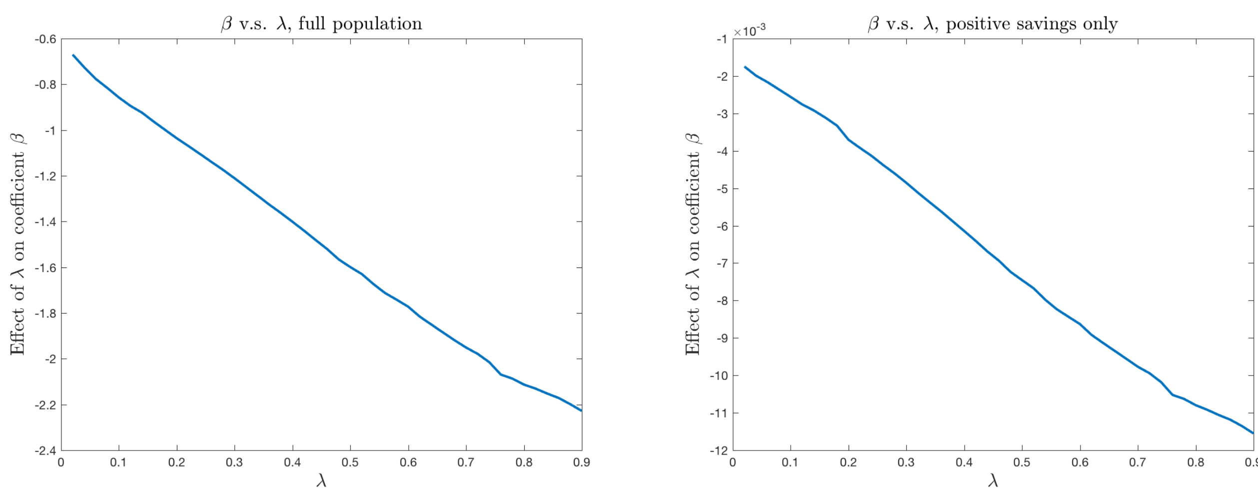
5.3.2 Frictions and the distribution of wealth and capital

As discussed earlier, capital illiquidity leads entrepreneurs to accumulate (relatively) more liquid assets as a buffer stock of savings. Moreover, this also drives low productivity entrepreneurs to operate larger-than-optimal businesses due to the options value effect. As a result, entrepreneurs who are wealthy also tend to operate larger firms that are less productive. Consequently, in a distributional sense, we will see that wealthier entrepreneurs are also operating firms that have lower average revenue product of capital.

A direct way to observe this effect is to consider the correlation between liquid wealth and ARPK. For instance, we could fit a regression of the form

$$\log ARPK_{i,t} = \beta_{j,0} + \beta_{j,1} \log b_{i,t} + \epsilon \tag{2}$$

, where $ARPK_{i,t}$ is the APRK of individual i's firm at time t, $b_{i,t}$ is the total bond holdings (including debt) of individual i at time t, and $\beta_{j,1}$ would reflect the strength of the correlation between bond holdings and the rate of return to capital for that entrepreneur. $\beta_{j,1}$ is indexed by $j \in \{\lambda, \varphi\}$, reflecting that the correlation depends on the extent of illiquidity or tightness of collateral constraints in the economy. This relationship can be re-estimated for multiple values of λ and φ, and the resulting relationship between $\beta_{j,1}$ and λ or φ can then be investigated. As suggested earlier, increasing illiquidity leads to wealthier entrepreneurs earning lower returns on capital. Indeed, this is reflect in figure 16 below, where we see as the extent of illiquidity increases in the economy (i.e. λ increases), wealth becomes increasingly negatively correlated with ARPK.



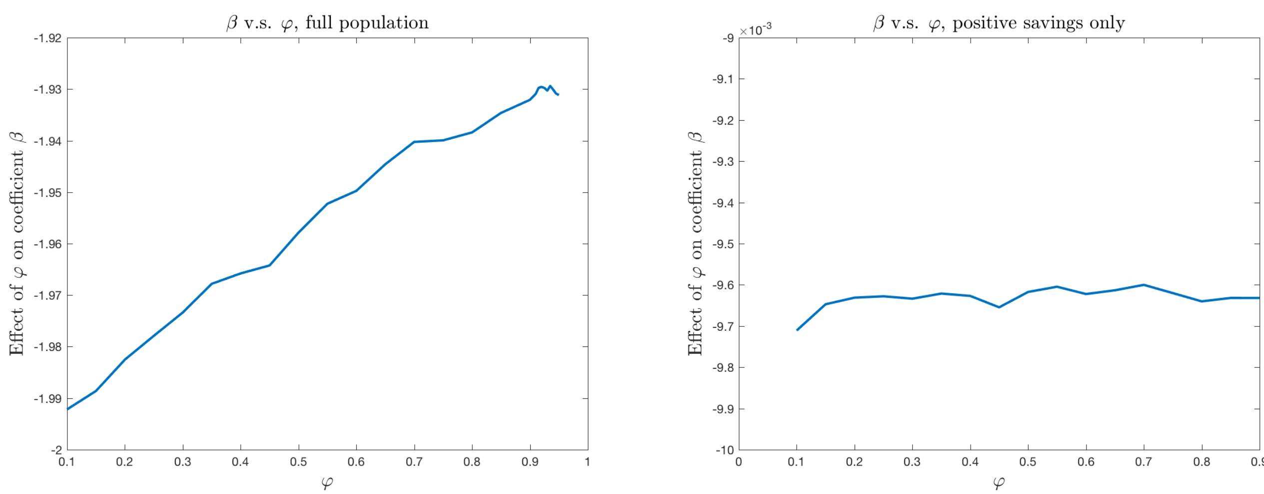
(a) Full population of entrepreneurs

(b) Only entrepreneurs with positive savings

Figure 16: Effect on β_1 for different λ. Confidence intervals are not provided as the regressions are computed using the population moments.

How does this contrast with an economy where the frictions are driven by changes borrowing conditions? When capital mis-allocation is driven by borrowing constraints, only low wealth entrepreneurs and potential entrants are affected. This happens because these entrepreneurs are unable to raise enough funds to target the optimal firm size, or in the case of potential entrants, operate at a size that is sufficiently profitable to warrant entry. In contrast, wealthy entrepreneurs and new entrants are completely unaffected by the borrowing constraints, and would see no correlation between their wealth and their firm's average revenue product of capital. As a result, when we consider the entire population of entrepreneurs, we should see that tightening collateral constraints would increase the correlation between wealth and ARPK. In contrast, when we consider only the sub-population of wealthy entrepreneurs, we should see no change in correlation between wealth and ARPK when collateral constraints change.

In fact, this is exactly what we see run the same regression (2) as above for different values of φ, the collateral constraint parameter. In figure 17a, we see that when we run the regression above for the whole population, wealth becomes less negatively autocorrelated with ARPK [28] when collateral constraints are relaxed. However, when we focus only on entrepreneurs with positive savings (i.e. $b > 0$), the correlation between wealth and ARPK essentially does not change. In contrast, referring back to figure 16b, we see that increasing illiquidity is still negatively correlated with the rate of return to capital for entrepreneurs with net positive wealth.



(a) Full population of entrepreneurs

(b) Only entrepreneurs with positive savings

Figure 17: Effect on β_1 for different φ. Confidence intervals are not provided as the regressions are computed using the population moments.

[28]the correlation is still negative, as the comparative statics exercises are done with all other parameters held at their benchmark calibration

5.4 Aggregate welfare and productivity

In this section, I discuss the effect of the distortions on two key aggregate outcomes: Welfare and Productivity. Prior to beginning this section, I first provide some definitions to give the reader a consistent reference to the terms I am referring to.

Welfare

Welfare here is measured as the consumption equivalent variation between a reference economy (given by a superscript "R") and the current economy of interest (given by a superscript "N"). In particular, for any household i, the consumption equivalent variation μ_i solves the following problem:

$$V((1+\mu_i)c_i^R) = V(c^N) \tag{3}$$

To compute μ_i for each household, I then simply solve the preceding equation for each point in the state space $\mathbb{S}$. Given this definition, $\mu_i > 0$ implies that the household prefers the new economy over the reference economy.

Having derive the entire distribution of μ_i, I then compute aggregate (average) welfare as

$$\widehat{\mu} = \int \mu_i d\Lambda \tag{4}$$

Moreover, I am also able to compute welfare changes for subsets of the economy $\Lambda_s \subseteq \Lambda$. For instance, if I wish to compute the average welfare change for only workers, I can compute

$$\widehat{\mu^w} = \frac{\int \mu_i \times \mathbb{I}_w d\Lambda}{\int \mathbb{I}_w d\Lambda} \tag{5}$$

where $\mathbb{I}_w$ is an indicator function that equates to 1 if that household is a worker, and 0 otherwise. The denominator is necessary in order to re-normalize the distribution (i.e. compute the conditional average), in order to avoid conflating the changes in the measure of workers (which is endogenous) and actual changes in the agent's welfare (or distribution of CEV).

Note that the consumption equivalent variation is computed taking into account of the full transitional dynamics of the economy. I refer the reader to the appendix for a full description of how this is computed.

Aggregate TFP

"Aggregate TFP" here refers specifically to the aggregate TFP in the entrepreneur sector. Since the corporate sector is composed of representative firms, there cannot be any TFP losses with respect to that sector, and hence I choose not to discuss it here[29].

Here, my concept of TFP stems from the perspective of a statistician who only observes the aggregate capital stock, labor input, and output of the entrepreneur sector. In this case, we can derive TFP $\overline{Z}$ as

$$\overline{Z} \equiv \frac{Y^e}{K_e^{\alpha_e \nu} L_e^{(1-\alpha_e)\nu}} \tag{6}$$

Comparisons of TFP are done across steady-states.

Average productivity of entrepreneur

As the model exhibits endogenous entry and exit, the steady-state distribution of *observed* idiosyncratic TFP is different from the underlying distribution of shocks. The selection effects can either increase (positive selection) or decrease (negative selection) the average productivity of entrepreneurs, and in turn, can have powerful effects on aggregate TFP. For this paper, I define the average productivity of an entrepreneur as

$$\overline{Z^e} = \frac{\int z_i \times \mathbb{I}_e d\Lambda}{\int \mathbb{I}_e d\Lambda} \tag{7}$$

where $\mathbb{I}_e$ is an indicator function that equates to 1 if that household is an entrepreneur, and 0 otherwise.

Comparisons of average productivity are also done across steady-states.

Having clarified these definitions, we can now examine the impact of these distortions on the aggregate economy.

5.4.1 Welfare and productivity

Table 10 reports the aggregate TFP, average productivity of entrepreneurs, and the average welfare change for the two asset economy, as well as the counter-factual economy with no resale frictions. Both the PE and GE results are reported for the counter-factual example. The welfare results are reported in terms of percentage consumption equivalent variation.

[29] This does not mean that the corporate sector is not important in this model. In fact, the amount of (effective) capital allocated to the corporate sector reflects the change in the precautionary savings behavior of the households, which is an important dimension of this model relative to models of complete markets

	Two assets, GE	No Frictions, fixed r	No Frictions, GE
TFP	1.21	1.44	1.34
Average productivity, entrepreneur	1.6	1.67	1.68
Average welfare (entrepreneur)	0	2.5	4.1
Average welfare (worker)	0	7.8	7.7
Average welfare (economy wide)	0	3.0	4.4

Table 10: Changes in TFP, average productivity of entrepreneur, and welfare, for counterfactual frictionless economy relative to benchmark. Welfare is computed as percentage consumption equivalent variation.

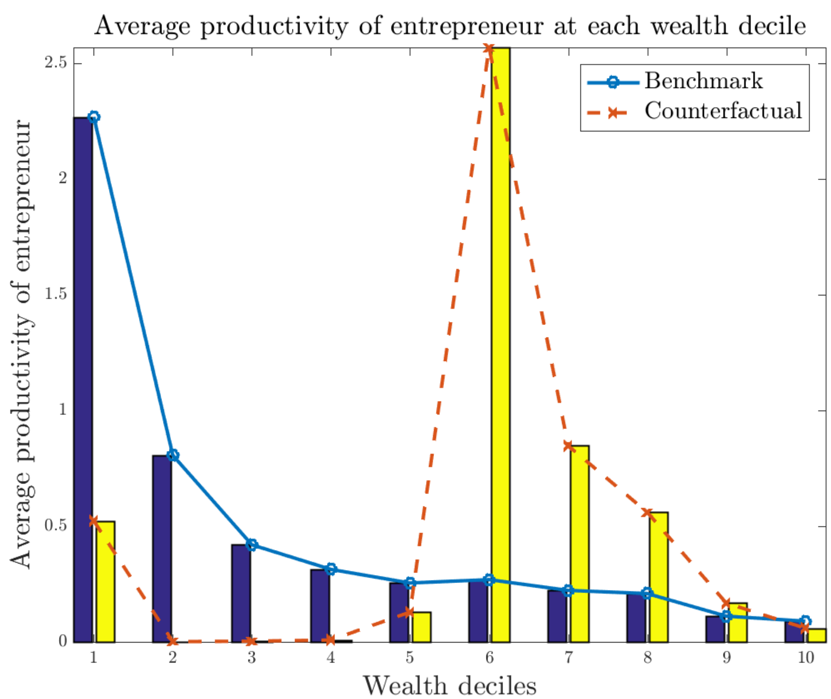


Figure 18: Average productivity of entrepreneurs for each 5% quantile along the wealth distribution, under the benchmark and counter-factual calibrations.

As we can see, there are substantial losses to TFP and welfare, regardless of whether we take into account of equilibrium effects or not. This happens exactly because of the effects discussed in section 5.2: The distortion along the external margin selects for low productivity entrepreneurs and against high productivity entrepreneurs; the distortion along the internal margin leads low productivity entrepreneurs to operate firms that are too large, and high productivity entrepreneurs to operate firms that are too small. Moreover, it also results in wealthy entrepreneurs who are less productive, as discussed in section 5.3.2 earlier, and also reflect in figure 18.

