LARGE AUCTIONS

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Abstract. We posit a standard model of an asymmetric double auction with interdependent values in which each trader observes a private signal about a hidden state before submitting a bid or ask price for a unit demand or supply. The state and signals are one-dimensional, traders' signals are independent conditional on the state, and their distributions have the strict monotone likelihood ratio property. The model encompasses auctions by allowing sellers to be non-strategic. We study a version in which there are n replicates of each type of trader, with each replicate observing a signal drawn independently from the same conditional distribution as the original trader of that type, and all traders of the same type using the same strategy. The limit economy with a countable set of traders has a unique Walrasian equilibrium, whose clearing price reveals the state. If this equilibrium is totally monotone in that each buyer's (resp. seller's) probability of trading decreases (resp. increases) with the state, then the limit auction has a monotone equilibrium yielding the Walrasian price as the clearing price. We present four asymptotic results as n grows large: (1) a sequence of monotone strategies comprises epsilon-equilibria iff limit points are monotone equilibria of the limit auction; (2) for a sequence of monotone strategy profiles converging to a monotone equilibrium, the Strong Law of Large Numbers for prices holds, in that the sequence of price functions converges a.s. to the price function of the limit equilibrium; (3) if the effect of the state on traders' valuations is symmetric (around the equilibrium) then large but finite auctions have monotone equilibria whose outcomes approximate the Walrasian equilibrium outcome when bidders are restricted to sufficiently fine bid-grids; and (4) the same conclusion holds true without the symmetry assumption when we discretize the state space as well. Total monotonicity seems to be crucial: an example has a Walrasian equilibrium that is not the outcome of a Nash equilibrium of an auction.

1. Introduction

Since the initial articles by Wilson [25] and Milgrom [16], the literature on large auctions has sought to establish foundations for rational expectations equilibria and information revelation in markets resulting from strategic behavior in auctions with large numbers of participants. Substantial positive results were obtained in the case of private values or symmetric players with interdependent values (building on the seminal contribution of Milgrom and

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Weber [17]) by, among others, Rustichini, Satterthwaite, and Williams [23], Wilson [26], Pesendorfer and Swinkels [18] and [19], Cripps and Swinkels [2], Fudenberg, Mobius, Szeidl [4], and Siga and Mihm [24].[1] The asymmetric case with interdependent values presented a technical obstacle: as shown by Reny and Zamir [22] and Reny and Perry [21], best-replies to monotone strategies need not be monotone due to a failure of the single-crossing property; and therefore one cannot establish the existence of monotone equilibria using standard fixed-point methods. This imposed a hurdle to obtaining asymptotic results that rely on limits of sequences of equilibria in monotone strategies. Reny and Perry [21] circumvented this problem by first taking the limit of the equilibria of a sequence of double auctions with symmetric buyers and symmetric sellers and increasingly finer discrete sets of bids, and then taking the limit as the set of bidders approaches the continuum, relying on the fact that failure of single crossing property becomes an increasingly less severe problem as the number of players grows large.[2] This is a remarkable achievement, but the assumption of symmetry among buyers and sellers is strong, and the question of whether large auctions provide foundations for rational expectations equilibria under more general conditions remained open. In this paper, we make some progress in addressing this question by allowing for heterogeneity among both buyers and sellers.

We consider an auction Γ with finite sets I_0 and I_1 of buyers and sellers, with unit demand and supply, respectively, and all trades are at the market clearing price selected by a fixed rule. The sellers are either all strategic as in a double auction or passive as in an auction. The set of states of the world is an interval Ω and each player i receives a signal x_i from an interval according to a probability distribution $P_i(\cdot\,|\,\omega)$ conditional on each state $\omega \in \Omega$. These conditionals are independent across players and each satisfies the strict monotone likelihood ratio property (MLRP), i.e. higher signals indicate higher states. Player i's valuation of a unit is a function v_i that depends only on the state ω and his signal x_i, and it is weakly increasing in ω and strictly increasing in x_i. By allowing for asymmetric players, this generalizes the model in Reny and Perry [21].

For each positive integer n, define the game Γ^n that is an n-fold replica of Γ, i.e., there are n agents of each type i. We consider two limit objects as n increases to infinity, both with a countable set of agents. One is a competitive economy E^∞ where the demand function is defined using the limit-of-means criterion; and the other is a Bayesian game Γ^∞. The competitive economy has a unique rational expectations Walrasian equilibrium (REE); and

[1] In a common-values setting with finitely many signals and states, [24] indicate how one would extend their analysis to allow for asymmetric distributions; their remarks do not apply to the case of interdependent values.

[2] See also Kazumori [11], who allows for multiple units of demand or supply.

under this equilibrium the clearing price $\phi^*(\omega)$ is fully revealing, i.e. the price is a strictly monotone function of the state ω. In general, the payoffs in the game Γ^∞ are not well-defined for arbitrary strategies, but if all players of the same type i employ the same monotone strategy then, given a profile of such strategies and given a bid for an individual player, each player's payoff is well-defined, which is all that is required to check if a strategy profile constitutes a Nash equilibrium. As an example in Section 8 shows, Γ^∞ may not have a Nash equilibrium. The reason is that while the REE price function is monotone, the cut-off signals for who gets to trade in each state need not be monotone. When the REE satisfies this stronger property, called total monotonicity, then Γ^∞ has a unique monotone Nash equilibrium σ^*, and the clearing prices coincide with ϕ^*. Existence of a Nash equilibrium in these limit auctions is robust, since a sufficient condition for total monotonicity is that traders' valuations satisfy an average-crossing property in the sense of Krishna [13].

The limit economy and the limit game invoke countable sets of players, rather than a continuum as in most prior work. This enables meaningful interpretation of convergence of the games Γ^n to Γ^∞ because Γ^n can be viewed as a game with the same player set as Γ^∞, but where the players in Γ^∞ who are not in Γ^n are dummy players.[3]

Linking the games Γ^n and Γ^∞ enables our main results. First, we get an upper semi-continuity property. The limit of (not necessarily strictly) monotone strategy profiles of Γ^n, under pointwise convergence, is a Nash equilibrium of Γ^∞ iff the sequence comprises epsilon-equilibria. Moreover, if the limit equilibrium induces the REE price function, we get a Strong Law of Large numbers (SSLN, henceforth) result for this convergence, namely for a.e. ω, the market clearing price of Γ^n, conditional on ω, represented as a random variable defined on the space of all signals in Γ^∞ to clearing prices, converges pointwise to the REE (in contrast with the usual statement of convergence in measure as in a weak law of large numbers). If the convergence of ε-equilibria is uniform and at a rate $o(n^{-\frac{1}{2}})$, we also obtain a Central Limit Theorem (CLT) for prices.

Restricting traders to finite grid of bids, we obtain, under total monotonicity, two results establishing existence of an equilibrium for Γ^n for sufficiently fine bid-grids. First, if the effect of the state on each agent's valuation is symmetric along the REE manifold, then for each sufficiently fine grid of bids, Γ^n has an equilibrium with this bid space. Second, we can dispense with this symmetry assumption by also discretizing the state space, provided that the grid on the signals of the players is somewhat finer than the grid on the state space.

[3] Our approach follows the tradition of replica economies of Debreu and Scarf [3] rather than Aumann's [1] formulation with a continuum of agents. Such a formulation appears to be better suited for the analysis of limits of finite-player Bayesian games, in light of the well-known difficulties associated with models involving a continuum of random variables.

Prices under these bid-grid equilibria converge to the REE as the grid sizes converge to zero, but unfortunately, for these equilibria, the rate of convergence falls outside the range for which the CLT holds.

To obtain the first result on the existence of bid-grid equilibria, we proceed as follows. When the bid grid is fine, the difference equations induced by the first-order conditions in Γ^∞ are perturbations of the first-order conditions when players can bid without this restriction. The fact that the limit equilibrium is unique allows us to apply degree theory to establish that these difference equations have a solution. For large n, the corresponding difference equations for Γ^n are a small perturbation of the ones for the limit, giving us a solution for these games. Finally, the fact that a solution of these equations induces an equilibrium follows from an argument that appeals to a single-crossing property.

For the second existence result, when the state and signal spaces are discretized as well, we first use a fixed-point map for the limit economy to identify a potential limit equilibrium. Then, for each n, using again a simple fixed-point argument, we obtain an equilibrium of a game where players are restricted to a subset of monotone strategies that are in a prescribed neighborhood of the identified limit. For large n, we then show that this restriction has no bite, giving us a kosher equilibrium of this finite approximation of Γ^n.

What prevents us from applying the logic of the approach used for our first result on bid-grid equilibria to directly solve the differential equations for Γ^n and thus obtain an equilibrium without restricting the strategy space? After all, the first-order conditions for Γ^n are, intuitively speaking, "small" perturbations of those for Γ^∞. The main problem is that the first-order conditions for Γ^∞ are functional equations in the inverse bidding strategies, while their counterparts for Γ^n are implicit differential equations involving the inverse bidding functions and their derivatives. Given that it is difficult to derive an explicit differential equation expressing the derivative of the signals (that is the derivative of the inverse bidding function) as a function of the bid—even in a neighborhood of the limit equilibrium, which is the logical place to search for an equilibrium—the technical tool to leverage the robustness idea is the implicit function theorem for functional spaces. But posed this way, we encounter what is well-known in the literature of differential equations as the loss-of-derivatives problem. The equations defining the first-order-conditions for Γ^n send functions that are r-times differentiable to those that are $(r-1)$-times differentiable. The solution to this problem, provided by the Nash-Moser Theorem (see Hamilton [8]), which is a version of the inverse function theorem for Fréchet spaces, is to work in the (Fréchet) space of smooth functions. Alas, in our case, even in this space, we cannot invoke the implicit function theorem: the (Gâteaux) derivative of the equations for Γ^n do not represent a small perturbation of the corresponding derivative for Γ^∞ unless we have a tame (i.e. a controlled) rate at which the norms of the derivatives of the functional variables in the equation system grow. Restricting the domain of the search to functions with such bounds—which is in effect a compact set—renders the implicit function theorem inapplicable.[4]

The assumption of total monotonicity seems key to establishing a link between strategic equilibria and competitive equilibria. However, there is an open set of economies on which this property is false, which seems to be a negative result for the project of providing strategic foundations for the formation of competitive prices. However, these results can also be viewed as negative for the theory of auctions, since some economies where equilibria fail to exist seem to be otherwise well-behaved (cf. Section 8). Perhaps, one could argue that the assumption of finitely many types, along with the focus on type-symmetric strategies, injects an element of atomicity into the game (even at the limit) and thus the strategic effects do not truly disappear in the limit. However, without the finiteness assumption, the stochastic model on which we build our theory of large auctions does not seem to be well-founded.

2. Model

We start with an auction or a double auction Γ and then we consider a sequence of auctions obtained by replicating the agents in Γ. The game Γ is as follows. The set of buyers is I_0 and the set of sellers is I_1. Buyers have unitary demand and each seller has a unit to sell. The buyers are all strategic agents. In the case of an auction, the sellers are non-strategic; and in the case of a double auction they are strategic. We let I be the set of strategic agents, called the players in Γ: thus I equals I_0 for an auction and it is $I_0 \cup I_1$ in a double auction. Using $|\cdot|$ to denote cardinality of a set, let $m_1 = |I_1|$ be the number of sellers, with $1 \leqslant m_1 < |I|$. Let $m_0 = |I| - m_1$, $\mu_1 = m_1 |I|^{-1}$, and $\mu_0 = 1 - \mu_1$.

