

Signaling with Private Monitoring*

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Abstract

We examine linear-quadratic signaling games in continuous time between a long-run player that has a normally distributed type and a myopic player who privately observes a noisy signal of the former's actions. A public signal of the myopic player's behavior is observed (i.e., there is two-sided signaling) and shocks are Brownian. We construct linear Markov equilibria using belief states up to the long-run player's *second-order* belief. The latter state is an explicit function of past play, reflecting that past behavior is used to forecast the continuation game. Via this higher-order belief channel, the informativeness of the long-run player's action is not only driven by the weight that the linear strategy attaches to her type, but also by how aggressively she signaled in the past. Applications to models of leadership, reputation, and trading are examined.

1 Introduction

The phenomenon of *signaling*—that is, the transmission of information through actions—is pervasive in economics, influencing areas as diverse as education (Spence, 1973), finance (Kyle, 1985) and leadership (Hermalin, 1998). Despite this breadth, the great majority of signaling games have a commonality: the sender is *certain* of the receiver's belief about the former's type at the moment of acting—the receiver's belief is *public*.¹ That economic agents

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¹We review the related literature later in this section.

operate under perfect knowledge of what others believe is questionable in many situations. In particular, *imperfect private signals of behavior* can arise in many ongoing interactions: when employers subjectively assess workers’ performances (MacLeod, 2003; Levin, 2003); when traders handle others’ orders (Yang and Zhu, 2019); or when data brokers collect consumer data (Bonatti and Cisternas, 2019). In those settings, receivers’ beliefs over fundamentals such as a worker’s ability, an asset’s value, or a consumer’s willingness to pay, are *private*.

Allowing for private monitoring of an informed player’s actions in dynamic settings can open the way for a whole new set of applied-theory questions to be analyzed. How do leaders influence their followers when they don’t know how their actions have been interpreted? Is an agent’s ability to manage a reputation hindered by not observing the signals generated by her actions? How is trading behavior affected by the possibility of hidden leakages to other traders? A comprehensive treatment of these questions is an important agenda, yet it presents at least three challenges. First, higher-order beliefs can arise: in most settings, senders will have to form a non-trivial belief about their receivers’ beliefs. Second, such games are inherently asymmetric: when facing a sender of a *fixed* type, the receiver develops *evolving* private information in the form of a belief. Third, most analyses will be non-stationary due to ongoing learning effects. Are there settings where the ‘beliefs about beliefs’ problem is manageable? Can we develop methods for a tractable treatment of asymmetric signaling games? What are the implications on behavior and economic outcomes?

Towards answering these questions, in this paper we examine linear-quadratic-Gaussian games of one-sided incomplete information and private monitoring. A long-run player (she) and a myopic one (he), both with linear-quadratic preferences, interact over a finite horizon. The long-run player has a normally distributed type. The key departure from the existing models lies in the myopic player privately observing a noisy signal of the long-run player’s action. To make this departure minimal, we let the long-run player learn about the myopic player’s private inferences from an imperfect public signal of the latter’s behavior. The shocks in both signals are additive and Brownian. Using continuous-time methods, we construct linear Markov perfect equilibria (LME) in which the players’ beliefs are the relevant states.

Equilibrium construction and signaling. The construction of non-trivial equilibria in games of private monitoring can be a daunting task. In fact, to estimate rivals’ continuation behavior under any strategy, players usually have to make an inference of the private histories of their rivals. Not knowing what their rivals have seen, the players will then rely on their past play; but this implies that players’ inferences will in turn vary with their *own* private histories. Thus, (i) probability distributions over histories must be computed, and (ii) the continuation game at an off-path history may differ from any on-path counterpart.

With incomplete information, one expects this statistical inference problem to become one of estimation of belief states that summarize the payoff-relevant aspects of the players’ private histories—our setup offers a parsimonious treatment of this issue. Specifically, the quadratic preferences open the possibility of our players using strategies that are linear in their beliefs’ posterior means (henceforth, beliefs). Conjecturing such linear strategies, learning is (conditionally) Gaussian: the myopic player’s belief is linear in the history of her private signals, and the long-run player’s *second-order belief*—her belief about the myopic player’s belief—is linear in the histories of the public signal and her past play. The estimation of histories described in (i) is thus simplified by the fact that these are aggregated *linearly*.

The long-run player’s second-order belief is private—even in equilibrium—as her actions depend on her type; hence, the myopic player must forecast this state. The problem of the state space expanding is circumvented by a key *representation* (Lemma 1) that expresses the (candidate, on path) second-order belief in terms of the long-run player’s type and the belief about it based on the public signal exclusively (and that makes this “public state” a relevant one). Thus, our analysis uses a novel additional state—a controlled second-order belief—that is redundant on the path of play (part (ii) above). Equipped with Markov belief states as sufficient statistics, we can write the long-run player’s best-response problem as one of stochastic control, and use dynamic programming for finding LME.²

Despite the complexities it introduces, private monitoring has natural implications for signaling. In fact, the notion that individuals rely on their *past* behavior to forecast what others *currently* know strongly resonates with reality: a leader relying on her past behavior for estimating an organization’s understanding of the environment in which it operates; a politician relying on her past actions for gauging people’s perception of her reputation; a trader using her past trades for estimating other investors’ perception of her private information. By contrast, in the knife-edge case in which all the signals are public, past behavior becomes irrelevant: current beliefs are fully determined by the realized public history.

Our representation result encodes the signaling implications of this explicit use of past play. Namely, since actions are used to forecast the myopic player’s belief, and different types behave differently in equilibrium, different types expect their “receivers” to have different beliefs. Consequently, the equilibrium informativeness of the long-run player’s action is determined not only by the weight that her strategy places on her type, but also by her past signaling behavior via the second-order belief channel. We refer to this as the *history-inference effect* on signaling. The potential amplitude of this effect is largest when the public signal is pure noise (the *no-feedback case*), and thus the reliance on past play to forecast the continuation game is strongest; conversely, it disappears when beliefs are public.

²The same states apply if the “receiver” is not myopic. The public state creates signal-jamming motives.

Applications. A key asset of our analysis is that we can compute behavior that explicitly conditions on beliefs. (This is done partly via ordinary differential equations (ODEs), which we discuss shortly.) To leverage this advantage, we examine one instance of our main model of Section 3, and two applications based on extensions of it.

In Section 2, we illustrate the history-inference effect and its implications for outcomes by means of a coordination game inspired by the linear-quadratic team theory of [Marschak and Radner \(1972\)](#)—this framework, along with its generalizations allowing for misaligned preferences, has become the canonical laboratory for studying organizations.³

In the setting we examine, a team is comprised of a leader and a follower. The team’s performance increases with the proximity of its members’ actions (coordination) and with the proximity of the leader’s actions to a newly realized state of the world (adaptation). The leader’s and team’s payoffs coincide, while the follower simply attempts to match the leader’s action at all times. Recognizing the existence of important structural barriers to the transmission of knowledge within organizations, we assume that the leader can convey the state of the world only gradually via an imperfect signal of her behavior privately observed by the follower (e.g., a subjective evaluation). In such a context, we show that the coordination motive leads the history-inference effect to result in more information being transmitted relative to the case in which the follower’s belief is public; yet the team’s performance is lower. Thus, organizations with a more shared understanding of the economic environment can in fact *underperform* their more heterogenous counterparts.

Uncertainty about others’ beliefs also arises in reputational settings. In Section 5.1, we examine a model of horizontal reputation based on an extension allowing for *terminal* payoffs: the long-run player suffers a terminal quadratic loss that increases in the distance between the myopic player’s belief and the type’s prior (e.g., a politician facing reelection trying to build a reputation for neutrality). In such a context, we show that not directly observing her reputation can benefit the long-run player—this is despite the negative direct effect from increased uncertainty over a concave objective. Indeed, since higher types take higher actions due to their larger biases, those types must offset higher beliefs to appear unbiased; the history-inference effect then reduces the informativeness of the long-run player’s action, making beliefs less sensitive to new information, a strategic effect that can dominate.

Finally, in Section 5.2 we exploit the presence of the public belief state in a trading model in which an informed trader faces both a myopic trader who privately monitors her orders and a competitive market maker who only observes the public total order flow. In this context, we show that there is no linear Markov equilibrium for any degree of noise of the private signal. Intuitively, the myopic player introduces momentum into the price, as the

³[Dessein and Santos \(2006\)](#), [Alonso et al. \(2008\)](#) and [Rantakari \(2008\)](#) are prominent recent examples.

information he obtains now gets distributed to the market maker through *all* future order flows. This causes prices to move against the insider and creates urgency—with an infinite number of opportunities to trade, the insider trades away all information in the first instant.

Existence of LME and technical contribution. The setting we examine is asymmetric, both in terms of the players’ preferences and their private information (a fixed state versus a changing one). In particular, the players can signal at substantially different rates, which is in stark contrast to a small literature on symmetric multi-sided learning. With different rates of learning, however, the equilibrium analysis can become severely complicated.

Specifically, our belief states depend on the myopic player’s posterior variance, which determines the sensitivity of the myopic player’s belief, and on the weight attached to the long-run player’s type in the representation result, which affects signaling via the history-inference effect. Moreover, both functions are deterministic due to the Gaussian learning. Using dynamic-programming, one can then show that the problem of existence of LME reduces to a *boundary value problem* (BVP) including ODEs for the two aforementioned functions of time and for the weights in the long-run player’s strategy. The two *learning* ODEs endow the BVP with exogenous initial conditions, while the rest carry terminal conditions arising from myopic play at the end of the game.

Determining the existence of a solution to such a BVP is challenging because it involves multiple ODEs in both directions. For this reason, we distinguish among two types of environments. In a *private value* setting, the myopic player’s best response does not directly depend on his belief about the type, but only indirectly via his expectation of the latter player’s action. In that context, we show that there is a one-to-one mapping between the solutions to the learning ODEs (Lemma 4), a consequence of the ratio of the signaling coefficients being constant. This, in turn, makes traditional *shooting* methods based on the continuity of the solutions applicable. Via this method, we show the existence of LME in the leadership model of Section 2 when the public signal is of intermediate quality for horizon lengths that are decreasing in the prior variance about the state of the world (Theorem 1).

In *common value* settings, the multidimensionality issue must be confronted. Building on the literature on BVPs with *intertemporal linear* constraints (Keller, 1968), we can show the existence of LME to our BVP with *intratemporal nonlinear* (terminal) constraints. Specifically, the multidimensional shooting problem can be formulated as a *fixed-point problem* for a suitable function derived from the BVP, which we tackle for a variation of the leadership model in which the follower cares about the state of the world directly (Theorem 2). Critically, the method is general: we show how to apply to the whole class of games under study, and it can open a way for examining other settings exhibiting learning and asymmetries.

Related Literature Early analyses of static noisy signaling are [Matthews and Mirman \(1983\)](#), [Carlsson and Dasgupta \(1997\)](#) and [Maggi \(1999\)](#). Recent dynamic signaling models with public beliefs include [Dilmé \(2019\)](#), [Gryglewicz and Kolb \(2019\)](#), [Kolb \(2019\)](#) and [Heinsalu \(2018\)](#). [Feltovich et al. \(2002\)](#) and [Voorneveld and Weibull \(2011\)](#) incorporate an *exogenous* private signal of the sender’s type in static contexts.

[Foster and Viswanathan \(1996\)](#) and [Bonatti et al. \(2017\)](#) examine multisided signaling in symmetric settings with imperfect public monitoring and dispersed fixed private information. In those models, beliefs are private, but the presence of a commonly observed public signal permits a representation of first-order beliefs that eliminates the need for higher-order ones.⁴ [Bonatti and Cisternas \(2019\)](#) in turn examine two-sided signaling in a setting where firms privately observe a summary statistic of a consumer’s past behavior to price discriminate; via the prices they set, however, firms perfectly reveal their information to the consumer. An additional difference from these papers is that we study a *class* of games.

Multisided private monitoring has been explored mostly in the context of repeated games, and hence with a focus on non-Markovian incentives. [Ely et al. \(2005\)](#) and [Hörner and Lovo \(2009\)](#) (the latter allowing for incomplete information) study equilibria in which inferences of others’ private histories are not needed. By contrast, [Mailath and Morris \(2002\)](#), [Hörner and Olszewski \(2006\)](#) and [Phelan and Skrzypacz \(2012\)](#) construct belief-based equilibria, the first two of these with almost-perfect information structures. In turn, [Levin \(2003\)](#) and [Fuchs \(2007\)](#) examine one-sided private monitoring in repeated principal-agent interactions.

Regarding our applications, the stage game of our leadership model is a simplified version of [Dessein and Santos \(2006\)](#).⁵ In turn, the value of public information has been studied in coordination games among infinitesimal agents, such as [Morris and Shin \(2002\)](#), [Angeletos and Pavan \(2007\)](#), and [Bolton et al. \(2012\)](#), the latter studying leader resoluteness. [Amador and Weill \(2012\)](#) study the gradual diffusion of private information when players see private signals of aggregate actions; with infinitesimal players, individual histories are irrelevant for forecasting aggregate behavior. Regarding trading models, [Yang and Zhu \(2019\)](#) show, in a richer two-period version of our model, that a linear equilibrium ceases to exist if a signal of an informed player’s last trade is too precise and privately observed by another player.

To conclude, this paper contributes to a growing literature employing continuous-time techniques to the analysis of dynamic incentives. [Sannikov \(2007\)](#) examines two-player games of imperfect public monitoring; [Faingold and Sannikov \(2011\)](#) reputation effects with behavioral types; [Cisternas \(2018\)](#) games of ex ante symmetric incomplete information; and [Hörner and Lambert \(2019\)](#) information design in career concerns settings.

⁴Likewise in [He and Wang \(1995\)](#), where infinitely many agents privately see dynamic exogenous signals.

⁵See [Bolton and Dewatripont \(2013\)](#) for such a static analysis with one round of pre-play communication.

2 Application: Leading Coordinated Adaptation

A team consisting of a leader (she) and a follower (he) operates in a setting parametrized by a state of the world $\theta \sim \mathcal{N}(\mu, \gamma^o)$. The leader’s (and team’s) payoff is given by

$$\int_0^T e^{-rt} \{-(a_t - \theta)^2 - (a_t - \hat{a}_t)^2\} dt, \quad (1)$$

where a_t denotes the leader’s action at time $t \in [0, T]$ and \hat{a}_t the follower’s counterpart, both taking values over \mathbb{R} ; in turn, $r \geq 0$ is a discount rate, and $T < \infty$. In this specification, the team’s performance increases with the proximity of the leader’s action to the state of the world (adaptation) and with the proximity of both players’ actions (coordination).⁶

We assume that the leader knows the value of θ , and that the follower only knows its distribution (and this is common knowledge). Critically, if leaders were able to easily induce their followers to take the right actions (in this case, θ), organizations would incur no losses while adapting to change. In reality, this does not occur largely for two reasons: misaligned objectives and information frictions. The former has been extensively studied strategic communication models (e.g., cheap talk). We are interested in the latter.

In this line, a vast literature on organizations has documented the importance of specialized knowledge as an input to production, and that much of that knowledge has a *tacit* form: “know-how” that resides in people’s minds and that is hard to codify and transfer.⁷ To focus on this transmission friction, we start by assuming that (i) the follower acts myopically by minimizing $\hat{\mathbb{E}}_t[(a_t - \hat{a}_t)^2]$ at all times, and that (ii) direct communication is shut down. The first assumption is for expositional purposes: it partially aligns the players’ objectives to avoid confounding structural transferability barriers from strategic counterparts.⁸ The second is a modeling choice: it is a dimensionality constraint that aims to capture knowledge that is richer than the communication channel or code available, thereby placing a barrier on an easy transfer via talk.

The leader’s knowledge of θ is then short for know-how relevant to the current economic conditions. The follower gradually learns about this know-how from privately observing a

⁶Achieving a coordinated adaptation to new economic conditions is a major concern in organizations; see, for instance, Chapter 4 in [Milgrom and Roberts \(1992\)](#) and Section 4.2 in [Williamson \(1996\)](#).

⁷Within economics, [Garicano \(2000\)](#) models tacit knowledge as a person’s ability to carry out complex tasks that is not transferable to others. Seminal articles in strategic management are [Kogut and Zander \(1992\)](#) and [Grant \(1996\)](#). Regarding coordination, the latter paper emphasizes the difficulties of integrating such knowledge across complementary divisions, such as marketing and R&D.

⁸In strict terms, the divergence in the players’ time preferences takes us away from Marschak and Radner’s framework. The analysis introduced in Section 3 can accommodate general forms of misalignment.

noisy signal of the leader’s actions of the form

$$dY_t = a_t dt + \sigma_Y dZ_t^Y$$

where $\sigma_Y > 0$ is a volatility parameter and Z^Y a Brownian motion. That the leader’s know-how is transferred via her actions is consistent with the key property of tacit knowledge “only being observed through its application”⁹ or being “deeply rooted in action and in an individual’s commitment to a specific context.”¹⁰ Finally, that the signal Y is hidden from the leader captures that leaders rarely know exactly how their messages or actions regarding change come across, which is a real and substantial concern for management.¹¹

We now revisit a classic topic in organizations: the impact of information channels on behavior and outcomes. We do so by varying the quality of the information fed to the leader while keeping the signal Y fixed (i.e., the difficulty in transferring knowledge is given). In the *perfect feedback* case, the leader observes the follower’s action, while in the *no-feedback* case, she observes nothing. These are two limit cases of the model studied in Section 3. Let $\mathbb{E}[\cdot]$ and $\hat{\mathbb{E}}[\cdot]$ denote the leader’s and follower’s expectation operator, respectively.

Perfect feedback (“public”) case. If the leader perfectly observes the follower’s action she can potentially infer the follower’s belief, rendering the environment essentially public.

In a *linear Markov equilibrium* (LME), therefore, the leader chooses actions that are linear in her type θ and the follower’s contemporaneous belief $\hat{M}_t := \hat{\mathbb{E}}_t[\theta]$, and that encode her adaptation and coordination motives; the coefficients on those states are deterministic, and we discuss them shortly. In turn, the follower’s action is his best prediction of the leader’s action, and hence it is a linear function of \hat{M}_t exclusively.¹²

The next result establishes the existence of a unique LME and its key properties. For consistency throughout the paper, we write β_{3t} for the weight on the type at $t \in [0, T]$.

Proposition 1 (LME—Public Case). *For all $r \geq 0$ and $T > 0$:*

- (i) *There exists a unique LME: $a_t = \beta_{3t}\theta + (1 - \beta_{3t})\hat{M}_t$ and $\hat{a}_t = \hat{\mathbb{E}}_t[a_t] = \hat{M}_t$, where $(\beta_{3t})_{t \in [0, T]}$ is deterministic.*

⁹Grant (1996), p. 111.

¹⁰Nonaka (1991), p. 98. He also discusses business examples of a slow transfer of tacit knowledge via example, and of a subsequent slow codification of it; he highlights the role of innate ability, intuition and experience at the origin of such knowledge. Nadler et al. (2003) provide experimental evidence on the advantages of example versus other means of communication.

¹¹Subjective interpretations and private experiences are key in this respect. In fact, Williamson (1996) points out, that “failures of coordination can arise because autonomous parties read and react to signals differently, even though their purpose is to achieve a timely and compatible combined response.”

¹²This notion of LME is *perfect* when Y is public, but only *Nash* when Y is private but the follower’s action is observed. We keep the LME abbreviation, despite the implicit ‘perfection’ qualifier used later on.

(ii) $\beta_{3t} \in (1/2, 1)$ for $t < T$, $\beta_{3T} = 1/2$, and β_3 is strictly decreasing.

(iii) $\gamma_t := \hat{\mathbb{E}}_t[(\theta - \hat{M}_t)^2]$ evolves according to $\dot{\gamma}_t = -\left(\frac{\gamma_t \beta_{3t}}{\sigma_Y}\right)^2$.

The coefficients on θ and \hat{M}_t in the leader's strategy are positive: leaders with higher types want to match higher states, and the coordination motive forces all types to take higher actions when facing followers with higher beliefs. From this perspective, the leader sacrifices adaptation to improve contemporaneous coordination: the weight on the type, β_{3t} , falls below 1 (its counterpart value in the full-information benchmark $a_t = \hat{a}_t = \theta$) to increase the weight assigned to \hat{M} by the same magnitude.¹³

The weight β_3 is the *signaling coefficient*, as it determines the follower's learning (part (iii)). Importantly, it remains above 1/2—its counterpart value in the static equilibrium $(\frac{1}{2}\theta + \frac{1}{2}\hat{M}, \hat{M})$ —except until the end of the game. In fact, by signaling her know-how more aggressively, the leader steers the follower's behavior toward the first-best action faster via the latter's learning. In other words, more adaptation today brings more coordination tomorrow. This incentive falls deterministically (β_{3t} is decreasing) because there is less time to enjoy future coordination and steering behavior is harder as the follower's learning progresses.¹⁴

No feedback case. Absent any information, the leader must perform an inference of the follower's private histories to forecast his belief and coordinate with him. Let $M_t := \mathbb{E}_t[\hat{M}_t]$ denote the leader's *second-order belief*.

In the public case, upon conjecturing a linear Markov strategy by the leader, the follower's learning is Gaussian. In particular, the follower's belief can be written as

$$\hat{M}_t = A_1(t) + \int_0^t A_2(t, s) dY_s \quad (2)$$

for some A_1 and A_2 deterministic. Crucially, the leader's past behavior is irrelevant for forecasting the follower's belief: when Y or \hat{M} are observable, the leader's forecast M_t is uniquely pinned down by the history $Y^t := (Y_s : 0 \leq s < t)$ via the linear formula above.

In the absence of feedback, the leader's forecasting problem becomes nontrivial. However, to the extent that \hat{M} is as above (for potentially different A_1 and A_2) the fact that $\mathbb{E}_t[dY_t] = a_t dt$ yields a second-order belief of the form

$$M_t = A_1(t) + \int_0^t A_2(t, s) a_s ds. \quad (3)$$

¹³It is easy to see that their sum is always one because of the follower always attempting to coordinate with the leader. In particular, the follower's belief can always be inverted from the action.

¹⁴In fact, $\hat{\mathbb{E}}_t[d\hat{M}_t] = \frac{\gamma_t \beta_{3t}^2}{\sigma_Y} (\theta - \hat{M}_t) dt$, so the sensitivity of the belief is lower as γ falls.

Unlike the public case, this forecast is now an *explicit* function of the leader’s past actions: in the absence of any information, the leader must reflect on how much she has emphasized her knowledge to assess how much of it has been transferred to the follower. By contrast, in the public case, a perfect feedback channel or a public signal Y perfectly reveal the follower’s knowledge at all times, rendering past play irrelevant.

The similarity between the cases is clear. Via (2) or (3), the leader evaluates how future beliefs respond to different continuation strategies, and in both cases this relationship is linear. This *forward-looking exercise* allows her to pin down her best response for fixed behavior of the follower. But the follower’s best response will depend on her assessment of the informational content behind the leader’s actions, and this is a *backward-looking exercise*: how do different types behave given their observed histories? Whether beliefs are an explicit function of commonly observed versus private information then makes a difference.

Critically, M is hidden from the follower in any equilibrium in which the leader’s actions carry her type—the follower must then forecast this second-order belief. Along the path of play of a linear strategy, however, (3) suggests a linear relationship between θ and M . To this end, suppose that the follower conjectures that, in equilibrium, M satisfies

$$M_t = \left(1 - \frac{\gamma_t}{\gamma^o}\right) \theta + \frac{\gamma_t}{\gamma^o} \mu \quad (4)$$

when the leader follows a strategy

$$a_t = \beta_{0t} \mu + \beta_{1t} M_t + \beta_{3t} \theta, \quad (5)$$

for some deterministic coefficients β_{it} , $i = 0, 1, 3$ (potentially different from those in the public case), and where $(\gamma_t)_{t \in [0, T]}$ encodes the follower’s posterior variance under (4)–(5). The *representation* (4) encodes two ideas. First, that there is no second-order uncertainty at time zero: $M_0 = \mu = \hat{M}_0$ follows from $\gamma_0 = \gamma^o$ in the right-hand side of (4). Second, if enough signaling has taken place, the leader would expect the follower to have learned the state: $\gamma_t \approx 0$ in the same expression leads to $M_t \approx \theta$.

To determine the follower’s learning, γ_t , we insert (4) into (5) to obtain a weight on θ of

$$\alpha := \beta_3 + \beta_1 \chi, \quad \text{where} \quad \chi_t := 1 - \frac{\gamma_t}{\gamma^o}.$$

The new signaling coefficient, α , consists now of the direct weight that the strategy attaches to the type, β_3 , plus the correction $\beta_1 \chi$ coming from the representation (4): we refer to this correction as the *history-inference effect* on signaling. In fact, because the leader uses her actions to forecast \hat{M} , the follower needs to infer the leader’s private histories

to extract the correct informational content of the signal Y . From the follower’s perspective, how differently would a leader of a marginally higher type behave *given* a history Y^t ? In the public case, the overall effect is β_3 , as all types agree on the value that \hat{M} takes (i.e., they pool along the belief dimension); this is not the case when there is no feedback, as their differing past actions also lead them to perceive a different continuation games via M .

Our players therefore need to conjecture the *outcome* of the game in a self-fulfilling way: the follower conjectures the second-order belief representation (4) to construct his first-order belief, from which the leader constructs her second-order belief, which in turn must deliver (4) under the linear strategy (5)—the proof that our conjecture works is provided in Lemma A.2 in the Appendix. Importantly, the representation does not hold after deviations from (5). More generally, the leader controls M as reflected by (3), and (θ, M, μ, t) summarizes all the payoff-relevant information for the leader’s decision-making.

Proposition 2 (LME—No Feedback Case). *For all $r \geq 0$ and $T > 0$:*

- (i) *There exists a LME. In any such equilibrium: $\beta_0 + \beta_1 + \beta_3 = 1$; $\beta_{3t} > 1/2$, $t \in [0, T)$; $\beta_{3T} = 1/2$; and $\beta_1 > 0$ over $[0, T]$.*
- (ii) *$\alpha := \beta_3 + \beta_1\chi$, where $\chi = 1 - \gamma_t/\gamma^o$ satisfies: $\alpha > 1/2$; $\alpha_T \rightarrow 1$ as $T \rightarrow \infty$, and $\alpha'_t \geq 0$, $t \in [0, T)$, with strict inequality if and only if $r > 0$.*
- (iii) *$\gamma_t := \hat{\mathbb{E}}_t[(\theta - \hat{M})^2]$ evolves as $\dot{\gamma}_t = -\left(\frac{\alpha_t \gamma_t}{\sigma_Y}\right)^2$.*

Part (ii) is key: private monitoring overturns standard decreasing signaling effects that we would expect under the traditional logic of public beliefs: the signaling coefficient α is non-decreasing, and its right endpoint approaches 1 as T grows. See Figure 1.¹⁵

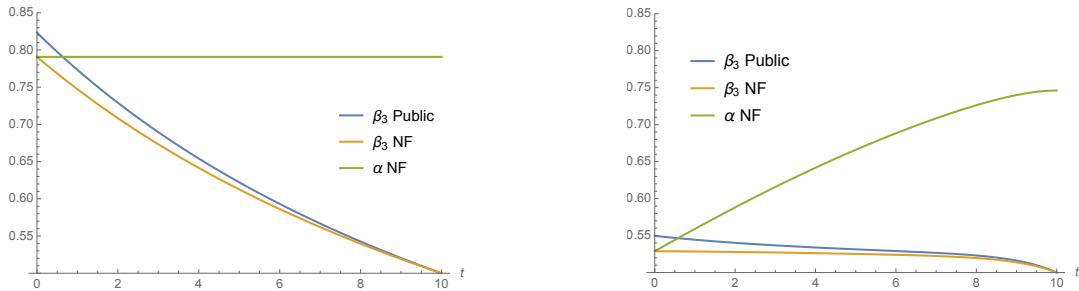


Figure 1: Left $r = 0$; Right: $r = 1$. Other parameter values: $\gamma^o = 1$, $\sigma_Y = 1.5$, $T = 10$.

Comparison across cases. Figure 1 plots the signaling *coefficients* in each LME. In the no-feedback case, the direct weight attached to the type in the linear strategy (5) is decreasing, so the fact that α is non-decreasing implies that the history-inference effect

¹⁵This a *private-value* setting—the method for showing existence is discussed in Section 4.3.

increases over time. Indeed, because higher types take higher actions holding everything else fixed, they will expect their followers to have higher beliefs. Moreover, this effect compounds over time as past play becomes more relevant for predicting the continuation game, which results in M attaching an increasing weight χ to θ in (4). With a positive coordination motive ($\beta_1 > 0$), this implies that higher types take even higher actions over time via this second-order belief channel, enhancing the informational content of the leader’s action.

What are the implications of private monitoring on learning and payoffs? In this coordination game, being forced to rely on her past actions to forecast the follower’s understanding essentially imposes discipline on the leader: she does not cater to the follower’s belief as she would do in the public case. In turn, this suggests that more knowledge is transferred to the follower. To assess the validity of this conjecture, we take advantage of the model’s analytic solutions in the patient ($r = 0$) and myopic ($r = \infty$) cases. Let γ^{Pub} and γ^{NF} denote the follower’s posterior variance in the public and no-feedback case, respectively.

Proposition 3 (Learning comparison). *For every $T > 0$:*

- (i) *Patient case: if $r = 0$, $\beta_{30}^{Pub} > \alpha_0$ and $\gamma_T^{Pub} > \gamma_T^{NF}$;*
- (ii) *Large r case: for every $\delta \in (0, T)$, $\gamma_t^{Pub} > \gamma_t^{NF}$ for $t \in [T - \delta, T]$ if r is large enough.*

Consequently, when the leader is either patient or very impatient, in the no-feedback case the follower always has learned more by the end of the interaction. When $r = 0$, this result is non-trivial due to an inter-temporal substitution effect: the leader, anticipating that the history-inference effect will eventually take place, decides to reduce $\alpha_0 = \beta_{30}^{NF}$ below the public counterpart, β_{30}^{Pub} . Part (ii) then states that the fraction of time over which the follower has a more accurate belief can converge to 1 as r grows large.¹⁶

Regarding payoffs, the leader clearly suffers by losing the ability to perfectly coordinate with the follower—this *direct* effect is the consequence of increased uncertainty over a concave payoff. The next result uncovers the *equilibrium* effects.

Proposition 4 (Team’s ex ante payoffs).

- (i) *Patient case: if $r = 0$, the team’s ex ante payoff is larger in the public case, all $T > 0$.*
- (ii) *For all $r > 0$, ex ante undiscounted coordination costs equal $\sigma_Y^2 \log\left(\frac{\gamma^o}{\gamma_T}\right)$ in each case.*

Part (ii) is essential: the extent of the follower’s learning, as measured by relative entropy, coincides with a metric of total coordination costs. This is due to information transmission occurring through *actions*: a more precise belief that is the consequence of more aggressive

¹⁶This also appears to hold for intermediate values of r . See Figure 1 in the online appendix.

signaling is necessarily the reflection of more transient miscoordination. Indeed, learning occurs only when Y is informative about the state of the world, and hence only when there is a mismatch between contemporaneous actions. From that perspective, the equilibrium effect of private monitoring is that it exacerbates such costs by making the follower’s actions more volatile in response to more informative, yet stable, behavior by the leader.¹⁷ In particular, the team is worse off in the no-feedback case when the leader is patient.