6 Other Implications

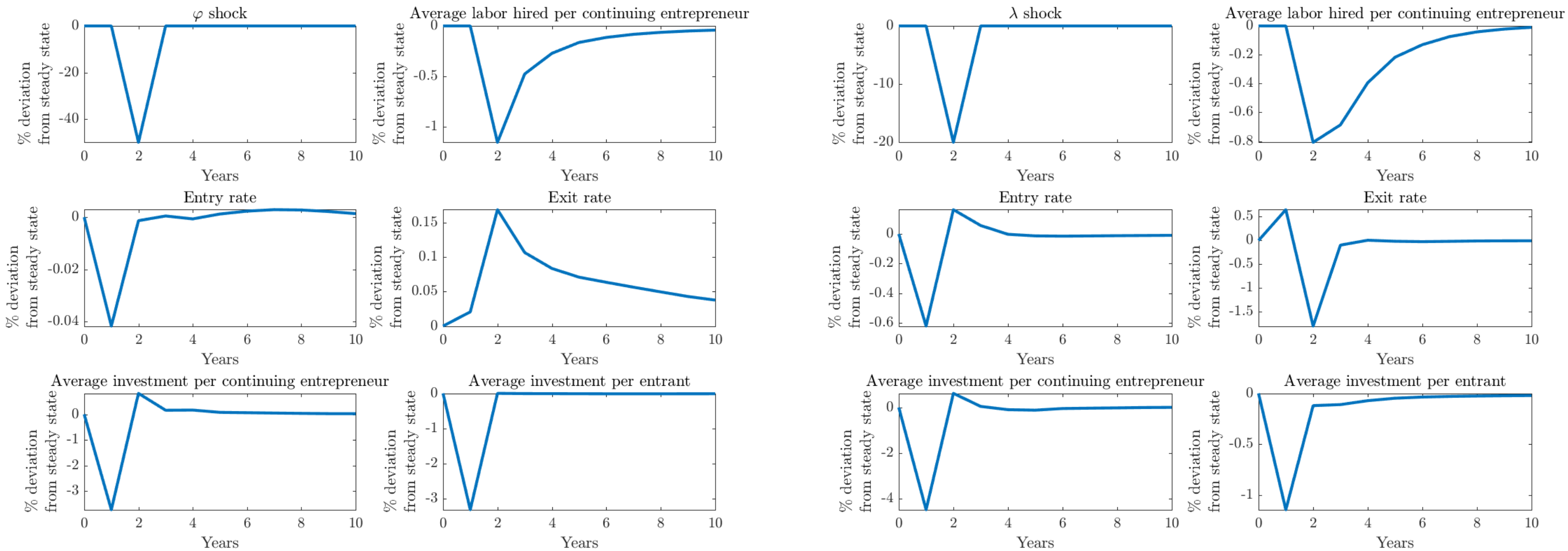
6.1 Implications for Finance Literature: Collateral Constraints or Illiquidity?

Recent research in the entrepreneurship literature has focused extensively on understanding the contribution of financing constraints to mis-allocation in entrepreneurial outcomes. At the micro-level, recent research have focused on testing for the existence of collateral constraints; at the macro-level, recent research have emphasized a collateralized borrowing channel to explain the propagation of credit shocks to capital or labor markets. In this section, I explain how my findings relate to this literature.

To begin the discussion, first consider two scenarios: (A) The economy suffers a negative temporary shock to the collateral constraint parameter, such that φ falls to φ' for one period before recovering; (B) The economy suffers a negative temporary shock to the resale value of used capital, where λ increases to λ' for one period before recovering. Scenario A corresponds to a sudden tightening of financial conditions, whereas scenario B corresponds to a sudden fall in the used price of capital. In the case of scenario B, φ is readjusted such that the net collateral constraint is fixed. Like in the earlier counter-factual exercises above, this ensures that the extent of financial frictions is held constant, allowing me to isolate the impact of falling resale prices. Figure 19 report the aggregate response of labor hiring, entry and exit rates, and investment, to a 50% decrease in φ and a 20% decrease in the resale price of capital (corresponding to an approximately 9% increase in λ).

The main thing that one notices is the remarkable similarity of the aggregate responses to a resale price shock and collateral constraint shock. In both cases, hiring, investment and entry rates fall upon impact, and exit rates increases. This finding is important, because the policy response to a financing constraint is very different to one where the value of capital has fallen. For instance, recent research studying the used capital market have shown that the resale price of used capital can be pro-cyclical. As such, this suggests that a portion of the decline in entrepreneurial startup rate or investment over the course of a recession can potentially be attributed to a natural response to the decline in the resale price of capital. Findings that attribute the decline in entrepreneurial startup or investment solely to financing constraints could therefore be over-estimating the extent to which financial constraints have tightened. Moreover, simply looking at aggregates such as the startup rate, exit rate or investment is insufficient to help researchers distinguish between the two sources of distortion.

Another important point relates to recent research that have attempted to estimate the existence and impact of financing constraints on entrepreneurial startup and investment, using exogenous regional variations in asset prices as an identifying instrument. In that literature, exogenous variations in house prices are often used as a proxy for variations in the extent of collateralizable assets, with the argument that houses serve as collateral for entrepreneurs who are taking out loans. A concern that this paper raises is that, to the extent that house prices are correlated with resale capital prices, the correlations detected by the preceding research might be reflective simply of the illiquidity effect rather than collateral constraints. Indeed, as figure 19 shows, liquidity shocks to the price of used capital can generate decreased economic activity independently of financial shocks.



(a) φ shock (b) λ shock

Figure 19: Effects of (a) a temporary tightening in collateral constraints (i.e. negative shock to φ), and (b) a temporary fall in resale value of capital (i.e. negative shock to λ). λ shock refers to a decline of 20% to resale value of capital $1-\lambda$, corresponding to a 9% decline in λ. All aggregate variables reported here are normalized by the appropriate measure, such that we can interpret the results as average variables.

6.2 Implications for Fiscal Policy

In section 4.1.2, I had discussed the fact that entrepreneurs do not necessarily face tight financial constraints arising from limited commitment; rather, the low resale value of capital induces a collateral constraint effect on top of the illiquidity effect. In this section, I show using a sketch, an example of how a straightforward policy that reduces the cost of reallocation can improve welfare and TFP.

In this example, the government commits to a fixed budget G to increase the resale price of capital from $1-\lambda$ to $1-\tilde{\lambda}$. In other words, the government commits to paying the difference $\lambda-\tilde{\lambda}$ for every unit of used capital transacted, and the total cost is paid for by G. From the perspective of households, this policy has two benefits: Firstly, it increases the resale value of capital, and therefore reduces the amount of illiquidity risk the individual faces; secondly, it relaxes the collateral constraint. To finance this policy, the government taxes bond returns. The cost is therefore only borne by relatively wealthier households who have positive liquid savings.

Using the notation established in section 3, the equilibrium of this model can be defined as follows:

Definition of equilibrium A stationary competitive equilibrium of the model consist of the interest rate r, wage rate w, used capital resale price $1-\tilde{\lambda}$, tax rate τ, value functions of households and firms $\{V_e, V_w, \Pi\}$, allocations $\{k', b', l\}$ and distribution of agents Λ over the state space $\mathbb{S}$ such that,

1. Taking r, w, $1-\tilde{\lambda}$ and τ as given, the households' and firms' choices are optimal.
2. The government's budget, where G is a parameter, balances:

$$G = \tau r \int b \times 1_{\{b\geq 0\}} d\Lambda = \left(\lambda - \tilde{\lambda}\right) \int \left(k' - (1-\delta)\, k\right) \times \mathbb{I}_{\{k'-(1-\delta)k<0\}} d\Lambda$$

$$\mathbb{I}_{\{k'-(1-\delta)k<0\}} = \begin{cases} 1 & \text{, if } k' - (1-\delta)\, k < 0 \\ 0 & \text{, if } k' - (1-\delta)\, k \geq 0 \end{cases}$$

3. Markets clear,
   (a) Bonds: $\int b' d\Lambda = K^c$
   (b) Labor: $\int \theta h d\Lambda = \int l d\Lambda + L^c$

4. The distribution Λ is time-invariant, given by

$$\Lambda = \Gamma(\Lambda)$$

Where Γ is the one-period transition operator on the distribution.

The policy proposed is motivated by two reasons. Firstly, recall that due to illiquidity risk, entrepreneurial capital is mis-allocated relative to the frictionless counter-factual economy. This suggests that a policy which reduces illiquidity risk can potentially bring sizable efficiency gains. Secondly, illiquidity risk drives entrepreneurs to "over-accumulate" liquid wealth in a fashion that is reminiscent of the effect that Aiyagari (1994) studies: that is, the effect that precautionary savings at the individual levels leads to larger aggregate savings relative to a complete markets economy. As such, the taxation on bond returns would serve as a simple and direct mechanism to address this effect.

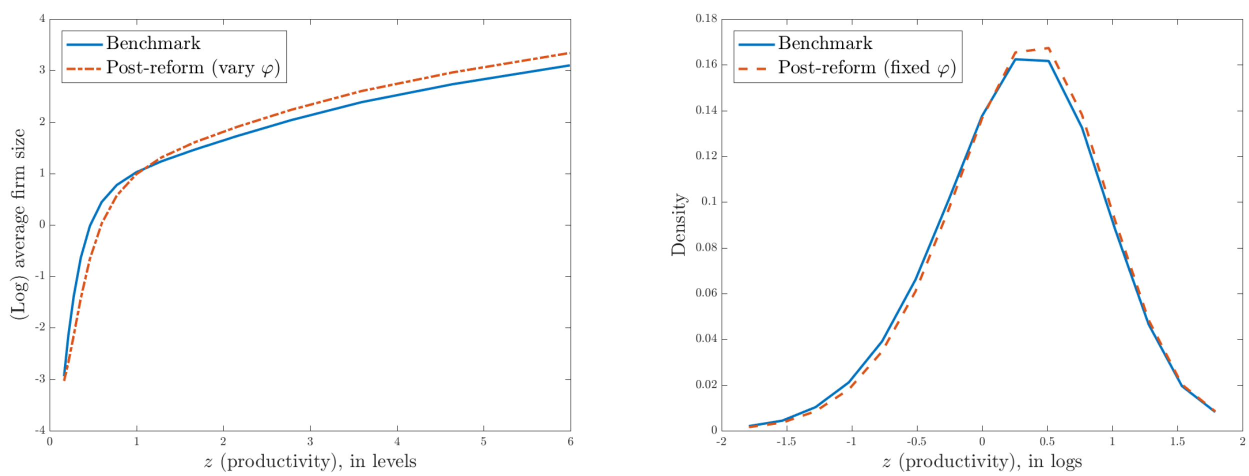
In table 11, I report the welfare gains and TFP of the preceding model relative to the benchmark calibration.

Model	**Welfare: W**	**Welfare: E**	**Welfare: All**	**TFP gains rel. to benchmark**	**Tax rate**	**Interest rate**	**Resale value rel. to benchmark**
Illiquidity + Collateral	0.54%	0.78%	0.56%	4.5%	8.3%	3.2%	1.82
Illiquidity only	0.36%	0.83%	0.40%	4.0%	8.2%	3.3%	1.86

Table 11: Effect of fiscal policy. Columns 1, 2 and 3 report the average percentage consumption equivalent variation for workers ("W"), entrepreneurs ("E") and the whole economy ("All"); column 4 report the percentage TFP gains relative to the benchmark economy; columns 5 and 6 report the tax rate and interest rate respectively; finally, column 7 reports the resale value under the policy relative to the benchmark, computed as $(1-\tilde{\lambda})/(1-\lambda)$.

Here, I solve two versions of the same model. In the first row ("Illiquidity + Collateral"), I report the results when λ is unconditionally lowered to $\tilde{\lambda}$; hence, the government's policy relaxes both the illiquidity and collateral constraint effect. In the second row ("Illiquidity only"), I report the same experiment, but holding the net collateral constraint fixed; that is, I reset φ such that $\varphi(1-\lambda)$ is fixed. This isolates the effect of this policy to only the illiquidity effect. This allows me to decompose the effect of fiscal policy into its effect through the illiquidity channel and the collateral constraint channel. I also report the average welfare gains for subsets of the population (i.e. workers and entrepreneurs), in order to decompose the effect of this policy across occupational types. For both exercises, I set $G = 0.01$, which amounts to about 0.5% of steady-state GDP under the benchmark calibration.

The first result one sees is that households are, on average, better off under this fiscal policy despite the relatively high tax rate they face. Moreover, TFP increases by about 4%. Given the substantial distortion arising from these resale frictions, this is not surprising. With the new policies, entrepreneurial talent and capital is now better allocated, despite the distortionary tax used to finance the program. As an example, in figure 20, I report (a) the average firm size as a function of productivity, and (b) the distribution of productivities, before and after the reform where both the illiquidity and collateral effects are alleviated. This figure is the counterpart to figures 11 and 14a from the earlier section. Here, we see that under the reform, low productivity firms are now smaller, and high productivity firms are larger. Moreover, we also see fewer low productivity firms under the reform.



(a) Average firm size (log scale) v.s. productivity (levels).

(b) Distribution of productivity (log scale)

Figure 20: Allocation of capital and productivity before and after the policy. After the reform, low productivity firms are smaller, and high productivity firms are larger. There are also fewer low productivity firms and more high productivity firms.

Going back to table 11, we see that these policies benefit both workers and entrepreneurs. Looking first to the workers, we see that even a reform that only alleviates the illiquidity effect also benefits workers, even though they do not directly utilize this policy. This happens because the lifetime utility of workers incorporate both the options to become an entrepreneur, as well as their option to stay a worker. Decreasing illiquidity increases the value of entrepreneurship, and therefore directly increases the welfare of workers. In addition, when we also factor in the collateral effect, we see that worker welfare improves further. This is because workers are now better able to enter into entrepreneurship now that they face looser borrowing constraints.