The set of unobserved states of the world is $\Omega \equiv [0,1]$. For each player i, let $X_i \equiv [0,1]$ be his space of signals with typical element x_i; let $X \equiv \prod_{i \in I} X_i$, with typical element x, and let ∂X denote the boundary of X. Let P be the probability distribution over $\Omega \times X$. For each $i \in I$ and $\omega \in \Omega$, let $P_i(\cdot \,|\, \omega)$ be the probability distribution over X_i conditional on ω, and let $P(\cdot \,|\, \omega)$ be the conditional distribution over X. Denote by P_0 the marginal on Ω. For each player i, the valuation of a unit is given by a function $v_i : \Omega \times X_i \to \mathbb{R}_+$. We make the following assumptions on P and v_i.

Assumption 2.1. The prior P satisfies the following conditions.

[4] There is a secondary problem that we have to confront: since in an asymmetric game, the strategies of the players typically do not have the same support, equilibria tend to be only piecewise-differentiable; but this issue has a fix, via piecewise-smooth maps.

(1) P has a continuously differentiable and strictly positive density p.
(2) The conditional distributions $P_i(\,\cdot\,|\,\omega)$ over the x_i's given ω are independent, i.e., $p(x\,|\,\omega)$ is the product of the densities $p_i(x_i\,|\,\omega)$, $i \in I$, of $P_i(x_i\,|\,\omega)$.
(3) $p_i(x_i\,|\,\omega)$ satisfies strict MLRP for each i.

Given the above assumption, we can view $P(\,\cdot\,|\,\omega)$ as the conditional CDF and thus write $P(x\,|\,\omega)$ for $P(\prod_i [0,x]\,|\,\omega)$, and $P_i(x_i\,|\,\omega)$ for $P_i([0,x_i]\,|\,\omega)$.

Assumption 2.2. For each i, the valuation v_i satisfies the following conditions:

(1) it is non-negative and twice-continuously differentiable;
(2) $\frac{\partial v_i(\omega,x_i)}{\partial \omega} \geqslant 0$ and $\frac{\partial v_i(\omega,x_i)}{\partial x_i} > 0$.

Loosely speaking, the next assumption, which is in two parts, is a non degeneracy assumption. The first part—which requires that there be some state of the world where for each player type, the probability of trade in that state is strictly between 0 and 1 in the limit economy—is assumed for simplicity in exposition; the second part, which stipulates that the probability of trade is nonconstant for at least one player type, is more substantive and will be shown to guarantee the existence of a non-trivial competitive equilibrium in the limit economy.

Assumption 2.3. The prior and the valuations jointly satisfy the following conditions.

(1) There exists $x \in X \setminus \partial X$ and $\omega \in \Omega$ such that:
    (a) $\sum_i P_i(x_i\,|\,\omega) = m_0$;
    (b) $v_i(\omega, x_i) = v_j(\omega, x_j)$ for all i, j.
(2) There do not exist ω, x and $J_1 \subseteq I$ such that $|J_1| = m_1$, and $v_i(\omega, 0) \geqslant v_j(\omega, 1)$ for all $i \in J_1$, $j \notin J_1$.

The game Γ is played as follows. A tuple (ω, x) of the state of the world (ω) and a profile of signals (x) is drawn according to P. Each player i is informed of his coordinate x_i and then submits a number $b_i \in \mathbb{R}_+$ simultaneously with the others. If i is a buyer, b_i is his bid, while if he is a seller, it represents his ask. The b_i's are then ordered $b_{(1)} \geqslant \cdots \geqslant b_{(|I|)}$, where $b_{(k)}$ is the k-th highest number. Each buyer with a bid that is $b_{(m_1)}$ or higher gets to buy an object; in a double auction, each seller with an ask below $b_{(m_1)}$ gets to sell an object. In the event of a tie, i.e. $b_{(m_1)} = b_{(m_1+1)}$, allocations are made randomly among those tied. The price at which trade occurs is $\alpha b_{(m_1)} + (1-\alpha) b_{(m_1+1)}$, where $0 \leqslant \alpha \leqslant 1$. Bidder i's ex post payoff if he wins an object at a profile $x \in X$ of signals is $v_i(\omega, x_i) - b$, where b is the price paid; and a seller i's ex post payoff is $b - v_i(\omega, x_i)$.

A pure strategy for a player $i \in I$ is a measurable map $\sigma_i : X_i \to \mathbb{R}_+$. It is said to be monotone if $\sigma_i(x_i) \geqslant \sigma_i(y_i)$ whenever $x_i > y_i$, and it is strictly monotone if the inequality is strict. By an equilibrium we mean a Nash equilibrium of the corresponding Bayesian game, and by a monotone equilibrium we mean an equilibrium in monotone strategies.[5] Formally, given a strategy profile σ, for each i and x_i, bidder i's expected payoff from bid b_i for him can be written as

$$\pi_i(b_i, \sigma; x_i) = \int_\Omega \tau_i(b, \omega, \sigma)[v_i(\omega, x_i) - \varrho_i(b_i, \omega, \sigma)]dP(\omega \,|\, x_i)$$

where $\tau_i(b, \omega, \sigma)$ is the probability that i trades in ω if he bids b and others play according to σ, with $\varrho_i(\cdot)$ being the expected clearing price for this event. The payoff for a seller is defined analogously. Now, σ is an equilibrium if for every i and for a.e. x_i under the marginal of P on X_i,

$$\pi_i(\sigma_i(x_i), \sigma; x_i) \geqslant \pi_i(b_i, \sigma; x_i)$$

for each $b_i \in \mathbb{R}_+$.

For each $n = 1, 2, \ldots$, we define an auction Γ^n as an n-fold replica of Γ. Specifically, the set of players is I^n, which has $n|I|$ players, where each player-type $i \in I$ has n players of that type, indexed by $(i, 1), \ldots, (i, n)$; and there are $\mu_1|I^n|$ objects for sale.[6] The set of states of nature remains Ω but the signal space is X^n, the n-fold product of X. The distribution P^n over $\Omega \times X^n$ is generated by the distribution P_0 on Ω, as in Γ, and the conditionally independent distributions $P_{(i,m)}(x_{(i,m)} \,|\, \omega)$, for $(i, m) \in I^n$, where for each i, the distributions are the same for all $1 \leqslant m \leqslant n$ and equal the distribution $P_i(x_i \,|\, \omega)$ of the game Γ. The rules of the game Γ^n are as in Γ.

3. The Limit Economy

At the limit we can define both a competitive economy, E^∞, as we do in this section, and an auction, Γ^∞, as in the next section. E^∞ has a denumerable set of agents, $I^\infty \equiv \lim_{n\uparrow\infty} I^n$. Let $X^\infty = X \times X \times \cdots$, where $X = \prod_{i\in I} X_i$ is the space of signals in the game Γ. An agent will be denoted by a pair (i, n), $i \in I$ and $n = 1, \ldots$, where i is his agent-type and n is the index of the agent in the infinite set of agents of type i. Each seller has one unit to sell and each buyer wants one unit. The valuation of agent (i, n) is given by the function $v_i(\omega, x_i)$. Let $\mathcal{O}$ be the Borel σ-algebra on Ω; and let $\mathcal{X}^\infty$ be the product σ-algebra on X^∞, using the Borel σ-algebra on each factor. Let P^* be the probability distribution over

[5]In this paper, we focus on monotone pure strategies, but we remark at appropriate points in the other sections on the implications of considering (mixed) behavioral strategies.

[6]We keep the fraction μ_1 independent of n for simplicity in notation. We could allow for a ratio $\mu_1(n)$ that depends on n, so long as there is a well-defined limit that is strictly between 0 and 1.

$(\Omega \times X^\infty, \mathcal{O} \otimes \mathcal{X}^\infty)$ for which P_0 is the marginal on Ω and for each ω, conditional on ω, the distribution over X^∞ is a product distribution with the distribution over $X_{i,n}$ being the same for all agents of type i and given by $P_i(\,\cdot\,|\,\omega)$.

A state of the economy is given by (ω, x^∞): it is a full description of the environment (the state ω) and traders' personal tastes (encoded in the signals x_i). A price map is a measurable function $\phi : \Omega \to \mathbb{R}_+$. Given ϕ, the valuation of the object for agent (i,n) with signal x_i is the measurable function $\mathbb{E}(v_i(\cdot, x_i) \,\|\, \phi) : \Omega \to \mathbb{R}_+$, where the expectation is w.r.t. $P_i(\,\cdot\,|\,x_i)$ and is conditional on the σ-algebra generated by the price function ϕ. Observe that this expectation is strictly monotone in x_i, since v_i is strictly increasing in x_i and $P_i(\,\cdot\,|\,x_i)$ satisfies MLRP; moreover, it is continuous differentiable in x_i as well. Consumer i's demand $D_i(\omega, x_i, \phi)$ is 1 or 0 depending on whether $\mathbb{E}(v_i(\cdot, x_i) \,\|\, \phi)(\omega)$ is greater or smaller than $\phi(\omega)$. (Indifference occurs for at most one signal x_i.) If the game Γ is a double auction, then $D_i(\omega, x_i, \phi)$ of a seller i is -1 or 0 depending on whether his expected value is smaller or greater than $\phi(\omega)$.

Given ϕ, excess demand is a function $Z(\omega, x^\infty, \phi)$ defined for each state of the economy (ω, x^∞) by

$$Z(\omega, x^\infty, \phi) = \lim_{n\to\infty} \frac{1}{n} \sum_{(i,k)\in I^n} D_i(\omega, x_{i,k}, \phi)$$

if Γ is a double auction, and

$$Z(\omega, x^\infty, \phi) = \lim_{n\to\infty} \frac{1}{n} \sum_{i\in I^n} D_i(\omega, x_{i,n}, \phi) - m_1$$

if Γ is an auction.

Lemma 3.1. *For each price function ϕ, the excess demand function is well-defined for P^*-a.e. (ω, x^∞).*

Proof. Fix a price function ϕ and a state ω. Let $\underline{x}_i(\phi, \omega)$ be zero (resp. one) if $v_i(\omega, 0) > \mathbb{E}(v_i(\cdot, 0) \,\|\, \phi)(\omega)$ (resp. $v_i(\omega, 1) < \mathbb{E}(v_i(\cdot, 1) \,\|\, \phi)(\omega)$); otherwise, let it be the unique x_i for which $v_i(\omega, x_i) = \mathbb{E}(v_i(\cdot, x_i) \,\|\, \phi)(\omega)$. For a buyer (resp. seller) i, the demand $D_i(\omega, x_i, \phi)$ is 1 (resp. 0) iff $x_i \geqslant \underline{x}_i(\phi, \omega)$. By SLLN, $Z(\omega, x^\infty, \phi)$ equals $m_0 - \sum_{i\in I} P_i(\underline{x}_i(\phi, \omega) \,|\, \omega)$ for P^*-a.e. (ω, x^∞). □

Say that a price function ϕ is a *Rational Expectations Equilibrium* (REE) if $Z(\omega, x^\infty, \phi) = 0$ for P^*-a.e. (ω, x^∞). Furthermore, it is *fully revealing* if ϕ is a strictly increasing function of ω. By MLRP of the prior, for each i and x_i, $P(x_i \,|\, \omega)$ is strictly decreasing in ω. Therefore, it follows from the computation of Z in the proof of the previous lemma that every REE is fully revealing. We will now show that E^∞ has a unique REE. To do that we need a preliminary lemma.