Organization scholars have stressed the importance to firms of having efficient ways of integrating knowledge when the transfer of such knowledge is costly and slow.¹⁸ The present setting of action-based information transmission conforms with this view. Specifically, the value of a better information channel feeding the leader is precisely that it allows the leader to reduce the amount of knowledge transfer that would otherwise be required to coordinate. Even more so, the application uncovers that an organization’s better understanding of its leadership’s goals need not be indicative of past or even future performance: in fact, it can be reflective of the organization’s painful struggle to coordinate.

The example in this section is just a first attempt at understanding organizations as *dynamic* enterprises, where decision makers can signal and learn information at the same time that decisions are being made, and where severe information frictions can be at play. From this standpoint, it is important to recognize that the information observed by leaders usually lies in between the two extreme cases analyzed: i.e., that public feedback channels are *partially* informative. This setting poses considerable conceptual and technical challenges; the next section introduces an operational framework for its analysis.

3 General Model

We consider two-player linear-quadratic-Gaussian games with one-sided private information and one-sided private monitoring in continuous time. The baseline model considered is introduced next, and extensions of it are presented in Section 5 via two further applications.

Players, Actions and Payoffs. A forward looking *long-run* player (she) and a myopic counterpart (he) interact in a repeated game that is played continuously over a time interval $[0, T]$, $T < \infty$. At each instant $t \in [0, T]$, the long-run player chooses an action a_t , while the myopic player chooses \hat{a}_t , both taking values over the real line. Given a profile of realized actions (a, \hat{a}) chosen at time t , the long-run player’s and myopic player’s realized flow payoffs

¹⁷We can show that ex ante *flow* payoffs can be higher in the absence of feedback. Indeed, for sufficiently large T and r , the history-inference effect in the no-feedback case can lead to substantially more adaptation than in the public case, in a way that it dominates the opposite ranking of flow coordination costs.

¹⁸See, for instance, [Garicano and Prat \(2013\)](#); in the strategy literature, see [Grant \(1996\)](#).

are given by

$$U(a, \hat{a}, \theta) \quad \text{and} \quad \hat{U}(a, \hat{a}, \theta), \quad (6)$$

respectively, where $U : \mathbb{R}^3 \rightarrow \mathbb{R}$ and $\hat{U} : \mathbb{R}^3 \rightarrow \mathbb{R}$ are quadratic functions. In this specification, θ denotes the value of a normally distributed random variable that parametrizes the economic environment; its mean and variance are denoted by $\mu \in \mathbb{R}$ and $\gamma^\circ > 0$, respectively. The long-run player discounts the future at a rate $r > 0$, while myopic player cares only about her instantaneous payoff at all times.

We now state our assumptions on the functions U and \hat{U} . Since these involve strategic considerations of the game, we introduce some shorthand notation. Specifically, for $x, y \in \{a, \hat{a}, \theta\}$, define the scalars

$$u_{xy} := \frac{\partial^2 U / \partial x \partial y}{|\partial^2 U / \partial a^2|} \quad \text{and} \quad \hat{u}_{xy} := \frac{\partial^2 \hat{U} / \partial x \partial y}{|\partial^2 \hat{U} / \partial \hat{a}^2|}.$$

Intuitively, best responses carry denominators as the ones above when the players' flow payoffs are concave in their respective actions.

Assumption 1.

- (i) *Strict concavity:* $u_{aa} = \hat{u}_{\hat{a}\hat{a}} = -1$;
- (ii) *Non-trivial signaling:* $u_{a\theta}(u_{a\theta} + u_{a\hat{a}}\hat{u}_{\hat{a}\theta}) > 0$;
- (iii) *Second-order inferences:* $|\hat{u}_{\hat{a}\theta}| + |\hat{u}_{a\hat{a}}| \neq 0$ and $|u_{a\hat{a}}| + |u_{\hat{a}\hat{a}}| \neq 0$.
- (iv) *Myopic best-replies intersect:* $u_{a\hat{a}}\hat{u}_{\hat{a}\hat{a}} < 1$;

We first require that the players' objectives are strictly concave in their respective choice variables. A second minimal requirement is that the long-run player strategically cares about θ , which is implied by (ii). Equipped with this, part (iii) says that second-order inferences matter. Specifically, the first condition states that the myopic player's first-order belief matters for his behavior, either directly because he cares about θ ($\hat{u}_{\hat{a}\theta}$ term) or because he wants to predict the long-run player's action ($\hat{u}_{a\hat{a}}$). The second condition then says that the long-run player needs to predict the myopic player's action, either due to an interaction term ($u_{a\hat{a}}$) or a nonlinear effect ($u_{\hat{a}\hat{a}}$), which calls for a second-order belief.¹⁹

The remaining parts are technical conditions pertaining to the static game with private beliefs that arises at the end of the interaction. Specifically, part (iv) ensures that a static

¹⁹Part (iii) allows us to focus on the more interesting cases; i.e., it is not a limitation of our analysis.

Nash equilibrium always exists, and part (ii) ensures that any such equilibrium involves non-trivial signaling. We revisit these in Section 4 when we discuss how to find LME.

Information. The long-run player observes θ before play begins, while the myopic player only knows its distribution $\theta \sim \mathcal{N}(\mu, \gamma^\circ)$ (and this is common knowledge). In addition, there are two signals X and Y that convey noisy information about the players' actions. In this baseline model, we work with a product-structure specification

$$dX_t = \hat{a}_t dt + \sigma_X dZ_t^X \quad (7)$$

$$dY_t = a_t dt + \sigma_Y dZ_t^Y, \quad (8)$$

where Z^X and Z^Y are independent Brownian motions, and σ_Y and σ_X are strictly positive volatility parameters; in particular, the players' actions affect the signals *linearly*.²⁰

Our key departure from traditional signaling games with public signals is that Y —which carries information about the long-run player's actions—is only observed by the myopic player, while X remains public. This mixed private-public information structure is important for our construction, but it is also natural for analyzing sender-receiver games, as it makes the departure minimal while still economically relevant.

In what follows, we let $\mathbb{E}_t[\cdot]$ denote the long-run player's conditional expectation operator, which can condition on the histories $(\theta, a_s, X_s : 0 \leq s \leq t)$ and on her conjecture of the myopic player's play. Likewise, $\hat{\mathbb{E}}_t[\cdot]$ denotes the myopic player's analog, which conditions on $(\hat{a}_s, X_s, Y_s : 0 \leq s \leq t)$ and on her belief about the long-run player's strategy, $t \geq 0$.

Strategies and Equilibrium Concept. With full-support monitoring, the only off-path histories for each player are those in which the player itself deviated from a candidate strategy. Since this implies that sequential rationality imposes no additional restrictions on the set of equilibrium outcomes relative to the Nash equilibrium concept, we content ourselves with the latter notion for defining an equilibrium of the game.

From a time-zero perspective, an admissible strategy for the long-run player is any square-integrable real-valued process $(a_t)_{t \in [0, T]}$ that is progressively measurable with respect to the filtration generated by (θ, X) . The analogous notion for the myopic player involves the identical integrability condition, but the measurability restriction is with respect to (X, Y) .²¹

²⁰Thus, flow payoffs do not convey any additional information to the players (i.e., they accrue after time T , or they can be written in terms of the actions and signals observed by each player).

²¹Square integrability is in the sense of the time-zero expectations of $\int_0^T a_t^2 dt$ and $\int_0^T \hat{a}_t^2 dt$ being finite. This ensures that a strong solution to (7)–(8) exists, and thus that the outcome of the game is well-defined.

Definition 1 (Nash equilibrium.). *An admissible pair $(a_t, \hat{a}_t)_{t \geq 0}$ is a Nash equilibrium if,*

(i) *given $(\hat{a}_t)_{t \in [0, T]}$, the process $(a_t)_{t \in [0, T]}$ maximizes*

$$\mathbb{E}_0 \left[\int_0^T e^{-rt} U(a_t, \hat{a}_t, \theta) dt \right]$$

among all admissible strategies, and

(ii) *\hat{a}_t solves $\max_{\hat{a}' \in \mathbb{R}} \hat{\mathbb{E}}_0[\hat{U}(a_t, \hat{a}', \theta)]$ for all $t \in [0, T]$.*

In the next section, we characterize Nash equilibria that are supported by linear Markov strategies that are sub-game perfect, i.e., that are sequentially rational on and off the path of play, thereby specifying optimal behavior after deviations. Such equilibria generalize that presented in Section 2 for the no-feedback case $\sigma_X = \infty$ to the whole range $0 < \sigma_X \leq \infty$.

Remark 1 (Extensions). *The baseline model can be generalized along two dimensions:*

(i) *Terminal payoffs: the long-run player's payoff can also carry a lump-sum component $\Psi(\hat{a}_T)$, with Ψ quadratic. See Section 5.1 for a reputation model with this property.*

(ii) *Long-run player affecting the public signal X : the drift of (7) can be generalized to $\hat{a}_t + \nu a_t$, where $\nu \in [0, 1]$ is a scalar.*

We exclude these from the baseline model purely for ease of exposition. An insider trading model with $\nu = 1$ (and $\partial^2 U / \partial a^2 = 0$, as in the literature) is explored in Section 5.2.

4 Equilibrium Analysis: Linear Markov Equilibria

In this section, we construct linear Markov perfect equilibria (henceforth, LME) that rely on the players' beliefs as the relevant states. Along the path of play of such equilibria, the players' actions are linear in the signals they observe due to the Gaussian structure of the environment. The appeal of such equilibria is twofold: first, the Markov restriction captures that behavior depends only on the aspects of the players' histories that they perceive to be payoff-relevant; second, the linear aggregation of signals is the natural generalization of linear equilibria studied in a wide body of applied-theory work of static nature.

The steps are as follows. We first postulate a minimal set of belief states up to the second order to be used by the players in any equilibrium of this kind. We then derive a *representation* of the long-run player's *private* second-order belief as a linear function of a subset of such belief states, when the players use the candidate belief states in a linear

fashion. This result generalizes the representation (4) obtained in Section 2, and is of major importance for our analysis: it encodes how the long-run player has relied on her past behavior to forecast the follower’s belief, and shows that the problem of “beliefs about beliefs” growing without bound can be circumvented (Section 4.1). Finally, Sections 4.2 and 4.3 are devoted to the question of finding LME. Our main contribution here is to offer methods for existence in *asymmetric* two-sided signaling games.

4.1 Belief States and Representation Lemma

The logic behind the need for a second-order belief is as follows. Since the myopic player must predict the long-run player’s action (and/or her type) to determine his best response, he will use the private signal Y to learn about θ whenever the long-run player signals her type. By part (iii) in Assumption 1, the long-run player is then forced to forecast the myopic player’s belief to determine her current actions, which makes such a forecast payoff-relevant.

The difficulty is that such second-order belief is privately observed by the long-run player. In fact, because of private monitoring, the long-run player will have to rely to some extent on her past behavior to forecast the myopic player’s belief. Thus, her second-order belief will depend explicitly on her past play, and the latter carries her type. The myopic player is then forced to perform an inference about such hidden second-order belief, and so forth.

Our key observation is that along the path of play of any *pure* strategy the outcome of the game should depend only on the tuple (θ, X, Y) . Intuitively, given any rule that specifies behavior as a function of past actions and information, following such a rule should lead realized outcomes to depend on the exogenous elements of the model only. In particular, the long-run player’s second-order belief should be a function of (θ, X) . Moreover, in this Gaussian environment, one would expect the relationship between M and (θ, X) to be linear if the rule that drives behavior is linear in some suitable belief states.

Let $\hat{M}_t := \hat{\mathbb{E}}_t[\theta]$ denote the mean of the myopic player’s posterior belief, and $M_t := \mathbb{E}_t[\hat{M}_t]$ denote the long-run player’s second-order counterpart. The previous discussion suggests the existence of a deterministic function χ and a process $(L_t)_{t \in [0, T]}$ that depends on the paths of the public signal X , such that M admits the representation

$$M_t = \chi_t \theta + (1 - \chi_t) L_t \tag{9}$$

when the players follow *linear Markov strategies* of the form

$$a_t = \beta_{0t} + \beta_{1t} M_t + \beta_{2t} L_t + \beta_{3t} \theta \tag{10}$$

$$\hat{a}_t = \delta_{0t} + \delta_{1t} \hat{M}_t + \delta_{2t} L_t, \tag{11}$$

where β_{it} and δ_{jt} , $i = 0, 1, 2, 3$ and $j = 0, 1, 2$, are deterministic. The reason for augmenting the strategies of the no-feedback case in Section 2 by the “public” state L is clear: if true, the myopic player uses (9) to forecast M , making L a payoff-relevant state for both players.

Lemma 1 below validates (9)–(11) by characterizing the pair (χ, L) . Before stating the result, let us elaborate on the intuition behind (9) and on how it can be derived in a constructive way, while abstracting from formal mathematical arguments.

Specifically, the representation (9) is a candidate for how, under linear Markov strategies, the long-run player has used her past play and the public signal to forecast the myopic player’s belief. Crucially, such a conjecture must be self-fulfilling: when forming their beliefs, the players recognize that different types take different actions through (9), which in turn must result in a second-order belief that coincides with (9) if (10)–(11) is followed.

In particular, the myopic player thinks that the long-run player behaves according to

$$a_t = \underbrace{\beta_{0t}}_{=:\alpha_{0t}} + \underbrace{(\beta_{2t} + \beta_{1t}(1 - \chi_t))}_{=:\alpha_{2t}} L_t + \underbrace{(\beta_{3t} + \beta_{1t}\chi_t)}_{=:\alpha_{3t}} \theta. \quad (12)$$

Because L is public, θ is the only unknown in the previous expression, so the myopic player can filter θ from observing Y when the latter is driven by (12). This learning problem is (conditionally) Gaussian, and hence the myopic player’s posterior belief is fully characterized by a mean process $(\hat{M}_t)_{t \geq 0}$, and a deterministic variance path

$$\gamma_t := \widehat{\text{Var}}_t = \hat{\mathbb{E}}_t[(\theta_t - \hat{M}_t)^2],$$

where we have omitted the “hat” symbol in γ_t for notational convenience. As in Section 2, this posterior variance will be determined by the signaling coefficient

$$\alpha_{3t} := \beta_{3t} + \beta_{1t}\chi_t,$$

with $\beta_{1t}\chi_t$ encoding the *history-inference effect*: how different types take different actions in equilibrium because their past actions have lead them to hold different beliefs today.

As is traditional, the long-run player will use the public signal X that carries the myopic player’s action to forecast \hat{M} . The novelty is that, due to the private monitoring, she will also use her past actions: statistically, higher action profiles lead to higher private observations by the myopic player, and vice versa. Crucially, the linearity of the signal structure renders the pair (\hat{M}, X) (conditionally) Gaussian again, and so the filtering equations deliver a second-order belief with (i) a mean M_t that is a linear function of past actions $(a_s)_{s < t}$ and of past signals $(X_s)_{s < t}$, and (ii) a deterministic posterior variance. One can then insert the linear

Markov strategy (10) into M_t to solve for the latter as a function of $\{\theta, (X_s)_{s<t}\}$, and then pin down (χ, L) . The precise mathematical steps can be found in the proof of the following:

Lemma 1 (Representation of second-order belief). *Suppose that (X, Y) is driven by (10)–(11) and that the myopic player believes that (9) holds. Then (9) holds at all times (path-by-path of X), if and only if*

$$\dot{\gamma}_t = -\frac{\gamma_t^2(\beta_{3t} + \beta_{1t}\chi_t)^2}{\sigma_Y^2}, \quad \gamma_0 = \gamma^o, \quad (13)$$

$$\dot{\chi}_t = \frac{\gamma_t(\beta_{3t} + \beta_{1t}\chi_t)^2(1 - \chi_t)}{\sigma_Y^2} - \frac{\gamma_t\chi_t^2\delta_{1t}^2}{\sigma_X^2}, \quad \chi_0 = 0, \quad (14)$$

$$dL_t = (l_{0t} + l_{1t}L_t)dt + B_t dX_t, \quad L_0 = \mu, \quad (15)$$

where l_{0t} and l_{1t} , and B_t are deterministic functions given in (B.6). Moreover, $L_t = \mathbb{E}[\hat{M}_t | \mathcal{F}_t^X] = \mathbb{E}[\theta | \mathcal{F}_t^X]$ and $\gamma_t\chi_t = \text{Var}_t = \mathbb{E}[(M_t - \hat{M}_t)^2]$.

In light of the lemma, the representation (9) reads

$$M_t = \frac{\text{Var}_t}{\text{Var}_t} \theta + \left(1 - \frac{\text{Var}_t}{\text{Var}_t}\right) \mathbb{E}[\theta | \mathcal{F}_t^X].$$

Indeed, in forecasting \hat{M} , the only informational advantage that the long-run player has relative to an outsider who observes X exclusively is that she knows what actions she has taken, and such actions carry her type. Under linear strategies, learning is Gaussian, so (i) M_t is a linear combination of θ and $\mathbb{E}[\hat{M}_t | \mathcal{F}_t^X]$, and (ii) the weights are deterministic. By the law of iterated expectations, $\mathbb{E}[\hat{M}_t | \mathcal{F}_t^X] = \mathbb{E}[\theta | \mathcal{F}_t^X]$, and the representation follows.

The χ -ODE (14) quantifies the dynamics of the importance of past behavior in this forecasting exercise. Indeed, by the common prior assumption, $\text{Var}_0 = 0$ and $\mathbb{E}[\theta | \mathcal{F}_0^X] = \mu$; thus, $M_0 = \mu$ above, and the χ -ODE starts at zero. As signaling progresses, the long-run player loses track of \hat{M} (i.e., $\text{Var}_t > 0$): this is captured in $\dot{\chi} > 0$ as soon as $\alpha_3 > 0$ in (14). In other words, the long-run player expects \hat{M} to gradually reflect her type θ , and so $\chi_t > 0$.

The relative importance of past play will naturally depend on the quality of the public information—this is captured by $-\gamma_t\chi_t^2\delta_{1t}^2/\sigma_X^2$ in (14). If $\sigma_X = \infty$ or $\delta_1 \equiv 0$ (the myopic player does not signal back) the public signal is uninformative: indeed, $L_t = L_0 = \mu$ and $\chi_t = 1 - \gamma_t/\gamma_0$ hold at all times as in the no-feedback in Section 2.²² Apart from this case, the public information is always useful. In particular, observe that as δ_1^2/σ_X^2 grows, there is more downward pressure on the growth of χ : as the signal-to-noise ratio in X improves, the

²²Setting $\delta_1/\sigma_X \equiv 0$ in (14) leads to the same ODE that χ satisfies in the no-feedback case. By uniqueness, the solution is $\chi = 1 - \gamma_t/\gamma^o$. See the proof of Lemma A.2.

long-run player relies less on her past actions, all else equal. In other words, the no-feedback case maximizes the potential amplitude of the history-inference effect.

Our subsequent analysis takes the system of ODEs (13)–(14) for (γ, χ) as an input. Thus, we require (13)–(14) to have a unique solution to ensure that the ODE-characterization is valid. To this end, notice that the signaling coefficient of the myopic player’s best reply can be written as $\delta_{1t} := \hat{u}_{\hat{a}\theta} + \hat{u}_{\hat{a}a}[\beta_{3t} + \beta_{1t}\chi_t]$.

Lemma 2. *Suppose that β_1 and β_3 are continuous, $\beta_3 \neq 0$ and $\delta_{1t} = \hat{u}_{\hat{a}\theta} + \hat{u}_{\hat{a}a}[\beta_{3t} + \beta_{1t}\chi_t]$. Then there is a unique solution to (13)–(14). The solution satisfies $0 < \gamma_t < \gamma^o$ and $0 < \chi_t < 1$, $t \in (0, T]$.*

The idea is that, under minimal integrability conditions on the coefficients in the linear Markov strategies, $\gamma_t = \hat{\mathbb{E}}_t[(\theta_t - \hat{\mathbb{E}}_t[\theta])^2]$ and $\chi_t = \text{Var}_t/\widehat{\text{Var}}_t = \mathbb{E}_t[(M_t - \hat{M}_t)^2]/\gamma_t$ are a solution to the system. A mild strengthening of the conditions ensures that a unique solution to (13)–(14) exists, and so we are allowed to use the latter system as a primitive object.

The belief representation (9) relies on the long-run player following the linear strategy (10); i.e., it does not hold off the path of play. The next result introduces the law of motion of M and L for an arbitrary strategy of the long-run player, which will allow us to state her best-response problem. Importantly, because deviations are hidden, the myopic player always assumes that (9) holds when constructing his belief. Thus, the pair (γ, χ) appears explicitly in the evolution of (M, L) through its appearance in the myopic player’s learning.

Lemma 3. *Suppose that the long-run player follows $(a'_t)_{t \geq 0}$ while the myopic player follows (11) and believes (9)–(10). Then from the long-run player’s perspective*

$$dM_t = \frac{\gamma_t \alpha_{3t}}{\sigma_Y^2} (a'_t - [\alpha_{0t} + \alpha_{2t}L_t + \alpha_{3t}M_t])dt + \frac{\chi_t \gamma_t \delta_{1t}}{\sigma_X} dZ_t \quad (16)$$

$$dL_t = \frac{\chi_t \gamma_t \delta_{1t}}{\sigma_X^2 (1 - \chi_t)} [\delta_{1t}(M_t - L_t)dt + \sigma_X dZ_t] \quad (17)$$

where (γ, χ) solves (13)–(14) and $(Z_t)_{t \geq 0}$ is a Brownian motion from her standpoint.

The dynamic (16) shows that long-run player’s choice of strategy a' affects M . In particular, she will revise her (second-order) belief upward when $a'_t > \mathbb{E}_t[\alpha_{0t} + \alpha_{2t}L_t + \alpha_{3t}\hat{M}_t]$, i.e., when she expects to beat the myopic player’s expectation of her behavior. The intensity of such a reaction is given by $\gamma_t \alpha_{3t}/\sigma_Y^2$, i.e., it is higher the more uncertain the myopic player is (higher γ) and the stronger the myopic player expects the long-run player to signal (larger α_3). Further, M evolves deterministically when $\delta_1/\sigma_X \equiv 0$, as in Section 2.²³

²³It is worth noting that $(M_t)_{t \geq 0}$ corresponds to a player’s non-trivial belief that is *controlled by the same player*. Unless there are experimentation effects, players’ own beliefs are usually affected by other players.

The drift of (17) demonstrates that the long-run player affects L only indirectly via changes in M ; this is because her actions do not enter the public signal. More interestingly, L always moves in the direction of M on average, reflecting that an outsider who only observes X does get to learn the long-run player's type over time. From this perspective, by leading to $L_t = \mu$ at all times, the no-feedback case ($\sigma_X = \infty$) misses a *signal-jamming effect*: the incentives that arise from the ability to influence a public belief (albeit only indirectly, in this case), with such incentives being perfectly accounted for in equilibrium.

4.2 Dynamic Programming and the Boundary-Value Problem

The long-run player's best-response problem. Let $\vec{\beta} := (\beta_0, \beta_1, \beta_2, \beta_3)$, $\vec{\delta} := (\delta_0, \delta_1, \delta_2)$, and $\vec{\alpha} := (\alpha_0, \alpha_2, \alpha_3)$, the latter defined in (12). Given a conjecture $\vec{\beta}$ by the myopic player, the coefficients $\vec{\delta}$ are found by matching coefficients in

$$\hat{a}_t := \delta_{0t} + \delta_{1t}\hat{M}_t + \delta_{2t}L_t = \arg \max_{\hat{a}'} \hat{\mathbb{E}}_t[\hat{U}(\alpha_{0t} + \alpha_{2t}L_t + \alpha_{3t}\theta, \hat{a}', \theta)]. \quad (18)$$

Using that $M_t = \mathbb{E}_t[\hat{M}_t]$, the long-run player's objective can be then written as

$$\mathbb{E}_0 \left[\int_0^T e^{-rt} U(a_t, \underbrace{\delta_{0t} + \delta_{1t}M_t + \delta_{2t}L_t}_{\mathbb{E}_t[\hat{a}_t]}, \theta) dt \right] + \frac{1}{2} \frac{\partial^2 U}{\partial \hat{a}^2} \mathbb{E}_0 \left[\int_0^T e^{-rt} \delta_{1t}^2 \mathbb{E}_t[(M_t - \hat{M}_t)^2] dt \right].$$

Importantly, due to the Gaussian learning structure, the variance terms $\mathbb{E}_t[(M_t - \hat{M}_t)^2]$, $t \in [0, T]$, are independent of the strategy followed.²⁴ Consequently, the relevant problem is

$$\begin{aligned} \max_{(a_t)_{t \in [0, T]} \text{ admissible}} \quad & \mathbb{E}_0 \left[\int_0^T e^{-rt} U(a_t, \delta_{0t} + \delta_{1t}M_t + \delta_{2t}L_t, \theta) dt \right] \\ \text{s.t.} \quad & (16) \text{ and } (17), \end{aligned} \quad (19)$$

where the laws of motion of M and L , (16) and (17), depend on (γ, χ) satisfying (13)–(14).²⁵

It is clear from (18) and (19) that the Markov states (t, θ, L, M) and (t, L, \hat{M}) summarize all the payoff-relevant information for our players on and off the path of play, with the time variable capturing both time-horizon and learning effects, the latter encoded in γ and χ .²⁶

²⁴In particular, by Lemma 1, $\mathbb{E}_t[(M_t - \hat{M}_t)^2] = \gamma_t \chi_t$.

²⁵Formally, the long-run player's problem is one of stochastic control of an *unobserved* state (\hat{M}), which introduces some subtleties in the joint filtering-optimization problem relative to standard control problems. By the *separation principle*, however, we can filter first by constructing M as a first step, and then optimize afterwards with the controlled process M as the state. The proof of Lemma 3 explains the details.

²⁶Deviations by the myopic player do affect L , but his flow payoff is fully determined by the current value of (t, L, \hat{M}) . It is easy to see that the same is true if instead this player is forward looking.

In particular, we can tackle the best-response problem (19) with dynamic programming.

Specifically, we postulate a quadratic value function

$$V(\theta, m, \ell, t) = v_{0t} + v_{1t}\theta + v_{2t}m + v_{3t}\ell + v_{4t}\theta^2 + v_{5t}m^2 + v_{6t}\ell^2 + v_{7t}\theta m + v_{8t}\theta\ell + v_{9t}m\ell,$$

where $v_i, i = 0, \dots, 9$ depend on time only. The Hamilton-Jacobi-Bellman (HJB) equation is

$$rV = \sup_{a'} \left\{ U(a', \mathbb{E}_t[\hat{a}_t], \theta) + V_t + \mu_M(a')V_m + \mu_L V_\ell + \frac{\sigma_M^2}{2} V_{mm} + \sigma_M \sigma_L V_{m\ell} + \frac{\sigma_L^2}{2} V_{\ell\ell} \right\},$$

where $\mu_M(a')$ and μ_L (respectively, σ_M and σ_L) are the drifts (respectively, volatilities) in (16) and (17), and where \hat{a}_t is determined via (18).

A Nash equilibrium in linear Markov strategies arises if $\beta_{0t} + \beta_{1t}M + \beta_{2t}L + \beta_{3t}\theta$ is an optimal policy for the long-run player. Along the path of play of such an equilibrium, the representation (9) holds by construction and so the long-run player's realized actions are given by $a_t = \alpha_{0t} + \alpha_{2t}L_t + \alpha_{3t}\theta$, where $(L_t)_{t \in [0, T]}$ is given by (15) in Lemma 1; i.e., actions are a function of θ and X exclusively. But conditioning explicitly on L and M can be optimal after deviations. An optimal policy $\beta_{0t} + \beta_{1t}M + \beta_{2t}L + \beta_{3t}\theta$ specifies how to behave at those off path histories, thereby inducing a linear Markov *perfect* equilibrium (LME).

The boundary-value problem. We briefly explain how to obtain a system of ordinary differential equations (ODEs) for $\vec{\beta}$. Letting $a(\theta, m, \ell, t)$ denote the maximizer of the right-hand side in the HJB equation, the first-order condition (FOC) reads

$$\frac{\partial U}{\partial a}(a(\theta, m, \ell, t), \delta_{0t} + \delta_{1t}m + \delta_{2t}\ell, \theta) + \frac{\gamma_t \alpha_{3t}}{\sigma_Y^2} \underbrace{[v_{2t} + 2v_{5t}m + v_{7t}\theta + v_{9t}\ell]}_{V_m(\theta, m, \ell, t)} = 0. \quad (20)$$

where $\gamma_t \alpha_{3t} / \sigma_Y^2$ in the second term captures the sensitivity of M to the long-run player's action at time t . Solving for $a(\theta, m, \ell, t)$ in the previous FOC, the equilibrium condition becomes $a(\theta, m, \ell, t) = \beta_{0t} + \beta_{1t}m + \beta_{2t}\ell + \beta_{3t}\theta$.

Because the latter condition is a linear equation, we can solve for (v_2, v_5, v_7, v_9) as a function of the coefficients $\vec{\beta}$. Inserting these into the HJB equation along with $a(\theta, m, \ell, t) = \beta_{0t} + \beta_{1t}m + \beta_{2t}\ell + \beta_{3t}\theta$ in turn allows us to obtain a system of ODEs that the $\vec{\beta}$ coefficients must satisfy. The resulting system is coupled with the ODEs that v_6 and v_8 satisfy (and that are obtained from the HJB equation): since M feeds into L , the envelope condition with respect to M is not enough to determine equations for the candidate equilibrium coefficients. Finally, since the pair (γ, χ) affects the law of motion of (M, L) , it also affects the evolution of $(\vec{\beta}, v_6, v_8)$, and so the ODEs (13)–(14) must be included.