Unsurprisingly, entrepreneurs also benefit from this policy, since this policy directly affects the allocation of capital amongst entrepreneurs. However, we see that a policy that addresses both the illiquidity and collateral effect actually delivers lower welfare gains for entrepreneurs than a policy that strictly reduces the resale frictions.

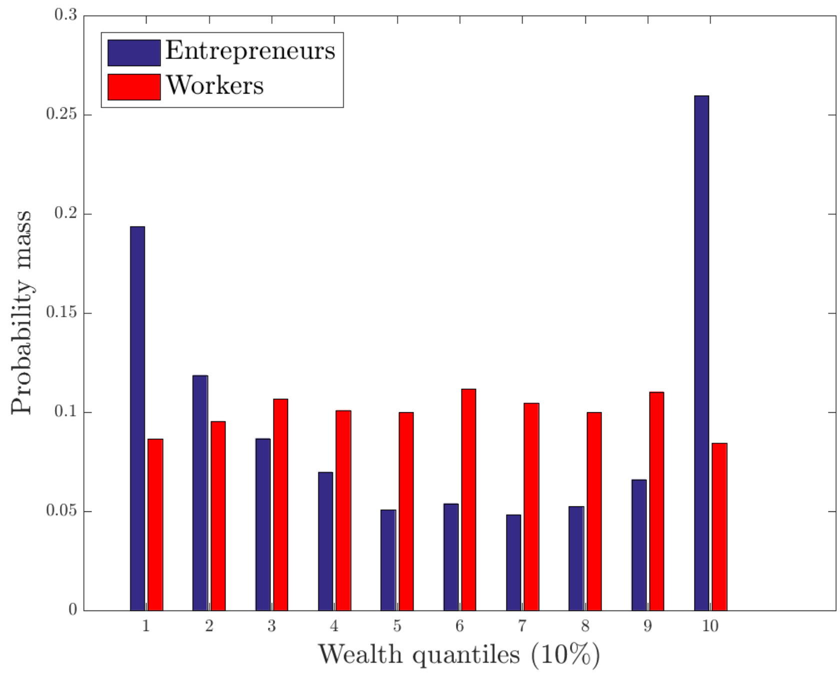


Figure 21: Distribution of wealth in deciles in the benchmark economy, for workers and entrepreneurs. The distribution is re-normalized by the appropriate measure (i.e. number of entrepreneurs and workers). Wealth is noticeably more concentrated amongst entrepreneurs.

Why is this so? This result is best understood when we put into context who the taxpayers are, and who benefits from the two policies. In figure 21, we see that wealth is substantially more concentrated amongst the entrepreneur population than worker population. Consequently, a disproportionate amount of the tax burden falls on entrepreneurs. However, rich entrepreneurs do not directly benefit from an alleviation of the collateral constraints, since they are already financially unconstrained. In contrast, entrepreneurs across the wealth distribution will benefit from alleviation of the illiquidity effect, since even very rich entrepreneurs can use this policy when they downsize. Consequently, they prefer a policy where all the tax receipts are put towards decreasing the level of illiquidity.

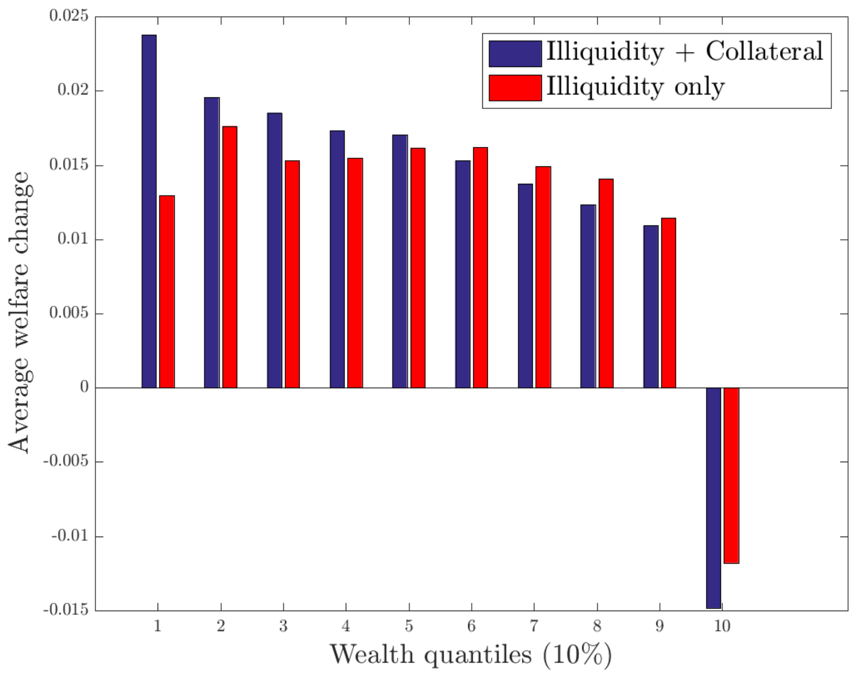


Figure 22: Distribution of welfare gains (in consumption equivalent variation) across the wealth distribution (in deciles) for entrepreneurs, under the two policies.

This effect can be seen when we overlay the distribution of welfare gains over the wealth distribution for entrepreneurs under the two policies, as in figure 22. We see that entrepreneurs in the bottom 50% generally prefer the "Illiquidity + Collateral" policy over the "Illiquidity only" policy, since these are the entrepreneurs who are most likely to be financially constrained. In contrast, entrepreneurs above the median prefer the "Illiquidity only" policy, reflecting the fact that they do not face especially tight borrowing constraints. The richest 10% of entrepreneurs suffer a welfare loss, as the bulk of this policy is financed by them. However, the welfare loss under the "Illiquidity only" policy is smaller, reflecting a preference for this policy. Combining this result with the wealth distribution in figure 21, we can see immediately why the average welfare gains are higher for an entrepreneur under the "Illiquidity only" policy.

Finally, in the third column, we see that average welfare gains are larger by about 0.16 percentage points if the same budget is used to alleviate both the collateral and illiquidity constraint effects, rather than just the illiquidity effects. A naive decomposition therefore shows that about 29% of the full impact of the policy can be attributed to the alleviation of collateral constraints, which is a sizable amount. However, the bulk of the policy's impact still derives through the illiquidity effect.

7 Conclusion

In this paper, I studied extensively the investment dynamics of entrepreneurs, and its relationship with the distribution of wealth, capital, and entrepreneurial productivity. There are four key contributions.

The first contribution is to document empirically that the distribution of the rates of return to capital among early stage entrepreneurs, as proxied by the log average revenue product of capital, is highly left-skewed. Moreover, entrepreneurs with low rates of return persist in this state for longer periods of time than their high return counterparts. This observation runs contrary to the preceding literature on entrepreneurial investment, which emphasizes collateral constraints as a key driver of entrepreneurial investment behavior. As discussed in this paper, financing constraints generate right tail skewness and right tail persistence, which is counter-factual to my empirical findings.

The second contribution of this paper is to build a parsimonious model of entrepreneurship to rationalize these findings. Specifically, I show that these facts can be easily reconciled within a standard macroeconomic model of entrepreneurship once we explicitly model entrepreneurial capital as an illiquid asset. When capital is no longer a perfect substitute for bonds, poor performing entrepreneurs hold on to capital for extended periods of time due to a real options value effect. At the aggregate level, this manifests as a left skewed distribution of returns to capital, as well as a left tail persistence.

The third contribution of this paper is to connect entrepreneurial investment, and the dispersion in returns to capital, back to the wealth distribution. This paper finds two important results that deviate from the prior literature. When entrepreneurial capital is illiquid, the average returns to wealth is lowered for all households. Consequently, the resulting wealth distribution features lower wealth inequality than which has been reported in the prior literature. Moreover, wealth dispersion in my model arises in part from a misallocation of capital. Relative to a counter-factual economy where capital is fully liquid, there are substantially more poor performing wealthy entrepreneurs. This leads to welfare and TFP losses relative to the counter-factual economy.

The last contribution relates to fiscal policy. Here, I show that a government program that subsidizes the sale of entrepreneurial capital can substantially improve welfare and TFP. Moreover, this program is fully funded by a tax on liquid bond returns, suggesting that a reallocation of wealth can bring about important welfare benefits.

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Appendix

I Data appendix I: Data construction

Following the literature, my key strategy is to document key moments for two measures of firm investment: The average revenue product of capital, and the investment rate. This section details how these two measures are constructed

I.a Capital stock

In order to construct the average revenue product of capital, I first need to construct the capital stock of the firm. The KFS provides the researcher the balance sheet of the firm, and it provides a breakdown of the type of capital asset that the entrepreneurial firm owns[30].

However, as in most standard models, I consider only a single generic capital asset of interest. As such, in order render the results comparable, I construct a representative single asset, real capital stock, $K_{i,t}$, using the nominal value of capital assets as follows:

$$K_{i,t} = \sum_s \frac{K_{i,s,t}}{P_{s,t}}$$

where $P_{s,t}$ is the relative price of each capital type s and vintage t. Subscript i indexes the firm. The relative prices are taken from the BEA. For the aggregated capital stock, I only consider the firm's holdings of product inventories, land and buildings and structures, vehicles, equipment or machinery, and other properties.

I.b Revenue and Value Added

Construction of the average revenue product of capital also necessitates the construction of a measure of the firm's real value added. A two step adjustment is used to transform nominal revenue into real value added. The first step is straightforward; nominal revenue is deflated by the GDP deflator to obtain real revenue.

[30] The full range of asset types are product inventories, land and buildings and structures, vehicles, equipment or machinery, other properties, cash, and "others".

Unfortunately, constructing real value added from real revenue is less straightforward. The KFS does not provide information on the firm's material expenses; consequently, one cannot simply subtract out the material cost to retrieve value added.

Instead, following the literature, I assume that the firm revenue production function is given by a Cobb-Douglas function of the form

$$Y^R_{i,j,t} = z_{i,j,t} K^{\beta^j_k}_{i,j,t} L^{\beta^j_l}_{i,j,t} M^{\beta^j_m}_{i,j,t}$$

where i indexes a firm, j indexes the industry, and t indexes time. z here refers to aggregated TPFR (i.e. it summarizes the firm, industry and aggregate level shocks). The capital (K), labor (L) and material (M) intensity parameters (β^j_k, β^j_l, β^j_m respectively) are allowed to vary across industries, but are restricted to sum to unity.

Let P^j_m denote the cost of materials for industry j. Then value added is

$$Y_{i,j,t} = Y^R_{i,j,t} - P^j_m M_{i,j,t}$$

As the KFS does not provide information on $P^j_m M_{i,j,t}$, the goal is to infer $P^j_m M_{i,j,t}$.

To do so, I follow the preceding literature by assuming that capital is fixed one period in advanced, but the firm is able to adjust it's labor and material inputs contemporaneously[31]. Given this, the firm's optimal material choice will yield the first order condition:

$$P^j_m = \beta^j_m \frac{Y^R_{i,j,t}}{M_{i,j,t}}$$
$$\implies Y_{i,j,t} = (1 - \beta^j_m) \times Y^R_{i,j,t}$$

Therefore, given the material share parameter, we can immediately back out value added. For the purposes of this paper, I assume that the entrepreneurial firms have the same material share as the industry that they operate in. The material share at the industry level can be estimated directly from national accounting data (the NIPA KLEMS tables). Denoting the estimated material share as $\hat{\beta}^j_m$, I back out an estimate of value added as[32]

$$Y_{i,j,t} = (1 - \hat{\beta}^j_m) \times Y^R_{i,j,t}$$

For the rest of this paper, revenue refers to this measure of value added, unless explicitly

[31] See for instance, Olley and Pakes (1996)

[32] One might justifiably be concerned that this measurement of value added is very noisy, and introduces extra measurement error. As a robustness check, I repeat the empirical exercises using the "raw" revenue measure. The qualitative results obtained using the value added measure is replicated when I use the raw revenue measure.

stated otherwise.

I.c Average Revenue Product of Capital

Having constructed both capital and revenue, the (log) average revenue product of capital is simply

$$\log ARPK \equiv \log\left(\frac{Y_{it}}{K_{i,t-1}}\right)$$

t indicates the survey year. This timing convention is adopted as the KFS only surveys the firm at the start of the following year for information regarding the current year. That is to say, if the survey report is for the year 2004, it was in fact surveyed in 2005. Therefore, K_{2004} is in fact the end of period capital stock, while Y_{2004} is the revenue for 2004. The convention chosen here hence matches the standard timing convention to construct average products of capital as the ratio of the total revenue to the start of period capital stock.

For the main body of the paper, the moments I report are constructed using a pooled measure of $\log ARPK$. In practice, moments constructed using pooled $\log ARPK$ might not be a good measure since there is likely to be large heterogeneity in capital share and returns to scale across industries [33]. However, due to the small sample size of the KFS relative to the number of industries[34], there is simply insufficient statistical power to draw any useful inference if one restricted analysis only to the 6 digit NAICS industry level. Instead, I address this issue through two methods.

Residualized ARPK All the benchmark empirical results utilize a residualized measure of $\log ARPK$ rather than the raw measure. Here, the pooled $\log ARPK$ variables are residualized by regressing the raw $\log ARPK$ on two digit NAICS industry level fixed effects and time dummies. The residuals of this regression then form my measure of $\log ARPK$ for analysis. This avoids the issue where permanent differences across industries (such as heterogeneity in capital share and returns to scale) would distort the distribution, and thus introduce spurious correlations and moments. Moreover, it also removes some of the common aggregate shocks that might distort the distribution of $\log ARPK$ over time. The results presented in the following sub-sections are constructed using this residualized measure[35].