Let $\mathcal{M}^*$ be the set of $(\omega, x, b, \lambda_0, \lambda_1) \in \Omega \times X \times \mathbb{R}_+ \times \mathbb{R}^I_+ \times \mathbb{R}^I_+$ such that:

$$\begin{array}{rcll} \sum_{i\in I} P_i(x_i \,|\, \omega) - m_0 & = & 0 & \\ v_i(\omega, x_i) - b - \lambda_{0,i} + \lambda_{1,i} & = & 0 & \forall i \\ x_i \lambda_{0,i} & = & 0 & \forall i \\ (1-x_i)\lambda_{1,i} & = & 0 & \forall i \end{array}$$

These equations characterize equilibrium prices. To see this, let ϕ be a price map. Fix $\omega \in \Omega$ and let $\phi(\omega) = b$. For each i, let x_i be the "cut-off" type determining demand (or supply). Let $\lambda_{0,i} = v_i(\omega, x_i) - b$ if $x_i = 0$ and zero otherwise; similarly, let $\lambda_{1,i} = b - v_i(\omega, x_i)$ if $x_i = 1$ and zero otherwise. Then it is simple to verify that b is a market clearing price iff $(\omega, x, b, \lambda_0, \lambda_1)$ solves the above system.

Lemma 3.2. *The natural projection from $\mathcal{M}^*$ to Ω is 1-1 and onto—thus, $\mathcal{M}^*$ is a connected 1-manifold. Moreover, $\mathcal{M}^*$ is monotonic in b, in the sense that if $(\omega, x, b, \lambda_0, \lambda_1)$ and $(\omega', x', b', \lambda_0', \lambda_1')$ belong to $\mathcal{M}^*$ and $\omega < \omega'$, then $b < b'$.*

Proof. We first show that the projection from $\mathcal{M}^*$ to Ω is one-to-one. Suppose not. Then we can take two different points $(\omega, x^k, b^k, \lambda_0^k, \lambda_1^k)$, $k = 1, 2$, in $\mathcal{M}^*$. Obviously $x^1 \neq x^2$. Assume w.l.o.g. that $b^2 \geqslant b^1$. By condition (2) of Assumption 2.3, the set J^1 of types j for whom $0 < x_j^1 < 1$ is nonempty. Now, $x_j^2 \geqslant x_j^1$ for all $j \in J^1$ as $b^2 \geqslant b^1$. Because $\sum_{i\in I} P_i(x_i^k \,|\, \omega) = m_0$ for each k, there exists some i for which $x_i^2 < x_i^1$. Necessarily this $i \notin J_1$ and thus $x_i^1 = 1$ but $x_i^2 < 1$. Then $v_i(\omega, x_i^2) \geqslant b^2 \geqslant b^1 \geqslant v_i(\omega, 1)$, which violates condition (2) of Assumption 2.2. Thus the projection map is one-to-one.

Next we show that $\mathcal{M}^*$ is monotone in b. Given two points $(\omega^k, x^k, b^k, \lambda_0^k, \lambda_1^k)$, $k = 1, 2$, in $\mathcal{M}^*$ with $\omega^1 < \omega^2$, observe that there exists some i such that $0 \leqslant x_i^1 < x_i^2 \leqslant 1$. For this i, then $b^1 \leqslant v_i(\omega^1, x_i^1) < v_i(\omega^2, x_i^2) \leqslant b^2$ and we get the monotonicity property for b.

$\mathcal{M}^*$ is nonempty by condition (1) of Assumption 2.3. Thus the lemma is proved if we show that $\mathcal{M}^*$ is a connected 1-manifold with boundary points projecting to the boundary of Ω. Assume to begin with that the valuations v_i and the prior P satisfy the following regularity condition, which holds generically in the space of valuations and priors. The set of solutions (ω, x) to every subcollection of equations derived from the following set of equations is a manifold, and these manifolds are pairwise transverse.

$$\begin{array}{rcl} v_i(\omega, x_i) - v_j(\omega, x_j) & = & 0 \\ \sum_{i\in I} P_i(x_i \,|\, \omega) - m_0 & = & 0 \\ x_i & \in & \{0, 1\} \\ \omega & \in & \{0, 1\} \end{array}$$

Take $(\omega, x, b, \lambda_0, \lambda_1) \in \mathcal{M}^*$. If for no i is it the case that $x_i = \lambda_{0,i} = 0$ or $1-x_i = \lambda_{1,i} = 0$, then the submatrix of the Jacobian at this point obtained by deleting the column corresponding to the derivative w.r.t. ω is nonsingular, with an upper bound on the norm of its inverse that depends only on the C^1 norm of the v_i's and P. This gives us, locally, the manifold property, and the inverse of the projection is locally C^1 with a bound on its derivative that depends only on the C^1 norm of (v, P).

Suppose now that we are at a singular point, where, say, $x_i = 0$ and $\lambda_{0,i} = 0$ for some i. (The other case where for some i, $x_i = 1$ and $\lambda_{1,i} = 0$ is similar.) By our regularity assumption this is the only coordinate where both x and λ are zero. Consider now the following two systems of equations derived from our original system: in the first, we set $x_i = \lambda_{1,i} = 0$ but retain $\lambda_{0,i}$ as a variable (along with the variables for the other types); and in the second, we set $\lambda_{0,i} = \lambda_{1,i} = 0$ but retain x_i as a variable. Ignoring the non-negativity constraints, each system has a unique 1-manifold of solutions, parametrized by the state in an ε-interval around ω. For each system, the continuation violates the non-negativity constraint (for $\lambda_{0,i}$ or x_i) on one half of the interval and satisfies it on the other half. Moreover, since the projection to Ω is one-to-one, the side on which the constraint is satisfied cannot be the same for the two systems. Thus the solution to one of the systems over $(\omega - \varepsilon, \omega]$ and the other over $[\omega, \omega + \varepsilon)$ gives us the manifold structure locally, and an upper bound on the derivatives for each of the two pieces. This proves the result in the generic case.

Now consider the general case. The inverse of the projection function in the generic case, which is piecewise differentiable, is a Lipschitz function with a constant that depends on the C^1 norm of the valuations and the prior. Take now a sequence (v^k, P^k) of functions that are generic in the above sense and converging in the C^1 norm to (v, P). For each k, there is a manifold $\mathcal{M}^k$ of solutions from the previous paragraph. The inverse of the projection is Lipschitz with a uniform bound on the constant (given the convergence of the sequence (v^k, P^k)). Hence there exists a convergent subsequence for which the inverse functions converge. The graph of the limit function is $\mathcal{M}$ and, of course, it projects onto Ω. □

The proof above shows that when valuations and the prior are generic, the inverse of the projection is piecewise-C^1, with a bound on the derivatives that depends continuously on (v, P). The manifold $\mathcal{M}^*$ is then piecewise-C^1 as well.

Define a function $\phi^* : \Omega \to \mathbb{R}_+$ as follows. For each ω, there exists a unique point $(\omega, x, b, \lambda_0, \lambda_1)$ in $\mathcal{M}^*$; $\phi^*(\omega) = b$ for this ω. Obviously ϕ^* is a continuous and monotone function. We now have the following theorem, whose proof is obvious.

Theorem 3.3. *ϕ^* is a fully-revealing REE; moreover it is the unique REE.*

From the equilibrium manifold, we can derive a function $\chi^* : \Omega \to X$ by setting it to be the unique x for which $(\omega, x, \phi^*(\omega), \lambda_0, \lambda_1) \in \mathcal{M}^*$ for some (λ_0, λ_1). For each ω, and i, $\chi_i^*(\omega)$ represents the cut-off type for whether i gets to trade or not. While ϕ^* is monotone in ω, the functions χ_i^* need not be. For the limit auction to have a monotone equilibrium, we need these functions χ_i^* to be monotone as well. And, for asymptotic results concerning auctions, we require a slightly stronger property to hold, which is basically that even at a point $(\omega, x, b, \lambda_0, \lambda_1) \in \mathcal{M}^*$ where some x_i is 0 or 1, the corresponding λ_i^0 (resp. λ_i^1) is locally strictly decreasing (resp. increasing). We now state this property formally below, where $>_L$ is the lexicographic ordering on vectors.

Definition 3.4. An REE ϕ^* is totally monotone if for each pair $(\omega^k, x^k, b^k, \lambda_0^k, \lambda_1^k)$, $k = 1, 2$ of points in $\mathcal{M}^*$ with $\omega^1 < \omega^2$, we have that for each i, $(x_i^2, -\lambda_0^2, \lambda_1^2) >_L (x_i^1, -\lambda_0^1, \lambda_1^1)$.

When valuations are private or if the game is symmetric, the REE of the economy is totally monotone. But more generally, we present a sufficient condition for totally monotone equilibria that is reminiscent of the average-crossing condition introduced by Krishna [13]. Define a function $Q^* : X \to \Omega$ as follows. $Q^*(x)$ is 0 (resp. 1) if $\sum_i P_i(x_i \,|\, 0)$ (resp. $\sum_i P_i(x_i \,|\, 1)$) is smaller (resp. greater) than m_0; otherwise it is the unique ω such that $\sum_i P_i(x_i \,|\, \omega) = m_0$. Q^* is a piecewise-C^2 function. Now we can define a "reduced form" valuation $v_i^* : X \to \mathbb{R}$ for each i by: $v_i^*(x) = v_i(Q^*(x), x)$. We say that v^* satisfies the average-crossing property if for all i, $j \neq i$ and x (s.t. Q^* is differentiable),

$$|I| \frac{\partial v_j^*(x)}{\partial x_i} < \sum_{k \in I} \frac{\partial v_k^*(x)}{\partial x_i}.$$

We now have the following proposition.

Proposition 3.5. *If v^* satisfies the average-crossing property then the REE is totally monotone.*

Proof. Fix $(\omega, x, b, \lambda) \in \mathcal{M}^*$. Let I^* be the set of i such that $v_i(\omega, x_i) = b$. Let A be the $I^* \times (I^* + 2)$ matrix obtained by the derivatives of the equations for players $i \in I^*$ with respect to the variables x_i for $i \in I^*$ along with ω and b. Let A^* be the matrix obtained by eliminating the column for ω and adding to each column i the column corresponding to ω scaled by $\frac{\partial P_i(x_i \,|\, \omega)}{\partial x_i} \left(\frac{\partial \sum_j P_j(x_j \,|\, \omega)}{\partial \omega} \right)^{-1}$. Then we solve for $A^*(\dot{x}_{I^*}, \dot{b}) = 0$ where $(\dot{x}_{I^*}, \dot{b})$ is an $(|I^*| + 1)$-column vector with $\dot{b} = 1$. By the average-crossing property, there is a unique solution, which has the property that $\dot{x}_{I^*} \gg 0$. Using the average-crossing property again, we get that for $i \notin I^*$, $\frac{\partial v_i(\omega,x_i)}{\partial\omega}\eta < 1$ where $\eta = \sum_{i\in I^*} \frac{\partial P_i(x_i \,|\, \omega)}{\partial x_i}\dot{x}_i\left(\frac{\partial \sum_j P_j(x_j \,|\, \omega)}{\partial\omega}\right)^{-1}$. Thus, over a small interval $[\omega, \omega+\varepsilon)$, the x_i^*'s are strictly increasing for $i \in I^*$, constant for $i \notin I^*$, and the λ_i's for $i \notin I^*$ have the required monotonicity property, which completes the proof. □

While it is simpler to state the average-crossing property as a global condition, what is required for the results to go through is that the property holds in a neighborhood of the manifold $\mathcal{M}^*$.