The boundary conditions for the system of ODEs that $(\beta_0, \beta_1, \beta_2, \beta_3, v_6, v_8, \gamma, \chi)$ satisfies are as follows. First, there are the exogenous initial conditions that γ and χ satisfy, i.e., $\gamma_0 = \gamma^o > 0$ and $\chi_0 = 0$. Second, there are terminal conditions $v_{6T} = v_{8T} = 0$ due to the absence of a lump-sum terminal payoff in the long-run player's problem.²⁷ Third, more interestingly, there are *endogenous* terminal conditions that are determined by the static Nash equilibrium that arises from myopic play at time T . In fact, letting

$$u_0 := \frac{\partial U / \partial a}{|\partial^2 U / \partial a^2|} \Big|_{(0,0,0)} \quad \text{and} \quad \hat{u}_0 := \frac{\partial \hat{U} / \partial \hat{a}}{|\partial^2 \hat{U} / \partial \hat{a}^2|} \Big|_{(0,0,0)}$$

we obtain

$$\beta_{0T} = \frac{u_0 + u_{a\hat{a}}\hat{u}_0}{1 - u_{a\hat{a}}\hat{u}_{\hat{a}a}}, \quad \beta_{1T} = \frac{u_{a\hat{a}}[u_{a\theta}\hat{u}_{\hat{a}a} + \hat{u}_{\hat{a}\theta}]}{1 - u_{a\hat{a}}\hat{u}_{\hat{a}a}\chi_T}, \quad \beta_{2T} = \frac{u_{a\hat{a}}^2\hat{u}_{\hat{a}a}[u_{a\theta}\hat{u}_{\hat{a}a} + \hat{u}_{\hat{a}\theta}](1 - \chi_T)}{(1 - u_{a\hat{a}}\hat{u}_{\hat{a}a})(1 - u_{a\hat{a}}\hat{u}_{\hat{a}a}\chi_T)}, \quad \beta_{3T} = u_{a\theta},$$

which are all well-defined thanks to part (iv) in Assumption 1 and the fact that $\chi \in (0, 1)$. Also, observe that by part (ii) in Assumption 1, $\alpha_{3T} = \beta_{3T} + \beta_{1T}\chi_T \propto u_{a\theta} + u_{a\hat{a}}\hat{u}_{\hat{a}\theta}\chi_T$ never vanishes for all $\chi_T \in [0, 1]$.²⁸

We conclude that $\mathbf{b} := (\beta_0, \beta_1, \beta_2, \beta_3, v_6, v_8, \gamma, \chi)'$ satisfies a *boundary-value problem* (BVP) of the form

$$\dot{\mathbf{b}}_t = \mathbf{f}(\mathbf{b}_t), \quad \text{s.t.} \quad \mathbf{D}_0\mathbf{b}_0 + \mathbf{D}_T\mathbf{b}_T = (\mathbf{B}(\chi_T)', \gamma^o, 0)' \quad (21)$$

where (i) $\mathbf{f} : \mathbb{R}^6 \times \mathbb{R}_+ \times [0, 1] \rightarrow \mathbb{R}^8$, (ii) \mathbf{D}_0 and \mathbf{D}_T are the diagonal matrices

$$\mathbf{D}_0 = \text{diag}(0, 0, 0, 0, 0, 0, 1, 1) \quad \text{and} \quad \mathbf{D}_T = \text{diag}(1, 1, 1, 1, 1, 1, 0, 0),$$

and where (iii) the function $\mathbf{B}(\chi) : [0, 1] \rightarrow \mathbb{R}$ defined by

$$\mathbf{B}(\chi) := \left(\frac{u_0 + u_{a\hat{a}}\hat{u}_0}{1 - u_{a\hat{a}}\hat{u}_{\hat{a}a}}, \frac{u_{a\hat{a}}[u_{a\theta}\hat{u}_{\hat{a}a} + \hat{u}_{\hat{a}\theta}]}{1 - u_{a\hat{a}}\hat{u}_{\hat{a}a}\chi}, \frac{u_{a\hat{a}}^2\hat{u}_{\hat{a}a}[u_{a\theta}\hat{u}_{\hat{a}a} + \hat{u}_{\hat{a}\theta}](1 - \chi)}{(1 - u_{a\hat{a}}\hat{u}_{\hat{a}a})(1 - u_{a\hat{a}}\hat{u}_{\hat{a}a}\chi)}, u_{a\theta}, 0, 0 \right) \in \mathbb{R}^6 \quad (22)$$

captures the form of the endogenous terminal conditions. The general expression that $\mathbf{f}(\cdot)$ takes for any given generic pair (U, \hat{U}) satisfying Assumption 1 is tedious and long, and can be found in `spm.nb` on our websites. (There, to simplify notation, we work with normalized payoffs $U/|\partial^2 U/\partial a^2|$ and $\hat{U}/|\partial^2 \hat{U}/\partial \hat{a}^2|$.) In the next subsection, we provide examples that exhibit all the relevant properties that any such $\mathbf{f}(\cdot)$ can satisfy.

²⁷The application studied in Section 5.1 relaxes this assumption.

²⁸This allows us to ensure that there is always non-trivial signaling in the game. Also, it is easy to see that $\delta_{0T} = \hat{u}_0 + \hat{u}_{\hat{a}a}\beta_{0T}$, $\delta_{1T} = \hat{u}_{\hat{a}\theta} + \hat{u}_{\hat{a}a}[\beta_{3T} + \beta_{1T}\chi_T]$ and $\delta_{2T} = \hat{u}_{\hat{a}a}[\beta_{2T} + \beta_{1T}(1 - \chi_T)]$.

The question of finding LME is then reduced to finding solutions to the BVP (21) (subject to the rest of the coefficients of the value function being well-defined). We turn to this issue in the next section.

4.3 Existence of Linear Markov Perfect Equilibria

In this section, we present two existence results for LME that are based on establishing the existence of solutions to BVPs. Underlying these results are two approaches that distinguish between *common* and *private-value* environments, and that permit dealing differently with two types of asymmetries in two-sided signaling games. We state the theorems in the context of variations of the coordination game of Section 2, and when the leader is patient. The approaches nevertheless apply to the whole class under study, and one of them is general: it applies to, and beyond, the whole class analyzed.

The shooting problem. The problem of finding a solution to any instance of the BVP (21) is complex because there are multiple ODEs in either direction: $(\beta_0, \beta_1, \beta_2, \beta_3, v_6, v_8)$ are traced backward from their (endogenous) terminal values, while (γ, χ) are traced forward using their initial (exogenous) ones—see Figure 2. This means that, one way or another, some notion of “shooting” must be involved: construct, say, a modified backward *initial value problem* (IVP) in which (γ, χ) has a parametrized initial condition at T , and find a way to ensure that the induced terminal values at 0 exactly match $(\gamma^o, 0)$. Attempting to apply traditional one-dimensional shooting arguments—i.e., tracing the initial parametrized condition over an interval so that the target is hit by continuity—to higher dimensions is hopeless: it essentially requires having an accurate knowledge of the relationship between γ and χ at T for all possible coefficients $\vec{\beta}$ for finding the right “tracing” path.

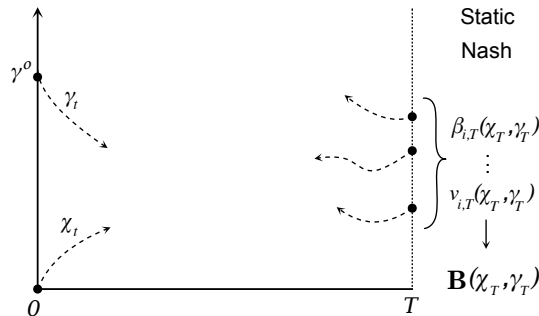


Figure 2: In the BVP, (γ, χ) has initial conditions, while $(\vec{\beta}, v_6, v_8)$ has terminal ones. We have allowed for non-zero v 's and for a dependence on γ_T , as this can occur with terminal payoffs.

The reason behind this dimensionality problem is the asymmetry in the environment:

the rate at which the long-run player signals her private information, $\alpha_3 := \beta_3 + \beta_1\chi$, can be substantially different than the rate at which the myopic player signals his private belief, δ_1 . This, in turn, potentially introduces a non-trivial history dependence between γ and χ , reflected in the coupled system of ODEs they satisfy. Two natural questions then arise: first, under which conditions such history dependence can be simplified; and second, how to tackle the issue of existence of LME when this simplification is not possible.

Private values: one-dimensional shooting. We say that an environment is one of *private values* if the myopic player's flow utility satisfies

$$\hat{u}_{\hat{a}\theta} = 0,$$

i.e., the myopic player's best-reply does not directly depend on his belief about θ , but only indirectly via the long-run player's action. Otherwise, we say that the environment is one of *common values* (despite the long-run player always knowing θ).

In a private-value setting, the myopic player's coefficient on \hat{M} is $\delta_1 = \hat{u}_{\hat{a}a}\alpha_3$. In this case, there is a one-to-one mapping between γ and χ :

Lemma 4. *Set $\sigma_X \in (0, \infty)$. Suppose that β_1 and β_3 are continuous and that $\delta_1 = \hat{u}_{\hat{a}a}\alpha_3$. If $\hat{u}_{\hat{a}a} \neq 0$, there are positive constants c_1, c_2 and d independent of γ° such that*

$$\chi_t = \frac{c_1 c_2 (1 - [\gamma_t / \gamma^\circ]^d)}{c_1 + c_2 [\gamma_t / \gamma^\circ]^d}.$$

Moreover, (i) $0 \leq \chi_t < c_2 < 1$ for all $t \in [0, T]$ and (ii) $c_2 \rightarrow 0$ as $\sigma_X \rightarrow 0$ and $c_2 \rightarrow 1$ as $\sigma_X \rightarrow \infty$. If instead $\hat{u}_{\hat{a}a} = 0$, $\chi_t = 1 - \gamma_t / \gamma^\circ$.

Whenever the players signal at proportional rates, there is always a decreasing relationship between χ and γ ; in particular, $\chi_0 = 0$ when $\gamma_0 = \gamma^\circ$. By part (i), as long as the public signal is informative, χ is always strictly below 1, reflecting that the scope for the history-inference effect is diminished relative to the no-feedback case; also, the public and no-feedback cases are recovered as we take limits. Further, the characterization of χ obtained in the latter case (4) is recovered when $\hat{u}_a = 0$, as the public signal is then uninformative.

Equipped with this result, the standard one-dimensional shooting method based on the continuity of the solutions is applicable. We state below the BVP for the leadership application of Section 2 for $\sigma_X \in (0, \infty)$ in its undiscounted version: recall that in that setting, the follower wants to match the leader's action, and so

$$\hat{a}_t = \hat{\mathbb{E}}_t[a_t] \Rightarrow \delta_{1t} = \alpha_{3t} \Leftrightarrow \hat{u}_{\hat{a}a} = 1.$$

We omit the β_0 -ODE (it is uncoupled from the rest and linear in itself):

$$\begin{aligned}
\dot{v}_{6t} &= \beta_{2t}^2 + 2\beta_{1t}\beta_{2t}(1 - \chi_t) - \beta_{1t}^2(1 - \chi_t)^2 + \frac{2v_{6t}\alpha_{3t}^2\gamma_t\chi_t}{\sigma_X^2(1 - \chi_t)} \\
\dot{v}_{8t} &= -2\beta_{2t} - 2(1 - 2\alpha_{3t})\beta_{1t}(1 - \chi_t) - 4\beta_{1t}^2\chi_t(1 - \chi_t) + \frac{v_{8t}\alpha_{3t}^2\gamma_t\chi_t}{\sigma_X^2(1 - \chi_t)} \\
\dot{\beta}_{1t} &= \frac{\alpha_{3t}\gamma_t}{2\sigma_X^2\sigma_Y^2(1 - \chi_t)} \left\{ 2\sigma_X^2(\alpha_{3t} - \beta_{1t})\beta_{1t}(1 - \chi_t) - \alpha_{3t}^2\beta_{1t}\gamma_t\chi_tv_{8t} \right. \\
&\quad \left. - 2\sigma_Y^2\alpha_{3t}\chi_t(\beta_{2t} - \beta_{1t}[1 - \chi_t - 2\beta_{2t}\chi_t]) \right\} \\
\dot{\beta}_{2t} &= \frac{\alpha_{3t}\gamma_t}{2\sigma_X^2\sigma_Y^2(1 - \chi_t)} \left\{ 2\sigma_X^2\beta_{1t}^2(1 - \chi_t)^2 + 2\sigma_Y^2\alpha_{3t}\beta_{2t}\chi_t^2(1 - 2\beta_{2t}) - \alpha_{3t}^2\gamma_t\chi_t(2v_{6t} + \beta_{2t}v_{8t}) \right\} \\
\dot{\beta}_{3t} &= \frac{\alpha_{3t}\gamma_t}{2\sigma_X^2\sigma_Y^2(1 - \chi_t)} \left\{ -2\sigma_X^2\beta_{1t}(1 - \chi_t)\beta_{3t} + 2\sigma_Y^2\alpha_{3t}\beta_{2t}\chi_t^2(1 - 2\beta_{3t}) - \alpha_{3t}^2\beta_{3t}\gamma_t\chi_tv_{8t} \right\} \\
\dot{\gamma}_t &= -\frac{\gamma_t^2\alpha_{3t}^2}{\sigma_Y^2}
\end{aligned}$$

with boundary conditions $v_{6T} = v_{8T} = 0$, $\beta_{1T} = \frac{1}{2(2-\chi_T)}$, $\beta_{2T} = \frac{1-\chi_T}{2(2-\chi_T)}$, $\beta_{3T} = \frac{1}{2}$ and $\gamma_0 = \gamma^\circ$, and where $\alpha_3 := \beta_3 + \beta_1\chi$ and χ_t is as in the previous lemma. We have the following:

Theorem 1. *Let $\sigma_X \in (0, \infty)$ and $r = 0$. There exists a strictly positive function $T(\gamma^\circ) \in O(1/\gamma^\circ)$ such that, for all $T < T(\gamma^\circ)$, there exists a LME based on the solution to the previous BVP that satisfies $\beta_{0t} = 0$, $\beta_{1t} + \beta_{2t} + \beta_{3t} = 1$ and $\alpha_{3t} > 0$, $t \in [0, T]$.*

The key step in the proof is to show that $(\beta_1, \beta_2, \beta_3, v_6, v_8, \gamma)$ can be bounded uniformly over $[0, T(\gamma^\circ))$, some $T(\gamma^\circ) > 0$, when $\gamma_t \in [0, \gamma^\circ]$ at all times. This implies that tracing the (parametrized) initial condition of γ in the (backward) IVP from 0 upwards as in Figure 3 will lead to at least one γ -path landing at γ° (while the rest of the ODEs still admit solutions), due to the continuity of the solutions with respect to the initial conditions.²⁹

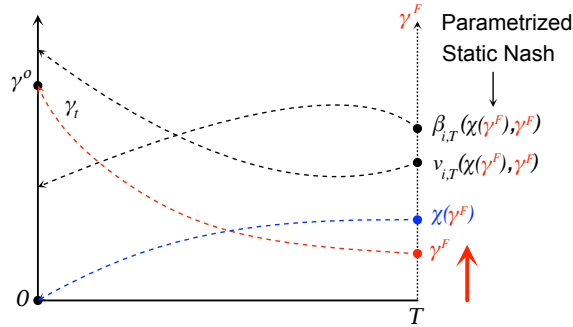


Figure 3: The one-dimensional shooting method.

²⁹See [Bonatti et al. \(2017\)](#) for an application of this method to a symmetric oligopoly model featuring dispersed fixed private information, imperfect public monitoring, and multiple long-run players.

The signaling coefficient for interior values $\sigma_X \in (0, \infty)$ lies “in between” those found for $\sigma_X = 0$ and $+\infty$ in Section 2, which validates the study of those extreme cases. Figure 4 illustrates: as σ_X increases, the signaling coefficient (dashed line) moves from the public benchmark to the no-feedback case counterpart.³⁰ With discounting and $\sigma_X < +\infty$, α_3 is *nonmonotonic*. Intuitively, the interior case combines the increasing history-inference effect of the no-feedback case (dominant early in the game) with the decreasing signaling motive driving the public case (dominant later): discounting then weakens the latter, while the former grows over time even with a myopic leader. Finally, as σ_X increases, the amplitude of the history inference effect increases, so the maximum of α_3 shifts to the right.

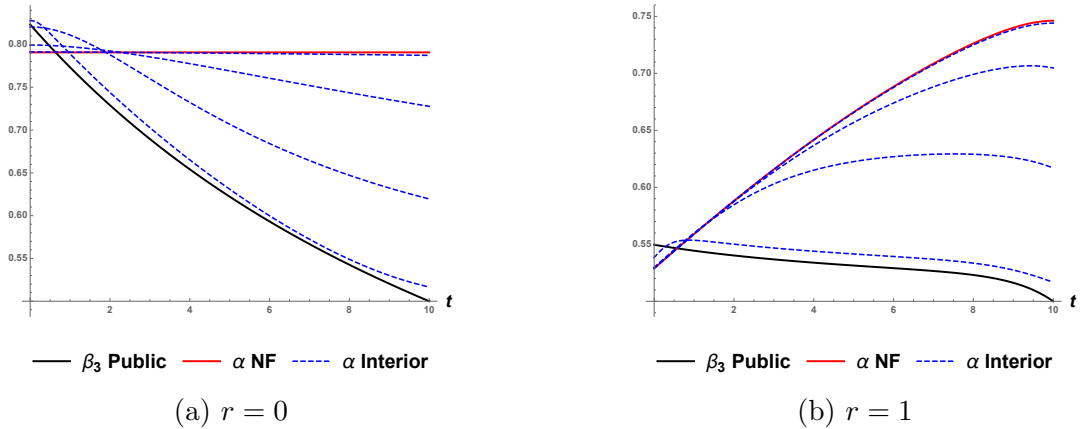


Figure 4: Signaling coefficients for $\sigma_X \in \{0, .1, .75, 2, 10, +\infty\}$; “ α Interior” denotes α_3 .

Common-value settings: fixed-point methods. When α_3 and δ are not proportional, χ can depend on both current and past values of γ , and the dimensionality problem resurfaces.

Our key observation is that finding a solution to any given instance of the BVP (21) is, mathematically, a *fixed-point problem*. Specifically, notice that the static Nash equilibrium at time T depends on the value that χ takes at that point. The latter value, however, depends on how much signaling has taken place along the way, i.e., on values of the coefficients $\vec{\beta}$ at times prior to T . Those values, in turn, depend on the value of the equilibrium coefficients at T by backward induction—thus, we are back to the same point where we started.

Our approach therefore applies a fixed-point argument adapted from the literature on BVPs with *intertemporal linear constraints* (Keller, 1968) to our problem with *intratemporal nonlinear constraints*. Because the method is novel and has the generality required to become useful in other settings, we briefly elaborate on how it works.³¹

³⁰To obtain sharper visual effects, we are potentially plotting beyond the interval of existence that the theorem guarantees (which is a lower bound). The discounted case can be treated with identical methods.

³¹Our adaptation is inspired by Theorem 1.2.7 in (Keller, 1968), which is stated without a proof.

Let $t \mapsto \mathbf{b}_t(s, \gamma^\circ, 0)$ denote the solution to the forward IVP version of (21) when the initial condition is $(s, \gamma^\circ, 0)$, $s \in \mathbb{R}^6$, provided a solution exists. From Lemma 2, the last two components of \mathbf{b} , i.e., γ and χ , always admit solutions as long as the others do; moreover, there are no constraints on their terminal values. Thus, for the fixed-point argument, we can focus on the first six components in $\mathbf{b} := (\beta_0, \beta_1, \beta_2, \beta_3, v_6, v_8, \gamma, \chi)$ by defining the *gap* function

$$g(s) = \mathbf{B}(\chi_T(s, \gamma^\circ, 0)) - D_T \int_0^T f(\mathbf{b}_t(s, \gamma^\circ, 0)) dt.$$

This function measures the distance between the total growth of $(\beta_0, \beta_1, \beta_2, \beta_3, v_6, v_8)$ (last term in the display), and its target value, $\mathbf{B}(\chi_T(s, \gamma^\circ, 0))$. By (22), $\mathbf{B}(\chi)$ is nonlinear: the static Nash equilibrium imposes nonlinear relationships across variables at time T .

By definition, $\mathbf{b}_0(s, \gamma^\circ, 0) = s$. Consequently, it follows that

$$g(s) = s \Leftrightarrow \mathbf{B}(\chi_T(s, \gamma^\circ, 0)) = s + D_T \int_0^T f(\mathbf{b}_t(s, \gamma^\circ, 0)) dt = D_T \mathbf{b}_T(s, \gamma^\circ, 0),$$

where the last equality follows from the definition of the ODE-system that $D_T \mathbf{b}$ satisfies. Thus, the shooting problem (i.e., finding s s.t. $\mathbf{B}(\chi_T(s, \gamma^\circ, 0)) = D_T \mathbf{b}_T(s, \gamma^\circ, 0)$) can be transformed to one of finding a fixed point of the function g .³²

The bulk of proof consists of finding a time $T(\gamma^\circ)$ and a compact set \mathcal{S} of values for s such that (i) for all $s \in \mathcal{S}$, a unique solution $(\mathbf{b}_t(s, \gamma^\circ, 0))_{t \in T(\gamma^\circ)}$ for the IVP with initial condition $(s, \gamma^\circ, 0)$ exists, and (ii) g is continuous map from \mathcal{S} to itself. The natural choice for \mathcal{S} is a ball with center $s_0 := \mathbf{B}(0)$, the terminal condition of the trivial game with $T = 0$.³³

We can now establish our main existence result for a variation of the leadership application in which the follower's best response is of the form

$$\hat{a}_t = \hat{u}_{\hat{a}\theta} \hat{\mathbb{E}}_t[\theta] + \hat{\mathbb{E}}_t[a_t] \Rightarrow \delta_{1t} = \hat{u}_{\hat{a}\theta} + \alpha_{3t}, \quad \text{where } \hat{u}_{\hat{a}\theta} > 0,$$

³²A BVP with intertemporal linear constraints differs from ours in that $D_0 \mathbf{b}_0 + D_T \mathbf{b}_T = (\mathbf{B}(\chi_T)', \gamma^\circ, 0)'$ becomes $A \mathbf{b}_0 + B \mathbf{b}_T = \alpha$, where α_3 is a constant vector and, critically, A and B are not necessarily diagonal matrices—thus, unlike in our analysis, one may not be able to dispense with a subset of the system. A complication that arises in our setting is that our version of α_3 is a nonlinear function of a subset of components of \mathbf{b}_T , which requires estimating $\mathbf{B}(\chi_T(s, \gamma^\circ, 0))$ for all values of s over which $g(\cdot)$ is a self-map.

³³It is useful to work with a change of variables that eliminates $1 - \chi_t$ from the denominator in the original system, and which reflects play when the state variable L is replaced by $(1 - \chi)L$. In the new system, part (i) can be accomplished by bounding solutions uniformly as in the one-dimensional shooting method, but now over $[0, T(\gamma^\circ)] \times \mathcal{S}$; in turn, the continuity requirement of (ii) is guaranteed by the regularity of $f(\cdot)$, while the self-map condition can be ensured due to the system scaling with γ° and T . Equipped with existence, we can then recover a solution to our original BVP by reversing the change of variables and applying Lemma 2 (which ensures that $1 - \chi_t > 0$ for all $t \in [0, T]$, and hence that the right-hand side of our system of interest is well-defined). This approach sidesteps finding a uniform upper bound for χ that is strictly less than 1, which would be required at the moment of bounding the system uniformly. In all cases, $\gamma_t \in [0, \gamma^\circ]$ due the IVP under consideration being in its forward version (Lemma 2).

i.e., the follower now has a positive bias relative to the original setting. The positivity constraint ensures that (ii) in Assumption 1 is satisfied, and clearly the rest of the assumption holds too. The associated BVP is given by (B.25)-(B.31) in the Appendix.

Theorem 2. *Set $\sigma_X \in (0, \infty)$, $\hat{u}_{a\theta} > 0$ and $r = 0$ in the leadership model. There is a strictly positive function $T(\gamma^\circ) \in O(1/\gamma^\circ)$ such that if $T < T(\gamma^\circ)$, there exists a LME based on the BVP (B.25)-(B.31). In such an equilibrium, $\alpha_3 > 0$.*

There are three immediate observations from this theorem. First, the self-map condition, while not affecting the order of $T(\gamma^\circ)$ relative to a traditional one-dimensional shooting case, is not vacuous either. In fact, since $s_0 = \mathbf{B}(0)$ is the center of \mathcal{S} , we have that

$$g(s) - s_0 = \mathbf{B}(\chi_T(s, \gamma^\circ, 0)) - \mathbf{B}(0) - D_T \int_0^T \mathbf{f}(\mathbf{b}_t(s, \gamma^\circ, 0)) dt.$$

Thus, bounding $\mathbf{B}(\chi_T(s, \gamma^\circ, 0)) - \mathbf{B}(0)$ imposes an additional constraint relative to those that ensure that the system is uniformly bounded (which in turn bound the last term in the previous expression), thereby shrinking the constant of proportionality in $T(\gamma^\circ) \in O(1/\gamma^\circ)$.

Second, the set of times for which a LME is guaranteed to exist increases without bound as $\gamma^\circ \searrow 0$: indeed, $\mathbf{f}(\cdot)$ naturally scales with this parameter, so the solutions converge to the full-information counterpart $(v_6, v_8, \beta_0, \beta_1, \beta_2, \beta_3, \chi, \gamma) = (0, 0, 0, 1/4, 1/4, 1/2, 0, 0)$, which is defined for all $T > 0$. Finally, the bound $T(\gamma^\circ)$ is obtained under minimal knowledge of the system: it imposes crude bounds that only use the degree of the polynomial vector $\mathbf{f}(\mathbf{b})$, and that do not exploit any relationship between the coefficients. Thus, the proof technique is both general and improvable, provided more is known about the system in specific settings.

In Appendix B.3 we sketch how the steps used in the proof of Theorem 2 apply to the whole class of games satisfying Assumption 1. Moreover, observe that this method, by being able to “shoot” multiple ODEs in either direction, is potentially applicable to other asymmetric games of learning beyond the class under study.

5 Extensions

As noted in Remark 1, our model can be generalized to accommodate a quadratic terminal payoff or to allow the long-run player to affect the public signal. To demonstrate, we first explore a career-concerns model, and then a trading model a la Kyle (1985) exhibiting private monitoring of an insider’s trades.

5.1 Reputation for Neutrality

Suppose that the long-run player is now an expert or politician with career concerns. This agent has a hidden ideological bias θ and takes repeated actions—for example, adopting positions on critical issues³⁴ or making campaign promises.³⁵ She receives utility from taking actions that conform to her bias but also from attaining a neutral reputation at the end of the horizon; hence, she must trade off her ideological desires with her career concerns.

The long-run player’s payoff is given by

$$-\int_0^T (a_t - \theta)^2 dt - \psi \hat{a}_T^2,$$

where $\psi > 0$ is a commonly known parameter, while the myopic player’s flow payoff is $\hat{U}(a_t, \hat{a}_t, \theta) = -(\hat{a}_t - \theta)^2$. In this specification, the myopic player chooses $\hat{a}_t = \hat{M}_t$ at all times, and so the termination payoff $-\psi \hat{M}_T^2$ is effectively a measure of career concerns; the parameter $\psi > 0$ governs the intensity of such motives.³⁶

We interpret the myopic player as a news outlet; Y defined in (8) is interpreted as the outlet having access to imperfect private sources regarding the long-run player’s actions. In turn, the outlet’s *news* process is given by $dX_t = \hat{M}_t dt + \sigma_X dZ_t^X$: the reporting on the perceived bias is fair on average, but imperfect.

When does the politician fare better? In settings where the reporting is precise (i.e., low σ_X), and hence she can tailor her actions to her reputation? Clearly, noisier environments as measured by σ_X entail a direct cost: they introduce increased uncertainty over a concave objective. The next result shows that increasing an agent’s uncertainty over her own reputation, thereby undermining her ability to take appropriate actions, can be beneficial:

Proposition 5. *(i) Suppose that $\sigma_X \in \{0, +\infty\}$. Then, for all $\psi, T > 0$ there exists an LME. Moreover, if $\psi < \sigma_Y^2/\gamma^0$, the LME is unique, and learning is lower and ex ante payoffs higher in the no feedback case.*

(ii) If $\sigma_X \in (0, \infty)$, there exists $T(\gamma^0) \in O(1/\gamma^0)$ such that a LME exists for all $T < T(\gamma^0)$.

All else equal, the long-run player prefers higher actions when her type is higher, and hence her equilibrium strategy attaches positive weight to her type. But because of career

³⁴Mayhew (1974) in a classic political science text describes the dynamic nature of position taking by congresspeople: “[. . .] it might be rational for members in electoral danger to resort to innovation. The form of innovation available is entrepreneurial position taking, its logic being that for a member facing defeat with his old array of positions, it makes good sense to gamble on some new ones.”

³⁵Campaign promises could be costly due to politicians’ honesty (Callander and Wilkie, 2007; Kartik et al., 2007; Kartik, 2009) or the electorate’s refusal to reelect politicians who renege (Aragonès et al., 2007).

³⁶A linear model a la Holmström (1999) makes the quality of feedback irrelevant. See Bouvard and Lévy (2019) for quadratic-based horizontal reputations under symmetric uncertainty.

concerns, the greater the perceived value of \hat{M} , the greater the long-run player's incentive to manipulate it downward. With private monitoring, higher types therefore must offset higher beliefs from their perspectives, leading to a history-inference effect that dampens the signaling coefficient α_3 . The belief is then less responsive from an ex ante perspective, which facilitates maintaining a reputation for neutrality.³⁷ Indeed, provided the objective is not too concave and the environment not too uncertain (which strengthen the direct cost), this strategic effect dominates.

Regarding part (ii), because the present environment is one of common values, one can establish the existence of a LME with minimal modifications to the method presented in Section 4.3. Indeed, the only difference is that our baseline model had terminal conditions that were a function of χ_T exclusively, whereas now we have an extra dependence on γ_T via

$$\beta_{1T} = -\frac{\psi\gamma_T}{\sigma_Y^2 + \psi\gamma_T\chi_T},$$

reflecting last-minute incentives to manipulate the myopic player's belief that decrease in the precision of such belief. Our approach does not vary with this extra dependence.

5.2 Insider Trading

An asset with fixed fundamental value θ , is traded in continuous time until date T , the time at which its true value is revealed, ending the game. A patient *insider* (the long-run player) privately observes θ prior to the start of the game. As in Yang and Zhu (2019), a second trader has a technology which allows him to privately observe imperfect signals of the insider's trades; this player is myopic. Both players and a flow of noise traders submit orders to a *market maker* who then executes those trades at a public price $L_t = \mathbb{E}[\theta|\mathcal{F}_t^X]$.

We depart from the baseline model along three dimensions. First, the players' flow payoffs depend directly on L , interpreted as the action taken by the market maker: the myopic player's flow payoff is given by $\xi(\theta - L)\hat{a} - \frac{\hat{a}^2}{2}$, where $\xi \geq 0$, while the long-run player's flow payoff is $(\theta - L_t)a_t$; we interpret the inverse of the parameter ξ as a measure of transaction costs for the myopic player.³⁸ Second, observe that the long-run player's flow payoff is linear in her action a_t at all instants $t \in [0, T]$. Finally, the public signal (total order flow) now includes the long-run player's action: $dX_t = (a_t + \hat{a}_t)dt + \sigma_X dZ_t^X$. Hence, the myopic player learns from both the private monitoring channel and the public price.

³⁷It is easy to show that the ex ante expectation of \hat{M}_T^2 is $\gamma^o - \gamma_T$, so that greater learning by the myopic player results in larger terminal losses for the long-run player. This reverses for slightly *negative* ψ , but so does the history-inference effect: there is *more* learning but again a higher payoff in the no feedback case.

³⁸The use of a quadratic loss term strengthens our non-existence result, as it limits the myopic player's ability to exploit the private information he acquires.

Following the literature, we seek an equilibrium in which the informed trader reveals her private information gradually over time through a linear strategy of the form (10). Hence, we require that the coefficients of the insider’s strategy be C^1 functions over strict compact subsets of $[0, T]$.³⁹ We can then apply Lemmas 1 and 2 to such sets.⁴⁰

Clearly, when $\xi = 0$ (or $\sigma_Y = \infty$), the model reduces to the classic model of Kyle (1985) (see also Back (1992)), and hence a LME with trading strategy of the form $\beta_3(\theta - L)$ always exists. This is not the case when $\xi > 0$.