[33] See for instance, Burnside (1993), who estimates the returns to scale for different industries in the United States

[34] There were 3140 firms in 2004 when the survey started; by 2011, there were only 1630 firms left. In contrast, there are 659 six digit level NAICS code industries.

[35] As a robustness check, I also construct extended residualized measures of $\log ARPK$ by using more regressors that could potentially systematically distort the distribution of $\log ARPK$ (for instance, the legal

Industry level moments The biggest concern about my findings relate to the fact that there is substantial permanent heterogeneity across industries that the residualization process is unable to fully purged. To address these concerns, I also directly investigate the relevant moments at the two digit industry level. Due to the smaller sample size, only one industry showed statistically significant results; however, most industries showed economically significant results. Moreover, the qualitative findings at the aggregate level also holds broadly across industries. The results at the industry level are reported in this appendix.

I.d Investment rates

To construct investment and investment rates, I use the perpetual inventory method as follows:

$$
\begin{aligned}
I_{i,s,t} &= \frac{K_{i,s,t}}{P_{s,t}} - (1-\delta_s)\frac{K_{i,s,t-1}}{P_{s,t-1}} \\
I_{i,s,t} &= \sum_s \frac{K_{i,s,t-1}}{\sum_s K_{i,s,t-1}} I_{i,s,t} \\
i_{i,s,t} &= \frac{I_{i,t}}{K_{i,t-1}}
\end{aligned}
$$

where $I_{i,s,t}$ refers to investment levels of firm i for capital type s and vintage t. I allow for depreciation of each type of capital (δ_s) to differ according to the BEA depreciation schedule. Gross investment at the firm level is then constructed as a weighted average of the firm's investment for each capital type. Investment rate is then constructed by scaling the gross investment by the total lagged capital stock.

Just as in the case of $\log ARPK$, using i_{it} "as is" poses a potential problem that the investment series is biased by unobserved aggregate shocks that are orthogonal to the idiosyncratic shocks or frictions that I am interested in studying. To purge the effect of aggregate shocks on investment, I construct a residualized investment series by regressing investment rates on year and two digit NAICS code fixed effects, and construct the relevant investment moments using the residuals.

II Data appendix II: Robustness checks

In this section, in an effort to ensure the robustness of my empirical findings, I re-examine the data moments I reported through multiple cuts of the data.

form of the firm, or the gender of the primary owner of the firm). I find that my benchmark results are robust to alternative measures. Results computed using these extended measures are available upon request.

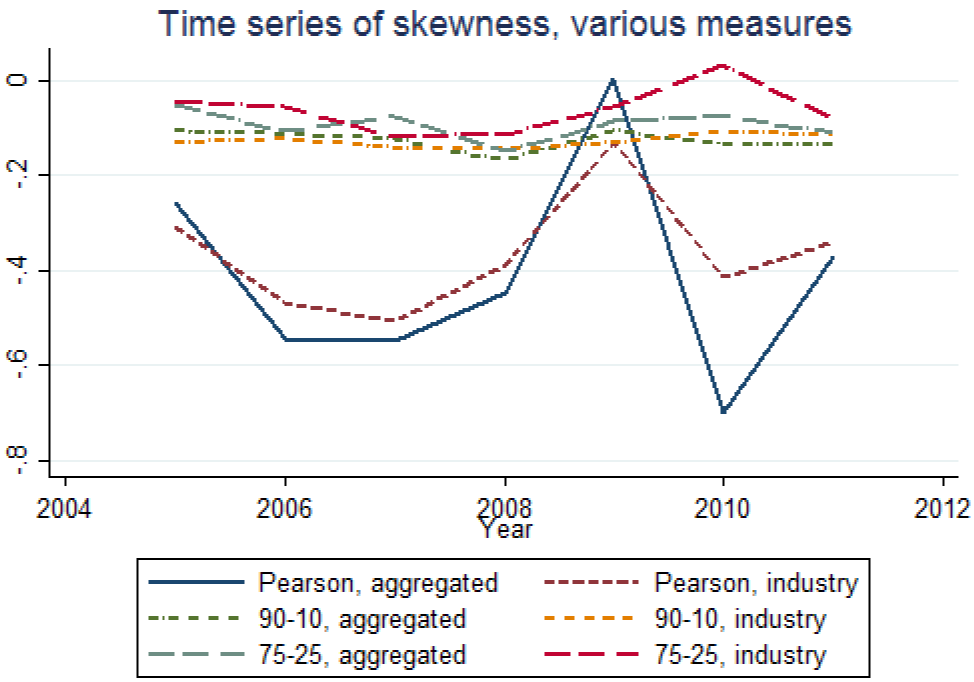

Figure 23: Time series of various measures of skewness.

II.a Skewness: Across time and definitions

The skewness moment that I report in the main text takes the entire distribution as a pooled sample, and reports the skewness using the Pearson's skewness estimator (i.e. the third central moment). There are two clear concerns. Firstly, the left skewness could be an artifact of survivorship bias. Secondly, the Pearson's estimator is sensitive to outliers. To ameliorate these concerns, I also estimated the skewness period-by-period over the entire sample; in addition, I also report the skewness using the Bowley and Kelley skewness measures. These results are reported in figure 23. From the graph, we see that regardless of the choice of skewness measure, the distribution of ARPK is left skewed.

II.b Asymmetric persistence: Using alternative definitions

The estimation of the persistence of relative rankings was estimated using the full sample, and the quantiles shift over time as I estimated the quantiles period-by-period[36]. This might lead the reader to raise a few concerns, specifically:

1. Quantile construction. The results could be an artifact of the shifting quantiles.

[36] The quantiles themselves are still estimated at the industry level, not across the pooled sample.

2. Sample selection. The results could be biased by very small firms. Very small firms could be operated by individuals with no desire to maximize profits[37], and these firms are also most likely to be in the left tail. As a result, the higher left tail persistence that I report in the main text could be driven by these firms.

To address these problems, I report in table 12 below a series of robustness checks. I address the first point by simply fixing the quantiles to the ones computed in period 1 (year 2005), and estimate the transition probabilities. To address the second point, I conduct the original estimation using an increasingly stricter cut off for firms. Specifically, I first conduct the estimation only for firms reporting more than \$5000 in assets. I then estimate the transition matrix again for firms with more than \$10,000 in assets. This process is repeated up to firms with \$50,000 in assets. In table 12, I only report the persistence in the left tail (i.e. probability of staying in rank 1) and the right tail (i.e. probability of staying in rank 5) since these are the key moments of interest.

Model	Fixed quantiles	Assets $\geq$ \$5000	Assets $\geq$ \$10000	Assets $\geq$ \$20000	Assets $\geq$ \$50000
$Pr(1 \to 1)$	0.61	0.56	0.57	0.58	0.60
	(0.017)	(.019)	(0.020)	(0.021)	(0.023)
$Pr(5 \to 15$	0.49	0.49	0.51	0.49	0.44
	(0.017)	(.026)	(.031)	(0.038)	(0.059)

Table 12: Persistence of relative rankings for firms in the first and fifth quintiles under different cut off assumptions. For fixed quintiles, the quintiles are fixed to the cutoffs from year 2005. For the other versions, the quintile is constructed using the full sample as in the main text, but the estimation is done using only firms that pass the asset cut-off. The standard errors are in parenthesis.

II.c Asymmetric persistence: Industry level

Here, I report the results for the persistence in relative rankings measure at the industry level (i.e. the results in section 3.3.2 at the industry level). I only report the probability of staying in the same quintile for the first and last quintile; the results of the full estimation of the transition matrices are available on request. The blue bars correspond to the probability of staying in the first quintile, given that the firm was in the first quintile. The red bars correspond to the probability of staying in the last quintile, given that the firm was in the last quintile. Each pair corresponds to one two digit NAICS industry code. As one can see, the asymmetric persistence does not just apply to the pooled sample. The vast majority of industries also feature economically significant asymmetry.

[37]See for instance, Hurst and Pugsley (2011) for a discussion on this class of entrepreneurs.

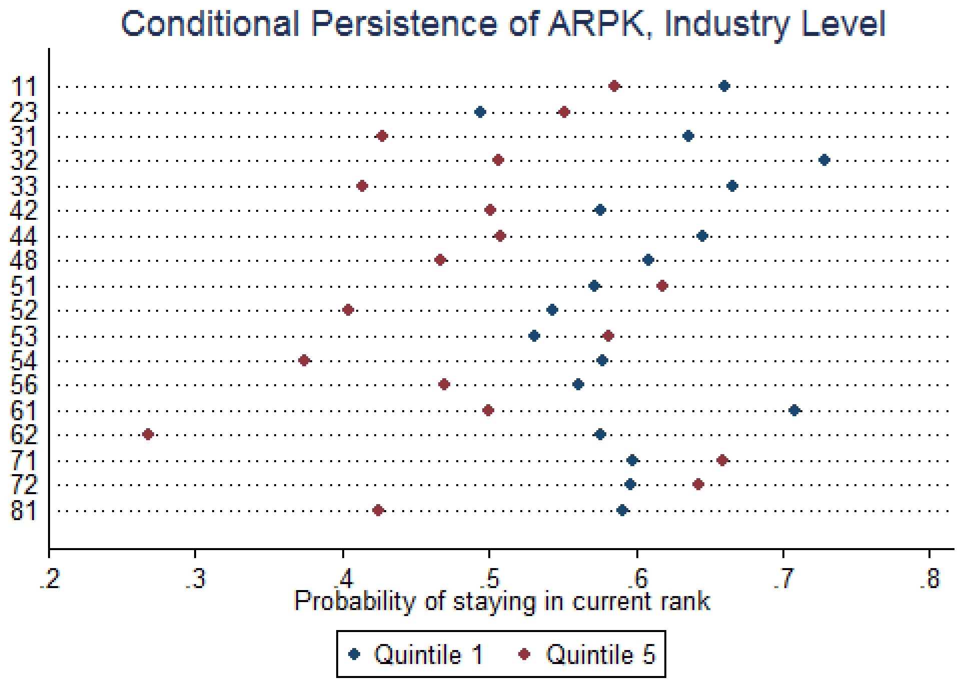

Figure 24: Probability of staying in same quintile. Blue corresponds to the first quintile, red to the last quintile.

II.d Asymmetric persistence in autocorrelation conditional of quintile

Here, I report the results for the conditional autocorrelation at the industry level (i.e. the results in section 3.3.3 at the industry level). I only report the autocorrelation for firms in the bottom and top quintiles; the results of the full estimation are available on request. The blue bars correspond to the autocorrelation when the firm is in the bottom quintile; the red bars to the autocorrelation when the firm is in the top quintile. Each pair corresponds to one two digit NAICS industry code. As in the result earlier, there exist also large differences in in conditional autocorrelation at the industry level.

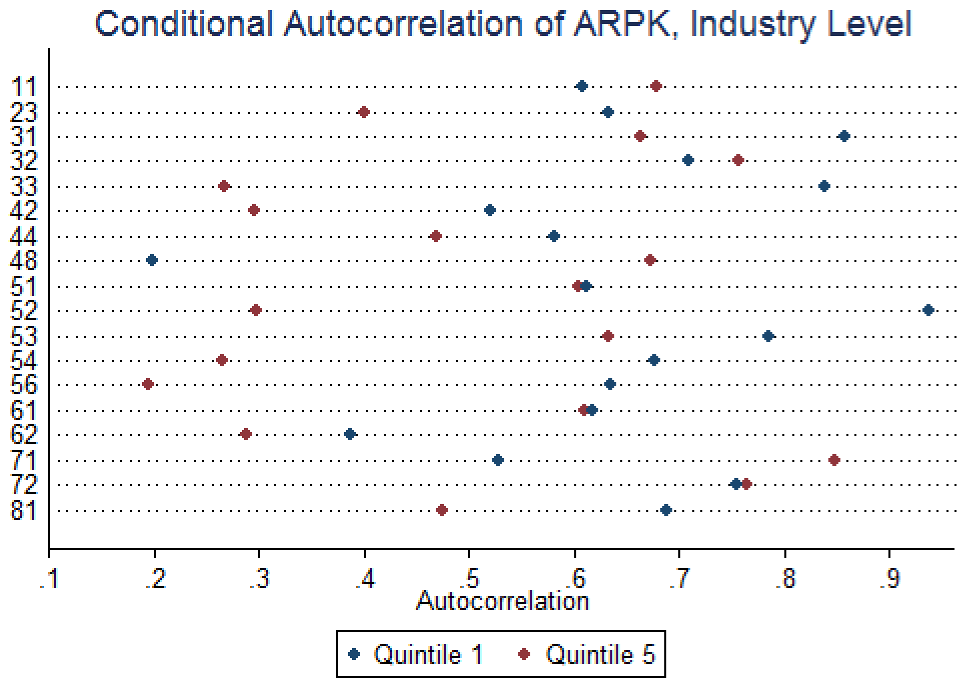

Figure 25: Autocorrelation conditional on being in the first quintile (blue), and the last quintile (red).

III Proofs of results in section 3

III.a Proof of result in section 3.7.2

Here, I show that the standard model of only time-to-build as a friction will not generate the left-skewness and conditional persistence of log ARPK as observed in the KFS. Consider the standard firm problem below (in recursive notation):

$$\begin{aligned}
\Pi\left(k,z\right) &= \max_{k'} D + \frac{1}{1+r}\mathbb{E}\left[\Pi\left(k',z'\right)|z\right] \\
&s.t. \\
D &= zk^{\alpha} + \left(1-\delta\right)k - k' \\
\log z' &= \rho \log z + \epsilon'
\end{aligned}$$

where $'$ denotes variables for the next period. ϵ' is any iid random variable and ρ captures persistence in TFP z. Here, investment has a "time-to-build' element', since the payoff to investment today (k') is only realized tomorrow. D here is per-period dividend flow, and the firm's problem is to maximize lifetime dividend flow.