4. The Limit Auction

We now define a limit-game, Γ^∞, with player set I^∞. The state space $\Omega \times X^\infty$ and the probability measure P^* are as in the previous section. A pure strategy for player (i,n) is a measurable map from X_i to bids. We restrict ourselves to type-symmetric strategies, i.e. players of the same player-type i play the same strategy. Payoffs will be defined for each player and each bid of this player against a profile of his opponents and that is enough to determine whether a profile is an equilibrium. We show that if the REE is totally monotone, it is also the outcome of a pure monotone equilibrium. Also, under average-crossing, this is the unique Nash equilibrium outcome in monotone pure strategies.

A type-symmetric strategy profile can be represented by $\sigma \equiv (\sigma_i)_{i\in I}$, where σ_i is strategy of all players of type i. Given a profile σ, we now define the payoffs to a type x_i of player i when he bids b. First, the profile σ induces, through σ, a distribution $\beta^\infty(\cdot \,|\, \omega, \sigma)$ over the bids by the formula:

$$\beta^\infty(B \mid \omega, \sigma) = |I|^{-1} \sum_i P_i(\sigma_i^{-1}(B) \,|\, \omega).$$

for every measurable set B of bids. Let

$$\varrho^\infty(\omega, \sigma) \equiv \sup\{\, b \mid \beta^\infty([0,b] \mid \omega, \sigma) \leqslant \mu_0 \,\},$$

with the convention that the supremum of the empty set is 0. If σ is monotone, then as P satisfies MLRP, $\varrho^\infty(\,\cdot\,, \sigma)$ is a weakly monotone function of ω. We could also use a "dual" formula for the market-clearing price that uses the infimum over all bids where supply exceeds μ_1 (or indeed any convex combination of those), without changing the results.

For a player i who is a buyer (resp. seller) define $\tau_i^\infty(b, \omega, \sigma)$ to be 1 (resp. -1) if b is strictly greater (resp. strictly smaller) than $\varrho^\infty(\omega, \sigma)$; if b is strictly smaller (resp. strictly larger) than $\varrho^\infty(\omega, \sigma)$ for a buyer (resp. seller) let $\tau_i^\infty(b, \omega, \sigma) = 0$; otherwise let it be defined by the equation $\beta^\infty(\{\, b \,\} \,|\, \omega, \sigma^\infty)\tau_i^\infty(b, \omega, \sigma) = \mu_1 - \beta^\infty((b, \infty) \,|\, \omega, \sigma)$ if i is a a buyer, and by the equation $\beta^\infty(\{\, b \,\} \,|\, \omega, \sigma)\tau_i^\infty(b, \omega, \sigma) = \beta^\infty([0, b) \,|\, \omega, \sigma) - \mu_0$ i is a seller. Observe that if b is not an atom of σ_i for any i then the choice of $\tau^\infty(\omega, \sigma)$ for $b = \varrho^\infty(\omega, \sigma)$ is payoff-irrelevant.

The payoff to player i of type x_i from a bid b against σ is given by:

$$\pi_i^\infty(b_i, \sigma; x_i) = \int_\Omega \tau_i^\infty(b_i, \omega, \sigma)[v_i(\omega, x_i) - \varrho^\infty(\omega, \sigma)]dP(\omega \mid x_i).$$

Call a strategy profile σ trivial if for each i, the function $\tau_i^\infty(\sigma^\infty(\cdot), \cdot, \sigma) : \Omega \times X_i \to [0, 1]$ is constant. If a strategy profile is monotone, then it is trivial iff there exists a subset J with $|J| = m_1$ such that player-types in J are allocated the object with probability one. In particular, if Γ is a double auction, nontriviality rules out the possibility that no trade occurs—as for instance a strategy profile where sellers have a very high ask and buyers make a very small bid and no trade takes place.

Our first result shows that when the economy E^∞ has a totally monotone REE, the corresponding game Γ^∞ has a monotone and nontrivial equilibrium with the same clearing prices as in the REE. To identify this Nash equilibrium, we use the manifold $\mathcal{M}^*$. If $(\omega, x, b, \lambda_0, \lambda_1) \in \mathcal{M}^*$, then we define the bid of type x_i to be b. We could already define an equilibrium by extending the strategy to signals x_i not covered by $\mathcal{M}^*$ in a monotone, but otherwise arbitrary, way. But anticipating our asymptotic results, we extend the manifold in a specific way by adding pieces that project to $\omega = 0, 1$ and then using this manifold to construct σ^*. Let $\mathcal{M}^0$ be the set of $(0, x, b, \lambda_0, 0) \in \Omega \times X \times \mathbb{R}_+ \times \mathbb{R}_+^I \times \mathbb{R}_+^I$ such that:

$$\begin{aligned} \textstyle\sum_i P_i(x_i \mid 0) - m_0 &\leqslant 0 \\ v_i(0, x_i) - b - \lambda_{0,i} &= 0 \quad \forall i \\ x_i \lambda_{0,i} &= 0 \quad \forall i \end{aligned}$$

$\mathcal{M}^1$ is defined analogously by replacing $\omega = 0$ with $\omega = 1$, reversing the first inequality, setting $\lambda_0 = 0$ and using λ_1. As in the previous section, it is easy to show that $\mathcal{M}^0$ is a 1-manifold that has two boundary points, one of which is the boundary point of $\mathcal{M}^*$ at $\omega = 0$ and the other is a boundary point $(0, 0, b, \lambda_0, 0)$ where $b = \min_i v_i(0, 0)$. A similar statement holds for $\mathcal{M}^1$. Moreover, $\mathcal{M}^0$ and $\mathcal{M}^1$ are totally monotone themselves (even without assuming average-crossing) since the valuations used in defining these sets have a fixed state ω.

Let $\mathcal{M} \equiv \mathcal{M}^0 \cup \mathcal{M}^* \cup \mathcal{M}^1$. Assume that ϕ^* is totally monotone. For each i and $0 < x_i < 1$, there exists a unique point $(\omega_i^*(x_i), x_{-i}, x_i, b^*(x_i), \lambda_0, \lambda_1)$ in $\mathcal{M}$. We define a strategy σ_i^* for player i by letting $\sigma_i^*(x_i)$ be this $b^*(x_i)$. The function extends continuously to the points $x_i = 0, 1$ and the resulting strategy is piecewise-differentiable. Also, the function $\omega_i^*(x_i)$ extends continuously as well.

Theorem 4.1. *If ϕ^* is totally monotone then σ^* is a nontrivial equilibrium of Γ^∞, and $\varrho^\infty(\cdot, \sigma^*) = \phi^*(\cdot)$.*

Proof. From the construction of σ^* it is obvious that σ^* is monotone and nontrivial, and that $\varrho^\infty(\cdot,\sigma^*) = \phi^*(\cdot)$. There remains to show that σ^* is an equilibrium. In the profile σ^*, we check the incentives for a buyer, leaving out the similar argument for a seller. Take a type x_i of buyer of type i. Observe first that x_i's payoff under σ^* is non-negative; indeed it is given by

$$\int_0^{\omega_i^*(x_i)} (v_i(\omega, x_i) - \varrho^\infty(\omega,\sigma^*))dP(\omega \mid x_i),$$

and the integrand is non-negative as the valuation is strictly increasing in i's signal.

As the payoff under σ^* is non-negative, there is no benefit to bidding outside the range $[\varrho^\infty(0,\sigma^*), \varrho^\infty(1,\sigma^*)]$. There remains to consider a deviation to a bid b in this range. Suppose first that b is the bid of some $y_i \in X_i$. If $x_i > y_i$, then the payoff difference between bidding $\sigma^*(x_i)$ and $\sigma^*(y_i)$ is

$$\int_{\omega_i^*(y_i)}^{\omega_i^*(x_i)} (v_i(\omega, x_i) - \varrho^\infty(\omega,\sigma^\infty))dP(\omega \mid x_i)$$

which is non-negative as before. A similar argument shows that x_i does not have an incentive to mimic a type $y_i > x_i$. If $\varrho^*(\omega) = b > \sigma_i^*(1)$ then $b > v_i(\omega, 1) > v_i(\omega, x_i)$ and there is no incentive to bid b; likewise, with the inequalities reversed we get that there is no incentive bid below $\sigma_i^*(0)$. Thus σ^* is an equilibrium. □

We now address the issue of uniqueness of equilibria.

Theorem 4.2. *Suppose that v^* satisfies the average-crossing condition and let σ be a nontrivial monotone equilibrium. Then for each $\omega \in \Omega$, $\varrho^\infty(\omega,\sigma) = \phi^*(\omega)$ and $\sigma_i(\chi_i^*(\omega)) = \phi^*(\omega)$ for each i.*

Proof. Let σ be a nontrivial and monotone equilibrium of Γ^∞. We claim that for each type i, σ_i restricted to $\sigma_i^{-1}(\varrho^\infty((0,1);\sigma))$, and hence also the function $\varrho^\infty(\cdot,\sigma)$, is strictly increasing. Indeed, otherwise, giving the nontriviality of σ, there exists a bid b, a type i, an interval X_i^* of X_i and an interval Ω^* of Ω such that types in X_i^* bid b, which is also the clearing price for states in Ω^*. The expectation of $v_i(x_i,\cdot)$ conditional on Ω^* is strictly increasing for i. Therefore, for a.e. signal x_i the conditional expectation is either strictly greater or smaller than b. But, the probability of trading in those states is strictly between zero and one, as σ is nontrivial. Therefore bidding up or down a little yields a higher payoff, contradicting the assumption that σ is an equilibrium.

We now show that, for each i, the restriction of σ_i to $\sigma_i^{-1}(\varrho^\infty((0,1));\sigma)$ and hence also ϱ^∞ is continuous. Indeed, otherwise, there exists x_i and $b^1 < b^2$ that are left and right limits of σ_i at x_i. For $k = 1, 2$, there exist states $\omega^1 \leqslant \omega^2$ such that b^1 and b^2 are the left and right limits of ϱ^∞ at these states. $v_i(\omega^k, x_i) = b^k$ for $k = 1, 2$, from the fact that σ is an equilibrium. Clearly then $\omega^1 < \omega^2$. It now follows from the average-crossing property that $\varrho^\infty(\omega, \sigma) > v_i(\omega, x_i)$ for any $\omega \in (\omega^1, \omega^2]$, implying that b^2 cannot be an optimal strategy for x_i, which is the desired contradiction.