Proposition 6. *Fix $\xi > 0$. For all $\sigma_Y > 0$, there does not exist a linear Markov equilibrium of the insider trading game.*

With linear Markov strategies, the myopic player acquires private information about θ over time by observing signals of the insider’s trades. Consequently, the myopic player’s own repeated trades carry further information to the market maker, beyond that which the market maker learns from the insider alone. This introduces momentum into the law of motion for the price from the insider’s perspective, measured by a term $\xi(m - l)$ in the drift of L ; future trades then become less attractive to the insider, thereby putting the insider in a race against herself which results in all her information being traded away in the first instant, regardless of the amount of noise in the private signal Y .⁴¹

In an intimately related result, Yang and Zhu (2019) show that a linear equilibrium ceases to exist in a two-period setting where a trader who only participates in the last round receives a sufficiently precise signal of an informed player’s first-period trade; a mixed-strategy equilibrium then emerges. More generally, the existence problem relates to how, with common information, an informed player’s rush to trade depends on the number of trading opportunities. The analysis of Foster and Viswanathan (1994) is illuminating in this respect: in a setting with nested information structures, the better informed trader quickly trades a commonly known piece of information (and exploits her superior information only later on). While there are important differences between our setups (the belief of the less informed player is, in their model, always known to the first, and their common information exogenous) there is a unifying theme: once common information is created, there is a pressure to trade quickly on it. Such pressure increases with the number of trading opportunities.⁴²

³⁹By not imposing this requirement over $[0, T]$, we maintain the possibility of full revelation of information near the end of the game, as is standard in insider trading models. In addition, this requirement ensures that the total order can be “inverted” from the price, and hence it is without loss to make X public.

⁴⁰Specifically, the proof of Lemma 1 provides the learning ODEs for the case $\nu > 0$, and it is easy to see that the steps of Lemma 2 (with $\hat{u}_{\hat{a}\theta} = \xi, \hat{u}_{\hat{a}a} = 0$) also go through for this case.

⁴¹Because this environment is linear, there is no well-defined Nash equilibrium at T . Thus, our argument is not based on a non-existence result for a BVP, but rather on an impossibility of indifference conditions for the long-run player to hold.

⁴²In symmetric settings, Holden and Subrahmanyam (1992) show that intense trading occurs in early pe-

6 Conclusion

We have examined a minimal departure from an extensive literature on signaling games: namely, allowing for imperfect private monitoring on the receiver’s side. In such settings, a “beliefs about beliefs” problem arises due to the private information obtained by the receiver and the sender’s need to rely on her past play to forecast the former’s belief. A key contribution of our analysis is to offer a framework where this problem is manageable, at the core of which is a novel and tractable representation of a second-order belief under linear Markov strategies. We explored the implications that such conditioning on past play has on signaling behavior and economic outcomes in applications, and we introduced an approach for establishing the existence of LME in asymmetric signaling games.

Let us conclude with a discussion of three assumptions of the model. First, the public-private signal structure studied indeed provides us with sufficient tractability: via the representation, it allows us to “close” the set of states at the second order. If instead the long-run player had a stochastic type, or access to an imperfect private signal, beliefs of even higher order would be payoff-relevant. While some economic environments may feature these assumptions, a natural question is whether we believe that economic behavior in such settings is substantially affected by those higher-order inferences.

Second, the presence of a myopic player is not a major limitation. In fact, most of the results are derived for, or can be generalized to, continuous coefficients $\vec{\delta}$ in the myopic player’s strategy. With a forward-looking receiver, such coefficients solve ODEs capturing optimal dynamic behavior, but crucially (i) no additional states are needed, and (ii) the fixed-point argument is applicable to an enlarged boundary value problem.

Finally, the linear-quadratic-Gaussian class is clearly a stylized one. Yet, its advantage lies in its tractability for uncovering economic effects that are likely to be key in other, more nonlinear, environments. From this perspective, (i) the way in which the history-inference effect interacts with payoffs and (ii) the time-effects arising from ongoing learning, seem to exhaust the effects expected to be of first-order importance when behavior depends on the payoff-relevant aspects of the players’ histories.

riods between two identically informed traders, and [Back et al. \(2000\)](#) obtain the corresponding nonexistence result with infinitely many rounds of trade directly in continuous time.

Appendix A: Proofs for Section 2

A.1: Preliminary Results

We begin by stating several standard results on ordinary differential equations which we use heavily in the proofs that follow. Let $f(t, x)$ be a continuous function from $[0, T] \times \mathbb{R}^n$ to \mathbb{R}^n , where $T > 0$. For any fixed $x_0 \in \mathbb{R}^n$, define an initial value problem (IVP)

$$\dot{x} = f(t, x), \quad x(0) = x_0.$$

We have the following.

- Peano’s Theorem (Teschl, 2012, p. 56, Theorem 2.19): For some $T' \in (0, T)$, there exists at least one solution to the IVP for $t \in [0, T']$.

If, moreover, f is locally Lipschitz continuous in x , uniformly in t , then the following apply:

- The Picard-Lindelöf Theorem (Teschl, 2012, p. 38, Theorem 2.2): For some $T' \in (0, T)$, there is a unique solution to the IVP for $t \in [0, T']$.
- “The” comparison theorem (Teschl, 2012, p. 27, Theorem 1.3): If $x(t), y(t)$ are two differentiable functions satisfying $x(t_0) \leq y(t_0)$ for some $t_0 \in [0, T)$ and $\dot{x}_t - f(t, x(t)) \leq \dot{y}_t - f(t, y(t))$ for all $t \in [t_0, T)$, then $x(t) \leq y(t)$ for all $t \in [t_0, T)$. If, in addition, $x(t) < y(t)$ for some $t \in [t_0, T)$, then $x(s) < y(s)$ for all $s \in [t, T)$.

A.2: Proofs for Public Case

Proof of Proposition 1. We aim to characterize a LME of the form

$$a_t = \beta_{0t} + \beta_{1t}\hat{M}_t + \beta_{3t}\theta \quad \text{and} \quad \hat{a}_t = \hat{\mathbb{E}}_t[a_t] = \beta_{0t} + (\beta_{1t} + \beta_{3t})\hat{M}_t \quad (\text{A.1})$$

where $\hat{M}_t := \hat{\mathbb{E}}_t[\theta]$, and β_{it} , $i = 0, 1, 3$, are functions of time satisfying $\beta_{1t} + \beta_{3t} \neq 0$, $t \in [0, T]$. Since the follower attempts to match the leader’s action, we have

$$\hat{a}_t = \hat{\mathbb{E}}_t[\beta_{0t} + \beta_{1t}M_t + \beta_{3t}\theta] = \underbrace{\beta_{0t}}_{\delta_{0t}} + \underbrace{(\beta_{1t} + \beta_{3t})}_{\delta_{1t}} M_t$$

in any equilibrium in which $\beta_{1t} + \beta_{3t} \neq 0$ at all times (and thus the follower’s belief can be inferred from the observation of the follower’s action, which leads to $M \equiv \hat{M}$). From standard results in filtering theory, if the follower expects $(a_t)_{t \geq 0}$ as in (A.1), then whenever

he is on path⁴³ (even if the long-run player is not) his time- t beliefs are $\theta \sim \mathcal{N}(\hat{M}_t, \gamma_t)$, where

$$d\hat{M}_t = \frac{\beta_{3t}\gamma_t}{\sigma_Y^2} [dY_t - \underbrace{\{\beta_{0t} + (\beta_{1t} + \beta_{3t})\hat{M}_t\}}_{\hat{\mathbb{E}}_t[a_t]} dt] \quad \text{and} \quad \dot{\gamma}_t = - \left(\frac{\gamma_t \beta_{3t}}{\sigma_Y^2} \right)^2, \quad (\text{A.2})$$

and where $\hat{M}_0 = \mu$ and $\gamma_0 = \gamma^o$.

Let $V : \mathbb{R}^2 \times [0, T] \rightarrow \mathbb{R}$ denote the leader's value function. The HJB equation is

$$rV(\theta, m, t) = \sup_a \left\{ -(a - \theta)^2 - (a - \hat{a}_t)^2 + \Lambda_t \mu_t(a) V_M(\theta, m, t) + \frac{\Lambda_t^2 \sigma_Y^2}{2} V_{MM}(\theta, m, t) + V_t(\theta, m, t) \right\}, \quad (\text{A.3})$$

where $\Lambda_t := \frac{\beta_{3t}\gamma_t}{\sigma_Y^2}$ and $\mu_t(a) := a - \beta_{0t} - (\beta_{1t} + \beta_{3t})m$.

We guess a quadratic value function $V(\theta, m, t) = v_{0t} + v_{1t}\theta + v_{2t}m + v_{3t}\theta^2 + v_{4t}m^2 + v_{5t}\theta m$. To obtain the maximizer of the RHS of (A.3), we impose the first-order condition

$$\begin{aligned} 0 &= -2(a - \theta) - 2(a - \beta_{0t} - (\beta_{1t} + \beta_{3t})m) + \frac{\beta_{3t}\gamma_t[v_{2t} + 2mv_{4t} + \theta v_{5t}]}{\sigma_Y^2} \\ \implies 0 &= -2(\beta_{0t} + \beta_{1t}m + \beta_{3t}\theta - \theta) - 2\beta_{3t}(\theta - m) + \frac{\beta_{3t}\gamma_t[v_{2t} + 2mv_{4t} + \theta v_{5t}]}{\sigma_Y^2}, \end{aligned} \quad (\text{A.4})$$

where in the second line we have used that the maximizer must be $a^* := \beta_{0t} + \beta_{1t}m + \beta_{3t}\theta$. Since (A.4) must hold for all $(\theta, m, t) \in \mathbb{R}^2 \times [0, T]$, the coefficients on θ and m and the constant term must vanish, and we obtain

$$(v_{2t}, v_{4t}, v_{5t}) = \left(\frac{2\sigma_Y^2\beta_{0t}}{\beta_{3t}\gamma_t}, \frac{\sigma_Y^2(\beta_{1t} - \beta_{3t})}{\beta_{3t}\gamma_t}, \frac{2\sigma_Y^2(2\beta_{3t} - 1)}{\beta_{3t}\gamma_t} \right). \quad (\text{A.5})$$

Since $v_{iT} = 0$ for all $i \in \{0, 1, \dots, 5\}$, (A.5) implies the terminal values

$$(\beta_{0T}, \beta_{1T}, \beta_{3T}) = (0, 1/2, 1/2), \quad (\text{A.6})$$

which are also the *myopic equilibrium coefficients*, that is, the coefficients that would arise were the players to act myopically at any instant of time.

⁴³If instead of $\sigma_X = 0$, Y is public, this holds also after deviations by the myopic player; see footnote 12.

Substituting a^* as above into (A.3) yields

$$\begin{aligned}
0 = & -r[v_{0t} + v_{1t}\theta + v_{2t}m + v_{3t}\theta^2 + v_{4t}m^2 + v_{5t}\theta m] \\
& - [\beta_{0t} + m\beta_{1t} + \theta(\beta_{3t} - 1)]^2 - (m - \theta)^2\beta_{3t}^2 - \frac{(m - \theta)[v_{2t} + 2mv_{4t} + \theta v_{5t}]\beta_{3t}^2\gamma_t}{\sigma_Y^2} \\
& + \frac{v_{4t}\beta_{3t}^2\gamma_t^2}{\sigma_Y^2} + \dot{v}_{0t} + \dot{v}_{1t}\theta + \dot{v}_{2t}m + \dot{v}_{3t}\theta^2 + \dot{v}_{4t}m^2 + \dot{v}_{5t}\theta m,
\end{aligned} \tag{A.7}$$

which again must hold for all $(\theta, m, t) \in \mathbb{R}^2 \times [0, T]$. Using (A.5), we can replace $(v_{2t}, v_{4t}, v_{5t}, \dot{v}_{2t}, \dot{v}_{4t}, \dot{v}_{5t})$ in (A.7) and obtain a new HJB equation as a function of $\vec{\beta} := (\beta_0, \beta_1, \beta_3)$ and $\dot{\vec{\beta}}$. As the constant term and the coefficients on θ, m, θ^2, m^2 and θm in this new equation must vanish, we obtain the following system of ODEs for $(v_0, v_1, v_3, \beta_0, \beta_1, \beta_3)$:

$$\dot{v}_{0t} = rv_{0t} + \beta_{3t}\gamma_t(\beta_{3t} - \beta_{1t}) \tag{A.8}$$

$$\dot{v}_{1t} = rv_{1t} - 2\beta_{0t}\beta_{3t} \tag{A.9}$$

$$\dot{v}_{3t} = 1 + rv_{3t} - 2\beta_{3t}^2 \tag{A.10}$$

$$\dot{\beta}_{0t} = 2r\beta_{0t}\beta_{3t} \tag{A.11}$$

$$\dot{\beta}_{1t} = \beta_{3t} \left[r(2\beta_{1t} - 1) + \frac{\beta_{1t}\beta_{3t}\gamma_t}{\sigma_Y^2} \right] \tag{A.12}$$

$$\dot{\beta}_{3t} = \beta_{3t} \left[r(2\beta_{3t} - 1) - \frac{\beta_{1t}\beta_{3t}\gamma_t}{\sigma_Y^2} \right] \tag{A.13}$$

with conditions $(v_{0T}, v_{1T}, v_{3T}, \beta_{0T}, \beta_{1T}, \beta_{3T}) = (0, 0, 0, 0, 1/2, 1/2)$. Observe that γ appears in all the previous equations, and its dynamic is given by (A.2). Furthermore, observe that solving the subsystem $(\beta_0, \beta_1, \beta_3, \gamma)$ delivers the remaining v_i : their ODEs (A.8)-(A.10) are uncoupled from one another and linear in themselves, and thus they have unique solutions. Hence, proving the existence of a linear Markov equilibrium reduces to solving the boundary value problem (A.2) and (A.11)-(A.13) with conditions $\gamma_0 = \gamma^o$ and (A.6).

We now show that a solution to this boundary value problem always exists. To do so, it is useful to transform this problem into backward form, i.e., reversing the direction of time and parameterizing the initial value of γ in that system. Specifically, we obtain

$$\dot{\beta}_{0t} = -2r\beta_{0t}\beta_{3t} \tag{A.14}$$

$$\dot{\beta}_{1t} = \beta_{3t} \left[r(1 - 2\beta_{1t}) - \frac{\beta_{1t}\beta_{3t}\gamma_t}{\sigma_Y^2} \right] \tag{A.15}$$

$$\dot{\beta}_{3t} = \beta_{3t} \left[r(1 - 2\beta_{3t}) + \frac{\beta_{1t}\beta_{3t}\gamma_t}{\sigma_Y^2} \right] \tag{A.16}$$

$$\dot{\gamma}_t = \frac{\beta_{3t}^2 \gamma_t^2}{\sigma_Y^2}, \quad (\text{A.17})$$

with initial conditions $\beta_{00} = 0$, $\beta_{10} = \beta_{30} = \frac{1}{2}$ and $\gamma_0 = \gamma^F \geq 0$.

Define $B_t^{pub} := \beta_{1t} + \beta_{3t}$.

Lemma A.1. *If a solution to the backward system exists over some interval $[0, T]$, then any such solution must have the following properties. If $\gamma^F > 0$, then (i) $B_t^{pub} = 1$ for all $t \in [0, T]$, (ii) $\beta_{3t} \in (1/2, 1)$ and $\beta_{1t} \in (0, 1/2)$ for all $t \in (0, T]$, (iii) β_3 is monotonically increasing while β_1 is monotonically decreasing, and (iv) γ is strictly increasing. If $\gamma^F = 0$, then $\beta_{1t} = \beta_{3t} = \frac{1}{2}$ and $\gamma_t = 0$ for all $t \in [0, T]$. For any $\gamma^F \geq 0$, $\beta_0 = 0$.*

Proof of Lemma A.1. We first claim that if a solution exists over some interval $[0, T]$, then $\beta_{3t} > 0$ for all $t \in [0, T]$. To see this let $f^{\beta_3}(t, \beta_{3t})$ denote the RHS of (A.16). Letting $x_t := 0$ for all $t \in [0, T]$, we have $\beta_{30} = 1/2 > x_0$ and $\dot{\beta}_{3t} - f^{\beta_3}(t, \beta_{3t}) = 0 = \dot{x}_t - f^{\beta_3}(t, x_t)$. By the comparison theorem in Teschl (2012, Theorem 1.3), the claim holds.

Next, we define $B_t^{pub} := \beta_{1t} + \beta_{3t}$ and show that $B^{pub} = 1$. Adding (A.15) and (A.16) yields $\dot{B}_t^{pub} = 2r\beta_{3t}(1 - B_t^{pub})$ which, given any β_{3t} , has solution of the form $B_t^{pub} = 1 - \tilde{C}e^{-\int_0^t 2r\beta_{3s} ds}$. Using the initial condition $B_0^{pub} = 1 = 1 - \tilde{C}$, we have $\tilde{C} = 0$ and $B^{pub} = 1$.

Hence we can rewrite the β_3 ODE as

$$\dot{\beta}_{3t} = \beta_{3t} \left[r(1 - 2\beta_{3t}) + \frac{\beta_{3t}(1 - \beta_{3t})\gamma_t}{\sigma_Y^2} \right]. \quad (\text{A.18})$$

We now show that $\beta_3 < 1$. Let $f^{\beta_3}(t, \beta_{3t})$ now denote the RHS of (A.18), and define $x_t := 1$ for all $t \in [0, T]$. Then $x_0 = 1 > \beta_{30} = \frac{1}{2}$, and $\dot{\beta}_{3t} - f^{\beta_3}(t, \beta_{3t}) = 0 \leq r = \dot{x}_t - f^{\beta_3}(t, x_t)$, so by the comparison theorem, the claim holds. Since $\beta_{1t} = 1 - \beta_{3t}$, we have $\beta_1 > 0$.

Consider the case $\gamma^F = 0$. Since $\beta_3 > 0$, $\gamma_t = 0$ is the unique solution to (A.17). Letting $z_t := \beta_{3t} - \beta_{1t}$, we have $\dot{z}_t = -2r\beta_{3t}z_t$. As this is a linear ODE with initial condition $z_0 = 0$, the unique solution is $z_t = 0$ for all $t \in [0, T]$. Since $\beta_{1t} + \beta_{3t} = 1$, we have $\beta_{1t} = \beta_{3t} = 1/2$ for all $t \in [0, T]$, proving the claim in the proposition statement.

Next consider the case $\gamma^F > 0$. Since $\beta_3 > 0$, (A.17) implies γ is strictly increasing, and hence $\gamma_t > 0$ for all $t \in [0, T]$. Now whenever $\beta_{3t} = \frac{1}{2}$, we have $\dot{\beta}_{3t} = \frac{1}{2} \left[0 + \frac{\gamma_t}{4\sigma_Y^2} \right] > 0$, and thus $\beta_{3t} > 1/2$ for all $t \in (0, T]$. Since $\beta_{1t} = 1 - \beta_{3t}$, we have $\beta_{1t} < 1/2$ for all such t .

We now turn to (iii). Since $\dot{\beta}_{1t} + \dot{\beta}_{3t} = 0$ for all $t \in [0, T]$, it suffices to show that $\dot{\beta}_{3t} > 0$ for all $t \in [0, T]$; in turn, it suffices to show that $H_t := r(1 - 2\beta_{3t}) + \frac{\beta_{3t}(1 - \beta_{3t})\gamma_t}{\sigma_Y^2} > 0$ for all $t \in [0, T]$, where $\dot{\beta}_{3t} = \beta_{3t}H_t$. Now $H_0 = \frac{\gamma_0}{4\sigma_Y^2} > 0$, and with algebra it can be shown that if $H_t = 0$, $\dot{H}_t = \frac{(1 - \beta_{3t})\beta_{3t}^3\gamma_t^2}{\sigma_Y^4} > 0$. It follows that $H_t > 0$ for all $t \in [0, T]$, as desired.

Finally, note that in all cases, we have $\beta_3 > 0$, so the unique solution to (A.14) consistent with the initial condition $\beta_{00} = 0$ is $\beta_0 = 0$. \square

Now we must show there exists a value of γ^F such that $\gamma_T = \gamma^o$ in the backward system while all the other ODEs admit solutions. As shown in Bonatti et al. (2017), a sufficient condition for this to occur is that the solutions are uniformly bounded when γ_t takes values in $[0, \gamma^o]$ when t is in $[0, T]$. By Lemma A.1, the following bounds hold as long as γ does not explode: $\beta_0, \beta_1, \beta_3 \in [0, 1]$. We conclude that there exists a solution to the BVP, and hence a LME exists.⁴⁴ Uniqueness is shown in the proof of Lemma A.3 for the case $r = 0$ and in the online appendix for the case $r \in (0, \infty)$.

For part (ii) of the proposition, $\beta_{3T} = 1/2$ has already been established, and the remaining claims have been shown (for the backward system) in Lemma A.1. \square

A.3: Proofs for No Feedback Case

Lemma A.2 (Belief Representation). *Suppose that the follower expects $a_t = [\beta_{0t} + \beta_{1t}(1 - \chi_t)]\mu + \alpha_t\theta$, where $\alpha = \beta_3 + \beta_1\chi$, $\chi = 1 - \gamma/\gamma^o$, and $\gamma_t := \hat{\mathbb{E}}_t[(\theta_t - \hat{M}_t)^2]$. Then $\dot{\gamma}_t = -\left(\frac{\gamma_t\alpha_t}{\sigma_Y^2}\right)^2$. Moreover, if the leader follows (5), $M_t = \chi_t\theta + (1 - \chi_t)\mu$ holds at all times.*

Proof of Lemma A.2. Let $\beta_{0t}\mu + \beta_{1t}M_t + \beta_{3t}\theta$ denote the long-run player's strategy. Thus, $\hat{\mathbb{E}}_t[a_t] = \alpha_{0t} + \alpha_t\hat{M}_t$, where $\alpha_0 = [\beta_0 + \beta_1(1 - \chi)]\mu$ and $\alpha = \beta_3 + \beta_1\chi$. Then, $d\hat{M}_t = \frac{\alpha_t\dot{\gamma}_t}{\sigma_Y^2}[dY_t - (\alpha_{0t} + \alpha_t\hat{M}_t)dt]$, so letting $R(t, s) = \exp(-\int_s^t \frac{\alpha_u^2\gamma_u}{\sigma_Y^2} du)$

$$\begin{aligned} \hat{M}_t &= \mu R(t, 0) + \int_0^t R(t, s) \frac{\alpha_s\gamma_s}{\sigma_Y^2} [(a_s - \alpha_{0s})ds + \sigma_Y dZ_s^Y] \\ \Rightarrow M_t &= \mu R(t, 0) + \int_0^t R(t, s) \frac{\alpha_s\gamma_s}{\sigma_Y^2} (a_s - \alpha_{0s})ds \\ \dot{\gamma}_t &= -\frac{\gamma_t^2\alpha_t^2}{\sigma_Y^2}. \end{aligned} \tag{A.19}$$

On the path of play, however, $a_t = \beta_{0t}\mu + \beta_{1t}M_t + \beta_{3t}\theta$, so we obtain

$$M_t = \mu R(t, 0) + \theta \int_0^t R(t, s) \frac{\alpha_s\gamma_s}{\sigma_Y^2} [-(1 - \chi_s)\beta_{1s}\mu + \beta_{1s}M_s + \beta_{3s}\theta] ds,$$

where we have used that $\beta_{0s}\mu - \alpha_{0s} = -(1 - \chi_s)\beta_{1s}\mu$. In particular,

$$dM_s = \left(-M_s \left[\frac{\alpha_s\gamma_s}{\sigma_Y^2} (\alpha_s + \beta_{1s}) \right] + \frac{\alpha_s\gamma_s}{\sigma_Y^2} [-\beta_{1s}(1 - \chi_s)\mu + \beta_{3s}\theta] \right) ds,$$

⁴⁴We elaborate on the specific details of this argument in the proof of Theorem 1.

and so, letting $\tilde{R}(t, s) = \exp(-\int_s^t \frac{\alpha_u \gamma_u}{\sigma_Y^2} (\alpha_u - \beta_{1u}) du)$,

$$M_t = \mu \left(\tilde{R}(t, 0) - \int_0^t \tilde{R}(t, s) \frac{\alpha_s \gamma_s}{\sigma_Y^2} \beta_{1s} (1 - \chi_s) ds \right) + \theta \int_0^t \tilde{R}(t, s) \frac{\alpha_s \gamma_s}{\sigma_Y^2} \beta_{3s} ds.$$

From here

$$\chi_t = \int_0^t \tilde{R}(t, s) \frac{\alpha_s \gamma_s}{\sigma_Y^2} \beta_{3s} ds \quad \text{and} \quad 1 - \chi_t = \tilde{R}(t, 0) - \int_0^t \tilde{R}(t, s) \frac{\alpha_s \gamma_s}{\sigma_Y^2} \beta_{1s} (1 - \chi_s) ds.$$

Critically, observe that the second constraint is a direct consequence of the first. Specifically, adding and subtracting β_{3s} and noticing that $\alpha_s - \beta_{1s} = \beta_{3s} - \beta_{1s}(1 - \chi_s)$ we can write

$$\begin{aligned} - \int_0^t \tilde{R}(t, s) \frac{\alpha_s \gamma_s}{\sigma_Y^2} \beta_{1s} (1 - \chi_s) ds &= \int_0^t \tilde{R}(t, s) \frac{\alpha_s \gamma_s}{\sigma_Y^2} [\alpha_s - \beta_{1s}] ds - \chi_t \\ &= \int_0^t \frac{d\tilde{R}(t, s)}{ds} ds - \chi_t \\ &= 1 - \tilde{R}(t, 0) - \chi_t \end{aligned} \tag{A.20}$$

where in the last equality we used that $\tilde{R}(t, t) = 1$.

With this in hand, the relevant constraint is the first. In differential form, and using that $\alpha_t = \beta_3 + \beta_1 \chi_t$,

$$\dot{\chi}_t = -\frac{\alpha_t \gamma_t}{\sigma_Y^2} [\alpha_t - \beta_{1t}] \chi_t + \frac{\alpha_t \beta_{3t} \gamma_t}{\sigma_Y^2} = \frac{(\beta_{3t} + \beta_{1t} \chi_t)^2 \gamma_t}{\sigma_Y^2} (1 - \chi_t) = \frac{\alpha_t^2 \gamma_t}{\sigma_Y^2} (1 - \chi_t).$$

Using the exact same arguments in the proof of Lemma 2, we conclude that the resulting (χ, γ) system admits a unique solution with the same properties as in that lemma. Furthermore, it is easy to check that $1 - \gamma_t / \gamma^o$ satisfies the χ -ODE, concluding the proof. \square

Proof of Proposition 2. From Lemma A.2, given a conjecture by the follower about $(\beta_0, \beta_1, \beta_3)$, the variance of the follower's belief evolves deterministically as $\dot{\gamma}_t = -\frac{\alpha_t^2 \gamma_t^2}{\sigma_Y^2}$ and $\chi \equiv 1 - \gamma / \gamma^o$.

The follower matches the expectation of the leader's action by playing

$$\begin{aligned} \hat{a}_t &= \hat{\mathbb{E}}_t[\beta_{0t} \mu + \beta_{1t} M_t + \beta_{3t} \theta] \\ &= \hat{\mathbb{E}}_t[\beta_{0t} \mu + \beta_{1t} (\mu[1 - \chi_t] + \chi_t \theta) + \beta_{3t} \theta] \\ &= \underbrace{(\beta_{0t} + \beta_{1t}[1 - \chi_t]) \mu}_{\delta_{0t}} + \underbrace{(\beta_{1t} \chi_t + \beta_{3t})}_{\delta_{1t}} \hat{M}_t. \end{aligned}$$

Ignoring the \hat{a}_t^2 term, the states (θ, M_t, t) capture the leader's expected flow payoff given an action a , on and off path. The \hat{a}_t^2 term introduces the term $\mathbb{E}_t[\hat{M}_t^2]$, but this can be

expressed as $(\mathbb{E}_t[\hat{M}_t])^2 + \mathbb{E}_t[(\hat{M}_t - M_t)^2] = M_t^2 + \gamma_t \chi_t$.⁴⁵

Defining $\mu_{1t} := \alpha_t \gamma_t / \sigma_Y^2$, $\mu_{0t} = -\mu_{1t}[\beta_{0t}\mu + \beta_{1t}\mu(1 - \chi_t)]$ and $\mu_{2t} = -\alpha_t \mu_{1t}$, where $\alpha_t = \beta_{1t}\chi_t + \beta_{3t}$, the HJB equation is

$$rV(\theta, m, t) = \sup_a \left\{ - \underbrace{(a^2 - 2a[\delta_{0t} + \delta_{1t}m] + \delta_{0t}^2 + 2\delta_{0t}\delta_{1t}m + \delta_{1t}^2[\gamma_t\chi_t + m^2])}_{=\mathbb{E}_t[(a - \hat{a}_t)^2]} - (a - \theta)^2 + (\mu_{0t} + a\mu_{1t} + m\mu_{2t})V_m(\theta, m, t) + V_t(\theta, m, t) \right\}. \quad (\text{A.21})$$

We guess a quadratic value function $V(\theta, m, t) = v_{0t} + v_{1t}\theta + v_{2t}m + v_{3t}\theta^2 + v_{4t}m^2 + v_{5t}\theta m$. Using a similar steps to those in the proof of Proposition 1,⁴⁶ we obtain a boundary value problem for $(\beta_0, \beta_1, \beta_3, \gamma)$. We now express this system in its backward form (the original system can be recovered by placing a ‘-’ sign in front of the RHS of (A.22)-(A.26) below). Using a guess $\gamma_0 = \gamma^F$, we formulate an initial value problem parameterized by γ^F :

$$\dot{\beta}_{0t} = \frac{\alpha_t}{2\sigma_Y^2} \left\{ -r\sigma_Y^2\beta_{0t}(2 - \chi_t) + r\sigma_Y^2(1 - \chi_t) - 2\gamma_t\beta_{1t}^2(1 - \chi_t) \right\} \quad (\text{A.22})$$

$$\dot{\beta}_{1t} = \frac{\alpha_t}{2\sigma_Y^2} \left\{ r\sigma_Y^2 - 2\beta_{1t}[\beta_{3t}\gamma_t + r\sigma_Y^2(2 - \chi_t)] + 2\beta_{1t}^2\gamma_t(1 - \chi_t) \right\} \quad (\text{A.23})$$

$$\dot{\beta}_{3t} = \frac{\alpha_t}{2\sigma_Y^2} \left\{ r\sigma_Y^2(2 - \chi_t) + 2\beta_{3t}[\beta_{1t}\gamma_t - r\sigma_Y^2(2 - \chi_t)] \right\} \quad (\text{A.24})$$

$$\dot{\alpha}_t = r\alpha_t[1 - \alpha_t(2 - \chi_t)] \quad (\text{A.25})$$

$$\dot{\gamma}_t = \frac{\alpha_t^2\gamma_t^2}{\sigma_Y^2} \quad (\text{A.26})$$

with boundary conditions $\beta_{00} = \frac{1-\chi_0}{2(2-\chi_0)}$, $\beta_{10} = \frac{1}{2(2-\chi_0)}$, $\beta_{30} = \frac{1}{2}$, $\alpha_0 = \frac{1}{2-\chi_0} > 0$ and $\gamma_0 = \gamma^F$.