The first order condition for capital yields

$$
\begin{aligned}
\mathbb{E}\left[z'\left(K'\right)^{\alpha-1}|z\right] &= \frac{r+\delta}{\alpha} \\
\implies \log K' &= \frac{1}{\alpha-1}\left(\log\left(\frac{r+\delta}{\alpha}\right) - \log \mathbb{E}\left[z'|z\right]\right) \\
\implies \log K' &= \frac{1}{\alpha-1}\left(\log\left(\frac{r+\delta}{\alpha}\right) - \rho \log z - \log \mathbb{E}\left[\exp\left(\epsilon'\right)|z\right]\right) \\
\Leftrightarrow \log K &= \frac{1}{\alpha-1}\left(\log\left(\frac{r+\delta}{\alpha}\right) - \rho \log z_{-1} - \log \mathbb{E}\left[\exp\left(\epsilon\right)|z_{-1}\right]\right)
\end{aligned}
$$

Where for the second last line, since $z' = z^{\rho}\exp\left(\epsilon'\right)$, I used the relationship $\log \mathbb{E}\left[z'|z\right] = \log\left(z^{\rho}\mathbb{E}\left[\exp\left(\epsilon'\right)|z\right]\right) = \rho \log z + \log \mathbb{E}\left[\exp\left(\epsilon'\right)|z\right]$. The last line is simply a change of time notation, with $_{-1}$ denoting "last period" variables.

Recall that log ARPK is also simply defined as

$$
\begin{aligned}
\log ARPK &= \log\left(\frac{Y}{K}\right) \\
&= \log z + (\alpha-1)\log K
\end{aligned}
$$

Combining the two equations, we get

$$
\log ARPK = \log z - \rho \log z_{-1} + \log\left(\frac{r+\delta}{\alpha}\right) - \log \mathbb{E}\left[\exp\left(\epsilon\right)|z_{-1}\right]
$$

Defining $\vartheta \equiv \log\left(\frac{r+\delta}{\alpha}\right) - \log \mathbb{E}\left[\exp\left(\epsilon\right)|z_{-1}\right]$, and recalling that $\log z = \rho \log z_{-1} + \epsilon$, we can simplify the above relation to

$$
\begin{aligned}
\log ARPK &= \rho \log z_{-1} + \epsilon + \log\left(\frac{r+\delta}{\alpha}\right) - \rho \log z_{-1} - \log \mathbb{E}\left[\exp\left(\epsilon\right)|z_{-1}\right] \\
&= \log\left(\frac{r+\delta}{\alpha}\right) - \log \mathbb{E}\left[\exp\left(\epsilon\right)|z_{-1}\right] + \epsilon \\
&= \vartheta + \epsilon \quad \square
\end{aligned}
$$

As discussed in the main text, the distribution of $\log ARPK$ simply inherits the distribution of ϵ, scaled by a constant.

III.a.1 Skewness

Since the distribution of $\log ARPK$ is simply the distribution of ϵ, the skewness of $\log ARPK$ will simply be equal to that of ϵ.

III.a.2 Persistence

Since ϵ is i.i.d, $\log ARPK$ is also i.i.d; that is, $\log ARPK$ exhibits no persistence. Moreover, although the result here is derived for homogeneous parameters, long run differences in α, δ or r across industries will not affect the persistence of $\log ARPK$.

III.b Proof of result in section 3.7.3

Here, I show that the common static investment framework with limited commitment in borrowing will not allow the model to match the left-skewness of the empirical $\log ARPK$ distribution.

Consider the following model:

$$\begin{aligned} \Pi\left(a,z\right) &= \max_k D + \frac{1}{1+r}\mathbb{E}\left[\Pi\left(a',z'\right)|z\right] \\ &s.t. \\ D &= Y-(r+\delta)k+(1+r)a \\ Y &= zk^{\alpha} \\ k &\leq (1+\varphi)a \\ \log z' &= \rho\log z+\epsilon' \end{aligned}$$

In this framework, there is a financial constraint as firms cannot rent as much capital as they desire. Instead, they either fully finance using internal assets (a), or they utilize the debt markets to finance their investments up to φ of their assets. $\varphi\in[0,\infty)$ captures how much the individual can leverage on his own wealth. If $\varphi=0$, no borrowing is possible; if $\varphi\to\infty$, there is no borrowing constraint.

Denote by μ the Lagrange multiplier on the borrowing constraint. The first order condition on capital yields

$$\begin{aligned} \alpha\frac{Y}{K} &= r+\delta+\mu \\ \implies \log ARPK &= \log(r+\delta+\mu)-\log(\alpha) \end{aligned}$$

Note that for firms that are unconstrained, $\mu = 0$. Therefore, we see that the support for $\log ARPK$ is given by $[\log \frac{r+\delta}{\alpha}, \infty)$ $\square$.

III.b.1 Skewness

As discussed in the main text, since the distribution is left-truncated, the distribution must naturally be right-skewed. This result is independent of the distribution of the underlying innovations.

III.b.2 Persistence

The distribution will exhibit greater persistence in the right tail than the left tail. To see this, consider, at time t, two firms i and j. For firm i, assume that the firm is unconstrained, so $\mu = 0$; consequently, the ARPK of firm i evaluates to

$$\log ARPK_{i,t} = \log(r + \delta) - \log(\alpha)$$

For firm j, assume that the firm is constrained, so $\mu > 0$; consequently, the ARPK of firm j evaluates to

$$\log ARPK_{j,t} = \log(r + \delta + \mu_{j,t}) - \log(\alpha)$$

Note that $\log ARPK_j > \log ARPK_i$; that is, firm j is in the right tail of the distribution, whereas firm i is in the left tail.

The autocorrelation of $logARPK$ for firm i is trivially 0:

$$\begin{aligned} corr(\log ARPK_{i,t}, \log ARPK_{i,t+1}) &= corr\left(\log(r + \delta) - \log(\alpha), \log(r + \delta + \mu_{i,t+1}) - \log(\alpha)\right) \\ &= 0 \end{aligned}$$

In contrast, for firm j, the autocorrelation is strictly positive:

$$\begin{aligned} corr(\log ARPK_{j,t}, \log ARPK_{j,t+1}) &= corr\left(\log(r + \delta + \mu_{j,t}) - \log(\alpha), \log(r + \delta + \mu_{j,t+1}) - \log(\alpha)\right) \\ &> 0 \end{aligned}$$

This holds true trivially because μ is always strictly bigger than 0.

Consequently, we see that there is greater persistence in the right tail than the left tail of the distribution.

III.c Proof of result in section 3.7.4

Using the same framework for section 3.7.2, we can extend it with collateralized borrowing, and write the firm's problem as

$$\begin{aligned}
\Pi(b,k,z) &= \max_{k',b'} Y + (1-\delta)k + (1+r)b - k' - b' + \frac{1}{1+r}\mathbb{E}\left[\Pi(b',k',z')\,|z\right] \\
&\quad s.t. \\
Y &= zk^{\alpha} \\
\log z' &= \rho \log z + \epsilon' \\
b' &\geq -\varphi k' \\
Y + (1-\delta)k + (1+r)b &\geq k' + b'
\end{aligned}$$

where $b < 0$ indicates that the firm is *borrowing*, and $\varphi < 1$ in order for the collateral constraint to be meaningful (if $\varphi \geq 1$, then the firm can always borrow enough to hit the optimal firm size). The last constraint indicates that the firm cannot issue equity. Re-stated in terms of Lagrange multipliers and defining net wealth $\omega \equiv Y + (1-\delta)k + (1+r)b$, we obtain the firm's problem as

$$\begin{aligned}
\Pi(\omega,z) &= \max_{k',b'} \omega - k' - b' + \frac{1}{1+r}\mathbb{E}\left[\Pi(\omega',z')\,|z\right] + \mu(b' + \varphi k') + \kappa(\omega - k' - b') \\
&\quad s.t. \\
\omega' &= z'k'^{\alpha} + (1-\delta)k' + (1+r)b' \\
\log z' &= \rho \log z + \epsilon'
\end{aligned}$$

The envelope condition is

$$\frac{\partial \Pi}{\partial \omega} = 1 + \kappa$$

and the first order conditions for capital and bond are (respectively):

$$\begin{aligned}
0 &= -1 + \frac{1}{1+r}\mathbb{E}\left[\frac{\partial \Pi'}{\partial \omega'}\left(\alpha z' k'^{\alpha-1} + (1-\delta)\right)|z\right] + \mu\varphi - \kappa \\
0 &= -1 + \frac{1}{1+r}\mathbb{E}\left[\frac{\partial \Pi'}{\partial \omega'}(1+r)\,|z\right] + \mu - \kappa
\end{aligned}$$

First, combining the first order conditions for bonds with the envelope conditions, we get

$$
\begin{aligned}
0 &= -1 + \frac{1}{1+r}\mathbb{E}\left[\left(1+\kappa'\right)\left(1+r\right)|z\right] + \mu - \kappa \\
\implies \mathbb{E}\left[\kappa'|z\right] &= \kappa - \mu
\end{aligned}
$$

Next, combining the first order conditions for capital with the envelope conditions, we get

$$
\frac{1}{1+r}\mathbb{E}\left[\left(1+\kappa'\right)\left(\alpha z' {k'}^{\alpha-1} + \left(1-\delta\right)\right)|z\right] = 1 + \kappa - \mu\varphi
$$

Next, we can express k' in terms of the Lagrange multipliers and z. For convenience, I will move all equations one "time step" back (so $'$ variables become "un-primed", and "un-primed" variables are now $_{-1}$).

I now re-write the first order conditions for capital as

$$
k^{\alpha-1}\left(\mathbb{E}\left[z|z_{-1}\right] + \mathbb{E}\left[\kappa z|z_{-1}\right]\right) = \frac{r+\delta}{\alpha} + \left(\kappa_{-1} - \mu_{-1}\varphi\right)\left(\frac{1+r}{\alpha}\right) - \mathbb{E}\left[\kappa|z_{-1}\right]\left(\frac{1-\delta}{\alpha}\right)
$$

Next, using the first order conditions for bonds, we note that

$$
\begin{aligned}
\left(\kappa_{-1} - \mu_{-1}\varphi\right)\frac{1+r}{\alpha} - \mathbb{E}\left[\kappa|z_{-1}\right]\left(\frac{1-\delta}{\alpha}\right) &= \left(\kappa_{-1} - \mu_{-1}\varphi\right)\frac{1+r}{\alpha} - \left(\kappa_{-1} - \mu_{-1}\right)\left(\frac{1-\delta}{\alpha}\right) \\
&= \kappa_{-1}\frac{r+\delta}{\alpha} - \mu_{-1}\frac{\varphi\left(1+r\right) - 1 + \delta}{\alpha} \\
&= \frac{r+\delta}{\alpha}\left(\kappa_{-1} - \mu_{-1}\frac{\varphi\left(1+r\right) - 1 + \delta}{r+\delta}\right) \\
&= \frac{r+\delta}{\alpha}\left(\kappa_{-1} - \mu_{-1}\tilde{\varphi}\right)
\end{aligned}
$$

where $\frac{\varphi(1+r)-1+\delta}{r+\delta} \equiv \tilde{\varphi} \in \left[\frac{-1+\delta}{r+\delta}, 1\right]$, as $0 \leq \varphi \leq 1$. Putting this relationship back into the first order conditions for capital, we get

$$
k^{\alpha-1} = \left[\frac{r+\delta}{\alpha} + \frac{r+\delta}{\alpha}\left(\kappa_{-1} - \mu_{-1}\tilde{\varphi}\right)\right]\left[\mathbb{E}\left[z|z_{-1}\right] + \mathbb{E}\left[\kappa z|z_{-1}\right]\right]^{-1}
$$

Where we have obtained an implicit solution for k in terms of z and the Lagrange multipliers. Substituting this back into the definition of $\log ARPK$,

$$
\begin{aligned}
\implies \log ARPK &= \log z + (\alpha - 1)\log k \\
&= \log z + \log\left(\frac{r+\delta}{\alpha} + \frac{r+\delta}{\alpha}\left(\kappa_{-1} - \mu_{-1}\tilde{\varphi}\right)\right) - \log\left(\mathbb{E}\left[z|z_{-1}\right] + \mathbb{E}\left[\kappa z|z_{-1}\right]\right) \\
&= \log z + \log\left[\left(\frac{r+\delta}{\alpha}\right)\left(1 + \kappa_{-1} - \mu_{-1}\tilde{\varphi}\right)\right] - \log\left[\left(\mathbb{E}\left[z|z_{-1}\right]\right)\left(1 + \frac{\mathbb{E}\left[\kappa z|z_{-1}\right]}{\mathbb{E}\left[z|z_{-1}\right]}\right)\right] \\
&= \log z + \log\left(\frac{r+\delta}{\alpha}\right) + \log\left(1 + \kappa_{-1} - \mu_{-1}\tilde{\varphi}\right) - \log\mathbb{E}\left[z|z_{-1}\right] - \log\left(1 + \frac{\mathbb{E}\left[\kappa z|z_{-1}\right]}{\mathbb{E}\left[z|z_{-1}\right]}\right) \\
&= \log z - \log\mathbb{E}\left[z|z_{-1}\right] + \log\left(\frac{r+\delta}{\alpha}\right) + \log\left(1 + \kappa_{-1} - \mu_{-1}\tilde{\varphi}\right) - ... \\
&\qquad \log\left(1 + \frac{\mathbb{E}\left[\kappa|z_{-1}\right]\mathbb{E}\left[z|z_{-1}\right] + cov_{z_{-1}}\left(\kappa, z\right)}{\mathbb{E}\left[z|z_{-1}\right]}\right) \\
&= \log z - \log\mathbb{E}\left[z|z_{-1}\right] + \log\left(\frac{r+\delta}{\alpha}\right) + \log\left(1 + \kappa_{-1} - \mu_{-1}\tilde{\varphi}\right) - ... \\
&\qquad \log\left(1 + \mathbb{E}\left[\kappa|z_{-1}\right] + \frac{cov_{z_{-1}}\left(\kappa, z\right)}{\mathbb{E}\left[z|z_{-1}\right]}\right) \\
&= \log z - \log\mathbb{E}\left[z|z_{-1}\right] + \log\frac{r+\delta}{\alpha} + \log\left(1 + \kappa_{-1} - \mu_{-1}\tilde{\varphi}\right) - \\
&\qquad \log\left(1 + \kappa_{-1} - \mu_{-1} + \frac{cov_{z_{-1}}\left(\kappa, z\right)}{\mathbb{E}\left[z|z_{-1}\right]}\right)
\end{aligned}
$$