Fix a state $\omega \in (0, 1)$ and let b be the clearing price in ω. For each i, if b is in the support of σ_i, let x_i be the signal that bids b. If b is below (resp. above) the support of σ_i, let x_i be 0 (resp. 1). Let λ be the vector of slack-variables solving the equations for $\mathcal{M}^*$. We will now show that (ω, x, b, λ) belongs to $\mathcal{M}$, which completes the proof. If b is in the support of σ_i, then as σ is an equilibrium and ϱ^∞ is continuous, $v_i(\omega, x_i) = b$ and $\lambda_i = 0$. If b is below the support of σ_i, we have to show that $\lambda_{i,0} \geqslant 0$. Indeed, otherwise, it follows from the average crossing property that $v_i(\omega, x_i) < \varrho^\infty(\omega, \sigma)$ on the interval $[\omega, \omega_0]$, where $\varrho^\infty(\omega_0, \sigma) = \sigma_i(0)$, making $\sigma_i(0)$ suboptimal. A similar argument applies when b is above the support of σ_i. □

In the game Γ^∞ the strategy sets can be expanded to allow for randomizations, i.e., for behavioral strategies. When the valuations are private, it is very easy to show that there is a unique equilibrium in behavioral strategies, which then induces the REE outcome. In the general case, however, even assuming the average-crossing property, it is not clear if the REE outcome is the unique Nash outcome in behavioral strategies. In fact, we do not even know if the REE outcome is isolated in the set of Nash outcomes. The one, more general, result is that with average-crossing, we can extend Theorem 4.2 to obtain uniqueness in pure strategies, i.e. dispense with the monotonicity requirement for σ in the statement of the theorem.

5. Asymptotic Analysis I

In this section, we study the asymptotic properties of ε-equilibria of the auctions Γ^n. First, given a strictly monotone equilibrium σ^∞ of Γ^∞ that is the limit of a sequence of monotone strategy profiles σ^n, for each $\varepsilon > 0$, there exists N such that for all $n \geqslant N$, σ^n is an ε-equilibrium of Γ^n; in particular, taking σ^n to be the constant sequence σ^∞, we get that σ^∞ is an epsilon-equilibrium of Γ^n for large n. Second, every limit as $\varepsilon_n \downarrow 0$ of a sequence of ε_n-equilibria of Γ^n in monotone strategies is a strictly monotone equilibrium of Γ^∞. Third, for such a sequence of ε_n-equilibria, we obtain a Strong Law of Large Numbers for the corresponding sequence of equilibrium prices, in that it converges a.s. to the clearing price under the limit; under uniform convergence of the ε_n-equilibria, we even get the weak-$*$ convergence of the distribution of prices to the limit price for a.e. state; and finally, under an assumption on the rate of convergence, we obtain a Central Limit Theorem for the equilibrium prices. The results of the previous section concerning the equivalence of

REE and Nash equilibria establish the appropriate connection between ε-equilibria of large auctions and the competitive limit.

Given the symmetry among players of the same type, we can now view the game Γ^n for each n as a game with player set I^∞, where the players (i,m), for $m > n$, are dummy players. A symmetric strategy profile σ^∞ in Γ^∞ induces a profile in Γ^n by projecting to the first n players of each type i and then the payoff $\pi_i^n(b,\sigma^\infty;x_i)$ to each i, x_i and bid b against σ^∞ is the payoff he would get in Γ^n by projecting σ^∞ to the first n factors. The payoff π_i^n of player-type i depends on the probability $\tau_i^n(b,\omega,\sigma)$ of trading and the expected clearing price function $\varrho_i^n(b,\omega,\sigma)$, which are as defined for Γ, only now with player set I^n. We also write $\beta^n(\cdot\,|\,\omega,\sigma)$ for the probability distribution over the clearing price in state ω under the profile σ; let $\varrho^n(\omega,x^\infty,\sigma)$ denote the clearing price under σ when the state of the world is (ω,x^∞). The next lemma is crucial in establishing a continuity property of payoffs.

Lemma 5.1. *Suppose σ^n is a monotone strategy profile converging pointwise to a nontrivial profile σ^∞. Let $\omega\in(0,1)$ be a point of continuity of $\varrho^\infty(\cdot,\sigma^\infty)$. Then:*

(1) *$\varrho^n(\omega,\cdot,\sigma^n)\to\varrho^\infty(\omega,\sigma^\infty)$ pointwise a.e. on X^∞;*
(2) *for each i, and each sequence b^n:*
  (a) *$\varrho_i^n(b^n,\omega,\sigma^n)\to\varrho^\infty(\omega,\sigma^\infty)$;*
  (b) *if $\lim b_n = b\neq\varrho^\infty(\omega,\sigma^\infty)$, then $\tau_i^n(b^n,\omega,\sigma^n)\to\tau^\infty(b,\omega,\sigma^\infty)$.*

Proof. Let $b_0=\varrho^\infty(\omega,\sigma^\infty)$. Take $\varepsilon>0$ such that the bids $b_0\pm\varepsilon$ are not atoms of the strategy profile σ^∞ (i.e., for each i a measure zero set of types x_i bid $b_0\pm\varepsilon$). We will show that for large n, the probability that the clearing price lies outside the interval $[b_0-\varepsilon,b_0+\varepsilon]$ is exponentially small in n. The Borel-Cantelli Lemma then proves (1).

For each i, let x_i be the supremum over y_i such that $\sigma_i^\infty(y_i)$ is less than $b_0-\varepsilon$, with the convention that the supremum of the empty set is 0. We claim that $\sum_i P_i(x_i\,|\,\omega)<m_0$. Indeed, since b_0 is the clearing price at ω, clearly $\sum_i P(x_i\,|\,\omega)\leqslant m_0$. If the strict inequality does not hold, then, because ω is a point of continuity of $\varrho^\infty(\cdot,\sigma^\infty)$, $b_0=\varrho^\infty(\omega',\sigma^\infty)$ for all $\omega'<\omega$ sufficiently close to ω, even though $\sum_i P_i(x_i\,|\,\omega')\geqslant m_0$ for all such ω', which is possible only if x_i is either 0 or 1 for all i, i.e. σ^∞ is a trivial profile, contrary to our assumption. Therefore, $\sum_i P_i(x_i\,|\,\omega)<m_0$. Choose now $\eta>0$ such that for each i, letting $x_i(\eta)=\min(x_i+\eta,1)$, $\sum_i P_i(x_i\,|\,\omega)<\sum_i P_i(x_i(\eta)\,|\,\omega)\equiv\underline{m}_0<m_0$; and choose N such that $\sigma_i^n(x_i(\eta))>b_0-\varepsilon$ for all $n\geqslant N$ if $x_i\neq 1$. Let $\delta=m_0-\underline{m}_0$. For $n\geqslant N$, a price $b\leqslant b_0-\varepsilon$ is a clearing price under σ^n in Γ^n only if at least $m_0 n$ agents have signals below $x_i+\eta$. Hoeffding's inequality shows that the probability of this event is at most $\exp(-2n\delta^2 I^{-2})$.

A similar argument shows that the probability of the clearing price being above $b_0+\varepsilon$ is exponentially small, which completes the proof of (1).

Both parts of point (2) follow if we show that given that player $(i,1)$ bids b, the clearing price, as a random variable on $X^{\infty}_{(i,1)}$, converges to b_0. The proof of this convergence follows the logic for the corresponding convergence of ϱ^n, with the following modification from the last paragraph. For each $n \geqslant N$, choose $\delta_n = \frac{m_0 n - I}{n-1} - \underline{m}_0$. (Of course we are assuming here that N is large enough such that $\delta_n > 0$.) The probability bounds hold when we use δ_n instead of δ and n is replaced with $n-1$. □

Remark 5.2. In point 2(b), we can dispense with the assumption $b \neq \varrho^{\infty}(\omega, \sigma^{\infty})$ if $\varrho^{\infty}(\cdot, \sigma^{\infty})$ is strictly increasing at ω. In other words, the problem arises only when there is an interval of states for which $\varrho^{\infty}(\omega, \sigma^{\infty})$ is a clearing price in Γ^{∞}.

To see the importance of the assumption of nontriviality in the lemma, take a trivial strategy profile σ^{∞} with a subset m_0 of traders bidding b_0 and the remaining traders bidding $b_1 > b_0$. Then the market price, using our formula, is b_1 while taking σ^n to be the constant sequence in the game Γ^n, the clearing price is $\alpha b_1 + (1-a)b_0$.

Our first result shows that equilibria of Γ^{∞} are ε-equilibria of Γ^n for large n.

Theorem 5.3. *Let σ^n be a sequence of strategy profiles converging to a nontrivial monotone equilibrium σ^{∞} of Γ^{∞}. Then, for each $\varepsilon > 0$, there exists N such that for each $n \geqslant N$, σ^n is an ε-equilibrium of Γ^n.*

Proof. As shown in the proof of Theorem 4.2, $\varrho^{\infty}(\cdot, \sigma)$ is strictly monotone. The result then follows from applying the conclusions of point (2) of the previous lemma along with Remark 5.2. □

We now have the following asymptotic result going in the other direction.

Theorem 5.4. *For each n, let σ^n be an ε_n-equilibrium of Γ^n in monotone pure strategies, where $\varepsilon_n \to 0^+$. Let σ^{∞} be a nontrivial strategy profile that is a limit point of the sequence under the topology of pointwise convergence. Then σ^{∞} is an equilibrium of Γ^{∞}.*

Proof. It is sufficient to prove that for each b, there is at most one state ω such that $b = \varrho^{\infty}(\omega, \sigma^{\infty})$. Indeed, it follows from point (2) of Lemma 5.1 that for each sequence $b^n \to b$, each i and x_i, $\pi_i^n(b^n, \sigma^n, x_i) \to \pi_i^{\infty}(b, \sigma^{\infty}, x_i)$ and therefore σ^{∞} must be an equilibrium. To prove this point, suppose to the contrary that there is some b and a nontrivial interval $\Omega^* \equiv [\underline{\omega}^*, \overline{\omega}^*]$ of states such that $b = \varrho^{\infty}(\omega, \sigma^{\infty})$ for all $\omega \in \Omega^*$. Since σ^{∞} is nontrivial, there exists a nonempty subset I^* of players i for whom a non-null subset $X_i^* = [\underline{x}_i^*, \overline{x}_i^*]$ of types bid b under σ_i^∞ with $\sum_i P_i((\sigma_i^\infty)^{-1}[0,b)\,|\,\omega) < m_0$ and $\sum_i P_i((\sigma_i^\infty)^{-1}(b,\infty)\,|\,\omega) < m_1$ for all ω in the interior of Ω^*.