Let $(\beta_{0t}^m, \beta_{1t}^m, \beta_{3t}^m, \alpha_t^m) = \left(\frac{1-\chi_t}{2(2-\chi_t)}, \frac{1}{2(2-\chi_t)}, \frac{1}{2}, \frac{1}{2-\chi_t} \right)$ denote the myopic coefficients, i.e. the myopic equilibrium given the variance γ induced by the original (dynamic) strategy.

By the comparison theorem, $\alpha > 0$ in any solution to the IVP. It follows that for $\gamma^F = 0$, the IVP has a unique solution $(\beta_0, \beta_1, \beta_3, \gamma, \chi) = (0, 1/2, 1/2, 0, 1)$. By continuity, suppose now that $\gamma^F > 0$ is sufficiently small that there exists a solution to the IVP. By the same argument as in the proof of Lemma A.1, γ is increasing.

Given such $(\gamma_t)_{t \in [0, T]}$, we can write the right-hand side of the α -ODE as a function of the form $f^\alpha(t, \alpha)$ that is of class C^1 . Using that $\chi_t = 1 - \gamma_t/\gamma^o$, observe that, in the backward

⁴⁵To see this, from the proof of Lemma A.2, $\mathbb{E}_t[(\hat{M}_t - M_t)^2] = \mathbb{E}_t[(\int_0^t R(t, s) \frac{\alpha_s \gamma_s}{\sigma_Y} dZ_s^Y)^2] = \int_0^t R(t, s)^2 \frac{\alpha_s^2 \gamma_s^2}{\sigma_Y^2} ds = \int_0^t \exp(-2 \int_s^t \frac{\dot{\gamma}_u}{\gamma_u} du) (-\dot{\gamma}_s) ds = \int_0^t (\gamma_t/\gamma_s)^2 (-\dot{\gamma}_s) ds = \gamma_t^2 (1/\gamma_t - 1/\gamma^o) = \gamma_t \chi_t$.

⁴⁶See the online appendix for the detailed steps.

system,

$$\frac{d}{dt} \left(\frac{1}{2 - \chi_t} \right) - f^\alpha(t, 1/(2 - \chi_t)) = -\frac{\dot{\gamma}_t}{\gamma^o(1 + \gamma_t/\gamma^o)^2} < 0 = \dot{\alpha}_t - f^\alpha(t, \alpha_t)$$

where the first inequality follows from γ_t being increasing in the backward system. Moreover, $\alpha_0 = \frac{1}{2 - \chi_0}$. The comparison theorem allows us to conclude that $\alpha_t \geq 1/(2 - \chi_t)$, and in turn $\dot{\alpha}_t \leq 0$ (and hence $\dot{\alpha}_t \geq 0$ in the forward system), for all $t \in [0, T]$ with both inequalities strict for $t \in (0, T]$ ($t \in [0, T)$ in the forward system) if and only if $r > 0$. It follows that for all $t \in (0, T]$, $\alpha_t < \alpha_0 = \frac{1}{2 - \chi_0} < 1$.

Now by simple addition of the ODEs, we obtain that $B_t^{NF} := \beta_0 + \beta_1 + \beta_3$ satisfies

$$\dot{B}_t^{NF} = \frac{\alpha_t}{2\sigma_Y^2} \{2r\sigma_Y^2(2 - \chi_t)[1 - B_t^{NF}]\}, \text{ with } B_0^{NF} = 1.$$

It is easy to see that an analogous argument to the one used for B^{pub} used in the proof of Lemma A.1 applies, yielding $B_t^{NF} = 1$ as the unique solution.

Next, we establish uniform bounds on β_1 and β_3 (and hence β_0). Toward showing $\beta_1 > 0$, observe that the RHS of the β_1 ODE can be written as $f^{\beta_1}(t, \beta_1)$ of class C^1 . Letting $x := 0$, we have $\beta_{10} > x_0 = 0$ and $\dot{x}_t - f^{\beta_1}(t, x_t) = 0 - \frac{\alpha_t}{2\sigma_Y^2} r\sigma_Y^2 \leq 0 = \dot{\beta}_{1t} - f^{\beta_1}(t, \beta_{1t})$ and thus by the comparison theorem, $\beta_1 > x = 0$. This implies that $\beta_3 = \alpha - \beta_1\chi \leq \alpha < 1$.

We now show $\beta_{3t} > 1/2$ and $\beta_{1t} < \beta_{1t}^m < 1$ for all $t \in (0, T]$. For the former, recall that $\beta_{30} = 1/2$, and whenever $\beta_{3t} = 1/2$, $\dot{\beta}_{3t} = \frac{\alpha\beta_{1t}\gamma}{2\sigma_Y^2} > 0$; it follows that $\beta_{3t} > 1/2$ for all $t \in (0, T]$. Now $\beta_{10} = \beta_{10}^m < 1$, and since

$$\begin{aligned} \dot{\beta}_{1t}^m - f^{\beta_1}(t, \beta_{1t}^m) &= \frac{\gamma_t}{4\sigma_Y^2(2 - \chi_t)^4} (\beta_{3t}[2 - \chi_t] - [1 - \chi_t]) (2\beta_{3t}[2 - \chi_t] + \chi_t) \\ &> 0 = \dot{\beta}_{1t} - f^{\beta_1}(t, \beta_{1t}), \end{aligned}$$

the comparison theorem implies $\beta_{1t} < \beta_{1t}^m < 1$ for all $t \in (0, T]$.

By a one-dimensional shooting argument as in the proof of Proposition 2, we obtain existence of a solution to the BVP for $(\beta_0, \beta_1, \beta_3, \gamma)$, from which the value function coefficients are characterized (see the online appendix), and hence existence of a LME.

The final claim to prove is that as $T \rightarrow \infty$, $\alpha_T \rightarrow 1$, for which we use the forward system. Recalling that $\alpha > 1/2$, we have $\gamma_T \rightarrow 0$ as $T \rightarrow \infty$, and thus $\chi_T \rightarrow 1$ and $\alpha_T = 1/(2 - \chi_T) \rightarrow 1$. \square

A.4: Proofs for Comparisons between Public and No Feedback Cases

To prove Propositions 3 and 4 we rely on two sets of results. The first set (Lemmas A.3-A.4 below) corresponds to closed-form solutions that are obtained in the fully patient and fully myopic cases. The second set of results establishes the uniform convergence of the equilibrium coefficients to the myopic counterparts as $r \nearrow \infty$ (Lemmas A.5-A.6). All four proofs are in the online appendix.

Lemma A.3 (Closed-form solutions when $r = 0$). *For $r = 0$, the leading by example game has a unique LME for the public case, and $(\beta_0, \beta_1, \beta_3, \gamma)$ satisfy $\beta_0 \equiv 0$, $\beta_1 \equiv 1 - \beta_3$,*

$$\gamma_t = \frac{\gamma_T}{2} + \frac{1}{\frac{2}{\gamma_T} - \frac{T-t}{\sigma_Y^2}}, \beta_{3t} = \frac{1}{2 - \frac{\gamma_T(T-t)}{2\sigma_Y^2}}, \text{ and } \gamma_T = \frac{\gamma^\circ T + 2\sigma_Y^2 - \sqrt{(\gamma^\circ T)^2 + 4\sigma_Y^4}}{T}. \quad (\text{A.27})$$

Lemma A.4 (Closed-form solution no-feedback case $r = 0$). *For $r = 0$, the leading by example game has a unique LME for the no feedback case:*

$$\beta_{1t} = \frac{\gamma^\circ [(\gamma^\circ + \gamma_T)^2 \sigma_Y^2 - (T-t)(\gamma^\circ)^2 \gamma_T]}{(\gamma^\circ + \gamma_T)[2\sigma_Y^2(\gamma^\circ + \gamma_T)^2 - (T-t)(\gamma^\circ)^2 \gamma_T]}, \quad \beta_{3t} = \frac{\sigma_Y^2(\gamma^\circ + \gamma_T)^2}{2\sigma_Y^2(\gamma^\circ + \gamma_T)^2 - (T-t)(\gamma^\circ)^2 \gamma_T}$$

$$\alpha_t = \frac{\gamma^\circ}{\gamma^\circ + \gamma_T}, \quad \gamma_t = \frac{\gamma_T \sigma_Y^2 (\gamma^\circ + \gamma_T)^2}{\sigma_Y^2 (\gamma^\circ + \gamma_T)^2 - (T-t)(\gamma^\circ)^2 \gamma_T},$$

for all $t \in [0, T]$, where $\chi_t = 1 - \gamma_t/\gamma^\circ$ and $\gamma_T \in (0, \gamma^\circ)$ is the unique solution in $(0, \gamma^\circ)$ to the cubic $q(\gamma) := \gamma T (\gamma^\circ)^3 + (\gamma - \gamma^\circ)(\gamma + \gamma^\circ)^2 \sigma_Y^2 = 0$, and $\beta_0 \equiv 1 - \beta_1 - \beta_3$.

Lemma A.5 (Closed-form solutions—myopic case). *Suppose the leader is myopic. In the LME for the public case, $\beta_3 = 1/2$ and $\gamma_t^{pub} = \frac{4\sigma_Y^2 \gamma^\circ}{4\sigma_Y^2 + \gamma^\circ t}$. In the LME for the no feedback case, $\alpha_t = \frac{\gamma^\circ}{\gamma^\circ + \gamma_t^{NF}}$, where γ_t^{NF} is defined implicitly as the unique solution in $(0, \gamma^\circ]$ of the equation $f(\gamma/\gamma^\circ) := 2 \ln(\gamma_t^{NF}/\gamma^\circ) - \gamma^\circ/\gamma_t^{NF} + \gamma_t^{NF}/\gamma^\circ = -\frac{\gamma^\circ t}{\sigma_Y^2}$.*

Lemma A.6 (Uniform Convergence as $r \rightarrow \infty$). *As $r \rightarrow \infty$, the solutions to the public and no feedback cases converge uniformly to their corresponding myopic solutions.*

Proof of Proposition 3

We first prove the learning comparison in (i). Recall that γ_T^{NF} is the unique positive root of the cubic equation $q(\gamma) = 0$ defined in Lemma A.4. Now q is increasing over $(0, \gamma^\circ)$, and hence $q(\gamma) > 0$ iff $\gamma > \gamma_T^{NF}$. Thus to prove the claim, it suffices to show that $q(\gamma_T^{pub}) > 0$; the proof of this algebraic result is in the online appendix.

Now we establish the ranking of signaling coefficients at time zero, i.e., that $\beta_{30}^{pub} > \alpha^{NF}$. Using the associated expressions from Lemmas A.3 and A.4, this is equivalent to

$$\frac{1}{2 - \frac{\gamma_T^{pub} T}{2\sigma_Y^2}} > \frac{\gamma^o}{\gamma^o + \gamma_T^{NF}} \iff \hat{\gamma} := \gamma^o \left(1 - \frac{\gamma_T^{pub} T}{2\sigma_Y^2} \right) < \gamma_T^{NF}.$$

It suffices to show that $q(\hat{\gamma}) = T\hat{\gamma}(\gamma^o)^3 + (\hat{\gamma} - \gamma^o)(\hat{\gamma} + \gamma^o)^2\sigma_Y^2 < 0$. Using the expression for γ_T^{pub} from Lemma A.3, one can show that

$$\begin{aligned} q(\hat{\gamma}) &= \frac{(\gamma^o)^4 T [-\gamma^o T + \sqrt{(\gamma^o T)^2 + 4\sigma_Y^4}]}{2\sigma_Y^2} \left[1 - \frac{2\sigma_Y^2 - (\gamma^o T - \sqrt{(\gamma^o T)^2 + 4\sigma_Y^4})}{2\sigma_Y^2} \right] \\ &= -\frac{T(\gamma^o)^4}{2\sigma_Y^4} \left[(T\gamma^o)^2 + 2\sigma_Y^4 - T\gamma^o \sqrt{(T\gamma^o)^2 + 4\sigma_Y^4} \right]. \end{aligned}$$

The expression in square brackets can be written as $\frac{x+y}{2} - \sqrt{xy} > 0$ where $x = (T\gamma^o)^2 > 0$ and $y = (T\gamma^o)^2 + 4\sigma_Y^4 > 0$, and thus $q(\hat{\gamma}) < 0$, concluding the proof.

Finally, to prove (ii), observe first that $\gamma_t^{Pub} > \gamma_t^{NF}$ for all $t \in (0, T]$ in the fully myopic case. Indeed, since $\gamma_0^{NF} = \gamma_0^{Pub} = \gamma^o$, solving the ODEs for γ^{Pub} and γ^{NF} by integration and using that $\alpha_t \geq \beta_{3t} = 1/2$ with strict inequality for all $t > 0$ delivers the result. This implies that for any $\delta \in (0, T)$, $0 < \bar{\gamma} := \min_{t \in [T-\delta, T]} (\gamma_t^{Pub, \infty} - \gamma_t^{NF, \infty})$, where we use $\gamma^{x,r}$ to denote the solution for the case $x \in \{Pub, NF\}$ and r is the discount rate. By Lemma A.6, $\gamma^{Pub,r} - \gamma^{NF,r}$ converges uniformly to $\gamma^{Pub, \infty} - \gamma^{NF, \infty}$ as $r \rightarrow \infty$, and thus for any $\epsilon \in (0, \bar{\gamma})$, there exists $\bar{r} > 0$ such that for all $r > \bar{r}$ and all $t \in [T - \delta, T]$, we have $\gamma_t^{Pub,r} - \gamma_t^{NF,r} > \gamma_t^{Pub, \infty} - \gamma_t^{NF, \infty} - \epsilon \geq \bar{\gamma} - \epsilon > 0$, as desired.

Proof of Proposition 4

We begin by calculating expected flow losses in the public and no feedback cases.

Lemma A.7. *The expected flow payoffs to the long-run player in LME for the public benchmark and no feedback case have magnitudes*

$$u_t^{pub} = \gamma_t^{pub} [(1 - \beta_{3t})^2 + \beta_{3t}^2] \text{ and } u_t^{NF} = (1 - \alpha_{3t})^2 \gamma^o + \alpha_{3t}^2 \gamma_t^{NF}.$$

Proof. Recall that $\hat{\mathbb{E}}_0$ is equivalent to the long-run player's ex ante expectation operator.

In the public benchmark, using $a_t = (1 - \beta_{3t})M_t + \beta_{3t}\theta$ and $\hat{a}_t = M_t$, we have

$$u_t^{pub} = \hat{\mathbb{E}}_0 [(\theta - a_t)^2 + (a_t - \hat{a}_t)^2] = \hat{\mathbb{E}}_0 [(\theta - M_t)^2 [(1 - \beta_{3t})^2 + \beta_{3t}^2]] = \gamma_t^{pub} [(1 - \beta_{3t})^2 + \beta_{3t}^2],$$

where in the last step we have used the law of iterated expectations and the fact that the variance is deterministic: $\hat{\mathbb{E}}_0 [(\theta - M_t)^2] = \hat{\mathbb{E}}_0 \left[\hat{\mathbb{E}}_t [(\theta - M_t)^2] \right] = \gamma_t^{pub}$.

In the no feedback case, using $a_t = (1 - \alpha_t)\mu + \alpha_t\theta$ and $\hat{a}_t = (1 - \alpha_t)\mu + \alpha_t\hat{M}_t$,

$$\begin{aligned} u_t^{NF} &= \hat{\mathbb{E}}_0 [(\theta - a_t)^2 + (a_t - \hat{a}_t)^2] = (1 - \alpha_t)^2 \hat{\mathbb{E}}_0 [(\theta - \mu)^2] + \alpha_t^2 \hat{\mathbb{E}}_0 [(\theta - \hat{M}_t)^2] \\ &= \gamma^\circ (1 - \alpha_t)^2 + \alpha_t^2 \gamma_t^{NF} \end{aligned}$$

where we have used the definition of γ° and $\hat{\mathbb{E}}_0 [(\theta - \hat{M}_t)^2] = \hat{\mathbb{E}}_0 \left[\hat{\mathbb{E}}_t [(\theta - \hat{M}_t)^2] \right] = \gamma_t^{NF}$. \square

To aid exposition, we subsume part (i) of the proposition in Lemma A.8 below. For $i \in \{pub, NF\}$, let V^i denote the long-run player's ex ante expected payoff and define $\tilde{T}^i := \frac{T\gamma_i^F}{\sigma_Y^2}$. Define $\rho := \gamma_T^{NF}/\gamma^\circ$.

Lemma A.8. *Assume $r = 0$. Then*

$$(i) \quad V^{pub} = -\sigma_Y^2 \left\{ \tilde{T}^{pub}/2 - \ln \left[\frac{16 - 8\tilde{T}^{pub}}{(4 - \tilde{T}^{pub})^2} \right] \right\} \quad \text{and} \quad V^{NF} = -\sigma_Y^2 \{ \rho(1 - \rho) - \ln \rho \}$$

$$(ii) \quad V^{pub} > V^{NF} \quad \text{for all } T, \gamma^\circ, \sigma_Y^2 > 0.$$

Proof. See the online appendix. \square

For part (ii) of the proposition, observe from the calculations in the proof of Lemma A.7 that for $i \in \{pub, NF\}$, the undiscounted coordination losses are $\int_0^T (x_t^i)^2 \gamma_t^i dt$, where $x^{pub} = \beta^{pub}$ and $x^{NF} = \alpha^{NF}$. Since $\dot{\gamma}_t^i = -(x_t^i \gamma_t^i)^2 / \sigma_Y^2$, such losses are $\int_0^T \sigma_Y^2 \frac{\dot{\gamma}_t^i}{\gamma_t^i} dt = \sigma_Y^2 \ln(\gamma_t^i)|_0^T = -\sigma_Y^2 \ln(\gamma^\circ / \gamma_T^i)$ as in the proposition.

Appendix B: Proofs for Section 4

In the proofs for this section, we denote the prior by $\hat{m}_0 := \mu$.

Proof of Lemma 1. We establish a more general version of the lemma for a drift of the form $\hat{a}_t + \nu a_t$, $\nu \in [0, 1]$, in X . We use “p1” and “p2” to refer to the long-run player and the myopic player, respectively. Without fear of confusion, we also use γ_{1t} for the posterior variance of p2 (γ_t in the main body), as this variance appears in the first filtering step of the proof. Likewise, p1's posterior variance will be denoted by γ_2 , as it is obtained from a second filtering step.

Inserting (9) into (10), we can write $a_t = \alpha_{0t} + \alpha_{2t}L_t + \alpha_{3t}\theta$, with

$$\alpha_{0t} = \beta_{0t}, \quad \alpha_{2t} = \beta_{2t} + \beta_{1t}(1 - \chi_t), \quad \text{and} \quad \alpha_{3t} = \beta_{3t} + \beta_{1t}\chi_t,$$

which p2 conjectures drives Y . (Since α_{3t} plays a key role in the economic analysis and appears frequently throughout the paper, we sometimes abbreviate it to α_t .)

With this in hand, p2's filtering problem can be obtained using the Kalman filter (Chapters 11 and 12 in [Liptser and Shiryaev, 1977](#)). Specifically, define

$$\begin{aligned} dX_t^2 &:= dX_t - [\hat{a}_t + \nu(\alpha_{0t} + \alpha_{2t}L_t)]dt = \nu\alpha_{3t}\theta dt + \sigma_X dZ_t^X \\ dY_t^2 &:= dY_t - [\alpha_{0t} + \alpha_{2t}L_t]dt = \alpha_{3t}\theta dt + \sigma_Y dZ_t^Y \end{aligned}$$

which are in p2's information set. Then, by Theorems 12.6 and 12.7 in [Liptser and Shiryaev \(1977\)](#), p2's posterior belief is normally distributed with mean \hat{M}_t and variance γ_{1t} , where,

$$\begin{aligned} d\hat{M}_t &= \frac{\nu\alpha_{3t}\gamma_{1t}}{\sigma_X^2} [dX_t^2 - \nu\alpha_{3t}\hat{M}_t dt] + \frac{\alpha_{3t}\gamma_{1t}}{\sigma_Y^2} [dY_t^2 - \alpha_{3t}\hat{M}_t dt], \\ \dot{\gamma}_{1t} &= -\gamma_{1t}^2 \alpha_{3t}^2 \Sigma, \end{aligned}$$

and $\Sigma := \left(\frac{\nu^2}{\sigma_X^2} + \frac{1}{\sigma_Y^2} \right)$. (These expressions hold for any (admissible) strategies of the players, as deviations go undetected.)

P1 can affect \hat{M}_t via her choice $(a_t)_{t \in [0, T]}$. It is easy to see that, from her perspective,

$$\begin{aligned} d\hat{M}_t &= \frac{\nu\alpha_{3t}\gamma_{1t}}{\sigma_X^2} [(\nu a_t - \nu\{\alpha_{0t} + \alpha_{2t}L_t + \alpha_{3t}\hat{M}_t\})dt + \sigma_X dZ_t^X] \\ &\quad + \frac{\alpha_{3t}\gamma_{1t}}{\sigma_Y^2} [(a_t - \{\alpha_{0t} + \alpha_{2t}L_t + \alpha_{3t}\hat{M}_t\})dt + \sigma_Y dZ_t^Y] \end{aligned}$$

under any admissible strategy $(a_t)_{t \in [0, T]}$. Rearranging terms we can write

$$d\hat{M}_t = (\mu_{0t} + \mu_{1t}a_t + \mu_{2t}\hat{M}_t)dt + B_t^X dZ_t^X + B_t^Y dZ_t^Y, \quad (\text{B.1})$$

where

$$\mu_{1t} = \alpha_{3t}\gamma_{1t}\Sigma, \quad \mu_{0t} = -\mu_{1t}[\alpha_{0t} + \alpha_{2t}L_t], \quad \mu_{2t} = -\alpha_{3t}\mu_{1t}, \quad B_t^X = \frac{\nu\alpha_{3t}\gamma_{1t}}{\sigma_X}, \quad B_t^Y = \frac{\alpha_{3t}\gamma_{1t}}{\sigma_Y}. \quad (\text{B.2})$$

This dynamic is linear in \hat{M} . Also, since L_t depends only on the paths of X , μ_{0t} is in p1's information set. Similarly with $(a_t)_{t \in [0, T]}$, which is measurable with respect to (θ, X) .

On the other hand, because p1 always thinks that p2 is on path, the public signal follows

$$dX_t = (\nu a_t + \delta_{0t} + \delta_{1t}\hat{M}_t dt + \delta_{2t}L_t)dt + \sigma_X dZ_t^X,$$

from her perspective. This dynamic is also affine in the unobserved state \hat{M} , with an

intercept that is again measurable with respect to (θ, X) . The hidden state \hat{M} along with the observation state (θ, X) is thus conditionally Gaussian. In particular, applying the filtering equations in Theorem 12.7 in [Liptser and Shiryaev \(1977\)](#) yields that $M_t := \mathbb{E}_t[\hat{M}_t]$ and $\gamma_{2t} := \mathbb{E}_t[(M_t - \hat{M}_t)^2]$ satisfy

$$dM_t = \underbrace{(\mu_{0t} + \mu_{1t}a_t + \mu_{2t}M_t)dt}_{=\mathbb{E}_t[(\mu_{0t} + \mu_{1t}a_t + \mu_{2t}\hat{M}_t)dt]} + \frac{\sigma_X B_t^X + \gamma_{2t}\delta_{1t}}{\sigma_X^2} [dX_t - (\nu a_t + \delta_{0t} + \delta_{1t}M_t + \delta_{2t}L_t)dt] \quad (\text{B.3})$$

$$\dot{\gamma}_{2t} = 2\mu_{2t}\gamma_{2t} + (B_t^X)^2 + (B_t^Y)^2 - \left(\frac{\sigma_X B_t^X + \gamma_{2t}\delta_{1t}}{\sigma_X} \right)^2, \quad (\text{B.4})$$

and where $dZ_t := [dX_t - (\nu a_t + \delta_{0t} + \delta_{1t}M_t + \delta_{2t}L_t)dt]/\sigma_X$ is a Brownian motion with respect to the long-run player's standpoint. (For notational simplicity, we have omitted the dependence of M and Z on the strategy $(a_t)_{t \in [0, T]}$ that is being followed.) Crucially, because (B.3) is linear, one can easily solve for M_t as an *explicit* function of past actions $(a_{s < t})$ and past realizations of the public history $(X_s)_{s < t}$.

Inserting $a_t = \beta_{0t} + \beta_{1t}M_t + \beta_{2t}L_t + \beta_{3t}\theta$ into the right-hand side of (B.3), and collecting terms we obtain

$$dM_t = [\hat{\mu}_{0t} + \hat{\mu}_{1t}M_t + \hat{\mu}_{2t}L_t + \hat{\mu}_{3t}\theta]dt + \hat{B}_t dX_t,$$

where

$$\begin{aligned} \hat{\mu}_{0t} &= \underbrace{-\alpha_{3t}\gamma_{1t}\alpha_{0t}\Sigma}_{\text{constant in } \mu_{0t}} + \underbrace{\alpha_{3t}\gamma_{1t}\beta_{0t}\Sigma}_{\mu_{1t}\beta_{0t}} + \underbrace{\frac{\nu\alpha_{3t}\gamma_{1t} + \gamma_{2t}\delta_{1t}}{\sigma_X^2}}_{=[\sigma_X B_t^X + \gamma_{2t}\delta_{1t}]/\sigma_X^2} [-\nu\beta_{0t} - \delta_{0t}] \\ \hat{\mu}_{1t} &= \underbrace{\alpha_{3t}\gamma_{1t}\beta_{1t}\Sigma}_{\mu_{1t}\beta_{1t}} + \underbrace{-\alpha_{3t}^2\gamma_{1t}\Sigma}_{\mu_{2t}} + \frac{\nu\alpha_{3t}\gamma_{1t} + \gamma_{2t}\delta_{1t}}{\sigma_X^2} [-\nu\beta_{1t} - \delta_{1t}] \\ \hat{\mu}_{2t} &= \underbrace{-\alpha_{3t}\gamma_{1t}\alpha_{2t}\Sigma}_{L_t \text{ term in } \mu_{0t}} + \underbrace{\alpha_{3t}\gamma_{1t}\beta_{2t}\Sigma}_{\mu_{1t}\beta_{2t}} + \frac{\nu\alpha_{3t}\gamma_{1t} + \gamma_{2t}\delta_{1t}}{\sigma_X^2} [-\nu\beta_{2t} - \delta_{2t}] \\ \hat{\mu}_{3t} &= \underbrace{\alpha_{3t}\gamma_{1t}\beta_{3t}\Sigma}_{\mu_{1t}\beta_{3t}} + \frac{\nu\alpha_{3t}\gamma_{1t} + \gamma_{2t}\delta_{1t}}{\sigma_X^2} [-\nu\beta_{3t}] = \left[\frac{\alpha_{3t}\gamma_{1t}}{\sigma_Y^2} - \frac{\nu\gamma_{2t}\delta_{1t}}{\sigma_X^2} \right] \beta_{3t} \\ \hat{B}_t &= \frac{\nu\alpha_{3t}\gamma_{1t} + \gamma_{2t}\delta_{1t}}{\sigma_X^2}. \end{aligned}$$

Let $R(t, s) = \exp(\int_s^t \hat{\mu}_{1u} du)$, and suppose and denote the prior distribution of θ by $\mathcal{N}(\hat{m}_0, \gamma_{10})$; in particular, $M_0 = \hat{m}_0$. Path-by-path of X , therefore,

$$M_t = R(t, 0)\hat{m}_0 + \theta \int_0^t R(t, s)\hat{\mu}_{3s} ds + \int_0^t R(t, s)[\hat{\mu}_{0s} + \hat{\mu}_{2s}L_s] ds + \int_0^t R(t, s)\hat{B}_s dX_s.$$

Imposing that this expression must coincide with (9) then yields a system of two equations:

$$\begin{aligned}\chi_t &= \int_0^t R(t, s) \hat{\mu}_{3s} ds, \\ L_t &= \frac{R(t, 0) \hat{m}_0 + \int_0^t R(t, s) [\hat{\mu}_{0s} + \hat{\mu}_{2s} L_s] ds + \int_0^t R(t, s) \hat{B}_s dX_s}{1 - \chi_t}.\end{aligned}$$

The validity of the construction boils down to finding a solution to the previously stated equation for χ that takes values in $[0, 1)$. In fact, when this is the case,

$$\begin{aligned}(1 - \chi_t) dL_t - L_t d\chi_t &= [\hat{\mu}_{0t} + (\hat{\mu}_{1t}(1 - \chi_t) + \hat{\mu}_{2t}) L_t] dt + \hat{B}_t dX_t \\ \implies dL_t &= \frac{L_t [\hat{\mu}_{1t} + \hat{\mu}_{2t} + \hat{\mu}_{3t}] dt + \hat{\mu}_{0t} dt + \hat{B}_t dX_t}{1 - \chi_t}.\end{aligned}$$

Thus, letting $R_2(t, s) := \exp\left(\int_s^t \frac{\hat{\mu}_{1u} + \hat{\mu}_{2u} + \hat{\mu}_{3u}}{1 - \chi_u} du\right)$, we obtain that

$$L_t = R_2(t, 0) \hat{m}_0 + \int_0^t R_2(t, s) \frac{\hat{\mu}_{0s}}{1 - \chi_s} ds + \int_0^t R_2(t, s) \frac{\hat{B}_s}{1 - \chi_s} dX_s, \quad (\text{B.5})$$

i.e., L is a (linear) function of the paths of X as conjectured. Moreover, in the particular case of $\nu = 0$, it is easy to verify that $dL_t = (\ell_{0t} + \ell_{1t} L_t) dt + B_t dX_t$, where

$$l_{0t} = -\frac{\gamma_t \chi_t \delta_{0t} \delta_{1t}}{\sigma_X^2 (1 - \chi_t)}, \quad l_{1t} = -\frac{\gamma_t \chi_t \delta_{1t} (\delta_{1t} + \delta_{2t})}{\sigma_X^2}, \quad B_t = \frac{\gamma_t \chi_t \delta_{1t}}{\sigma_X^2 (1 - \chi_t)}. \quad (\text{B.6})$$

We will ultimately find a solution to the equation for χ that is of class C^1 and that takes values in $[0, 1)$. In particular, if χ is differentiable,

$$\begin{aligned}\dot{\chi}_t &= \hat{\mu}_{1t} \chi_t + \hat{\mu}_{3t} \\ &= \alpha_{3t} \gamma_{1t} \underbrace{\Sigma [\chi_t \beta_{1t} - \alpha_{3t} \chi_t]}_{= \alpha_{3t} (1 - \chi_t) - \beta_{3t}} + \chi_t \frac{\nu \alpha_{3t} \gamma_{1t} + \gamma_{2t} \delta_{1t}}{\sigma_X^2} [-\nu \beta_{1t} - \delta_{1t}] \\ &\quad + \alpha_{3t} \gamma_{1t} \beta_{3t} \Sigma + \frac{\nu \alpha_{3t} \gamma_{1t} + \gamma_{2t} \delta_{1t}}{\sigma_X^2} [-\nu \beta_{3t}].\end{aligned} \quad (\text{B.7})$$

Using that $\alpha_{3t} = \beta_{3t} + \beta_{1t} \chi_t$, we obtain the following system of ODEs

$$\begin{aligned}\dot{\gamma}_{1t} &= -\gamma_{1t}^2 (\beta_{3t} + \beta_{1t} \chi_t)^2 \Sigma \\ \dot{\gamma}_{2t} &= -2\gamma_{2t} \gamma_{1t} (\beta_{3t} + \beta_{1t} \chi_t)^2 \Sigma + \gamma_{1t}^2 (\beta_{3t} + \beta_{1t} \chi_t)^2 \Sigma - \left(\frac{\nu \gamma_{1t} (\beta_{3t} + \beta_{1t} \chi_t) + \gamma_{2t} \delta_{1t}}{\sigma_X} \right)^2\end{aligned}$$

$$\dot{\chi}_t = \gamma_{1t}(\beta_{3t} + \beta_{1t}\chi_t)^2 \Sigma(1 - \chi_t) - (\nu[\beta_{3t} + \beta_{1t}\chi_t] + \delta_{1t}\chi_t) \left(\frac{\nu\gamma_{1t}(\beta_{3t} + \beta_{1t}\chi_t) + \gamma_{2t}\delta_{1t}}{\sigma_X^2} \right).$$

In the proof of the next lemma we establish that $\chi = \gamma_2/\gamma_1 \in [0, 1)$. After replacing $\nu = 0$ and $\gamma_2 = \chi\gamma$ in the third ODE, and writing γ for γ_1 , the first and third equations of the previous system correspond to (13)–(14) as desired. The representation $L_t = \mathbb{E}[\theta | \mathcal{F}_t^X]$ is proved in the Online Appendix. \square

Proof of Lemma 2. Consider the system $(\gamma_1, \gamma_2, \chi)$ from the proof of the previous lemma when $\nu = 0$ (in particular, Σ becomes $1/\sigma_Y^2$). Also, let $\delta_{1t} := \hat{u}_{\hat{a}\theta} + \hat{u}_{\hat{a}\hat{a}}\alpha_{3t}$.⁴⁷ The local existence of a solution follows from continuity of the associated operator. Suppose that the maximal interval of existence is $[0, \tilde{T})$, with $\tilde{T} \leq T$.