Where I went from the second last line to the last line using the first order conditions for bonds ($\mathbb{E}\left[\kappa|z_{-1}\right] = \kappa_{-1} - \mu_{-1}$)

Finally, noting that

$$
\begin{aligned}
\log z - \log\mathbb{E}\left[z|z_{-1}\right] &= \rho\log z_{-1} + \epsilon - \log\mathbb{E}\left[z_{-1}^{\rho}\exp\left(\epsilon\right)|z_{-1}\right] \\
&= \rho\log z_{-1} + \epsilon - \rho\log z_{-1} - \log\mathbb{E}\left[\exp\left(\epsilon\right)|z_{-1}\right] \\
&= \epsilon - \log\mathbb{E}\left[\exp\left(\epsilon\right)|z_{-1}\right]
\end{aligned}
$$

and that under the model of only time-to-build and no collateral constraints,

$$
\log ARPK^{TTB} = \log\frac{r+\delta}{\alpha} - \log\mathbb{E}\left[\exp\left(\epsilon\right)|z_{-1}\right] + \epsilon
$$

We obtain for $\log ARPK$:

$$
\log ARPK = \log ARPK^{TTB} + \log\left(1 + \kappa_{-1} - \mu_{-1}\tilde{\varphi}\right) - \log\left(1 + \kappa_{-1} - \mu_{-1} + \frac{cov_{z_{-1}}\left(\kappa, z\right)}{\mathbb{E}\left[z|z_{-1}\right]}\right)
$$

Now, note that since $\frac{-1+\delta}{r+\delta} \leq \tilde{\varphi} \leq 1$, we see that $\kappa_{-1} - \mu_{-1}\tilde{\varphi} \geq \kappa_{-1} - \mu_{-1}$. Moreover, since $cov_{z_{-1}}(\kappa, z) < 0$ and $\mathbb{E}[z|z_{-1}] > 0$, $\frac{cov_{z_{-1}}(\kappa,z)}{\mathbb{E}[z|z_{-1}]} < 0$. As such, we see that

$$\log(1 + \kappa_{-1} - \mu_{-1}\tilde{\varphi}) - \log\left(1 + \kappa_{-1} - \mu_{-1} + \frac{cov_{z_{-1}}(\kappa, z)}{\mathbb{E}[z|z_{-1}]}\right) \equiv \xi_{-1} > 0$$

where I write ξ_{-1} to note that this is a pre-determined variable.

Rewriting the equation for $\log ARPK$ again, and defining $\log ARPK^{TTB} \equiv A$ for notational ease, we get the following simple linear relation between the "time to build" ARPK and the actual ARPK:

$$\log ARPK = A + \xi_{-1}$$

□

III.c.1 Skewness

To derive the unconditional skewness of $\log ARPK$, note that (and using the notation $\mathcal{S}(X, Y, Y)$ for the co-skewness between X and Y):

$$\begin{aligned} skewness(\log ARPK) &= skewness(A + \xi_{-1}) \\ &= \frac{1}{\sigma_A^3 + \sigma_{\xi_{-1}}^3}\left[\sigma_A^3\mathcal{S}(A) + 3\sigma_A^2\sigma_\xi\mathcal{S}(A, A, \xi_{-1}) + 3\sigma_A\sigma_\xi^2\mathcal{S}(A, \xi_{-1}, \xi_{-1}) + \sigma_\xi^3\mathcal{S}(\xi_{-1})\right] \end{aligned}$$

Note that while A depends on ϵ ("today's innovation"), ξ_{-1} depends on ϵ_{-1} ("yesterday's innovation"). As such, $A \perp \xi_{-1}$, which implies that $3\sigma_A^2\sigma_\xi\mathcal{S}(A, A, \xi) + 3\sigma_A\sigma_\xi^2\mathcal{S}(A, \xi, \xi) = 0$: i.e. they have 0 co-skewness, since they are independent. As such, the skewness of $\log ARPK$ reduces to

$$skewness(\log ARPK) = \frac{1}{\sigma_A^3 + \sigma_{\xi_{-1}}^3}\left[\sigma_A^3\mathcal{S}(A) + \sigma_\xi^3\mathcal{S}(\xi_{-1})\right]$$

Finally, recall that $\mathcal{S}(A) = \mathcal{S}(\epsilon)$ (and likewise for the standard deviation); therefore, we obtain the final relation:

$$skewness(\log ARPK) = \frac{1}{\sigma_\epsilon^3 + \sigma_{\xi_{-1}}^3}\left[\sigma_\epsilon^3\mathcal{S}(\epsilon) + \sigma_\xi^3\mathcal{S}(\xi_{-1})\right]$$

Note that if $\mathcal{S}(\xi_{-1}) > -\frac{\sigma_\epsilon^3}{\sigma_{\xi_{-1}}^3}\mathcal{S}(\epsilon)$, then the distribution will always be right-skewed. This is trivially true for all $\mathcal{S}(\epsilon) > 0$, since $\mathcal{S}(\xi_{-1}) > 0$ (ξ_{-1} has a left-truncated distribution since it is strictly bigger than 0). Moreover, if $\mathcal{S}(\xi_{-1}) < -\frac{\sigma_\epsilon^3}{\sigma_{\xi_{-1}}^3}\mathcal{S}(\epsilon)$, then the skewness of log ARPK is trivially higher than the skewness of the innovations, although the distribution of log ARPK will be left-skewed. □

III.c.2 Persistence

Here, I show how the model with collateral constraints produces asymmetric persistence in ARPK. To proceed, I first show how there is unconditional autocorrelation, and then show how this autocorrelation is asymmetric.

Persistence We can compute the autocovariance of $\log ARPK$ to understand its persistence. Specifically,

$$\begin{aligned}
cov(\log ARPK, \log ARPK_{-1}) &= cov\left(A + \xi_{-1}, A_{-1} + \xi_{-2}\right) \\
&= cov\left(A, A_{-1}\right) + cov\left(A, \xi_{-2}\right) + cov\left(\xi_{-1}, A_{-1}\right) + cov\left(\xi_{-1}, \xi_{-2}\right) \\
&= cov\left(\xi_{-1}, A_{-1}\right) + cov\left(\xi_{-1}, \xi_{-2}\right) \\
&> 0
\end{aligned}$$

So we see that $\log ARPK$ exhibits persistence (positive autocorrelation).

Asymmetric Persistence We can now show that the model with collateral constraints delivers asymmetric persistence, where the right tail exhibits greater persistence than the left tail.

To do so, recall that $\xi \in [0, \infty)$, while $A \in \mathbb{R}$. As such, we can always define some arbitrary quantile of the distribution q^- such that the following holds:

1. $\forall \log ARPK < q^-$, $Pr(\xi_{-1} = 0) = 1$ such that $\log ARPK = A$

2. $\forall \log ARPK > 1 - q^-$, $Pr(\xi_{-1} = 0) < 1$

Moreover, by the definition of a quantile, $Pr(\log ARPK < q^-) = Pr(\log ARPK > 1 - q^-)$.

Next, consider 2 types of firms: i and j, where $\log ARPK_i < q^-$ and $\log ARPK_j > 1 - q^-$.

Note that for firm i,

$$
\begin{aligned}
cor(\log ARPK_t, \log ARPK_{t+1}) &= cor(A_t, A_{t+1}) + cor(A_t, A_{t+1} + \xi_t) \\
&= cor(A_t, \xi_t)
\end{aligned}
$$

where this correlation term reflects the probability that a firm becomes financially constrained given it's productivity shock. This reflects that fact that even for large firms, there is a non-zero probability that they will become financially constrained.

For firm j,

$$
\begin{aligned}
cor(\log ARPK_t, \log ARPK_{t+1}) &= cor(A_t + \xi_{t-1}, A_{t+1} + \xi_t) \\
&= cor(A_t, A_{t+1}) + cor(A_t, \xi_t) + cor(\xi_{t-1}, A_{t+1}) + cor(\xi_{t-1}, \xi_t) \\
&= cor(A_t, \xi_t) + cor(\xi_{t-1}, \xi_t)
\end{aligned}
$$

Here, $cor(\xi_{t-1}, A_{t+1}) = 0$ since the probability of being financially constrained "yesterday" does not depend on future productivity shocks. However, $cor(\xi_{t-1}, \xi_t) > 0$ as noted earlier. Therefore, we see that firms of type i exhibit lower autocorrelation than firms of type j. In other words, the autocorrelation in the right tail of the distribution is greater than that in the left tail. □

IV Computational Appendix

In this section, I detail in subsection IV.a the solution method used in solving the individual's problem, and in subsection IV.b the method used in solving for the stationary equilibrium. A novel contribution to the literature is the algorithm used in solving for the stationary equilibrium.

IV.a Solving the individual's problem

The household's problem is broken down into a multi-stage problem in order to render it numerically feasible to solve. The steps are detailed below.

IV.a.1 First stage: Occupational choice

Define the intermediate value functions V_{ee}, V_{ew}, V_{we}, V_{ww} as the value functions of (1) an entrepreneur who has chosen to stay as an entrepreneur, (2) an entrepreneur who decides to exit, (3) a worker who decides to become an entrepreneur, and (4) a worker who decides to stay a worker; moreover, let this be the value function associated with the household who has already made its optimal portfolio-savings choice.

Given this definition, we see that the optimal occupational choice of an entrepreneur is given by

$$V_e = \max_{h'} \left(1 - h'\right) V_{ee} + h' \times V_{ew}$$

and the optimal occupational choice of a worker is given by

$$V_w = \max_{h'} \left(1 - h'\right) V_{we} + h' \times V_{ww}$$

where $h' = 1$ if the household chooses to become a worker tomorrow, and $h' = 0$ if she chooses to become a entrepreneur.

IV.a.2 Second stage: Savings-investment

The investment-savings problem is described for each for the four groups of households below. Here, for convenience, I shall define the interest rate function $\tilde{r} \equiv r \times 1_{\{b \geq 0\}} + r_d \times 1_{\{b<0\}}$.

Entrepreneur: Exit

For an entrepreneur who decides to exit, her problem is simply

$$\begin{aligned} V_{ew}\left(\psi^{\theta}, z, k, b\right) &= \max_{b'} U\left(c\right) + \beta \mathbb{E}\left[V_w\left(\psi^{z'}, \theta', b'\right) | \psi^{\theta}\right] \\ &s.t. \\ c + b' &= \pi^* + \left(1 + \tilde{r}\right) b + \left(1 - \zeta\right)\left(1 - \lambda\right)\left(1 - \delta\right) k \\ b' &\geq \underline{b} \end{aligned}$$

To reduce the dimensionality of this problem, I approximate

$$\hat{V}_{ew}\left(\psi^{\theta}, b'\right) \equiv \mathbb{E}\left[V_w\left(\psi^{z'}, \theta', b'\right) | \psi^{\theta}\right]$$

using a cubic spline in the b' direction. I then solve the individual's problem using the monotone condition.

Entrepreneur: Stay

For the entrepreneur who stays an entrepreneur, it is more convenient to define another set of intermediate value functions. Let V_{ee}^I, V_{ee}^D and V_{ee}^0 denote the value functions of an entrepreneur who has decided to (1) invest in his business, (2) disinvest, and (3) keep the same capital stock into the next period.

For an investing entrepreneur, his problem is

$$V_{ee}^{I}\left(\psi^{\theta},z,k,b\right)=\max_{k',b'}U\left(c\right)+\beta\mathbb{E}\left[V_{e}\left(\psi^{\theta'},z',k',b'\right)|z\right]$$

$$s.t.$$

$$c+\left(k'-\left(1-\delta\right)k\right)+b'+f_{s}k=\pi^{*}+\left(1+\tilde{r}\right)b$$

$$k'>\left(1-\delta\right)k$$

$$b'\geq-\varphi\left(1-\lambda\right)\left(1-\delta\right)k'-\underline{b}$$

For a disinvesting entrepreneur, his problem is

$$V_{ee}^{D}\left(\psi^{\theta},z,k,b\right)=\max_{k',b'}U\left(c\right)+\beta\mathbb{E}\left[V_{e}\left(\psi^{\theta'},z',k',b'\right)|z\right]$$

$$s.t.$$

$$c+\left(1-\lambda\right)\left(k'-\left(1-\delta\right)k\right)+b'=\pi^{*}+\left(1+r\right)b$$

$$0<k'<\left(1-\delta\right)k$$

$$b'\geq-\varphi\left(1-\lambda\right)\left(1-\delta\right)k'-\underline{b}$$

Finally, for an entrepreneur who is holding the same (depreciated) capital stock, his problem is

$$V_{ee}^{0}\left(\psi^{\theta},z,k,b\right)=\max_{k',b'}U\left(c\right)+\beta\mathbb{E}\left[V_{e}\left(\psi^{\theta'},z',k',b'\right)|z\right]$$

$$s.t.$$

$$c+b'=\pi^{*}+\left(1+r\right)b$$

$$k'=\left(1-\delta\right)k$$

$$b'\geq-\varphi\left(1-\lambda\right)\left(1-\delta\right)k'-\underline{b}$$

This formulation posses three problems. Firstly, the state space spanned by $\mathbb{K}\times\mathbb{B}$ is not orthogonal due to the financing constraints, which makes standard numerical methods not feasible. Secondly, as the portfolio choice here is non-trivial, computational time is substantially increased. Moreover, due to the asymmetric adjustment cost, the type of adjustment itself is conditional on the current state of capital holdings. Finally, the state space of the entrepreneur is spanned by four variables, which also increases computational time substantially. To address these issues, I consider the following modifications to the problem.