For each $i \in I^*$, let $\tilde{\tau}_i^\infty : \Omega^* \times X_i^* \to [0,1]$ be the weak-$*$ limit in $L_\infty(\Omega^* \times X_i^*, [0,1])$ of the function $\tilde{\tau}_i^n(\omega, x_i) \equiv \tau_i^n(\sigma_i^n(x_i), \omega, \sigma^n)$. $\tilde{\tau}_i^\infty$ is weakly increasing in x_i, as this property holds along the sequence. Also, clearly for some $i \in I^*$, there exists a subset of X_i^* with positive measure such that $\int_{\Omega^*} \tilde{\tau}_i^\infty(\omega, x_i) d\omega \neq 0, 1$ for all $x_i \in X_i^*$. Fix such an i. Point (2) of Lemma 5.1 now shows that for a.e. $x_i \in X_i^*$:

$$\begin{aligned}\lim_n \pi_i^n(\sigma_i^n(x_i), \sigma^n; x_i) &= \int_{\Omega \setminus \Omega^*} \tau_i^\infty(b, \omega, x_i)[v_i(\omega, x_i) - \varrho^\infty(\omega, \sigma^\infty)] dP(\omega\,|\,x_i) \\ &+ \int_{\Omega^*} [v_i(\omega, x_i) - b] \tilde{\tau}_i^\infty(\omega, x_i) dP(\omega\,|\,x_i).\end{aligned}$$

Assume for the moment that $\tilde{\tau}_i^\infty$ is weakly monotonically decreasing in ω. Suppose that i is a buyer (the argument for a seller is symmetric). Then,

$$\frac{\int_{\Omega^*} [v_i(\omega, x_i) - b](1 - \tilde{\tau}_i^\infty(\omega, x_i)) dP(\omega\,|\,x_i)}{\int_{\Omega^*} (1 - \tilde{\tau}_i^\infty(\omega, x_i)) dP(\omega\,|\,x_i)}$$

is strictly increasing in x_i. If for some $x_i \in X_i^*$ this is non-negative (resp. non-positive), then it is strictly positive (resp. negative) for all higher (resp. lower) x_i, which means that by bidding up (resp. down) to some $b + \delta$ (resp. $b - \delta$), for some sufficiently small $\delta > 0$, x_i can increase its payoff in Γ^n by more than ε_n for large n, which is impossible.

To finish the proof, it remains to be shown that $\tilde{\tau}_i^\infty$ is weakly monotonically decreasing in ω for each $i \in I^*$. Fix $i \in I^*$. For each $j \in I^*$ and n, define $\underline{\gamma}_{ij}^n, \overline{\gamma}_{ij}^n : X_i^* \to X_j^*$ by $\underline{\gamma}_{ij}^n(x_i) = \sup\{\, x_j \,|\, \sigma_j^n(x_j) < \sigma_i^n(x_i) \,\}$ and $\overline{\gamma}_{ij}^n(x_i) = \inf\{\, x_j \,|\, \sigma_j^n(x_j) > \sigma_i^n(x_i) \,\}$, with the convention that the supremum (resp. infimum) of the empty set is 0 (resp. 1). $\underline{\gamma}_{ij}^n \leqslant \overline{\gamma}_{ij}^n$ are monotonic functions and by going to a subsequence they have limits $\underline{\gamma}_{ij}^\infty$ and $\overline{\gamma}_{ij}^\infty$. Take any x_i that is a point of continuity of $\underline{\gamma}_{ij}^\infty$ and $\overline{\gamma}_{ij}^\infty$. It follows from the SLLN that $\tilde{\tau}_i^\infty(\omega, x_i)$ equals the allocation x_i would receive if he bids b and each j played a strategy where types in $[\underline{\gamma}_{ij}^\infty, \overline{\gamma}_{ij}^\infty]$ bid b, while types above (resp. below) bid above (resp. below) b. Clearly this is weakly decreasing in ω. □

The nontriviality condition can be dispensed with when $\alpha > 0$ and Γ is an auction (and not a double auction). Otherwise, there are trivial equilibria of Γ^n where a subset J_1 of types bid a high number and the other types bid a small number. This strategy profile cannot be an equilibrium of Γ^∞, where bids affect only the probability $\tau^\infty(\cdot)$ of winning and not the price $\varrho^\infty(\cdot)$.

Remark 5.5. If valuations are private, we can actually obtain a convergence result for equilibria in behavioral strategies as well: limits of equilibria of Γ^n are equilibria of Γ^∞. In this case, since Γ^n has an equilibrium in behavioral strategies (cf. Jackson and Swinkels [14]), and since it can be shown that the REE outcomes is the unique nontrivial equilibrium in Γ^∞, we get asymptotic efficiency for all sequences of equilibria, replicating Cripps and Swinkels [2].

We have the following Strong Law of Large Numbers for equilibrium prices, whose proof follows from point (1) of Lemma 5.1.

Corollary 5.6. *Let σ^n and σ^∞ be as in Theorem 5.4. Then for a.e. (ω, x^∞), $\varrho^n(\omega, x^\infty, \sigma^n)$ converges to $\varrho^\infty(\omega, \sigma^\infty)$.*

The above corollary implies, in particular, the weak-$*$ convergence of $\beta^n(\cdot \,|\, \omega, \sigma^n)$ to the point-mass concentrated at $\varrho^\infty(\omega, \sigma^\infty)$. Under stronger assumptions on the convergence of σ^n, we can draw stronger conclusions about the convergence of β^n to ϱ^∞, as we demonstrate now through two theorems. The stated assumptions on convergence is a bit stronger than what we need, in that we posit this convergence on the whole domain of the signal space, though what we need is that it hold only in a neighborhood of the inverse image of $[\phi^*(0), \phi^*(1)]$, i.e. only for bids that matter for determining the price.

Theorem 5.7. *For each n, let σ^n be an ε_n-equilibrium of Γ^n in monotone pure strategies, where $\varepsilon_n \to 0^+$. Let σ^∞ be a nontrivial and continuous strategy profile such that σ^n converges uniformly to σ^∞. Then the weak-$*$ convergence of $\beta^n(\cdot \,|\, \omega, \sigma^n)$ to $\varrho^\infty(\omega, \sigma^\infty)$ is uniform in ω.*

Proof. By Theorem 5.4, σ^∞ is an equilibrium of Γ^∞. Therefore, $\varrho^\infty(\cdot, \sigma^\infty)$ is strictly monotone. As σ^∞ is continuous, so is $\varrho^\infty(\cdot, \sigma^\infty)$. For each $\varepsilon > 0$, the construction in the proof of Lemma 5.1 to obtain the Hoeffding bounds on the probability of the clearing price lying outside $\varrho^\infty(\omega, \sigma^\infty) \pm \varepsilon$ for large n can be done independently of ω thanks to the assumption of uniform convergence of σ^n to σ^∞, which then proves the result. □

Finally, we obtain a Central Limit Theorem when the limit equilibrium is differentiable. Suppose that σ^∞ is a strictly monotone and a.e. differentiable strategy profile. Let B be an interval of bids that contains the range of σ_i^∞ for all i. The inverse bidding function ς_i^∞ of σ_i^∞ extends to a function over B by letting $\varsigma_i^\infty(b)$ be either zero or one depending on whether b is below or above the range of σ_i^∞ and it is differentiable a.e. Define $G(b \,|\, \omega, \sigma^\infty) = \sum_i P_i(\varsigma_i^\infty(b) \,|\, \omega)$. For a.e. ω, $G(b \,|\, \omega, \sigma^\infty)$ is differentiable in b at $\varrho^\infty(\omega, \sigma^\infty)$; denote by $g(\omega, \sigma^\infty)$ this derivative. Also let $h(\omega, \sigma^\infty) = \sum_i P_i(\varsigma_i^\infty(\varrho^\infty(\omega, \sigma^\infty)) \,|\, \omega)(1 - P_i(\varsigma_i^\infty(\varrho^\infty(\omega, \sigma^\infty)) \,|\, \omega))$.

Theorem 5.8. *Suppose that the rate of convergence of σ^n to σ^∞ in Theorem 5.7 is in $o(n^{-\frac{1}{2}})$. Suppose further that σ^∞ is strictly monotone and differentiable a.e. Then for a.e.*

ω, $\sqrt{n}[\varrho^n(\omega,\cdot,\sigma^n)-\varrho^\infty(\omega,\sigma^\infty)]$ converges weak-$*$ to the Normal distribution with mean zero and variance $h(\omega,\sigma^\infty)(g(\omega,\sigma^\infty))^{-2}$.

Proof. Using the Central Limit Theorem for quantiles, $\sqrt{n}[\varrho^n(\omega,\cdot,\sigma^\infty)-\varrho^\infty(\omega,\sigma^\infty)]$ converges to the Normal distribution with mean zero and variance $h(\omega,\sigma^\infty)(g(\omega,\sigma^\infty))^{-2}$. By our assumption on the rate of convergence of σ^n to σ^∞, $\sqrt{n}[\varrho^n(\omega,\cdot,\sigma^\infty)-\varrho^n(\omega,\cdot,\sigma^n)]$ converges to zero a.s. The result then follows from Slutsky's Theorem. □

We remark that when $\varrho^\infty(\cdot,\sigma^\infty)$ coincides with $\phi^*(\cdot)$ and the data is generic, it is differentiable outside of a finite set, and the convergence in the theorem above is uniform on compact subsets of the open set where it is differentiable.

6. Asymptotic Analysis II

Theorems 4.1 and 5.3 already establish the existence of ε-equilibria of Γ^n for large n, namely σ^* itself viewed as a strategy profile in Γ^n. Assuming that the REE is totally monotone and satisfies a second condition, to be specified shortly, we now provide another lower semicontinuity result. If we restrict the strategy spaces to bid grids that are sufficiently fine, then we can prove the existence of a sequence of exact equilibria of the corresponding game with finitely many agents. These bid-grid equilibria converge to the equilibrium associated with the REE and are thus ε-equilibria of Γ^n for large n.

Assume that the REE is totally monotone. Also we make the following assumption on the equilibrium manifold $\mathcal{M}^*$. This assumption is satisfied, a.o., by the private values model and the model where all types are symmetric. But it is implied by conditions that are slightly weaker than those, since it invokes symmetry only around the equilibrium manifold, and it does not invoke symmetry of distributions.

Assumption 6.1. $\mathcal{M}^*$ is a C^2-manifold with boundary; and for each $(\omega,x,b)\in\mathcal{M}^*$ and $i,j\in I$, $\frac{\partial v_i(\omega,x_i)}{\partial\omega}=\frac{\partial v_j(\omega,x_j)}{\partial\omega}$.

The first part of the assumption is made only for convenience of exposition. Similar to the proof of Lemma 3.2, it can be dispensed with because we can approximate $\mathcal{M}^*$ with one that is C^2.

Let σ^* be the Nash equilibrium of Γ^∞ defined in Section 4, which induces the REE outcome. Assume without loss of generality that the valuations are strictly positive so that there exists $\delta_x>0$ such that we can extend each v_i to a monotone C^2 function from $\Omega\times[-\delta_x,1+\delta_x]$ to $\mathbb{R}_+$ such that $v_i(0,-\delta_x)=v_j(0,-\delta_x)$ and $v_i(1,1+\delta_x)=v_i(1,1+\delta_x)$ for all i,j. Also extend the distributions $P_i(\omega\,|\,x_i)$ to the larger interval of types still satisfying MLRP and still C^2.

Use X_i, still, to denote $[-\delta_x, 1+\delta_x]$. $P_i(x_i \,|\, \omega)$ is now used to represent the probability $P_i([0,1] \cap X_i] \,|\, \omega)$, i.e., $P_i(x_i \,|\, \omega)$ is zero for $x_i < 0$ and one for $x_i > 1$. $P_i(\cdot \,|\, \omega)$ is then piece-wise differentiable. $Q^*(x)$ is defined as previously in Section 3, only now it uses the modified definition of the P_i's. Then $v_i(Q^*(x), x_i)$ is differentiable at a.e. x.