Since the system is locally Lipschitz continuous in $(\gamma_1, \gamma_2, \chi)$ uniformly in $t \in [0, T]$ for given continuous coefficients, its solution is unique over the same interval (Picard-Lindelöf). In particular, observe that $(\gamma_{1t}, \gamma_{2t}, \chi) = (\gamma^\circ, 0, 0)$ solves the system as long as $\beta_3 = 0$. Without loss of generality then, assume $\beta_{30} \neq 0$.

Observe that γ_1 is (weakly) decreasing over $[0, \tilde{T})$, so $\gamma_{1t} \leq \gamma^\circ$. Suppose there is a time at which γ_1 is strictly negative. Let $s < t$ be the first time γ_1 crosses zero, and notice that for $t > s$ close to s ,

$$0 > \gamma_{1t} = \int_s^t \dot{\gamma}_{1u} du = - \int_s^t \gamma_{1u}^2 [\beta_{3u} + \beta_{1u}\chi_u]^2 \Sigma ds \geq 0,$$

which is a contradiction. Thus, $\gamma_{1t} \in [0, \gamma^\circ]$ for all $t \in [0, \tilde{T})$. Moreover, if $\gamma_t > 0$, straightforward integration shows that

$$\gamma_{1t} = \frac{\gamma^\circ}{1 + \int_0^t [\beta_{3s} + \beta_{1s}\chi_s]^2 \Sigma ds}.$$

Since $\vec{\beta}$ is continuous over $[0, T]$, if γ ever vanishes in $[0, \tilde{T})$ we must have that χ diverges at such a point; by definition of \tilde{T} , however, that point must be \tilde{T} . Thus, $\gamma_{1t} > 0$ in $[0, \tilde{T})$ (regardless of whether χ diverges at \tilde{T} or not). Also, by continuity of $\vec{\beta}$ and the strict positivity of $(\beta_{30} + \beta_{10}\chi_0)^2 = \beta_{30}^2$, we get $\gamma_t < \gamma^\circ$ from the previous expression for γ_t .

We now show that $0 < \gamma_{2t} < \gamma_{1t}$ for $t > 0$. In fact, since $\gamma_{20} = 0$, $\gamma_{10} > 0$ and $\beta_{30} > 0$, we

⁴⁷All the results in this proof extend to a generic continuous function δ_1 over $[0, T]$ in which the explicit dependence on $\vec{\beta}$ and χ is not recognized, which happens when the myopic player becomes forward looking.

have $\gamma_{2t} > 0$ for $t \in (0, \epsilon)$ for small $\epsilon > 0$. Consider now $[\epsilon, \tilde{t}]$ with $\tilde{t} \in (\epsilon, \tilde{T})$. Then

$$f^{\gamma_2}(t, x) := -2x \frac{\gamma_{1t}(\beta_{3t} + \beta_{1t}\chi_{1t})^2}{\sigma_Y^2} + \frac{\gamma_{1t}^2(\beta_{3t} + \beta_{1t}\chi_{1t})^2}{\sigma_Y^2} - \left(\frac{x\delta_{1t}}{\sigma_X} \right)^2$$

is locally Lipschitz continuous with respect to x uniformly in $t \in [\epsilon, \tilde{t}]$. Since $0 - f^{\gamma_2}(t, 0) \leq 0 = \dot{\gamma}_{2t} - f^{\gamma_2}(t, \gamma_{2t})$ and $0 < \gamma_{2\epsilon}$, we obtain that $\gamma_{2t} > 0$ for all $t \in [0, \tilde{t}]$ by the comparison theorem, and hence over $(0, \tilde{T})$ as well.

Now, let $z_t := \gamma_{2t} - \gamma_{1t}$, $t < \tilde{T}$. Using the ODEs for γ_1 and γ_2 we deduce that

$$\dot{z}_t < -\frac{2\gamma_{1t}(\beta_{3t} + \beta_{1t}\chi_{1t})z_t}{\sigma_Y^2}, \quad z_0 = \gamma_{20} - \gamma_{10} = -\gamma^o < 0.$$

It is then easy to conclude that by Grönwall's inequality,

$$z_t < z_0 \exp\left(-\int_0^t \frac{2\gamma_{1s}(\beta_{3s} + \beta_{1s}\chi_{1s})}{\sigma_Y^2} ds\right) < 0, \quad t < \tilde{T},$$

as $\gamma_{1t}(\beta_{3t} + \beta_{1t}\chi_{1t})$ is continuous over $[0, \tilde{t}]$, $t < \tilde{T}$. Thus, $\gamma_{2t} < \gamma_{1t}$ for all $t \in [0, \tilde{T})$.

With this in hand, $\gamma_{2t}/\gamma_{1t} \in (0, 1)$ for all $t \in (0, \tilde{T})$, and $\gamma_{20}/\gamma_{10} = 0$. Moreover, it is easy to verify that the previous ratio solves the χ -ODE. By uniqueness, $\chi = \gamma_2/\gamma_1$. Replacing $\gamma_2 = \chi\gamma_1$ and $\nu = 0$ in the χ -ODE above yields (14), i.e.,

$$\dot{\chi}_t = \gamma_{1t}(\beta_{3t} + \beta_{1t}\chi_t)^2 \frac{(1 - \chi_t)}{\sigma_Y^2} - \gamma_{1t} \frac{(\delta_{1t}\chi_t)^2}{\sigma_X^2}, \quad t \in [0, \tilde{T}).$$

By the previous analysis, $(\gamma_1, \gamma_2, \chi)$ is bounded over $[0, \tilde{T})$. If $\tilde{T} < T$, the solution can be extended strictly beyond \tilde{T} thanks to the continuity of the associated operator, contradicting the definition of \tilde{T} . Thus, the only option is that $\tilde{T} = T$; since the solution remains bounded, the system admits a continuous extension to T .⁴⁸ By continuity, such an extension is unique, and the desired properties ($\chi = \gamma_2/\gamma_1$ stated in Lemma 1; χ solves (14); and $\chi \in (0, 1)$; and $\gamma^o \in (0, \gamma^o)$) hold up to T by the exact same arguments now applied over $[0, T]$ (as opposed to over strict compact subsets of $[0, \tilde{T})$).⁴⁹ \square

Proof of Lemma 3. The long-run player's problem is to choose an admissible $a := (a_t)_{t \in [0, T]}$

⁴⁸For a generic system $\dot{z}_t = f(t, z_t)$, if z is bounded over $[0, T)$ and f continuous, there exists K s.t. $|x_t - x_s| < K|t - s|$; but this implies that $(x_s)_{s \nearrow T}$ is Cauchy, and hence the limit exists. Having extended the solution to $[0, T) \cup \{T\}$, one can then further extend it to the right by applying Peano's theorem.

⁴⁹An alternative way of seeing that $\chi < 1$ is that $\dot{\chi} \leq \gamma_{1t}(\beta_{3t} + \beta_{1t}\chi_t)^2(1 - \chi_t)/\sigma_Y^2$, and so $\chi_t \leq 1 - \gamma_t/\gamma^o$ by the comparison theorem, as the latter function satisfies $\dot{z}_t = \gamma_{1t}(\beta_{3t} + \beta_{1t}z_t)^2(1 - z_t)/\sigma_Y^2$, $z_0 = 0$. This shows that the amplitude of the history-inference effect is maximized in the no-feedback case.

that maximizes

$$\mathcal{U}(a) := \mathbb{E}_0 \left[\int_0^T e^{-rt} U(a_t, \delta_{0t} + \delta_{1t} \hat{M}_t + \delta_{2t} L_t, \theta) dt \right]$$

where $(\hat{M}_t)_{t \geq 0}$ is given by (B.1) and $(L_t)_{t \geq 0}$ by (B.5). Using that the flow is quadratic, we obtain that

$$\mathcal{U}(a) = \mathbb{E}_0 \left[\int_0^T e^{-rt} U(a_t, \delta_{0t} + \delta_{1t} M_t^a + \delta_{2t} L_t, \theta) dt \right] + \frac{1}{2} \frac{\partial^2 U}{\partial \hat{a}^2} \mathbb{E}_0 \left[\int_0^T e^{-rt} \delta_{1t}^2 \mathbb{E}_t[(M_t^a - \hat{M}_t^a)^2] dt \right]$$

with $M_t^a := \mathbb{E}_t[\hat{M}_t^a]$, where we have made explicit the dependence of both processes on the strategy followed. By the proof of Lemma 1, $(M_t^a)_{t \in [0, T]}$ evolves as in (B.3), i.e.,

$$dM_t^a = (\mu_{0t} + \mu_{1t} a_t + \mu_{2t} M_t^a) dt + \frac{\sigma_X B_t^X + \gamma_{2t} \delta_{1t}}{\sigma_X} dZ_t^a$$

where $dZ_t^a := [dX_t - (\nu a_t + \delta_{0t} + \delta_{1t} M_t^a + \delta_{2t} L_t) dt] / \sigma_X$ is a Brownian motion from the long-run player's standpoint, $(\mu_0, \mu_1, \mu_2, B_t^X)$ are given by (B.2), and where γ_{2t} evolves as in (B.4). Moreover, from the same filtering equations (B.3)–(B.4) we know that $\mathbb{E}_t[(M_t^a - \hat{M}_t^a)^2]$ is independent of the strategy followed, and that it coincides with γ_{2t} , $t \in [0, T]$. Thus, the long-run player's problem reduces to

$$\max_{(a_t)_{t \geq 0} \text{ admissible}} \mathbb{E}_0 \left[\int_0^T e^{-rt} U(a_t, \delta_{0t} + \delta_{1t} M_t^a + \delta_{2t} L_t, \theta) dt \right]$$

where $(M_t^a)_{t \in [0, T]}$ is as above, and $(L_t)_{t \geq 0}$ is linear in the paths of X according to (B.5). In differential form, the latter process can be written as

$$dL_t = \frac{1}{1 - \chi_1} \left\{ L_t [\hat{\mu}_{1t} + \hat{\mu}_{2t} + \hat{\mu}_{3t}] + \hat{\mu}_{0t} + \hat{B}_t [\nu a_t + \delta_{0t} + \delta_{1t} M_t^a + \delta_{2t} L_t] \right\} dt + \frac{\sigma_X \hat{B}_t}{1 - \chi_1} dZ_t^a.$$

where we used that $dX_t = (\nu a_t + \delta_{0t} + \delta_{1t} M_t^a + \delta_{2t} L_t) dt + \sigma_X dZ_t^a$ from the long-run player's standpoint. (Refer to the proof of Lemma 1 for the expressions for $(\hat{\mu}_{0t}, \hat{\mu}_{1t}, \hat{\mu}_{2t}, \hat{\mu}_{3t}, \hat{B}_t^X)$.)

So far, we have fixed an admissible strategy $(a_t)_{t \in [0, T]}$ (in the sense of Section 3) for the long-run player, and then obtained processes M^a and Z^a that potentially depend on that choice. The above problem thus differs from traditional control problems with perfectly observed states in that the Brownian motion is, in principle, affected by the choice of strategy.

With linear dynamics, however, the *separation principle* (e.g., Liptser and Shiryaev, 1977, Chapter 16), applies. In fact, the solution to the long-run player's problem can be found by first fixing a Brownian motion, say, $Z_t := Z_t^0$ (i.e., Z_t^a when $a \equiv 0$), and then solving the optimization problem that replaces Z^a by Z in the laws of motion of \hat{M}^a and L . The

method works to the extent that $Z^a \equiv Z$ for all $(a_t)_{t \geq 0}$: it is easy to conclude from (B.1) and (B.3) that the process $\hat{M}_t^a - M_t^a$ is independent of the strategy followed, and hence so is Z_t^a , given that $\sigma_X dZ_t^a = dX_t - (\nu a_t + \delta_{0t} + \delta_{1t} M_t^a + \delta_{2t} L_t) dt = \delta_{1t} (\hat{M}_t^a - M_t^a) dt + \sigma_X dZ_t^X$ under the true data-generating process, thanks to the linearity of the dynamics. In this procedure, therefore, one filters as a first step, and then optimizes afterwards using the posterior mean as a controlled state.⁵⁰

Returning to the $\nu = 0$ case, we can then insert Z_t in the dynamic of M_t^a . Omitting the dependence of the resulting process on a (as any control problem does), it is easy to see that

$$dM_t = \frac{\gamma_t \alpha_{3t}}{\sigma_Y^2} (a_t - [\alpha_{0t} + \alpha_{2t} L_t + \alpha_{3t} M_t]) dt + \frac{\chi_t \gamma_t \delta_{1t}}{\sigma_X} dZ_t.$$

As for the expression for L (display (17)), this one follows from (15) using that $dX_t = (\delta_{0t} + \delta_{2t} L_t + \delta_{1t} M_t) dt + \sigma_X dZ_t$ from the long-run player's perspective. In fact, it is easy to see from (B.6) that

$$l_{0t} + B_t \delta_{0t} + (l_{1t} + B_t \delta_{1t}) L_t + B_t \delta_{1t} M_t = \frac{\gamma_t \chi_t \delta_{1t}}{\sigma_X^2 (1 - \chi_t)} (M_t - L_t).$$

This concludes the proof. □

Proof of Lemma 4. Suppose $\delta_1 = \hat{u}_{a\hat{a}} \alpha_3$, $\hat{u}_{a\hat{a}} \neq 0$. The χ -ODE for $\nu \in [0, 1]$ boils down to

$$\dot{\chi}_{1t} = \gamma_{1t} \alpha_{3t}^2 \left(\left[\frac{\nu^2}{\sigma_X^2} + \frac{1}{\sigma_Y^2} \right] (1 - \chi_{1t}) - \frac{(\nu + \hat{u}_{a\hat{a}} \chi_t)^2}{\sigma_X^2} \right) =: -\gamma_{1t} \alpha_{3t}^2 Q(\chi_t).$$

The goal is to find a function $f : [0, \bar{\chi}] \rightarrow [0, \gamma^o]$, some $\bar{\chi} \in (0, 1)$, such that $f(\chi_t) = \gamma_t$ for all $t \geq 0$. When this is the case, and such f is differentiable, $f'(\chi_t) \dot{\chi}_t = \dot{\gamma}_t$. Thus, if $\alpha_{3t} > 0$,

$$\frac{f'(\chi_t)}{f(\chi_t)} = \frac{\Sigma}{Q(\chi_t)}.$$

⁵⁰Relative to Chapter 16 in [Liptser and Shiryaev \(1977\)](#), our problem is more general in that it allows for a linear component in the flow, and the public signal can be controlled (when $\nu \neq 0$). The first generalization is clearly innocuous. As for the second, the key behind the separation principle is that the innovations $dX_t - \mathbb{E}_t[dX_t]$ are independent of the strategy followed, which also happens when $\nu \neq 0$. Given any admissible strategy $(a_t)_{t \geq 0}$, therefore, the fact that the filtrations of Z , Z^a and X^a satisfy $\mathcal{F}_t^Z = \mathcal{F}_t^{Z^a} \subseteq \mathcal{F}_t^{X^a}$, $t \geq 0$, means the optimal control found by using Z is weakly better than any such $(a_t)_{t \geq 0}$. See p.183 in section 16.1.4 in [Liptser and Shiryaev \(1977\)](#) for more details in a context of a quadratic regulator problem.

Thus, we aim to solve the ODE

$$\frac{f'(\chi)}{f(\chi)} = \frac{\Sigma}{Q(\chi)}, \quad \chi \in (0, \bar{\chi}), \quad \text{and } f(0) = \gamma^o,$$

over some domain $[0, \bar{\chi})$, with the property that $f(\chi) > 0$ if $\chi > 0$.

To this end, let

$$c_2 = \frac{\sqrt{b^2 + 4(\hat{u}_{a\hat{a}})^2/[\sigma_X\sigma_Y]^2} - b}{2(\hat{u}_{a\hat{a}}/\sigma_X)^2} \quad \text{and} \quad -c_1 = \frac{-\sqrt{b^2 + 4(\hat{u}_{a\hat{a}})^2/[\sigma_X\sigma_Y]^2} - b}{2(\hat{u}_{a\hat{a}}/\sigma_X)^2},$$

where $b := [\nu^2/\sigma_X^2 + 1/\sigma_Y^2] + 2\nu\hat{u}_{a\hat{a}}/\sigma_X^2$, be the roots of the quadratic

$$Q(\chi) = \left(\frac{\hat{u}_{a\hat{a}}}{\sigma_X}\right)^2 \chi^2 + \chi \left(\left[\frac{\nu^2}{\sigma_X^2} + \frac{1}{\sigma_Y^2} \right] + \frac{2\nu\hat{u}_{a\hat{a}}}{\sigma_X^2} \right) - \frac{1}{\sigma_Y^2}.$$

Clearly, $-c_1 < 0 < c_2$. Also, it is easy to verify that $c_2 < 1$.⁵¹ Thus,

$$\frac{\Sigma}{Q(\chi)} = -\frac{\sigma_X^2 \Sigma}{(\hat{u}_{a\hat{a}})^2 (c_1 + c_2)} \left[\frac{1}{\chi + c_1} - \frac{1}{\chi - c_2} \right]$$

is well-defined (and negative) over $[0, c_2)$ with $1/(\chi + c_1) > 0$ and $-1/(\chi - c_2) > 0$ over the same domain. We can then set $\bar{\chi} = c_2$ and solve

$$\int_0^\chi \frac{f'(s)}{f(s)} ds = -\frac{\sigma_X^2 \Sigma}{(\hat{u}_{a\hat{a}})^2 (c_1 + c_2)} \log \left(\frac{\chi + c_1 c_2}{c_2 - \chi c_1} \right) \Rightarrow f(\chi) = f(0) \left(\frac{c_1}{c_2} \right)^{1/d} \left(\frac{c_2 - \chi}{\chi + c_1} \right)^{1/d}$$

where $1/d = \sigma_X^2 \Sigma / [(\hat{u}_{a\hat{a}})^2 (c_1 + c_2)] > 0$. We then impose $f(0) = \gamma^o$, thus obtaining a strictly positive and decreasing function that has the initial condition we look for. Moreover, letting $\gamma = f(\chi)$, its inverse is decreasing and given by

$$\chi(\gamma) = f^{-1}(\gamma) = c_1 c_2 \frac{1 - (\gamma/\gamma^o)^d}{c_1 + c_2 (\gamma/\gamma^o)^d}.$$

When $\gamma_1 = \gamma^o$, we have that $\chi = 0$, whereas when $\gamma = 0$, it follows that $\chi = c_2$ as desired.

We verify that this candidate satisfies the χ -ODE; we do this for the $\nu = 0$ case only. To this end, it is easy to verify that

$$\frac{d(\chi(\gamma_t))}{dt} = \frac{\alpha_{3t}^2 \gamma_t}{\sigma_Y^2 [c_1 + c_2 (\gamma/\gamma^o)^d]^2} c_1 c_2 d [c_1 + c_2] \left(\frac{\gamma_t}{\gamma^o} \right)^d.$$

⁵¹This follows from squaring both sides of $\sqrt{b^2 + 4(\hat{u}_{a\hat{a}})^2/[\sigma_X\sigma_Y]^2} < b + 2(\hat{u}_{a\hat{a}})^2/\sigma_X^2$ using that $b + 2(\hat{u}_{a\hat{a}})^2/\sigma_X^2 > 0$ and $b = [\nu^2/\sigma_X^2 + 1/\sigma_Y^2] + 2\nu\hat{u}_{a\hat{a}}/\sigma_X^2$.

By construction, moreover,

$$c_1 c_2 = c_1 - c_2 = \frac{\sigma_X^2}{\sigma_Y^2 (\hat{u}_{a\hat{a}})^2}$$

which follows from equating the first- and zero-order coefficients in $Q(\chi) = \hat{u}_{a\hat{a}}^2 \chi^2 / \sigma_X^2 + \chi / \sigma_Y^2 - 1 / \sigma_Y^2 = \hat{u}_{a\hat{a}}^2 (\chi - c_2)(\chi + c_1) / \sigma_X^2$. Thus, $dc_1 c_2 = c_1 + c_2$.

On the other hand,

$$\frac{[\hat{u}_{a\hat{a}} \chi(\gamma)]^2}{\sigma_X^2} = \frac{\hat{u}_{a\hat{a}}^2}{\sigma_X^2} \left[c_1 c_2 \frac{1 - (\gamma/\gamma^o)^d}{c_1 + c_2 (\gamma/\gamma^o)^d} \right]^2 = \frac{c_1^2 (1 - c_2)}{\sigma_Y^2} \left[\frac{1 - (\gamma/\gamma^o)^d}{c_1 + c_2 (\gamma/\gamma^o)^d} \right]^2$$

where we used that $c_1^2 c_2^2 / \sigma_X^2 = c_1^2 (1 - c_2) / \sigma_Y^2$ follows from $\hat{u}_{a\hat{a}}^2 c_2^2 / \sigma_X^2 = (1 - c_2) / \sigma_Y^2$ by definition of c_2 . Thus, the right-hand side of the χ -ODE evaluated at our candidate $\chi(\gamma)$ satisfies

$$\gamma_1 \alpha_3^2 \left(\frac{1 - \chi}{\sigma_Y^2} - \frac{(\hat{u}_{a\hat{a}} \chi)^2}{\sigma_X^2} \right) \Big|_{\chi=\chi(\gamma)} = \frac{\alpha_3^2 \gamma_1}{\sigma_Y^2} \left(1 - \chi - c_1^2 (1 - c_2) \left[\frac{1 - (\gamma/\gamma^o)^d}{c_1 + c_2 (\gamma/\gamma^o)^d} \right]^2 \right).$$

Thus, using that $c_1 c_2 d = c_1 + c_2$ in our expression for $d(\chi(\gamma_t))/dt$, it suffices to show that

$$[c_1 + c_2]^2 \left(\frac{\gamma_t}{\gamma^o} \right)^d = (1 - \chi) [c_1 + c_2 (\gamma/\gamma^o)^d]^2 - c_1^2 (1 - c_2) [1 - (\gamma/\gamma^o)^d]^2.$$

Using that $\chi [c_1 + c_2 (\gamma/\gamma^o)^d] = 1 - (\gamma/\gamma^o)$, it is easy to conclude that this equality reduces to three equations

$$\begin{aligned} 0 &= c_1^2 - c_1^2 c_2 - c_1^2 + c_1^2 c_2 \\ (c_1 + c_2)^2 &= 2c_1 c_2 - c_1 c_2 (c_2 - c_1) + 2c_1^2 (1 - c_2) \\ 0 &= c_2^2 + c_1 c_2^2 - c_1^2 (1 - c_2) \end{aligned}$$

capturing the conditions on the constant, $(\gamma/\gamma^o)^d$ and $(\gamma/\gamma^o)^{2d}$, respectively. The first condition is trivially satisfied. As for the third, by the definition of c_1 and c_2 we have that $c_2^2 / (1 - c_2) = \sigma_X^2 / (\hat{u}_{a\hat{a}} \sigma_Y)^2 = c_1^2 / (1 + c_1)$. Thus, $c_1^2 (1 - c_2) = c_2^2 (1 + c_1)$, and the result follows. For the second, use that $c_1 c_2 (c_2 - c_1) = -(c_1 - c_2)^2$ and that $c_1^2 (1 - c_2) = c_2^2 (1 + c_1)$ to conclude

$$2c_1 c_2 - c_1 c_2 (c_2 - c_1) + 2c_1^2 (1 - c_2) = c_1^2 + c_2^2 + 2c_2^2 (1 + c_1) = c_1^2 + c_2^2 + 2c_2^2 + 2c_2 \underbrace{c_2 c_1}_{=c_1 - c_2} = (c_1 + c_2)^2.$$

Thus, $\chi(\gamma)$ as postulated satisfies the χ -ODE. We then conclude by uniqueness of any such solution.

Finally, when $\hat{u}_{a\hat{a}} = 0$, we have that $\delta_1 \equiv 0$, and the χ -ODE reduces to $\dot{\chi} = \alpha_{3t}^2 \gamma_t (1 - \chi_t) / \sigma_Y^2$, $\chi_0 = 0$. It is then easy to verify that $\chi(\gamma) = 1 - \gamma_t / \gamma^\circ$ satisfy the ODE, and hence we conclude using the same uniqueness argument. \square

B.1: Proof of Theorem 1

Our main task is to prove that a solution to the BVP exists, and any solution has the stated properties; from there we will establish the rest of the solution. Recall that $\alpha_t = \alpha_{3t} = \beta_{1t}\chi_t + \beta_{3t}$. We begin by reversing time and posing the associated IVP, together with the α -ODE, parameterized by an initial guess $\gamma_0 = \gamma^F \in [0, \gamma^\circ]$:

$$\dot{v}_{6t} = -\beta_{2t}^2 - 2\beta_{1t}\beta_{2t}(1 - \chi_t) + \beta_{1t}^2(1 - \chi_t)^2 - \frac{2v_{6t}\alpha_t^2\gamma_t\chi_t}{\sigma_X^2(1 - \chi_t)} \quad (\text{B.8})$$

$$\dot{v}_{8t} = 2\beta_{2t} + 2(1 - 2\alpha_t)\beta_{1t}(1 - \chi_t) + 4\beta_{1t}^2\chi_t(1 - \chi_t) - \frac{v_{8t}\alpha_t^2\gamma_t\chi_t}{\sigma_X^2(1 - \chi_t)} \quad (\text{B.9})$$

$$\begin{aligned} \dot{\beta}_{1t} = & \frac{\alpha_t\gamma_t}{2\sigma_X^2\sigma_Y^2(1 - \chi_t)} \left\{ -2\sigma_X^2(\alpha_t - \beta_{1t})\beta_{1t}(1 - \chi_t) \right. \\ & \left. + 2\sigma_Y^2\alpha_t\chi_t(\beta_{2t}[1 + 2\beta_{1t}\chi_t] - \beta_{1t}[1 - \chi_t]) + \alpha_t^2\beta_{1t}\gamma_t\chi_t v_{8t} \right\} \end{aligned} \quad (\text{B.10})$$

$$\begin{aligned} \dot{\beta}_{2t} = & \frac{\alpha_t\gamma_t}{2\sigma_X^2\sigma_Y^2(1 - \chi_t)} \left\{ -2\sigma_X^2\beta_{1t}^2(1 - \chi_t)^2 - 2\sigma_Y^2\alpha_t\beta_{2t}\chi_t^2(1 - 2\beta_{2t}) \right. \\ & \left. + \alpha_t^2\gamma_t\chi_t(2v_{6t} + \beta_{2t}v_{8t}) \right\} \end{aligned} \quad (\text{B.11})$$

$$\begin{aligned} \dot{\beta}_{3t} = & \frac{\alpha_t\gamma_t}{2\sigma_X^2\sigma_Y^2(1 - \chi_t)} \left\{ 2\sigma_X^2\beta_{1t}(1 - \chi_t)\beta_{3t} - 2\sigma_Y^2\alpha_t\beta_{2t}\chi_t^2(1 - 2\beta_{3t}) \right. \\ & \left. + \alpha_t^2\beta_{3t}\gamma_t\chi_t v_{8t} \right\} \end{aligned} \quad (\text{B.12})$$

$$\dot{\gamma}_t = \frac{\gamma_t^2\alpha_t^2}{\sigma_Y^2} \quad (\text{B.13})$$

$$\dot{\alpha}_t = \frac{\alpha_t^3\gamma_t\chi_t}{2\sigma_X^2\sigma_Y^2(1 - \chi_t)} \left\{ 4\sigma_Y^2\beta_{2t}\chi_t + \alpha_t\gamma_t v_{8t} \right\} \quad (\text{B.14})$$

with initial conditions $v_{60} = v_{80} = 0$, $\beta_{10} = \frac{1}{2(2-\chi_0)}$, $\beta_{20} = \frac{1-\chi_0}{2(2-\chi_0)}$, $\beta_{30} = \frac{1}{2}$ and $\gamma_0 = \gamma^F$, where $\chi_t = \chi(\gamma_t)$ using the function $\chi : \mathbb{R}_+ \rightarrow (-\infty, c_2)$ as defined in Lemma 4. Hereafter, we use the notation $\bar{\chi} := c_2$. Recall also that $c_2 < 1$, and note by inspection that $\chi(\gamma) \searrow -c_1 < 0$ as $\gamma \rightarrow +\infty$, and since χ is decreasing, it has range $(-c_1, \bar{\chi})$. The right hand sides of the equations above are of class C^1 over the domain $\{(v_6, v_8, \beta_1, \beta_2, \beta_3, \gamma) \in \mathbb{R}^5 \times \mathbb{R}_+\}$, and hence, if a solution exists over $[0, T]$, it is unique. To solve the BVP, our goal is to show that there exists $\gamma^F \in (0, \gamma^\circ)$ such that a (unique) solution to the IVP above exists and it satisfies $\gamma_T(\gamma^F) = \gamma^\circ$.