Orthogonality problem

Firstly, I address the orthogonality problem by defining a new variable, net liquid asset, a as follows:

$$a \equiv b + \varphi\left(1-\delta\right)\left(1-\lambda\right)k + \underline{b} \geq 0$$

The state space of the entrepreneur $\mathbb{S}^E$ is now defined by $\left(\psi^\theta, z, k, a\right) \in \Psi^\theta \times \mathbb{Z} \times \mathbb{K} \times \mathbb{A} = \mathbb{S}^E$, which is orthogonal, since both a and k are only restricted by $a \geq 0$ and $k \geq 0$.

Using this new definition, I address the numerical challenges arising from the non-trivial portfolio choice problem and dimensionality problem using the following strategy.

Portfolio choice and Dimensionality problem

Firstly, note that conditioned on deciding to *stay* as an entrepreneur (i.e. V_{ee}), the current signal shock ψ^θ is not informative of the next period's signal as it is IID. As such, the relevant state space for the entrepreneur who is staying as an entrepreneur is simply the subspace $\mathbb{S}^{EE} = \mathbb{Z} \times \mathbb{K} \times \mathbb{A} \subset \mathbb{S}^E$. This cuts down on one dimension.

Secondly, note that because of the IID nature of ψ^θ, the conditional expectation $\mathbb{E}\left[V_e\left(\psi^{\theta\prime}, z', k', a'\right)|z\right]$ is defined only over the three dimensional subspace $\mathbb{S}^{EE}$. Specifically, we can write the following definition:

$$\hat{V}_{ee}\left(z, k', a'\right) \equiv \mathbb{E}\left[V_e\left(\psi^{\theta\prime}, z', k', a'\right)|z\right]$$

To evaluate the continuation value of k' and a', I then approximate $\hat{V}_e$ using a two dimensional cubic spline along the dimensions of k' and a'.

Thirdly, we can take advantage of the linear separability of the cost function to reduce the dimensionality of the state space. Specifically, note that conditional on either form of adjustment, we can rewrite the household's budget as

$$c + b' + k' + \mathcal{C}_1\left(k'\right) = \pi^* + \left(1+\tilde{r}\right)b + \left(1-\delta\right)k - \mathcal{C}_2\left(k\right)$$
$$\Leftrightarrow c + a' - \varphi\left(1-\delta\right)\left(1-\lambda\right)k' + k' + \mathcal{C}_1\left(k'\right) = \pi^* + \left(1+\tilde{r}\right)\left(a - \varphi\left(1-\delta\right)\left(1-\lambda\right)k\right) + \left(1-\delta\right)k - \mathcal{C}_2\left(k\right)$$

where $\mathcal{C}\left(k', k\right) = \mathcal{C}_1\left(k'\right) + \mathcal{C}_2\left(k\right)$ is the total cost associated with this adjustment.

When the firm is adjusting upwards (i.e. investing, denoted henceforth by I), we get

$$\mathcal{C}_1\left(k'\right) = 0$$
$$\mathcal{C}_2\left(k\right) = f_s k$$

When the firm is adjusting downwards (i.e. disinvesting, denoted henceforth by D), we get

$$\mathcal{C}_1\left(k'\right) = -\lambda k'$$
$$\mathcal{C}_2\left(k\right) = \lambda\left(1-\delta\right)k$$

Therefore, we have two budget constraints given by

$$c + a' - \varphi\left(1-\delta\right)\left(1-\lambda\right)k' + k' = \pi^* + \left(1+\tilde{r}\right)a + \left(1-\delta-f_s-(1+\tilde{r})\varphi(1-\delta)(1-\lambda)\right)k \quad (\text{ if } I)$$
$$c + a' + \left(1-\varphi(1-\delta)\right)\left(1-\lambda\right)k' = \pi^* + \left(1+\tilde{r}\right)a + \left(1-(1+\tilde{r})\varphi\right)(1-\lambda)(1-\delta)k \quad (\text{ if } D)$$

Also, note that the budget for no adjustment (i.e. $k' = \left(1-\delta\right)k$) is given by

$$\begin{aligned}
c + b' &= \pi + \left(1+\tilde{r}\right)b \\
c + b' &= \pi + (1+\tilde{r})a - (1+\tilde{r})\varphi(1-\delta)\left(1-\lambda\right)k \\
\Leftrightarrow c + b' + \varphi(1-\delta)\left(1-\lambda\right)k' - \varphi(1-\delta)\left(1-\lambda\right)k' &= \pi + (1+\tilde{r})a - (1+\tilde{r})\varphi(1-\delta)\left(1-\lambda\right)k \\
\Leftrightarrow c + a' - \varphi(1-\delta)\left(1-\lambda\right)\left(1-\delta\right)k &= \pi + (1+\tilde{r})a - (1+\tilde{r})\varphi(1-\delta)\left(1-\lambda\right)k \\
\Leftrightarrow c + a' &= \pi^* + (1+\tilde{r})a - \left(\tilde{r}+\delta\right)\varphi(1-\delta)\left(1-\lambda\right)k
\end{aligned}$$

Next, we can define by x the total liquid value of the portfolio. That is, let

$$\begin{aligned}
x^I &\equiv \pi^* + \left(1+\tilde{r}\right)a + \left(1-\delta-f_s-(1+\tilde{r})\varphi(1-\delta)(1-\lambda)\right)k \quad (\text{ if } I) \\
x^D &\equiv \pi^* + \left(1+\tilde{r}\right)a + \left(1-(1+\tilde{r})\varphi\right)(1-\lambda)(1-\delta)k \quad (\text{ if } D) \\
x^O &\equiv \pi^* + (1+\tilde{r})a - \left(\tilde{r}+\delta\right)\varphi(1-\delta)\left(1-\lambda\right)k
\end{aligned}$$

At this point, it is important to note that because $x = x\left(z,k,a\right)$, for any generic (z,k,a)-triple, it is unlikely that $x^I = x^D$ numerically, due to their differing definitions. However, because this total liquid value is a sufficient state variable for the (z,k,a)-triple, it does not matter that $x^I \neq x^D$ for the same generic (z,k,a)-triple.

Now, define by $\left\{g_I^a\left(x\right), g_I^k\left(x\right)\right\}$ and $\left\{g_D^a\left(x\right), g_D^k\left(x\right)\right\}$ the optimal *unconstrained* savings-investment policy (i.e. choice of a' and k') when the household is faced with the budget constraints defined by x^I and x^D respectively (and $\tilde{V}^I_{ee}$ and $\tilde{V}^D_{ee}$ the value functions associated with this policy). That is, let them solve the two problems

$$\tilde{V}^I_{ee} = \max_{g^a_I, g^k_I} U(c) + \beta \mathbb{E}\left[V_e\left(\psi^{\theta'}, z', k', a'\right)|z\right]$$

$$s.t.$$

$$c + +g^a_I + (1 - \varphi(1-\delta)(1-\lambda))\, g^k_I = x^I$$

$$g^k_I > 0$$

$$g^a_I \geq 0$$

and

$$\tilde{V}^D_{ee} = \max_{g^a_D, g^k_D} U(c) + \beta \mathbb{E}\left[V_e\left(\psi^{\theta'}, z', k', a'\right)|z\right]$$

$$s.t.$$

$$c + g^a_D + (1 - \lambda - \varphi(1-\delta)(1-\lambda))\, g^k_D = x^D$$

$$g^k_D > 0$$

$$g^a_D \geq 0$$

Notice that for this sub-problem, the household does not care about the individual components of the current portfolio, and only cares about its total value x. Consequently, $\left\{g^a_I(x), g^k_I(x)\right\}$ and $\left\{g^a_D(x), g^k_D(x)\right\}$ are only functions of x.

Finally, let $\left\{g^a_0, g^k_0\right\}$ denote the optimal savings policy when the household is not adjusting. In particular $g^k_0 = (1-\delta)\, k$.

First, for convenience, denote the following indicator functions as follows:

$$\mathcal{I}^I = \begin{cases} 1 & (\text{if } g^k_I > (1-\delta)\, k) \\ 0 & (\text{otherwise}) \end{cases}$$

$$\mathcal{I}^D = \begin{cases} 1 & (\text{if } g^k_I < (1-\delta)\, k) \\ 0 & (\text{otherwise}) \end{cases}$$

$$\mathcal{I}^v = \begin{cases} 1 & (\text{if } \tilde{V}^I_{ee} > \tilde{V}^D_{ee}) \\ 0 & (\text{otherwise}) \end{cases}$$

$$\mathcal{I}^0 = \begin{cases} 1 & (\text{otherwise}) \\ 0 & (\text{if } V^0_{ee} \geq \max\left\{V^I_{ee}, V^D_{ee}\right\} \end{cases}$$

Now, denote by $g^a\left(\psi^{\theta'}, z, k, a\right)$ and $g^k\left(\psi^{\theta'}, z, k, a\right)$ the optimal *constrained* savings-investment policy functions. They must satisfy the following condition: For any portfolio state (z, k, a) [38], we have:

$$\begin{bmatrix} g^a \\ g^k \end{bmatrix} = \left[\mathcal{I}^I \times \mathcal{I}^v \times \begin{bmatrix} g_I^a \\ g_I^k \end{bmatrix} \circ \left(x^I\right) + \mathcal{I}^D \times (1-\mathcal{I}^v) \times \begin{bmatrix} g_D^a \\ g_D^k \end{bmatrix} \circ \left(x^D\right)\right] +$$
$$\left(1 - \mathcal{I}^0 \times \left[\mathcal{I}^I \times \mathcal{I}^v + \mathcal{I}^D \times (1-\mathcal{I}^v)\right]\right) \times \begin{bmatrix} g_0^a \\ g_0^k \end{bmatrix} \circ \left(\psi^{\theta'}, z, k, a\right)$$

Similarly then, the value function for the entrepreneur who chooses to stay an entrepreneur is

$$V_{ee} = \left[\mathcal{I}^I \times \mathcal{I}^v \times \tilde{V}_{ee}^I + \mathcal{I}^D \times (1-\mathcal{I}^v) \times \tilde{V}_{ee}^D\right] +$$
$$\left(1 - \mathcal{I}^0 \times \left[\mathcal{I}^I \times \mathcal{I}^v + \mathcal{I}^D \times (1-\mathcal{I}^v)\right]\right) \times V_{ee}^0$$

Worker: Entry into entrepreneurship

Conditional on choosing to become an entrepreneur, the worker's problem is

$$V_{we}\left(\psi^z, \theta, a\right) = \max_{k', a'} U(c) + \beta \mathbb{E}\left[V_e\left(\psi^{\theta'}, z', k', a'\right) | \psi^z\right]$$
$$s.t.$$
$$c + k' + a' - \varphi(1-\delta)(1-\lambda) k' = \theta w + (1+r) a$$
$$k' \geq k_{min}^w$$
$$a' \geq 0$$

Note that in this case, $a = b$ since $k = 0$, but $a' = b' + \varphi(1-\lambda) k'$.

As in the entrepreneur who stays an entrepreneur, I approximate

$$\hat{V}_{we}\left(\psi^z, k', a'\right) \equiv \mathbb{E}\left[V_e\left(\psi^{\theta'}, z', k', a'\right) | \psi^z\right]$$

using a two dimensional cubic spline in the k' and a' directions.

[38] Note the dependence on the portfolio state, not just x. In general, for some (z, k, a), $x_I \neq x_D \neq x_0$.

Worker: Stay

Conditional on staying a worker, the household's problem is

$$\begin{aligned} V_{ww}\left(\psi^{z},\theta,b\right) &= \max_{b'} U\left(c\right)+\beta\mathbb{E}\left[V_{w}\left(\psi^{z'},\theta',b'\right)|\theta\right] \\ &s.t. \\ c+b' &= \theta w+\left(1+r\right)b \\ b' &\geq 0 \end{aligned}$$

Note that in this case, $a=b$ and $a=b'$ since $k=k'=0$.

As in the entrepreneur who exits, I approximate

$$\hat{V}_{ww}\left(\theta,b'\right)\equiv\mathbb{E}\left[V_{w}\left(\psi^{z'},\theta',b'\right)|\theta\right]$$

using a cubic spline along the b' direction.

IV.b Solving for the stationary distribution

Given the policy functions and law of motion of exogenous processes from the preceding section, the time-t distribution evolves as

$$\lambda_{t+1}(x')=\int T_{t}(x'|x)\lambda_{t}(x)dx$$

where I use x to denote the full vector of state variables. T here is the Markov kernel induced by the policy functions and exogenous Markov processes. In the stationary distribution, this reduces to

$$\lambda(x')=\int T(x'|x)\lambda(x)dx$$

where the only difference is that the time subscripts are dropped: i.e. λ is an invariant measure. A direct method to solve for the invariant distribution is to approximate both λ and T on a grid; in particular, Young (2010) proposes a straightforward method to approximate T as a Markov transition matrix $\hat{T}$.

Unfortunately, a direct construction of $\hat{T}$ is not feasible when the state space is too large. In particular, this model features many points of non-convexity (discrete choices, non-convex adjustment costs). As a result, a large number of nodes are required in order to solve this model to a sufficient degree of accuracy. To that end, I approximate the distribution using

200 grid points in a and b, and 15 nodes in z, θ, ψ^z and ψ^θ. Coupled with the two discrete states in occupational choice, this comes up to $\approx 9 \times 10^6$ grid points. The matrix $\hat{T}$ is therefore $\approx 9 \times 10^6 \times 9 \times 10^6$ with a density of about 0.25% (which amounts to around $15^2 \times 9 \times 10^6$ non-zero grid points; or about 16.2 gb in RAM). Given that the power method used to compute λ involves iterating on $T \times \lambda$, this method is both extremely memory intensive and slow.