It is convenient to represent the manifold $\mathcal{M}$ from Section 4 as triples (ω, x, b) by dropping the slack variables so that $\mathcal{M}$ is the set of (ω, x, b) such that $v_i(\omega, x) - b = 0$ for all i and $\omega - Q^*(x) = 0$. (Extending the signals to a neighborhood of $[0,1]$ allows this representation.) We can take an $\varepsilon > 0$ such that over the ε-neighborhood U of $\mathcal{M}$, the Jacobian of the system defining $\mathcal{M}$ has full row rank; and, in fact, deleting any column corresponding to x_i or b, or even ω when ω is locally nonconstant, yields a non-singular square matrix. We can also extend the equilibrium bidding function σ^* to the larger set of signals $[-\delta_x, 1+\delta_x]$ using the same formula, i.e., $\sigma_i^*(x) = b$ for the unique (ω, x_{-i}, b) such that $(\omega, x, b) \in \mathcal{M}$.

Take a sequence of positive numbers $\zeta \to 0$. For each ζ in the sequence, and for each n, including for $n = \infty$, let $\Gamma^{n,\zeta}$ be the game where the set of admissible bids is $\{\, 0, \zeta, 2\zeta, \dots, \,\}$. We use $b^k(\zeta)$ to denote $k\zeta$, and when the ζ we are using is unambiguous, we simply write b^k. Let b^{k_0} (resp. b^{k_1}) be the highest (resp. lowest) bid in $\Gamma^{n,\zeta}$ that is below (resp. above) $v_i(0,0)$ (resp. $v_i(1,1)$) for all i. If necessary by dropping finitely many terms of the sequence of ζ's, we can assume that for each ζ, and i, $b^{k_0} > v_i(0, -\delta_x)$ and $v_i(1, 1+\delta_i) > b^{k_1}$. Let $B(\zeta)$ be the bids $\{\, k_0\zeta, \dots, k_1\zeta \,\}$. There is no loss in restricting players to choosing bids in $B(\zeta)$.

Let $\theta^{*,\zeta} : B(\zeta) \to \Omega \times X$ be the function defined by letting $\theta^{*,\zeta}(b^k)$ be the unique vector $(\omega^*(k), x^*(k))$ such that $(\omega^*(k), x^*(k), b^k)$ belongs to $\mathcal{M}$. Let $b^{k_0^*}$ (resp. $b^{k_1^*}$) be the highest (resp. lowest) bid in $B(\zeta)$ that is below (resp. above) $\phi^*(0)$ (resp. $\phi^*(1)$). For the sequence of ζ's we are considering, we will assume that there exist $0 < \lambda_0, \lambda_0^*, \lambda_1, \lambda_1^* < \frac{1}{2}$ such that $b^{k_0+1}(\zeta) - \min_i v_i(0,0) = 2\lambda_0\zeta + O(\zeta^2)$, $\max_i v_i(1,1) - b^{k_1-1}(\zeta) = 2\lambda_1\zeta + O(\zeta^2)$ and similarly with $(k_0^*, k_1^*, \lambda_0^*, \lambda_1^*)$ replacing $(k_0, k_1, \lambda_0, \lambda_1)$, using $\phi^*(0)$ in the place of $\min_i v_i(0,0)$ and $\phi^*(1)$ for $\max_i v_i(1,1)$. Thus, we are considering a "generic" sequence of ζ's. Fix $0 < \eta < \min\{\, \frac{1}{11}, \frac{4}{3}\lambda_0, \frac{4}{3}\lambda_1, \frac{4}{3}\lambda_0^*, \frac{2}{3}(1-2\lambda_0^*), \frac{1}{2}\lambda_1^*(1-\lambda_1^*) \,\}$. For each $k_0^* + 1 < k \leqslant k_1^*$, let $\underline{\theta}_0(k) = (\phi^*)^{-1}(b^{k-1} + 2\eta\zeta)$; and for $k_0^* + 1 \leqslant k < k_1^*$, let $\overline{\theta}_0(k) = (\phi^*)^{-1}(b^k + \eta\zeta)$. For each i: let $\underline{\theta}_i(k)$ be the unique x_i for which: (1) $\sigma_i^*(x_i) = \max(b^{k-1}, \min_j v_j(0,0)) + \frac{3}{2}\eta\zeta$ if $k_0 < k \leqslant k_1^*$; (2) $\sigma_i^*(x_i) = b^{k-1} - \frac{1}{2}\eta\zeta$ if $k > k_1^*$; let $\overline{\theta}_i(k)$ be the unique x_i for which: (3) $\sigma_i^*(x_i) = b^k + \frac{1}{2}\eta\zeta$ if $k_0 \leq k < k_1^*$; (4) $\sigma_i^*(x_i) = b^k - \frac{3}{2}\eta\zeta$ if $k \geqslant k_1^*$.

Let Θ^ζ be the closure of the set of all functions $\theta : B(\zeta) \to \Omega \times X$ such that for each k, writing $\theta(k)$ as short for $\theta(b^k)$, with $\theta(k) = (\theta_0(k), \theta_{-0}(k))$: (0) $\theta(k_0) = \theta^*(k_0)$; (1a) $\theta_0(k) \in (\underline{\theta}_0(k), \overline{\theta}_0(k))$ if $k_0^* + 2 \leqslant k \leqslant k_1^* - 1$; (1b) $\theta_0(k) \in [0, \overline{\theta}_0(k))$ if $k = k_0^* + 1$ and $\theta_0(k) \in (\underline{\theta}_0(k), 1]$ if $k = k_1^*$; (1c) $\theta_0(k)$ is zero (resp. one) if $k \leqslant k_0^*$ (resp. $k \geqslant k_1^* + 1$);

(2) $\theta_i(k) \in (\underline{\theta}_i(k), \overline{\theta}_i(k))$ for all i and $k_0 < k \leqslant k_1$; (3) $\theta_0(k) = Q^*(\theta_{-0}(k))$. If necessary by dropping finitely many ζ's from the sequence, we now have that $(\theta(k), b^k) \in U$ for each θ in the closure of Θ^ζ.

For $k_0^* + 1 \leqslant k \leqslant k_1^*$, let Ω^k be the interval of ω's specified in sub (1) above. Θ^ζ is a compact $|I|(|B(\zeta)| - 1)$-dimensional manifold with boundary points consisting of θ where either: $\theta_0(k)$ belongs to $\partial\Omega^k \setminus \Omega^k$ for some $k_0^* + 1 \leqslant k \leqslant k_1^*$; or for some i, $\theta_i(k)$ is in $\{ \underline{\theta}_i(k), \overline{\theta}_i(k) \}$.

For each $\theta \in \Theta^\zeta$, the i-th coordinate of θ helps define a monotone strategy σ_i for i with values in $B(\zeta)$ by letting $\theta_i(k)$ be the cut-off type for switching from b^{k-1} to b^k. Fix $\theta \in \Theta^\zeta$ and let σ be the strategy profile induced by θ. We define a function $\bar{\pi}_i^{n,\zeta,k} : \Theta^\zeta \times X_i \to \mathbb{R}$ for each n (including $n = \infty$), ζ and $b^k \in B(\zeta) \setminus \{ b^{k_0} \}$ as follows. For each i, $k_0 + 1 \leqslant k \leqslant k_1$, x_i, and $n \neq \infty$, define

$$\bar{\pi}_i^{n,\zeta,k}(\theta, x_i) = \frac{1}{\int_\Omega [\tau_i^n(b^k, \omega, \sigma) - \tau_i^n(b^{k-1}, \omega, \sigma)]\, p(\omega \mid x_i) d\omega} \left[\pi_i^n(b^k, \sigma; x_i) - \pi_i^n(b^{k-1}, \sigma; x_i)\right]$$

which is the payoff difference between bidding b^k and b^{k-1} for type x_i conditional on the event that if i is a buyer (resp. seller) bid b^k (resp. b^{k-1}) clinches trade while b^{k-1} (resp. b^k) does not. For $n = \infty$, we can use the same formula if with positive probability, either b^{k-1} or b^k is a clearing price under σ. Otherwise, either b^k is below the clearing price at state $\omega = 0$ and we let this difference be $v_i(0, x_i^k) - b^k$; or b^{k-1} is above the clearing price at state $\omega = 1$ and we let this formula be $v_i(1, x_i^k) - b^{k-1}$.

The following lemma, whose proof can be found in the Appendix, gives a continuity property for the conditional expectations $\bar{\pi}_i^{n,\zeta,k}(\theta, x_i)$.

Lemma 6.2. *For each k, $\bar{\pi}_i^{n,\zeta,k}(\theta, y_i)$ converges to $\bar{\pi}_i^{\infty,\zeta,k}(\theta, y_i)$ uniformly in $(\theta, y_i) \in \Theta^\zeta \times X_i$; and the same is true of its derivative w.r.t. y_i.*

We now compute a good approximation for $\bar{\pi}_i^{\infty,\zeta,k}$ when either b^k or b^{k-1} is a clearing price. Let $t^k = \theta_0(k+1) - \theta_0(k)$, $t^{k-1} = \theta_0(k) - \theta_0(k-1)$, α^k equals $\tau_i^\infty(b^k, \theta_0(k+1), \sigma)$ if i is a buyer and $1 - |\tau_i^\infty(b^k, \theta_0(k+1), \sigma)|$ if i is a seller; α^{k-1} equals $1 - \tau_i^\infty(b^{k-1}, \theta_0(k-1), \sigma)$ for a buyer i and $|\tau_i^\infty(b^{k-1}, \theta_0(k-1), \sigma)|$ for a seller i. Either t^k or t^{k-1} will be positive by assumption (and then in $O(\zeta)$). Also α^k is zero if $\theta_0(k+1) < 1$ and α^{k-1} is zero if $\theta_0(k-1) > 0$, so that one of these two variables will be zero for small ζ. We can approximate $\tau^\infty(b^k, \omega, \sigma)$ on $[\theta_0(k), \theta_0(k+1)]$ by a linear function that is 1 at $\theta_0(k)$ and α^k at $\theta_0(k+1)$, and $\tau^\infty(b^{k-1}, \omega, \sigma)$ on $[\theta_0(k-1), \theta_0(k)]$ by a linear function that is $1 - \alpha^{k-1}$ at $\theta_0(k-1)$ and 0 at $\theta_0(k)$. Also, using $v_i(\theta_0(k) + s, x_i) = v_i(\theta_0(k), x_i) + \frac{\partial v_i(\theta_0(k), x_i)}{\partial \omega} s + O(s^2)$

and $p(\theta_0(k)+s \mid x_i) = p(\theta_0(k) \mid x_i) + \frac{\partial p(\theta_0(k) \mid x_i)}{\partial \omega} s + O(s^2)$, we can then write

$$\bar{\pi}_i^{\infty,\zeta,k}(\theta, x_i) = [v_i(w^-(k), x_i) - b^{k-1}]\bar{t}^{k-1} + [v_i(w^+(k), x_i) - b^k]\bar{t}^k + O(\zeta^2) \tag{1}$$

where

$$w^-(k) = \theta_0(k) - \frac{t^{k-1}(1+2\alpha^{k-1})}{3+3\alpha^{k-1}}$$

if $\frac{\partial v_i(\theta_0(k), x_i)}{\partial \omega} > 0$ and $w^-(k) = \theta_0(k)$ otherwise;

$$\bar{t}^{k-1} = \frac{t^{k-1}(1+\alpha^{k-1})}{t^k(1+\alpha^k) + t^{k-1}(1+\alpha^{k-1})};$$

$$w^+(k) = \theta_0(k) + \frac{t^k(1+2\alpha^k)}{3+3\alpha^k}$$

if $\frac{\partial v_i(\theta_0(k), x_i)}{\partial \omega} > 0$ and $w^+(k) = \theta_0(k)$ otherwise; and

$$\bar{t}^k = \frac{t^k(1+\alpha^k)}{t^k(1+\alpha^k) + t^{k-1}(1+\alpha^{k-1})}.$$

Since the integrals above using ω can also be performed by a change of variable using b, we have:

$$\bar{t}^{k-1} = \frac{\hat{t}^{k-1}(1+\alpha^{k-1})}{\hat{t}^k(1+\alpha^k) + \hat{t}^{k-1}(1+\alpha^{k-1})} + O(\zeta),$$

where $\hat{t}^{k-1} = \phi^*(\theta_0(k)) - \phi^*(\theta_0(k-1))$ and $\hat{t}^k = \phi^*(\theta_0(k+1)) - \phi^*(\theta_0(k))$, and similarly for $\bar{t}^k$.