Note that if $\gamma^F = 0$, then the IVP has the following (unique) solution: for all $t \in [0, T]$,

$\chi_t = \bar{\chi}$, $\beta_{1t} = \frac{1}{2(2-\bar{\chi})}$, $\beta_{2t} = \frac{1-\bar{\chi}}{2(2-\bar{\chi})}$ and $\beta_{3t} = \frac{1}{2}$ (so that $\alpha_t = \frac{1}{2-\bar{\chi}}$) and $\gamma_t = 0$, with v_6 and v_8 obtained from integration of their ODEs. (Clearly, this does not correspond to a solution to the BVP, since $\gamma_T(0) = 0 < \gamma^o$.)

We now consider $\gamma^F > 0$. Given the C^1 property mentioned above, and the existence of a solution to the IVP for $\gamma^F = 0$, there exists $\epsilon > 0$ such that a (unique) solution to the IVP exists over $[0, T]$ for each $\gamma^F \in (0, \epsilon)$ (see Theorem on page 397 in [Hirsch et al. \(2004\)](#)).

Observe that for $\gamma^F > 0$, we can change variables using $\tilde{v}_i := \gamma v_i$ for $i = 6, 8$ and create a new IVP in $(\tilde{v}_6, \tilde{v}_8, \beta_1, \beta_3, \beta_3, \gamma)$. We label this System 1, and it consists of (B.10)-(B.13) (replacing all instances of γv_i with \tilde{v}_i , $i = 6, 8$) together with

$$\dot{\tilde{v}}_{6t} = \gamma_t \left\{ -\beta_{2t}^2 - 2\beta_{1t}\beta_{2t}(1 - \chi_t) + \beta_{1t}^2(1 - \chi_t)^2 + \tilde{v}_{6t}\alpha_t^2 \left[\frac{1}{\sigma_Y^2} - \frac{2\chi_t}{\sigma_X^2(1 - \chi_t)} \right] \right\} \quad (\text{B.15})$$

$$\dot{\tilde{v}}_{8t} = \gamma_t \left\{ 2\beta_{2t} + 2(1 - 2\alpha_t)\beta_{1t}(1 - \chi_t) + 4\beta_{1t}^2\chi_t(1 - \chi_t) + \tilde{v}_{8t}\alpha_t^2 \left[\frac{1}{\sigma_Y^2} - \frac{\chi_t}{\sigma_X^2(1 - \chi_t)} \right] \right\} \quad (\text{B.16})$$

subject to $\tilde{v}_{60} = \tilde{v}_{80} = 0$ and the remaining initial conditions above. We will argue later that when this system has a solution over $[0, T]$, $v_6 = \tilde{v}_6/\gamma$ and $v_8 = \tilde{v}_8/\gamma$ are well-defined, and hence a solution to the original IVP can be recovered.

System 1 has the property that in its solution, \tilde{v}_6 and β_2 can be expressed directly as functions of the other variables. In anticipation of this property (which we will soon verify), it is convenient to work with a reduced IVP in $(\tilde{v}_8, \beta_1, \beta_3, \gamma)$, which we call System 2:

$$\dot{\tilde{v}}_{8t} = \gamma_t \left\{ 2[1 - \beta_{1t} - \beta_{3t}] + 2(1 - 2\alpha_t)\beta_{1t}(1 - \chi_t) + 4\beta_{1t}^2\chi_t(1 - \chi_t) + \tilde{v}_{8t}\alpha_t^2 \left[1/\sigma_Y^2 - \chi_t/(\sigma_X^2[1 - \chi_t]) \right] \right\} \quad (\text{B.17})$$

$$\dot{\beta}_{1t} = \frac{\alpha_t\gamma_t}{2\sigma_X^2\sigma_Y^2(1 - \chi_t)} \left\{ -2\sigma_X^2(\alpha_t - \beta_{1t})\beta_{1t}(1 - \chi_t) + 2\sigma_Y^2\alpha_t\chi_t([1 - \beta_{1t} - \beta_{3t}][1 + 2\beta_{1t}\chi_t] - \beta_{1t}[1 - \chi_t]) + \alpha_t^2\beta_{1t}\chi_t\tilde{v}_{8t} \right\} \quad (\text{B.18})$$

$$\dot{\beta}_{3t} = \frac{\alpha_t\gamma_t}{2\sigma_X^2\sigma_Y^2(1 - \chi_t)} \left\{ 2\sigma_X^2\beta_{1t}(1 - \chi_t)\beta_{3t} - 2\sigma_Y^2\alpha_t[1 - \beta_{1t} - \beta_{3t}]\chi_t^2(1 - 2\beta_{3t}) + \alpha_t^2\beta_{3t}\chi_t\tilde{v}_{8t} \right\} \quad (\text{B.19})$$

and (B.13), subject to the initial conditions $\tilde{v}_{80} = 0$, $\beta_{10} = \frac{1}{2(2-\chi(\gamma^F))}$, $\beta_{30} = \frac{1}{2}$ and $\gamma_0 = \gamma^F$. Observe that in this system, we have substituted $1 - \beta_1 - \beta_3$ for all instances of β_2 in the original ODEs. We will show that there exists a $\gamma^F \in (0, \gamma^o)$ such that a (unique) solution to System 2 exists and satisfies $\gamma_T(\gamma^F) = \gamma^o$, and then we will show that given this solution, we can construct the solutions to (B.11) (which will be $1 - \beta_1 - \beta_3$) and (B.15) directly, solving System 1 (and hence the BVP).

Based on System 2, define

$$\bar{\gamma} := \sup\{\gamma^F > 0 \mid \text{a solution to System 2 with } \gamma_0 = \gamma^F \text{ exists over } [0, T]\},$$

with respect to set inclusion. Since the RHS of the equations that comprise System 2 are of class C^1 , the solution is unique when it exists, and there is continuous dependence of the solution on the initial conditions; in particular, the terminal value γ_T is continuous in γ^F (see Theorem on page 397 in [Hirsch et al. \(2004\)](#)). Hence if there exists $\gamma^F \in (0, \bar{\gamma})$ such that $\gamma_T(\gamma^F) \geq \gamma^o$, by the intermediate value theorem there exists a $\gamma^F \in (0, \bar{\gamma})$ such that $\gamma_T(\gamma^F) = \gamma^o$, allowing us to construct a solution to System 1. We rule out the alternative case by contradiction, and then we return to the case just described to complete the construction.

Suppose by way of contradiction that for all $\gamma^F \in (0, \bar{\gamma})$, $\gamma_T(\gamma^F) < \gamma^o$. In particular, because γ_t is nondecreasing in the backward system for any initial condition, we have that $\gamma_t \in (0, \gamma^o)$ and by Lemma 4, $\chi_t \in (0, \bar{\chi})$ for all $t \in [0, T]$ when $\gamma^F \in (0, \bar{\gamma})$.

To reach a contradiction, it suffices to show that the solution to System 2 can be bounded uniformly over $\gamma^F \in (0, \bar{\gamma})$, as this would imply that the solution can be extended strictly to the right of $\bar{\gamma}$ in this case, violating the definition of $\bar{\gamma}$.

To establish uniform bounds, we decompose β_1 and β_3 as sums of forward-looking and myopic components and show that both of these components are uniformly bounded; we bound \tilde{v}_8 without such a decomposition. Specifically, define $\beta_{1t}^m := \frac{1}{2(2-\chi_t)}$, $\beta_{3t}^m = \frac{1}{2}$ and $\beta_{it}^f := \beta_{it} - \beta_{it}^m$ for all $t \in [0, T]$, $i = 1, 3$. Observe that for $\gamma^F \in (0, \bar{\gamma})$, β_{1t}^m is uniformly bounded by $[1/4, 1/2] \subset [0, 1]$, as $\chi_t = \chi(\gamma_t) \in [0, 1]$ for all $t \in [0, T]$, and trivially $\beta_{3t}^m \in [0, 1]$. Hence, we define one final system, System 3, in $(\tilde{v}_8, \beta_1^f, \beta_3^f, \gamma)$ to be (B.13), (B.17), and

$$\begin{aligned} \dot{\beta}_{1t}^f &= -\frac{\dot{\chi}_t}{2(2-\chi_t)^2} + \frac{\alpha_t \gamma_t}{2\sigma_X^2 \sigma_Y^2 (1-\chi_t)} \left\{ -2\sigma_X^2 (\alpha_t - [\beta_{1t}^f + \beta_{1t}^m]) [\beta_{1t}^f + \beta_{1t}^m] (1-\chi_t) \right. \\ &\quad + 2\sigma_Y^2 \alpha_t \chi_t \left([1 - (\beta_{1t}^f + \beta_{1t}^m) - (\beta_{3t}^f + \beta_{3t}^m)] [1 + 2(\beta_{1t}^f + \beta_{1t}^m) \chi_t] \right) \\ &\quad \left. - 2\sigma_Y^2 \alpha_t \chi_t (\beta_{1t}^f + \beta_{1t}^m) (1-\chi_t) + \alpha_t^2 (\beta_{1t}^f + \beta_{1t}^m) \chi_t \tilde{v}_{8t} \right\}, \end{aligned} \quad (\text{B.20})$$

$$=: h^{\beta_1^f}(\beta_{1t}^f, \beta_{1t}^m, \beta_{3t}^f, \beta_{3t}^m, \tilde{v}_{8t}, \gamma_t)$$

$$\begin{aligned} \dot{\beta}_{3t}^f &= \frac{\alpha_t \gamma_t}{2\sigma_X^2 \sigma_Y^2 (1-\chi_t)} \left\{ 2\sigma_X^2 (\beta_{1t}^f + \beta_{1t}^m) (1-\chi_t) (\beta_{3t}^f + \beta_{3t}^m) + \alpha_t^2 (\beta_{3t}^f + \beta_{3t}^m) \chi_t \tilde{v}_{8t} \right. \\ &\quad \left. + 4\sigma_Y^2 \alpha_t \beta_{3t}^f \chi_t^2 [1 - (\beta_{1t}^f + \beta_{1t}^m) - (\beta_{3t}^f + \beta_{3t}^m)] \right\} \end{aligned} \quad (\text{B.21})$$

$$=: h^{\beta_3^f}(\beta_{1t}^f, \beta_{1t}^m, \beta_{3t}^f, \beta_{3t}^m, \tilde{v}_{8t}, \gamma_t)$$

subject to initial conditions $\tilde{v}_{80} = 0$, $\beta_{10}^f = 0$, $\beta_{30}^f = 0$ and $\gamma_0 = \gamma^F \in (0, \bar{\gamma})$, where $\alpha_t = [\beta_{3t}^f + \beta_{3t}^m] + [\beta_{1t}^f + \beta_{1t}^m] \chi_t$. Define $h^{\tilde{v}_8}(\beta_{1t}^f, \beta_{1t}^m, \beta_{3t}^f, \beta_{3t}^m, \tilde{v}_{8t}, \gamma_t)$ as the RHS of (B.17) with $\beta_{it}^f + \beta_{it}^m$

substituted for β_{it} , $i = 1, 3$.

Given that β_1^m and β_2^m are uniformly bounded, it suffices to show that the solutions $(\tilde{v}_8, \beta_1^f, \beta_3^f)$ are uniformly bounded by some $[-K, K]^3$. (Recall that γ is already bounded by $[0, \gamma^o]$.)

Define $\bar{\alpha} = (K + 1)\bar{\chi} + (K + 1)$, where we suppress dependence on K . Next, for $x \in \{\tilde{v}_8, \beta_1^f, \beta_3^f\}$ define $\bar{h}^x : \mathbb{R}_{++}^2 \rightarrow \mathbb{R}_{++}$ as follows:

$$\begin{aligned} \bar{h}^{\tilde{v}_8}(\gamma^o; K) &:= \gamma^o \{2[1 + 2(K + 1)] + 2(1 + 2\bar{\alpha})(K + 1) \\ &\quad + 4(K + 1)^2\bar{\chi} + K\bar{\alpha}^2 [1/\sigma_Y^2 + \bar{\chi}/(\sigma_X^2[1 - \bar{\chi}])]\} \end{aligned} \quad (\text{B.22})$$

$$\begin{aligned} \bar{h}^{\beta_1^f}(\gamma^o; K) &:= \frac{\bar{\alpha}^2\gamma^o [1/\sigma_Y^2 + \bar{\chi}/(\sigma_X^2[1 - \bar{\chi}])]}{2(2 - \bar{\chi})^2} + \frac{\bar{\alpha}\gamma^o}{2\sigma_X^2\sigma_Y^2(1 - \bar{\chi})} \{2\sigma_X^2[\bar{\alpha} + K + 1](K + 1) \\ &\quad + 2\sigma_Y^2\bar{\alpha}\bar{\chi}([1 + 2(K + 1)][1 + 2(K + 1)\bar{\chi}] + K + 1) + \bar{\alpha}^2K(K + 1)\bar{\chi}\} \end{aligned} \quad (\text{B.23})$$

$$\begin{aligned} \bar{h}^{\beta_3^f}(\gamma^o; K) &:= \frac{\bar{\alpha}\gamma^o}{2\sigma_X^2\sigma_Y^2(1 - \bar{\chi})} \{2\sigma_X^2(K + 1)^2 + \bar{\alpha}^2\bar{\chi}K(K + 1) \\ &\quad + 4\sigma_Y^2\bar{\alpha}\bar{\chi}^2K[1 + 2(K + 1)]\}, \end{aligned} \quad (\text{B.24})$$

Define

$$T(\gamma^o) := \max_{K' > 0} \min_{x \in \{\tilde{v}_8, \beta_1^f, \beta_3^f\}} \frac{K'}{\bar{h}^x(\gamma^o; K')},$$

and let K denote the arg max.⁵² We now show that given $T < T(\gamma^o)$, $(\tilde{v}_8, \beta_1^f, \beta_3^f)$ are uniformly bounded by $[-K, K]^3$. Suppose otherwise, and define $\tau = \inf\{t > 0 : (\tilde{v}_{8t}, \beta_{1t}^f, \beta_{3t}^f) \notin [-K, K]^3\}$; by supposition and continuity of the solutions, $\tau \in (0, T)$ and $|x_\tau| = K$, some $x \in \{\tilde{v}_8, \beta_1^f, \beta_3^f\}$. Now by construction of the $\bar{h}^x(\gamma^o; K)$, for all $t \in [0, \tau]$ and for each $x \in \{\tilde{v}_8, \beta_1^f, \beta_3^f\}$ we have

$$|\dot{x}_t| = |h^x(\beta_{1t}^f, \beta_{1t}^m, \beta_{3t}^f, \beta_{3t}^m, \tilde{v}_{8t}, \gamma_t)| < \bar{h}^x(\gamma^o; K)$$

and thus by the triangle inequality,

$$|x_\tau| < 0 + \tau \cdot \bar{h}^x(\gamma^o; K) < T(\gamma^o)\bar{h}^x(\gamma^o; K) \leq K,$$

a contradiction. We conclude that the solutions $(\tilde{v}_8, \beta_1^f, \beta_3^f, \gamma)$ to System 3 are uniformly bounded by $[-K, K]^3 \times [0, \gamma^o]$, and by another application of the triangle inequality the solutions $(\tilde{v}_8, \beta_1, \beta_3, \gamma)$ to System 2 are uniformly bounded by $[-K, K] \times [-K, K + 1]^2 \times [0, \gamma^o]$. This gives us the desired contradiction on the definition of $\bar{\gamma}$ from before, so we conclude

⁵²Note that $T(\gamma^o), K < \infty$ as the \bar{h}^x grow faster than linearly in K .

that for $T < T(\gamma^\circ)$, there exists $\gamma^F \in (0, \bar{\gamma})$ such that the solution to System 2 satisfies $\gamma_T(\gamma^F) = \gamma^\circ$. (Note that any such γ^F lies in $(0, \gamma^\circ)$, as γ is nondecreasing in the backward system.)

For the remainder of the proof, assume $T < T(\gamma^\circ)$ and consider any such γ^F as above and its induced solution $(\tilde{v}_8, \beta_1, \beta_3, \gamma)$ to System 2 with $\gamma_T(\gamma^F) = \gamma^\circ$.

We claim that $\alpha > 0$ over $[0, T]$. First, we have $\alpha_0 > 0$, as $\alpha_0 = \frac{1}{2 - \chi(\gamma^F)}$, and by Lemma 4, $0 \leq \chi(\gamma^F) < \bar{\chi} < 1$. Now the RHS of (B.14) is locally Lipschitz continuous in α , uniformly in t , as the remaining coefficients appearing in that ODE are bounded (being continuous functions of time over the compact set $[0, T]$). By standard application of the comparison theorem, we have $\alpha_t > 0$ for all $t \in [0, T]$.

Hence, we can define

$$\tilde{v}_{6t}^{cand} := \frac{\sigma_Y^2[-1 + 2\beta_{1t}(1 - \chi_t) + \alpha_t]}{\alpha_t} - \frac{\tilde{v}_{8t}}{2}, \quad \beta_{2t}^{cand} := 1 - \beta_{1t} - \beta_{3t}.$$

Observe that in System 2, (B.17)-(B.19) was obtained by replacing β_{2t} with β_{2t}^{cand} in (B.16), (B.10) and (B.12), so $(\tilde{v}_8, \beta_1, \beta_2^{cand}, \beta_3, \gamma)$ solves (B.16), (B.10), (B.12) and (B.13). It is tedious but straightforward to verify that \tilde{v}_{6t}^{cand} and β_{2t}^{cand} solve (B.15) and (B.11), respectively. Now the RHS of the system (B.15) with (B.11), given the solutions $(\tilde{v}_8, \beta_1, \beta_3, \gamma)$, is locally Lipschitz continuous (\tilde{v}_6, β_2) , uniformly in t , and thus $(\tilde{v}_6^{cand}, \beta_2^{cand})$ are the *unique* solutions to (B.15) and (B.11) given the other variables. To summarize, $(\tilde{v}_6^{cand}, \tilde{v}_8, \beta_1, \beta_2^{cand}, \beta_3, \gamma)$ solves System 1.

We have $\gamma_t \geq \gamma^F > 0$ for all $t \in [0, T]$. Thus, we can recover $v_6 = \tilde{v}_6/\gamma$ and $v_8 = \tilde{v}_8/\gamma$ as the solutions to (B.8) and (B.9). Hence, we have established the existence of $\gamma^F \in (0, \gamma^\circ)$ and solution to the associated IVP posed at the beginning of the proof such that $\gamma_T(\gamma^F) = \gamma^\circ$. By reversing the direction of time, this is a solution to the BVP in the theorem statement.

For the remainder of the proof, we refer to the forward system. Now in the full system of equations for the equilibrium learning and value function coefficients, we have

$$(v_{2t}, v_{5t}, v_{7t}, v_{9t}) = \left(\frac{2\sigma_Y^2\beta_{0t}}{\gamma_t\alpha_t}, -\frac{\sigma_Y^2[\beta_{3t} - \beta_{1t}(2 - \chi_t)]}{\gamma_t\alpha_t}, \frac{2\sigma_Y^2(2\beta_{3t} - 1)}{\gamma_t\alpha_t}, \frac{2\sigma_Y^2[\beta_{2t} - \beta_{1t}(1 - \chi_t)]}{\gamma_t\alpha_t} \right)$$

and a system of ODEs for $v_0, v_1, v_3, v_4, \beta_0$:

$$\begin{aligned} \dot{v}_{0t} &:= \beta_{0t}^2 + \frac{\alpha_t\gamma_t\chi_t}{\sigma_X^2(1 - \chi_t)^2} \left\{ \sigma_Y^2[-2\beta_{2t}\chi_t(1 - \chi_t) + \alpha_t\chi_t(1 - \chi_t)^2] \right. \\ &\quad \left. + \alpha_t\sigma_X^2(1 - \chi_t)^2 - \alpha_t\gamma_t\chi_tv_{6t} \right\} \\ \dot{v}_{1t} &:= -2\beta_{0t} \end{aligned}$$

$$\begin{aligned}\dot{v}_{3t} &:= \frac{\alpha_t^2 \gamma_t \chi_t v_{3t}}{\sigma_X^2 (1 - \chi_t)} + 2\beta_{0t} [\beta_{1t} (1 - \chi_t) + \beta_{2t}] \\ \dot{v}_{4t} &:= 1 - 2\beta_{3t}^2 \\ \beta_{0t} &:= -\frac{\alpha_t^2 \gamma_t \chi_t [\alpha_t \gamma_t (v_{3t} + \beta_{0t} v_{8t}) + 4\sigma_Y^2 \beta_{0t} \beta_{2t} \chi_t]}{2\sigma_X^2 \sigma_Y^2 (1 - \chi_t)},\end{aligned}$$

all of which have terminal values 0. Since the system of ODEs above is locally Lipschitz continuous in $(v_0, v_1, v_3, v_4, \beta_0)$, uniformly in t , it has a unique solution (given a solution to the BVP), and hence v_2, v_5, v_7 and v_9 are well-defined by the above formulas since $\alpha, \gamma > 0$. Furthermore, by inspection, $(v_3, \beta_0) = (0, 0)$ solves the pair of ODEs for (v_3, β_0) . We conclude that for $T < T(\gamma^\circ)$, there exists a LME with the stated properties.

B.2: Proof of Theorem 2

Let

$$\tilde{\beta}_2 = \beta_2 / (1 - \chi); \quad \tilde{v}_6 = v_6 \gamma / (1 - \chi)^2; \quad \tilde{v}_8 = v_8 \gamma / (1 - \chi).$$

The boundary value problem is

$$\dot{\tilde{v}}_{6t} = \gamma_t \left\{ -\beta_{1t}^2 + 2\beta_{1t} \tilde{\beta}_{2t} + \tilde{\beta}_{2t}^2 + \tilde{v}_{6t} \left(\frac{\alpha_t^2}{\sigma_Y^2} + \frac{2(\hat{u}_{a\theta} + \alpha_t)^2 \chi_t}{\sigma_X^2} \right) \right\} \quad (\text{B.25})$$

$$\dot{\tilde{v}}_{8t} = \gamma_t \left\{ (-2 + 4\alpha_t) \beta_{1t} - 2\tilde{\beta}_{2t} + \frac{\tilde{v}_{8t} (\hat{u}_{a\theta} + \alpha_t)^2 \chi_t}{\sigma_X^2} - 4\beta_{1t}^2 \chi_t \right\} \quad (\text{B.26})$$

$$\begin{aligned}\dot{\beta}_{1t} &= \frac{\gamma_t}{4\sigma_X^2 \sigma_Y^2 (1 + \hat{u}_{a\theta} \chi_t)} \times \left\{ 2\sigma_X^2 \alpha_t (\hat{u}_{a\theta}^2 - 2\beta_{1t}^2 + \alpha_t (\hat{u}_{a\theta} + 2\beta_{1t})) \right. \\ &\quad \left. + \tilde{v}_{8t} \alpha_t (\hat{u}_{a\theta} + \alpha_t)^2 (\hat{u}_{a\theta} - 2\beta_{1t}) \chi_t \right. \\ &\quad \left. + 4\beta_{1t} \chi_t [\hat{u}_{a\theta}^2 \sigma_Y^2 + (2\hat{u}_{a\theta} \sigma_X^2 + \sigma_Y^2) \alpha_t^2 + \hat{u}_{a\theta} \alpha_t (\hat{u}_{a\theta} \sigma_X^2 + 2\sigma_Y^2 - \sigma_X^2 \beta_{1t})] \right. \\ &\quad \left. - 4\sigma_Y^2 (\hat{u}_{a\theta} + \alpha_t)^2 \tilde{\beta}_{2t} \chi_t + 4\sigma_Y^2 (\hat{u}_{a\theta} + \alpha_t)^2 \beta_{1t} (\hat{u}_{a\theta} - 2\tilde{\beta}_{2t}) \chi_t^2 \right\} \quad (\text{B.27})\end{aligned}$$

$$\begin{aligned}\dot{\tilde{\beta}}_{2t} &= \frac{\gamma_t}{4\sigma_X^2 \sigma_Y^2 (1 + \hat{u}_{a\theta} \chi_t)} \times \left\{ 2\sigma_X^2 \alpha_t [\hat{u}_{a\theta}^2 + 2\beta_{1t}^2 + \alpha_t (\hat{u}_{a\theta} + 2\tilde{\beta}_{2t})] \right. \\ &\quad \left. + \alpha_t \chi_t (\hat{u}_{a\theta} + \alpha_t)^2 [-4\tilde{v}_{6t} + \tilde{v}_{8t} (\hat{u}_{a\theta} - 2\tilde{\beta}_{2t})] \right. \\ &\quad \left. + 4\alpha_t \chi_t \hat{u}_{a\theta} \sigma_X^2 [\beta_{1t}^2 + (\hat{u}_{a\theta} + 2\alpha_t) \tilde{\beta}_{2t}] \right. \\ &\quad \left. - 4(\hat{u}_{a\theta} + \alpha_t)^2 [\hat{u}_{a\theta} \tilde{v}_{6t} \alpha_t + \sigma_Y^2 \tilde{\beta}_{2t} (-\hat{u}_{a\theta} + 2\tilde{\beta}_{2t})] \chi_t^2 \right\} \quad (\text{B.28})\end{aligned}$$

$$\begin{aligned} \dot{\beta}_{3t} = & \frac{\gamma_t}{4\sigma_X^2\sigma_Y^2(1+\hat{u}_{\hat{a}\theta}\chi_t)} \times \left\{ -4\sigma_X^2\alpha_t^2\beta_{1t} \right. \\ & + 2\alpha_t\chi_t(\hat{u}_{\hat{a}\theta} + \alpha_t) \left[-\hat{u}_{\hat{a}\theta}\sigma_X^2 + 2\hat{u}_{\hat{a}\theta}\sigma_X^2\alpha_t - \tilde{v}_{8t}\alpha_t(\hat{u}_{\hat{a}\theta} + \alpha_t) \right] \\ & - 2\alpha_t\chi_t[2\hat{u}_{\hat{a}\theta}\sigma_X^2\alpha_t\beta_{1t} - 2\sigma_X^2\beta_{1t}^2] \\ & - \chi_t^2[\tilde{v}_{8t}\alpha_t(\hat{u}_{\hat{a}\theta} + \alpha_t)^2(\hat{u}_{\hat{a}\theta} - 2\beta_{1t}) + 4\hat{u}_{\hat{a}\theta}\sigma_X^2\alpha_t(\hat{u}_{\hat{a}\theta} + \alpha_t - \beta_{1t})\beta_{1t}] \\ & \left. + 4\sigma_Y^2\chi_t^2(\hat{u}_{\hat{a}\theta} + \alpha_t)^2(-1 + 2\alpha_t)\tilde{\beta}_{2t} + 8\sigma_Y^2(\hat{u}_{\hat{a}\theta} + \alpha_t)^2\beta_{1t}\tilde{\beta}_{2t}\chi_t^3 \right\} \end{aligned} \quad (\text{B.29})$$

$$\dot{\gamma}_t = -\frac{\alpha_t^2\gamma_t^2}{\sigma_Y^2} \quad (\text{B.30})$$

$$\dot{\chi}_t = \gamma_t \left\{ \alpha_t^2(1 - \chi_t)/\sigma_Y^2 - (\hat{u}_{\hat{a}\theta} + \alpha_t)^2\chi_t^2/\sigma_X^2 \right\}. \quad (\text{B.31})$$

with boundary conditions $(\gamma_0, \chi_0, \tilde{v}_{6T}, \tilde{v}_{8T}, \beta_{1T}, \tilde{\beta}_{2T}, \beta_{3T}) = (\gamma^\circ, 0, 0, 0, \frac{1+2\hat{u}_{\hat{a}\theta}}{2(2-\chi_T)}, \frac{1+2\hat{u}_{\hat{a}\theta}}{2(2-\chi_T)}, 1/2)$.

The proof proceeds in several steps. The main task is to establish the existence of a solution $(\tilde{v}_6, \tilde{v}_8, \beta_1, \tilde{\beta}_2, \beta_3, \gamma, \chi)$ to the boundary value problem for all $T < T(\gamma^\circ)$; from there, it is straightforward to verify that the remaining equilibrium coefficients are well-defined, as we do at the end of the proof.

Step 1: Convert BVP to fixed point problem in terms of a parameterized IVP. It is useful to introduce $z = (\tilde{v}_6, \tilde{v}_8, \beta_1, \tilde{\beta}_2, \beta_3, \gamma, \chi)$ and write the system of ODEs (B.25)-(B.31) as $\dot{z}_t = F(z_t)$. We write $\tilde{z} = (z_1, z_2, \dots, z_5)$ and $\tilde{F}(z) = (F_1(z), F_2(z), \dots, F_5(z))$.

Define $\mathbf{B} : \mathbb{R}_+ \rightarrow \mathbb{R}^5$ by $\mathbf{B}(\chi) = \left(0, 0, \frac{1+2\hat{u}_{\hat{a}\theta}}{2(2-\chi)}, \frac{(1+2\hat{u}_{\hat{a}\theta})}{2(2-\chi)}, 1/2\right)$, formed by writing the terminal value of \tilde{z} as a function of χ . Define $s_0 \in \mathbb{R}^5$ by $s_0 = \mathbf{B}(0) = \left(0, 0, \frac{1+2\hat{u}_{\hat{a}\theta}}{4}, \frac{1+2\hat{u}_{\hat{a}\theta}}{4}, 1/2\right)$. For $x \in \mathbb{R}^n$, let $\|x\|_\infty$ denote the sup norm, $\sup_{1 \leq i \leq n} |x_i|$. For any $\rho > 0$, let $\mathcal{S}_\rho(s_0)$ denote the ρ -ball around s_0

$$\mathcal{S}_\rho(s_0) := \{s \in \mathbb{R}^5 \mid \|s - s_0\|_\infty \leq \rho\}.$$

For all $s \in \mathcal{S}_\rho(s_0)$, let IVP- s denote the initial value problem defined by (B.25)-(B.31) and initial conditions $(\tilde{v}_{60}, \tilde{v}_{80}, \beta_{10}, \tilde{\beta}_{20}, \beta_{30}, \gamma_0, \chi_0) = (s, \gamma^\circ, 0)$. Whenever a solution to IVP- s exists, it is unique as F is of class C^1 ; denote it by $z(s)$, where $z(s) = (\tilde{z}(s), \gamma(s), \chi(s)) = (\tilde{v}_6(s), \tilde{v}_8(s), \beta_1(s), \tilde{\beta}_2(s), \beta_3(s), \gamma(s), \chi(s))$, where we suppress additional dependence on $(\gamma^\circ, 0)$ which remain fixed. Note that such a solution solves the BVP if and only if

$$\tilde{z}_T(s) = \mathbf{B}(\chi_T(s)), \quad (\text{B.32})$$

as the initial values $\gamma_0(s) = \gamma^\circ$ and $\chi_0(s) = 0$ are satisfied by construction. Note also that $\tilde{z}_T(s) = s + \int_0^T \tilde{F}(z_t(s)) dt$; hence (B.32) is satisfied if and only if s is a fixed point of the

function $g : \mathcal{S}_\rho(s_0) \rightarrow \mathbb{R}^5$ defined by

$$g(s) := \mathbf{B}(\chi_T(s)) - \int_0^T \tilde{F}(z_t(s)) dt. \quad (\text{B.33})$$

Note, moreover, that for any solution, we have by Lemma 2 that $\chi_t \in [0, \bar{\chi})$ where we define $\bar{\chi}$ as 1 for the purpose of this proof.