Instead of approximating T, I follow the method documented in a companion paper (c.f. Tan (2018b)). There, I utilize the fact that the kernel T is simply the product of two independent Markov kernels G and H, where H iterates on the exogenous state variables (z), and G iterates on the endogenous state variables (y):

$$\begin{aligned} T &= HG \\ \implies \lambda(y', z') &= \int_z H(z'|z) \int_y G(y'|y, z)\lambda(y, z) dy dz \end{aligned}$$

G is then approximated using the method in Young (2009), whereas H is the transition matrix associated with all the exogenous forcing processes. G has the same dimensions as $\hat{T}$, but now only has a density of $\approx 0.001\%$. This method greatly reduces the computational burden to around 2 gb in RAM. The stationary distribution can be computed in under 100 seconds (holding fixed the same convergence tolerance).

V Parameters estimated from the data

In this section, I explain how the depreciation rate, capital intensity, and returns to scale are estimated from the data.

V.a Depreciation rate

The depreciation rate of capital is different for an entrepreneur and for the corporate sector. The rationale here is that the depreciation rate of the corporate sector is a weighted average of the entire range of capital types that exist in the economy. This leads most papers to typically set δ to numbers between 6% to 10%. In contrast, the illiquid capital asset data drawn from the KFS only encompasses a few classes of capital: Structures/buildings/land, vehicles, equipment, inventory and accounts receivables. As such, I construct δ_k as a weighted average of the stock of these capital, and arrive at 15% depreciation (with the individual depreciation rates taken from the BEA).

V.b Capital intensity and returns to scale

To construct α_e and ν, I utilize a two-step approach. Firstly, I followed the cost shares approach utilized in Asker et al (2014) to compute a consistent estimator for the labor share. When labor is a variable input and chosen contemporaneously, the first order condition for labor yields (using notation from section 3)

$$\beta^l \equiv (1-\alpha_e)\nu = \frac{wL}{Y}$$

where β_l is the labor share. Following Asker et al (2014), I first construct a data set of labor shares at the individual level i, and estimate, for each two digit NAICS industry j, the *median* labor share:

$$\hat{\beta}^l_j = median\{\beta^l_{i,j}\}$$

Given a value for ν, I can then back out the capital share using the identity:

$$\hat{\beta}^k_j = \nu - \hat{\beta}^l_j$$

which in turn gives the capital intensity:

$$\hat{\alpha_e} = \frac{\hat{\beta}^k_j}{\nu}$$

To compute the returns to scale ν, note that for an employer firm, the capital stock and revenue is connected through the following equation:

$$\log Y = \Theta_0 + \Theta_k \log K + \Theta_z z \log z$$
$$\Theta_k \equiv \frac{\alpha_e \nu}{1-(1-\alpha_e)\nu}$$

Θ_k can be consistently estimated using a quantile regression (at the median). This estimate, along with the labor share, then returns us estimates for α_e and ν.

As discussed in the main body of the paper, Y corresponds to an imputed measure of value added.

VI Returns to scale and wealth dispersion

In the main text, I discussed the effect of capital illiquidity on the wealth dispersion. Here, I explain why the returns to scale is also an important factor in understanding the contribution of entrepreneurial investment and risk taking to the wealth distribution.

To understand this, we have to first understand what constitutes capital income risk in a model of entrepreneurship. Recall that gross capital income in a model of entrepreneurship is simply production income. For a simplified model without labor, gross capital income is then simply $y = zk^{\nu}$. In a static one asset model without any frictions, we know that optimal choice of k is simply $(\frac{z}{r+\delta})^{\frac{1}{1-\nu}}$, so gross capital income is just $z^{1+\frac{\nu}{1-\nu}}(\frac{1}{r+\delta})^{\frac{\nu}{1-\nu}}$. The dispersion and persistence in (log) capital income is then:

$$
\begin{aligned}
var(\log(y_t)) &= var(\log(z^{1+\frac{\nu}{1-\nu}})) \\
&= \left(1 + \frac{\nu}{1-\nu}\right)^2 var(\log(z)) \\
cov(\log(y_t), \log(y_{t-1})) &= \left(1 + \frac{\nu}{1-\nu}\right)^2 cov(\log(z_t), \log(z_{t-1}))
\end{aligned}
$$

where we see right away that (i) the **volatility** of capital income depends on *both* the volatility of TFP as well as the returns to scale, and (ii) the **persistence** of capital income depends only on the persistence of TFP, but it's covariance depends on *both* the persistence of TFP and returns to scale. Crucially, ν inflates the volatility and covariance of capital income at an exponential rate. For instance, consider a TFP process with a small unconditional variance of 0.0335 and a returns to scale of 0.88[39]: The resulting volatility of capital income is inflated to almost 1.34, a nearly 40 times increase over the underlying productivity process. On one level, this is in fact an important reason why models of entrepreneurship have been so successful in matching the wealth distribution: For even small values of underlying productivity risk, a positive returns to scale can greatly inflate the dispersion of capital income, thus also increasing the dispersion in wealth.

As an example of the importance ν plays in determining the dispersion of wealth, I report in figure 26 below a comparative statics exercise that reports the Gini coefficient for every $\nu \in (0.68, 0.88)$. The vertical dashed line is a reference line corresponding to the benchmark ν. Notice that the returns to scale for the benchmark calibration is too small, relative to the standard calibration such as that in Cagetti and De Nardi (2006), to generate the high empirical wealth inequality.

[39] These numbers in fact correspond to the calibration chosen by Cagetti and De Nardi (2006)

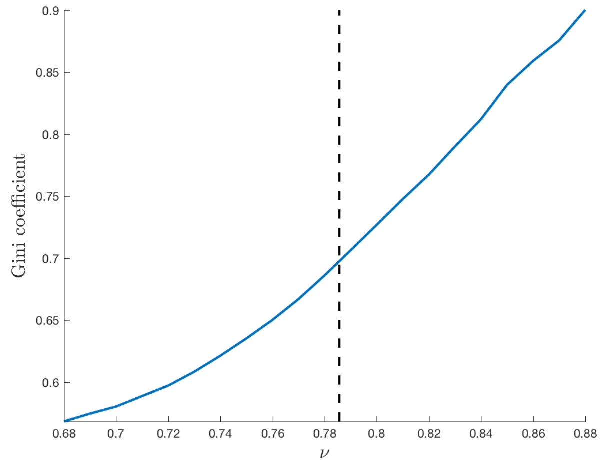

Figure 26: Effect of ν on wealth inequality.

VII Model extensions

VII.a Computing Welfare Changes Along Transition Path

This subsection describes how welfare is computed along the transition path in section 5.4.1.

To do so, I first solve for the steady-state distribution of households and value functions under the benchmark calibration. For notational convenience, denote the initial steady-state distribution as Λ_0, and the value functions as $V_e(a, k, z, \psi_\theta; 0)$ and $V_w(a, \theta, \psi_z; 0)$, where the "0" denotes that the distribution and value functions are at "time-0". Finally, denote the initial adjustment cost parameters as $\bar{\lambda}$, $\bar{\zeta}$, and $\bar{f}_s$, and the collateral constraint as $\bar{\varphi}$.

Next, at time $t = 1$, I assume that households receive information that from $t \geq 2$ onwards, all the frictions that render capital illiquid is suddenly and costlessly removed. In other words, I set $\lambda = 0$, $\zeta = 0$ and $f_s = 0$. In addition, I assume that this change is entirely unforeseen by the households, prior to the release of this information. The economy is then allowed to transition to the new steady-state where capital is fully liquid; moreover, households now have perfect foresight over the entire transition path of the economy.

Stated in recursive terms, this amounts to the following dynamic program. For entrepreneurs,

$$
\begin{aligned}
V_e\left(\psi^{\theta}, z, k, b; t\right) = & \max_{h', k', b', l} U\left(\tilde{c}\right) \\
& + \left(1 - h'\right) \times \beta \int_{\psi'_{\theta}} \int_{z'} V_e\left(\psi^{\theta'}, z', k', b'; t+1\right) dP_{z'|z} dF_{\psi^{\theta}} \\
& + h' \times \beta \int_{\psi'_{z}} \int_{\theta'} V_w\left(\psi^{z'}, \theta', b'; t+1\right) dP_{\theta'|\psi^{\theta}} dF_{\psi^{z}} \\
& s.t. \\
& \hat{\pi} \equiv zf\left(k, \iota\bar{l} + l\right) - wl + \left(1 + r \times 1_{\{b \geq 0\}} + r_d \times 1_{\{b<0\}}\right) b - \mathcal{C}\left(k', k, h', h; f_{s_t}, \lambda_t, \zeta_t, \varphi_{t-1}\right) \\
& c = \max\left\{\hat{\pi}, 0\right\} - k' - b' \geq 0 \\
& k' \begin{cases} > 0 & \text{if } h' = 0 \\ = 0 & \text{if } h' = 1 \end{cases} \\
& h' = 1 \quad (\text{ if } \hat{\pi} < 0) \\
& b' \geq -\varphi_t \left(1 - \lambda_{t+1}\right)\left(1 - \delta\right) k' - \underline{b} \\
& \tilde{c} = c + \bar{c}
\end{aligned}
$$

where

$$
\lambda_t = \begin{cases} \bar{\lambda} & \text{if } t \leq 1 \\ 0 & \text{if } t \geq 2 \end{cases}
$$

$$
\zeta_t = \begin{cases} \bar{\zeta} & \text{if } t \leq 1 \\ 0 & \text{if } t \geq 2 \end{cases}
$$

$$
f_{s_t} = \begin{cases} \bar{f}_s & \text{if } t \leq 1 \\ 0 & \text{if } t \geq 2 \end{cases}
$$

$$
\varphi_t = \bar{\varphi} \frac{\left(1 - \bar{\lambda}\right)}{\left(1 - \lambda_{t+1}\right)}
$$

Note the explicit definition of the adjustment function $\mathcal{C}$ on the time-varying parameters, as well as the adjustment of φ to fix the level of collateral constraint. For workers,

$$
\begin{aligned}
V_w\left(\psi^z,\theta,b;t\right) = & \max_{h',k',b'} U\left(\tilde{c}\right) \\
& + \left(1-h'\right)\times\beta\int_{\psi'_\theta}\int_{z'} V_e\left(\psi^{\theta'},z',k',b';t+1\right) dP_{z'|\psi^z} dF_{\psi^\theta} \\
& + h'\times\beta\int_{\psi'_z}\int_{\theta'\in\Theta} V_w\left(\psi^{z'},\theta',b';t+1\right) dP_{\theta'|\theta} dF_{\psi^z} \\
& s.t. \\
c+k'+b' = & \ \theta w\bar{l} + \left(1 + r\times 1_{\{b\geq 0\}} + r_d \times 1_{\{b<0\}}\right) b \\
& k' \begin{cases} >0 & \text{if } h'=0 \\ =0 & \text{if } h'=1 \end{cases} \\
& b' \geq -\varphi_t\left(1-\lambda_{t+1}\right)\left(1-\delta\right)k' - \underline{b} \\
& \tilde{c} = c + \bar{c}
\end{aligned}
$$

The associated equilibrium conditions follows: A sequential competitive equilibrium of the model consist of the path of interest and wage rates $\{r_t, w_t)_{t\geq 0}$, value functions of households and firms $\{V_e(\cdot;t), V_w(\cdot;t), \Pi(\cdot;t)\}_{t\geq 0}$, allocations $\{k_t, b_t, l_t\}_{t\geq 0}$ and distribution of agents Λ_t over the state space $\mathbb{S}$ such that,

1. Taking $\{r_t, w_t)_{t\geq 0}$ as given, the households' and firms' choices are optimal.
2. Markets clear, such that for every period t
   (a) Bonds: $\int b_t d\Lambda_t = K_t^c$
   (b) Labor: $\int \theta h_t d\Lambda_t = \int l d\Lambda + L_t^c$
3. The distribution Λ_t evolves as

$$\Lambda_t = \Gamma_t\left(\Lambda_{t+1}\right)$$

   Where Γ_t is the one-period transition operator on the distribution at time t induced by the policy functions of the households and firms.

For the partial equilibrium example, I fix interest rates and wages to the time 0 prices.

To compute welfare, I then compare the time 0 value functions with the time 1 value functions; that is to say, I compare the welfare of a household that lives in the benchmark economy forever with that of an equivalent household who lives in the counter-factual economy, taking into account the entire transitional dynamics associated with moving from the initial economy to the new economy. The individual consumption equivalent variation, welfare changes for sub-populations, as well as the aggregate welfare changes are computed as discussed in the main text, using the time 0 distribution of households for aggregation.

VII.b Aggregate Shocks: Credit Tightening

To construct the dynamics in figure 19a, I assume that at time $t = 0$, $\varphi_t = \bar{\varphi}$ is held at its steady state. At $t = 1$, the household's collateral constraint suddenly tightens unexpectedly, such that $\varphi_t < \bar{\varphi}$. However, at time $t \geq 2$, the collateral constraint returns back to it's steady-state value. The households have full information of this reversion once the shock hits at $t = 1$. This, in a stylized fashion, replicates an exogenous credit tightening.

VII.c Aggregate Shocks: Increasing Illiquidity

To construct the dynamics in figure 19b, I assume that at time $t = 0$, $\lambda_t = \bar{\lambda}$ is held at its steady state. At $t = 1$, the price of used capital suddenly falls unexpectedly, such that $1 - \lambda < 1 - \bar{\lambda}$. In order to avoid sudden default, I assume that λ_1 stays at it's steady-state value, but λ_2 falls[40]. At $t \geq 3$, the value of capital reverts back to it's steady-state. Along the same time path, the collateral constraint also adjusts such that the net collateral constraint is held fixed, as in section VII.a.This entire path is fully known to the household on e the shock hits at $t = 1$.

[40]Note that this does not mean that the shock is expected. Rather, one can consider this as similar to a “news shock".