Step 2: Obtain sufficient conditions for IVP- s to have unique and uniformly bounded solutions for all $s \in \mathcal{S}_\rho(s_0)$, any $\rho > 0$. Specifically, for arbitrary $K > 0$, we ensure that the solution $\tilde{z}_t(s)$ varies at most K from its starting point s for all $t \in [0, T]$, and thus by the triangle inequality, this solution varies most $\rho + K$ from s_0 . These bounds will be used later in the proof.

Lemma B.1. *Fix $\gamma^\circ > 0$, $\rho > 0$ and $K > 0$. Then there exists a threshold $T^{SBC}(\gamma^\circ; \rho, K) > 0$ such that if $T < T^{SBC}(\gamma^\circ; \rho, K)$, then for all $s \in \mathcal{S}_\rho(s_0)$ a unique solution to IVP- s exists over $[0, T]$. Moreover, for all $t \in [0, T]$, $\tilde{z}_t(s) \in \mathcal{S}_{\rho+K}(s_0)$. We call this property the System Bound Condition (SBC).*

Proof. Recall that \tilde{F} is of class C^1 , and hence given $s \in \mathcal{S}_\rho(s_0)$, the solution $z(s)$ is unique whenever it exists. Toward the SBC, note that it suffices to ensure that for all $\|\tilde{z}(s) - s\|_\infty < K$, since then by the triangle inequality, $\|\tilde{z}(s) - s_0\|_\infty \leq \|\tilde{z}(s) - s\|_\infty + \|s - s_0\|_\infty < \rho + K$. In what follows, we construct bounds on \tilde{F} by writing $\tilde{F}(z(s)) = \tilde{F}(z(s) - s_0 + s_0)$ and using the conjectured bounds $\|\tilde{z}(s) - s_0\|_\infty < \rho + K$, $\gamma \in (0, \gamma^\circ]$, $\chi \in [0, \bar{\chi})$ for the solution, when it exists. Using these bounds on \tilde{F} , we identify a threshold $T^{SBC}(\gamma^\circ; \rho, K)$ such that for all $t < T^{SBC}(\gamma^\circ; \rho, K)$ the solution to IVP- s (exists and) satisfies the conjectured bounds.

Note that the desired component-wise inequalities $|z_{it}(s) - s_{i0}| < \rho + K$, $i \in \{1, 2, \dots, 5\}$, imply the further bounds

$$\begin{aligned} |\tilde{v}_{6t}|, |\tilde{v}_{8t}| &< \rho + K \\ |\beta_{1t}| &< \bar{\beta}_1(\rho, K) := \frac{1 + 2\hat{u}_{\hat{a}\theta}}{4} + \rho + K \\ |\tilde{\beta}_{2t}| &< \bar{\beta}_2(\rho, K) := \frac{1 + 2\hat{u}_{\hat{a}\theta}}{4} + \rho + K \\ |\beta_{3t}| &< \bar{\beta}_3(\rho, K) := 1/2 + \rho + K \\ |\alpha_t| &< \bar{\alpha}(\rho, K) := \bar{\beta}_1(\rho, K)\bar{\chi} + \bar{\beta}_3(\rho, K). \end{aligned}$$

Hereafter, we suppress the dependence of $\bar{\beta}_i$, $i \in \{1, 2, 3\}$, and $\bar{\alpha}$ on (ρ, K) .

Define functions $h_i : \mathbb{R}_{++}^3 \rightarrow \mathbb{R}_{++}$ as follows:⁵³

$$\begin{aligned}
h_1(\gamma^\circ; \rho, K) &:= \gamma^\circ \{ (\bar{\beta}_1 + \bar{\beta}_2)^2 + \bar{v}_6 (\bar{\alpha}^2 / \sigma_Y^2 + 2(\hat{u}_{\hat{a}\theta} + \bar{\alpha})^2 \bar{\chi} / \sigma_X^2) \} \\
h_2(\gamma^\circ; \rho, K) &:= \gamma^\circ \{ (2 + 4\bar{\alpha})\bar{\beta}_1 + 2\bar{\beta}_2 + \bar{v}_8 (\hat{u}_{\hat{a}\theta} + \bar{\alpha})^2 \bar{\chi} / \sigma_X^2 + 4\bar{\beta}_1^2 \bar{\chi} \} \\
h_3(\gamma^\circ; \rho, K) &:= \frac{\gamma^\circ}{4\sigma_X^2 \sigma_Y^2} \times \{ 2\sigma_X^2 \bar{\alpha} (\hat{u}_{\hat{a}\theta}^2 + 2\bar{\beta}_1^2 + \bar{\alpha}(\hat{u}_{\hat{a}\theta} + 2\bar{\beta}_1)) \\
&\quad \bar{v}_8 \bar{\alpha} (\hat{u}_{\hat{a}\theta} + \bar{\alpha})^2 (\hat{u}_{\hat{a}\theta} + 2\bar{\beta}_1) \bar{\chi} \\
&\quad + 4\bar{\beta}_1 \bar{\chi} [\hat{u}_{\hat{a}\theta}^2 \sigma_Y^2 + (2\hat{u}_{\hat{a}\theta} \sigma_X^2 + \sigma_Y^2) \bar{\alpha}^2 + \hat{u}_{\hat{a}\theta} \bar{\alpha} (\hat{u}_{\hat{a}\theta} \sigma_X^2 + 2\sigma_Y^2 + \sigma_X^2 \bar{\beta}_1)] \\
&\quad + 4\sigma_Y^2 (\hat{u}_{\hat{a}\theta} + \bar{\alpha})^2 [\bar{\beta}_2 \bar{\chi} + \bar{\beta}_1 (\hat{u}_{\hat{a}\theta} + 2\bar{\beta}_2) \bar{\chi}^2] \} \\
h_4(\gamma^\circ; \rho, K) &:= \frac{\gamma^\circ}{4\sigma_X^2 \sigma_Y^2} \times \{ 2\sigma_X^2 \bar{\alpha} [\hat{u}_{\hat{a}\theta}^2 + 2\bar{\beta}_1^2 + \bar{\alpha}(\hat{u}_{\hat{a}\theta} + 2\bar{\beta}_2)] \\
&\quad \bar{\alpha} \bar{\chi} (\hat{u}_{\hat{a}\theta} + \bar{\alpha})^2 [4\bar{v}_6 + \bar{v}_8 (\hat{u}_{\hat{a}\theta} + 2\bar{\beta}_2)] + 4\bar{\alpha} \bar{\chi} \hat{u}_{\hat{a}\theta} \sigma_X^2 [\bar{\beta}_1^2 + (\hat{u}_{\hat{a}\theta} + 2\bar{\alpha})\bar{\beta}_2] \\
&\quad + 4(\hat{u}_{\hat{a}\theta} + \bar{\alpha})^2 \bar{\chi}^2 [\hat{u}_{\hat{a}\theta} \bar{v}_6 \bar{\alpha} + \sigma_Y^2 \bar{\beta}_2 (\hat{u}_{\hat{a}\theta} + 2\bar{\beta}_2)] \} \\
h_5(\gamma^\circ; \rho, K) &:= \frac{\gamma^\circ}{4\sigma_X^2 \sigma_Y^2} \times \{ 4\sigma_X^2 \bar{\alpha}^2 \bar{\beta}_1 + 2\bar{\alpha} \bar{\chi} (\hat{u}_{\hat{a}\theta} + \bar{\alpha}) [\hat{u}_{\hat{a}\theta} \sigma_X^2 + 2\hat{u}_{\hat{a}\theta} \sigma_X^2 \bar{\alpha} + \bar{v}_8 \bar{\alpha} (\hat{u}_{\hat{a}\theta} + \bar{\alpha})] \\
&\quad + 2\bar{\alpha} \bar{\chi} [2\hat{u}_{\hat{a}\theta} \sigma_X^2 \bar{\alpha} \bar{\beta}_1 + 2\sigma_X^2 \bar{\beta}_1^2] \\
&\quad \bar{\chi}^2 [\bar{v}_8 \bar{\alpha} (\hat{u}_{\hat{a}\theta} + \bar{\alpha})^2 (\hat{u}_{\hat{a}\theta} + 2\bar{\beta}_1) + 4\hat{u}_{\hat{a}\theta} \sigma_X^2 \bar{\alpha} \bar{\beta}_1 (\hat{u}_{\hat{a}\theta} + \bar{\alpha} + \bar{\beta}_1)] \\
&\quad + 4\sigma_Y^2 \bar{\chi}^2 (\hat{u}_{\hat{a}\theta} + \bar{\alpha})^2 (1 + 2\bar{\alpha}) \bar{\beta}_2 + 8\sigma_Y^2 (\hat{u}_{\hat{a}\theta} + \bar{\alpha})^2 \bar{\beta}_1 \bar{\beta}_2 \bar{\chi}^3 \}.
\end{aligned}$$

Now for arbitrary $(\rho, K) \in \mathbb{R}_{++}^2$, define

$$T^{SBC}(\gamma^\circ; \rho, K) := \min_{i \in \{1, 2, \dots, 5\}} \frac{K}{h_i(\gamma^\circ; \rho, K)}. \quad (\text{B.34})$$

We claim that, for any $t < T^{SBC}(\gamma^\circ; \rho, K)$, if a solution exists at time t , then $\|\tilde{z}_t(s) - s\|_\infty < K$, $\gamma_t \in (0, \gamma^\circ]$ and $\chi_t \in [0, \bar{\chi})$. To see this, suppose by way of contradiction that there is some $s \in \mathcal{S}_\rho$ and some $t < T^{SBC}(\gamma^\circ; \rho, K)$ at which a solution to IVP-s exists but either $|z_{it}(s) - s_i| \geq K$ for some $i \in \{1, 2, \dots, 5\}$, $\gamma_t \notin (0, \gamma^\circ]$ or $\chi_t \notin [0, \bar{\chi})$; let τ be the infimum of such times. Now by Lemma 2, it cannot be that $\gamma_t \notin (0, \gamma^\circ]$ or $\chi_t \notin [0, \bar{\chi})$ while $\tilde{z}_t(s)$ exists, so (by continuity of $z(s)$ w.r.t. time) it must be that for some $i \in \{1, 2, \dots, 5\}$, $|z_{i\tau}(s) - s_i| \geq K$, and the bounds $\gamma_t \in (0, \gamma^\circ]$ and $\chi_t \in [0, \bar{\chi})$ hold for $t \in [0, \tau]$.

By construction of $h_i(\gamma^\circ; \rho, K)$, for all $t \in [0, \tau]$ we have $|F_i(z_t(s))| \leq h_i(\gamma^\circ; \rho, K)$ and thus

$$|z_{i\tau}(s) - s_i| \leq \int_0^\tau |F_i(z_t(s))| dt \leq \tau \cdot h_i(\gamma^\circ; \rho, K) < T^{SBC}(\gamma^\circ; \rho, K) h_i(\gamma^\circ; \rho, K) \leq K,$$

⁵³We use \mathbb{R}_{++} to denote $(0, +\infty)$.

where the last step uses the definition of $T^{SBC}(\gamma^\circ; \rho, K)$; but via the strict inequality, this contradicts the definition of τ , proving the claim. By the triangle inequality, it follows that $z_t(s) \in \mathcal{S}_{\rho+K}(s_0)$ if a solution exists at time $t < T^{SBC}(\gamma^\circ; \rho, K)$. Together, these bounds imply that the solution cannot explode prior to time $T^{SBC}(\gamma^\circ; \rho, K)$. In other words, a unique solution must exist over $[0, T]$ for any $T < T^{SBC}(\gamma^\circ; \rho, K)$ and it satisfies the SBC. \square

Step 3: Establish that g is a well-defined, continuous self-map on \mathcal{S}_ρ when T is below a threshold $T(\gamma^\circ; \rho, K)$. The expression for the latter is shown in the proof Lemma B.2 below.

Lemma B.2. *Fix $\gamma^\circ > 0$, $\rho > 0$ and $K > 0$. There exists $T(\gamma^\circ; \rho, K) \leq T^{SBC}(\gamma^\circ; \rho, K)$ such that for all $T < T(\gamma^\circ; \rho, K)$, g is a well-defined, continuous self-map on \mathcal{S}_ρ .*

Proof. First, the inequality $T(\gamma^\circ; \rho, K) \leq T^{SBC}(\gamma^\circ; \rho, K)$, which holds by construction (as carried out below), ensures that a unique solution to IVP- s exists for all $s \in \mathcal{S}_\rho$. Next, we argue that g is continuous. Note that $g(s)$ can be written as $\mathbf{B}(\chi_T(s)) - [z_T(s) - s]$. Since F is of class C^1 on the domain $\mathcal{S}_{\rho+K} \times (0, \gamma^\circ] \times [0, \bar{\chi})$, $z_t(s)$ (which includes γ and χ) is locally Lipschitz continuous in s , uniformly in $t \in [0, T]$,⁵⁴ and \mathbf{B} is continuous, and thus continuity of g follows readily.

To complete the proof, we show that if $T < T(\gamma^\circ; \rho, K)$, g satisfies the condition

$$\|g(s) - s_0\|_\infty \leq \rho \quad \text{for all } s \in \mathcal{S}_\rho,$$

which we refer to as the Self-Map Condition (SMC).

Note that $g(s) - s_0 = \Delta(s) - \int_0^T \tilde{F}(z_t(s)) dt$, where

$$\begin{aligned} \Delta(s) &:= \mathbf{B}(\chi_T(s)) - \mathbf{B}(0) \\ &= \left(0, 0, \frac{1 + 2\hat{u}_{a\theta}}{2} \left[\frac{1}{2 - \chi_T(s)} - \frac{1}{2} \right], \frac{1 + 2\hat{u}_{a\theta}}{2} \left[\frac{1}{2 - \chi_T(s)} - \frac{1}{2} \right], 0 \right). \end{aligned}$$

The $h^i(\rho, K)$ constructed in the proof of the previous lemma will provide us a bound for the components of $\int_0^T \tilde{F}(z_t(s)) dt$, but we must also bound $\Delta(s)$, and in particular, $\Delta_3(s)$ and $\Delta_4(s)$. Note that $\Delta_3(s) = \Delta_4(s)$.

Recalling that $\chi \in [0, 1)$, the ODE for χ implies that

$$\dot{\chi}_t \leq \gamma_t \{ \alpha_t^2 (1 - \chi_t) / \sigma_Y^2 \} \leq \gamma^\circ \bar{\alpha}^2 / \sigma_Y^2,$$

which depends on (ρ, K) through $\bar{\alpha}$. Hence by the fundamental theorem of calculus, we have $\chi_t = \int_0^t \dot{\chi}_s ds \leq (\gamma^\circ \bar{\alpha}^2 / \sigma_Y^2) t$.

⁵⁴See Theorem on page 397 in [Hirsch et al. \(2004\)](#).

Hence, using $\chi_T(s) \leq 1$ to bound $(2 - \chi_T(s))$ in the denominators from below by 1, we have the following bound for $\Delta_3(s) = \Delta_4(s)$:

$$|\Delta_3(s)| = \left| \frac{1 + 2\hat{u}_{\hat{a}\theta}}{2} \left[\frac{1}{2 - \chi_T(s)} - \frac{1}{2} \right] \right| = \frac{1 + 2\hat{u}_{\hat{a}\theta}}{2} \left| \frac{\chi_T(s)}{2(2 - \chi_T(s))} \right| \leq \frac{1 + 2\hat{u}_{\hat{a}\theta}}{4} (\gamma^o \bar{\alpha}^2 / \sigma_Y^2) T$$

For arbitrary $(\rho, K) \in \mathbb{R}_{++}^2$, define $\bar{\Delta}_i(\gamma^o; \rho, K) = \frac{1+2\hat{u}_{\hat{a}\theta}}{4} (\gamma^o \bar{\alpha}^2 / \sigma_Y^2)$ for $i \in \{3, 4\}$ and define $\bar{\Delta}_i(\gamma^o; \rho, K) = 0$ for $i \in \{1, 2, 5\}$. Note that for all $i \in \{1, 2, 3, 4, 5\}$, $\bar{\Delta}_i(\rho, K)$ is proportional to γ^o , and by construction, $|\Delta_i(s)| \leq T \bar{\Delta}_i(\gamma^o; \rho, K)$.

Now for arbitrary $(\rho, K) \in \mathbb{R}_{++}^2$, define

$$T(\gamma^o; \rho, K) := \min \left\{ T^{SBC}(\gamma^o; \rho, K), \min_{i \in \{1, 2, \dots, 5\}} \frac{\rho}{\bar{\Delta}_i(\gamma^o; \rho, K) + h_i(\gamma^o; \rho, K)} \right\}. \quad (\text{B.35})$$

To establish the SMC, it suffices to establish for each $i \in \{1, 2, \dots, 5\}$ that $|g_i(s) - s_{i0}| \leq \rho$ for all $s \in \mathcal{S}_\rho(s_0)$. We begin by calculating

$$\begin{aligned} |g_i(s) - s_{i0}| &= \left| \underbrace{\mathbf{B}_i(\chi_T(s)) - \mathbf{B}_i(0)}_{=\Delta_i(s)} - \int_0^T F_i(z_t(s)) dt \right| \\ &\leq |\Delta_i(s)| + \int_0^T |F_i(z_t(s))| dt \leq T \bar{\Delta}_i(\gamma^o; \rho, K) + T h_i(\gamma^o; \rho, K) < \rho, \end{aligned}$$

where we have used the definition of $\bar{\Delta}_i(\gamma^o; \rho, K)$ and that $|F_i(z_t(s))| \leq h_i(\gamma^o; \rho, K)$; and (ii) in the last line we have used that $T < T(\gamma^o; \rho, K) \leq \frac{\rho}{\bar{\Delta}_i(\gamma^o; \rho, K) + h_i(\gamma^o; \rho, K)}$ by construction. Hence, for all $i \in \{1, 2, \dots, 5\}$ we have $|g_i(s) - s_{i0}| \leq \rho$, completing the proof. \square

Step 4: Apply a fixed point theorem to g to find s such that solution to IVP- s solves the BVP. Note that by Lemma B.2, g is a well-defined, continuous self-map on the compact set \mathcal{S}_ρ . By Brouwer's Theorem, there exists s^* such that $s^* = g(s^*)$, and hence the solution to IVP- s^* is a solution to the BVP. To see that $T(\gamma^o) \in O(1/\gamma^o)$, note simply that γ^o appears as an outside factor in the denominators of the expressions defining $T^{SBC}(\gamma^o; \rho, K)$ and $T(\gamma^o; \rho, K)$. Moreover, since ρ, K have been chosen arbitrarily, we can then optimize $T(\gamma^o; \rho, K)$ over choices of $(\rho, K) \in \mathbb{R}_{++}^2$ to obtain $T(\gamma^o)$.

Step 5: Show that given a solution to the BVP as above, the remaining coefficients are well-defined and thus a LME exists. We argue that α is finite and that γ and α are strictly positive. Finiteness comes directly from the definition $\alpha = \beta_1 \chi + \beta_3$ and the finiteness of the

underlying variables. This implies that $\gamma_t > 0$ for all $t \in [0, T]$. The ODE for α is

$$\dot{\alpha}_t = \frac{\alpha_t(\hat{u}_{\hat{a}\theta} + \alpha_t)\gamma_t\chi_t}{2\sigma_X^2\sigma_Y^2(1 + \hat{u}_{\hat{a}\theta}\chi_t)} \left\{ 2\hat{u}_{\hat{a}\theta}\sigma_X^2\alpha_t - \tilde{v}_{8t}\alpha_t(\hat{u}_{\hat{a}\theta} + \alpha_t) - 4\sigma_Y^2(\hat{u}_{\hat{a}\theta} + \alpha_t)\tilde{\beta}_{2t}\chi_t \right\}. \quad (\text{B.36})$$

By continuity of the solution to the BVP, the RHS of the equation above is locally Lipschitz continuous in α , uniformly in t . Moreover, $\alpha_T = \beta_{1T} + \beta_{3T}\chi_T = \frac{1 + \hat{u}_{\hat{a}\theta}\chi_T}{2 - \chi_T} > 0$. By a standard application of the comparison theorem to the backward version of the previous ODE, it must be that $\alpha_t > 0$ for all $t \in [0, T]$.

Using the solution to the BVP and the facts above, we solve for the rest of the equilibrium coefficients. First, we have directly

$$(v_{2t}, v_{5t}, v_{7t}, v_{9t}) = \left(\frac{2\sigma_Y^2\beta_{0t}}{\gamma_t\alpha_t}, \frac{\sigma_Y^2[\beta_{1t}(2 - \chi_t) - \beta_{3t} - \hat{u}_{\hat{a}\theta}]}{\gamma_t\alpha_t}, -\frac{2\sigma_Y^2(1 - 2\beta_{3t})}{\gamma_t\alpha_t}, \frac{2\sigma_Y^2[\beta_{2t} - \beta_{1t}(1 - \chi_t)]}{\gamma_t\alpha_t} \right).$$

The last three are clearly well-defined due to $\alpha, \gamma > 0$. The remaining ODEs are

$$\begin{aligned} \dot{\beta}_{0t} &= -\frac{(\hat{u}_{\hat{a}\theta} + \alpha_t)\gamma_t\chi_t}{2\sigma_X^2\sigma_Y^2(1 - \chi_t)(1 + \hat{u}_{\hat{a}\theta}\chi_t)} \left\{ 4\hat{u}_{\hat{a}\theta}\sigma_Y^2\beta_{0t}\tilde{\beta}_{2t}(1 - \chi_t)\chi_t \right. \\ &\quad + \alpha_t^2[\tilde{v}_{8t}\beta_{0t}(1 - \chi_t) + v_{3t}\gamma_t(1 + \hat{u}_{\hat{a}\theta}\chi_t)] \\ &\quad \left. + \alpha_t \left[\hat{u}_{\hat{a}\theta}v_{3t}\gamma_t(1 + \hat{u}_{\hat{a}\theta}\chi_t) + \beta_{0t}(1 - \chi_t) \left(-2\hat{u}_{\hat{a}\theta}\sigma_X^2 + \hat{u}_{\hat{a}\theta}\tilde{v}_{8t} + 4\sigma_Y^2\tilde{\beta}_{2t}\chi_t \right) \right] \right\}, \quad \beta_{0T} = 0, \\ \dot{v}_{0t} &= \beta_{0t}^2 + (\hat{u}_{\hat{a}\theta} + \alpha_t)^2\gamma_t\chi_t \\ &\quad + \frac{(\hat{u}_{\hat{a}\theta} + \alpha_t)^2\gamma_t\chi_t^2}{\sigma_X^2} \left[-\tilde{v}_{6t} + \sigma_Y^2(\hat{u}_{\hat{a}\theta} + \alpha_t - 2\tilde{\beta}_{2t})/\alpha_t \right], \quad v_{0T} = 0, \\ \dot{v}_{1t} &= -2\beta_{0t}, \quad v_{1T} = 0, \\ \dot{v}_{3t} &= 2\beta_{0t}(\beta_{1t} + \tilde{\beta}_{2t})(1 - \chi_t) + \frac{v_{3t}(\hat{u}_{\hat{a}\theta} + \alpha_t)^2\gamma_t\chi_t}{\sigma_X^2(1 - \chi_t)}, \quad v_{3T} = 0, \quad \text{and} \\ \dot{v}_{4t} &= 1 - 2\beta_{3t}^2, \quad v_{4T} = 0. \end{aligned}$$

Observe that the system for (β_0, v_1, v_3) is uncoupled from (v_0, v_4) . By inspection, the former has solution $(\beta_0, v_1, v_3) = (0, 0, 0)$, and uniqueness follows from the associated operator being locally Lipschitz continuous in (β_0, v_1, v_3) uniformly in time. It follows that $v_2 = 0$, and the solutions for (v_0, v_4) can be obtained directly by integration, given their terminal values. \square

B.3: Existence Proof Sketch for the General Model

In what follows, we refer the reader to the Mathematica file `spm.nb` available on our websites. Since scaling flow payoffs by a constant does not affect incentives, the file works under the

normalization $\partial^2 U / \partial a^2 = \partial^2 \hat{U} / \partial \hat{a}^2 = -1$; consequently, $U_{xy} = u_{xy}$, for $x, y \in \{a, \hat{a}, \theta\}$ in that file. We show below that under the conditions of Assumption 1 our method works, and so the method carries out to any unnormalized version satisfying the assumptions.

As outlined in Section 4, the problem of finding a quadratic value function V characterized by time-varying coefficients $\vec{v} = (v_0, \dots, v_9)$, and optimal policy (10) characterized by coefficients $\vec{\beta}$, reduces to a BVP in $(\vec{v}, \vec{\beta}, \gamma, \chi)$. Under Assumption 1, a static Nash equilibrium exists of the static game played at time T after any history, given that players have conjectured linear strategies as in (9)-(11); hence the terminal conditions of this boundary value problem are well-defined.⁵⁵ These terminal conditions are parameterized by χ_T .

For any payoff specification satisfying Assumption 1, it can be seen from `spm.nb` that the vector (v_2, v_5, v_7, v_9) can be written in terms of $\vec{\beta}$, and that (v_2, v_5, v_7, v_9) is well-defined as long as $\alpha \neq 0$ and $\gamma > 0$. Hence, we eliminate (v_2, v_5, v_7, v_9) from the BVP and check afterward that any solution to the resulting BVP, which we label BVP', satisfies the conditions on γ and α . From `spm.nb`, the ODEs in BVP' are well-defined as their denominators are nonzero; we have $\chi_t \in [0, 1)$, and the terms $(u_{a\theta} + u_{a\hat{a}}\hat{u}_{\hat{a}\theta}\chi_t)$ and $(1 - u_{a\hat{a}}\hat{u}_{a\hat{a}})$ are nonzero by Assumption 1 parts (ii) and (iv), respectively. Further, BVP' contains a subsystem in $(v_6, v_8, \beta_1, \beta_2, \beta_3, \gamma, \chi)$, which does not contain any coefficients in $\{v_0, v_1, v_3, v_4, \beta_0\}$. After the change of variables $(\tilde{v}_6, \tilde{v}_8, \tilde{\beta}_2) = (v_6\gamma/(1-\chi)^2, v_8\gamma/(1-\chi), \beta_2/(1-\chi))$, defined in the proof of Theorem 2, we obtain a core BVP which we label BVP'', in $(\tilde{v}_6, \tilde{v}_8, \beta_1, \tilde{\beta}_2, \beta_3, \gamma, \chi)$. The ODEs in BVP'' are again well-defined by Assumption 1, and by the change of variables, the term $(1 - \chi_t)$ is absent from their denominators.

The existence proof now follows the same steps as in the proof of Theorem 2: we show in Steps 1-4 that there is $T(\gamma^o)$ such that, a solution to BVP'' exists for all $T < T(\gamma^o)$ and then show in Step 5 that we can recover a solution to the original BVP.

In Step 1, we convert BVP'' to a fixed point problem w.r.t. the initial conditions of an IVP. Write the system of ODEs as $\dot{z}_t = F(z_t)$, and write $\tilde{z} = (z_1, \dots, z_5)$. Define $s_0 \in \mathbb{R}^5$ as $(0, 0, \beta_{10}^{st}, \beta_{20}^{st}, \beta_{30}^{st})$, where $(\beta_{10}^{st}, \beta_{20}^{st}, \beta_{30}^{st})$ are the equilibrium coefficients of the static game played at time 0, and let $\mathcal{S}_\rho(s_0)$ denote the ρ -ball centered at s_0 for arbitrary $\rho > 0$. For $s \in \mathcal{S}_\rho(s_0)$, define the parameterized initial value problem IVP- s by the ODEs in BVP'' together with initial conditions $z_0 = (s, \gamma^o, 0)$. We can define a function $g : \mathcal{S}_\rho(s_0) \rightarrow \mathbb{R}^5$ as in the proof of Theorem 2 such that, if a solution to IVP- s exists and $g(s) = s$, then this solution also solves BVP''.

In Step 2, we construct a function $T^{SBC} : \mathbb{R}_{++}^3 \rightarrow \mathbb{R}_+$ such that if $T < T^{SBC}(\gamma^o; \rho, K)$,

⁵⁵To see this, note that the coefficients on the players' (purported) static equilibrium strategies, displayed in `spm.nb`, contain only the terms $(1 - u_{a\hat{a}}\hat{u}_{a\hat{a}})$ and $(1 - u_{a\hat{a}}\hat{u}_{a\hat{a}}\chi_T)$ in their denominators; by Assumption 1 part (iv), and since $\chi_T \in [0, 1)$, these terms are nonzero.

then for all $s \in \mathcal{S}_\rho(s_0)$, a unique solution to IVP- s exists over $[0, T]$ and has the property that $\tilde{z}(s) \in \mathcal{S}_{\rho+K}(s_0)$. Intuitively, we use these conjectured bounds on z to bound the growth rate of the system and show that for small T , the conjectured bounds must hold.

In Step 3, we note that g is well-defined when $T < T^{SBC}(\gamma^\circ; \rho, K)$; g is also continuous, owing to the fact that F is of class C^1 and the terminal conditions in BVP'' are continuous in χ_T , each on the appropriate domain. We then show that for T below a threshold $T(\gamma^\circ; \rho, K) \leq T^{SBC}(\gamma^\circ; \rho, K)$, g is a self-map. We observe here that $T(\gamma^\circ; \rho, K) \in O(1/\gamma^\circ)$; we can then define $T(\gamma^\circ)$ by optimizing over (ρ, K) . In Step 4, then, we apply Brouwer's Theorem to establish the existence of a fixed point $s^* = g(s^*)$ for such $T < T(\gamma^\circ)$. By construction, the unique solution to IVP- s^* , for any such s^* , is a solution to BVP''.

In Step 5, all that remains is to recover a solution to the original BVP from the solution to BVP''. Now $\alpha = \beta_1\chi + \beta_3$ is well-defined and finite, and hence $\gamma > 0$ at all times. Hence, (v_6, v_8, β_2) is recovered from $(\tilde{v}_6, \tilde{v}_8, \tilde{\beta}_2)$. Moreover, $\alpha_T \neq 0$ by Assumption 1 parts (ii) and (iv), and since the right hand side of the ODE for α contains a factor of α , we have $\alpha \neq 0$ by the comparison theorem. Consequently, given a solution to BVP'', the ODEs for (β_0, v_3) are well-defined and form a linear system in (β_0, v_3) that does not contain any coefficients in $\{v_0, v_1, v_4\}$ and has known terminal conditions since χ_T is known. The associated operator for this system is Lipschitz continuous in (β_0, v_3) uniformly in $t \in [0, T]$ (as the solutions to the previous ODEs are continuous), and so there exists a unique solution (β_0, v_3) . The coefficients (v_2, v_5, v_7, v_9) are also well-defined as $\alpha, \gamma \neq 0$. Finally, given a solution for the coefficients determined thus far, the remaining ODEs for v_0, v_1 and v_4 are linear in themselves and uncoupled, so they have unique solutions.

Proofs for Section 5: Refer to the online appendix.

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