Finally, by the assumption on the range of the functions θ, we have that $|\chi_i^*(\theta_0(k)) - \theta_i(k)| \in O(\zeta)$ for all i, k, where χ_i^* is the cutoff type defined in Section 3. Therefore, if t^k is nonzero, $v_i(w^+(k), \theta_i(k)) - v_j(w^+(k), \theta_j(k)) = v_i(\theta_0(k), \theta_i(k)) - v_j(\theta_0(k), \theta_j(k)) + O(\zeta^2)$ for all i, j by Assumption 6.1. A similar statement holds for t^{k-1} as well, with the conclusion that $\bar{\pi}_i^{\infty,\zeta,k}(\theta, \theta_i(k)) - \bar{\pi}_j^{\infty,\zeta,k}(\theta, \theta_j(k)) = v_i(\theta_0(k), \theta_i(k)) - v_j(\theta_0(k), \theta_j(k)) + O(\zeta^2)$.

Lemma 6.3. *For all sufficiently small ζ, there exists $N(\zeta)$ with the following property. For $n \geqslant N(\zeta)$ if there is θ in Θ^ζ such that $\bar{\pi}_i^{n,\zeta,k}(\theta, \theta_i(b^k)) = 0$ for each k and i, then the corresponding strategy profile σ is an equilibrium of $\Gamma^{n,\zeta}$.*

Proof. The result follows if we establish a single-crossing property for the payoffs, i.e. if we show the existence of $\delta > 0$ such that $\frac{\partial \bar{\pi}_i^{n,\zeta,k}(\theta, x_i)}{\partial x_i} \geqslant \delta$ for all $\theta \in \Theta^*$, $i \in I$, $x_i \in X_i$ and k, if n is large. In light of Lemma 6.2, it is sufficient to get this bound when $n = \infty$. Using the approximation of $\bar{\pi}_i^{\infty,\zeta,k}$ above, the derivative of $\bar{\pi}_i^{\infty,\zeta,k}$ is strictly positive with a lower bound that is independent of ζ or θ, and the conclusion follows. □

Theorem 6.4. *For each sufficiently small $\zeta > 0$, there exists $N(\zeta)$ such that for each $n \geqslant N(\zeta)$, the game $\Gamma^{n,\zeta}$ has an equilibrium $\sigma^{n,\zeta}$.*

Proof. Fix ζ. For each n, define a map $\Upsilon^{n,\zeta} : \Theta^\zeta \to \mathbb{R}^{I\times(B(\zeta)\setminus\{\,b^{k_0}\,\})}$ by:

$$\Upsilon^{n,\zeta}_{i,k}(\theta) = \bar{\pi}^{n,\zeta,k}_i(\theta, \theta_i(k)).$$

for each $i \in I$ and $b^k \in B(\zeta) \setminus \{\,b^{k_0}\,\}$. We will now show that for all small ζ, $\Upsilon^{\infty,\zeta}$ has no zeros on the boundary of Θ^ζ and that the degree of zero over Θ^ζ is one. The result then follows. Indeed, by Lemma 6.2, $\Upsilon^{n,\zeta}$ has a zero $\theta^{n,\zeta} \in \Theta^\zeta$ for large n; and Lemma 6.3 shows that $\theta^{n,\zeta}$ induces an equilibrium of $\Gamma^{n,\zeta}$.

To prove that the degree of zero over Θ^ζ under the map $\Upsilon^{\infty,\zeta}$ is one, we proceed as follows. Define $\Upsilon^{*,\zeta} : \Theta^\zeta \to \mathbb{R}^{I\times(B(\zeta)\setminus\{\,b^{k_0}\,\})}$ by:

$$\Upsilon^{*,\zeta}_{i,k}(\theta) = \begin{cases} v_i(\theta_0(k), \theta_i(k)) - b^k & \text{if } k < k^*_1 \\ v_i(\theta_0(k), \theta_i(k)) - \phi^*(1) & \text{if } k = k^*_1 \\ v_i(\theta_0(k), \theta_i(k)) - b^{k-1} & \text{o.w.} \end{cases}$$

Obviously $\Upsilon^{*,\zeta}$ has a unique zero. Moreover this map is a homeomorphism onto its image and hence has degree one (for an appropriate orientation of Θ^ζ). To obtain our result, we show that for each $\lambda \in [0, 1)$, $\lambda\Upsilon^{*,\zeta} + (1 - \lambda)\Upsilon^{\infty,\zeta}$ has no zero on the boundary of Θ^ζ. Take θ in the boundary of Θ^ζ. Let ϑ^* and ϑ^∞ be its image under $\Upsilon^{*,\zeta}$ and $\Upsilon^{\infty,\zeta}$ respectively; we will show that $\lambda\vartheta^\infty + (1 - \lambda)\vartheta^* \neq 0$ for all $\lambda \in [0, 1)$.

Since $\theta \in \partial\Theta^\zeta$, there exists some k such that one of the following holds: (1) $k^*_0+1 \leqslant k \leqslant k^*_1$ and $\theta_0(k) \in \partial\Omega^k \setminus \Omega^k$; (2) $\theta_i(k) \in \{\,\underline{\theta}_i(k), \overline{\theta}_i(k)\,\}$ for some i.

Start with possibility (1). Suppose $\theta_0(k) = \underline{\theta}_0(k)$ for some $k^*_0 + 1 < k \leqslant k^*_1$. For each b in the range of ϕ^*, let $(\omega(b), x(b))$ be such that $\phi^*(\omega(b)) = b$ and $(\omega(b), x(b), b) \in \mathcal{M}$. With $c^k = \phi^*(\theta_0(k))$, there must exist i with $\theta_i(k) \leqslant x_i(c^k)$. For this i, we have $\vartheta^*_{i,k} \leqslant c^k - \min(b^k, \phi^*(1)) \leqslant -(2\lambda^*_1 - 2\eta)\zeta < 0$. To prove that θ cannot be a zero along the homotopy, it is now sufficient to show that $\vartheta^\infty_{i,k}$ is also negative with $\zeta \in O(\vartheta^\infty_{i,k})$. Suppose $k < k^*_1$. Then:

$$v_i(w^+(k), \theta_i(k)) - b^k \leqslant \frac{\int_{\phi^*(\theta_0(k))}^{\phi^*(\theta_0(k+1))} v_i(\omega(b), x_i(b))\tau(b^k, \omega(b), \sigma)[\frac{d\phi^*}{db}]^{-1} db}{\int_{\phi^*(\theta_0(k))}^{\phi^*(\theta_0(k+1))} \tau(b^k, \omega(b), \sigma)[\frac{d\phi^*}{db}]^{-1} db} - b^k \leqslant -\frac{1 - 5\eta}{3}\zeta + O(\zeta^2)$$

while $v_i(w^-(k), \theta_i(k)) - b^{k-1} \leqslant 2\eta\zeta + O(\zeta^2)$. Also

$$\frac{\bar{t}^k}{\bar{t}^{k-1}} = \frac{\hat{t}^k}{\hat{t}^{k-1}} + O(\zeta^2) \geqslant 1 + O(\zeta^2),$$

where $\hat{t}^k$ was as defined following our approximation for $\bar{\pi}_i^{\infty,\zeta,k}$. Therefore, $\vartheta_{i,k}^{\infty} \leqslant -\frac{1-11\eta}{6}\zeta + O(\zeta^2)$, giving us the result in this case. If $k = k_1^*$, then

$$v_i(w^+(k), \theta_i(k)) - b^k \leqslant \frac{\int_{\phi^*(\theta_0(k))}^{\phi^*(1)} v_i(\omega(b), x_i(b))\tau(b^k, \omega(b), \sigma)[\frac{d\phi^*}{db}]^{-1} db}{\int_{\phi^*(\theta_0(k))}^{\phi^*(1)} \tau(b^k, \omega(b), \sigma)[\frac{d\phi^*}{db}]^{-1} db} - b^k \leqslant (-1 + \lambda_1^* + \eta)\zeta + O(\zeta^2).$$

As before, we have the bound $v_i(w^-(k), \theta_i(k)) - b^{k-1} \leqslant 2\eta\zeta + O(\zeta^2)$. Also

$$\frac{\bar{t}^k}{\bar{t}^{k-1}} \geqslant 2\lambda_1^* - 2\eta + O(\zeta^2).$$

Hence, $\vartheta_{i,k}^{\infty} \leqslant \frac{2\eta(2-\eta) - 2\lambda_1^*(1-\lambda_1^*)}{1+2\lambda_1^* - 2\eta} + O(\zeta^2)$, and this inequality, by our assumption on η, completes the proof that rules out the left endpoint. The case where $\theta_0(k)$ is the right endpoint of the interval is similar, and in some ways easier too, so we omit it.

We turn now to possibility (2). Suppose $\theta_i(k) = \underline{\theta}_i(k)$ for some k. If $k < k_0^*$, we have $\vartheta_{i,k}^* = \vartheta_{i,k}^{\infty} = \max(b^{k-1}, v_i(0, \underline{\theta}_i(k))) - b^k \leqslant -(2\lambda_0 - \frac{3}{2}\eta)\zeta < 0$. Thus, all along the homotopy the value is negative. For $k = k_0^*$, $\vartheta_{i,k}^* = b^{k_0^*-1} + \frac{3}{2}\eta\zeta - b^{k_0^*} < 0$; if b^k is not a clearing price under θ with positive probability, then $\vartheta_{i,k}^{\infty} = \vartheta_{i,k}^*$ and we are done. Otherwise, $\vartheta_{i,k}^{\infty} \leqslant b^{k_0^*-1} + \frac{3}{2}\eta\zeta + \frac{2}{3}[\lambda_0^* + \eta]\zeta - b^{k_0^*} + O(\zeta^2)$, which is again negative and in $O(\zeta)$, wrapping up the case $k = k_0^*$. Now consider the case $k_0^* < k \leqslant k_1^*$. Then, $\chi_i^*(\theta_0(k)) - \theta_i(k)$ is positive and in $O(\zeta)$. Therefore, there exists $j \neq i$ such that $\theta_j(k) \geqslant \chi_j^*(\theta_0(k))$. For the same reason, $\phi^*(\theta_0(k)) - v_i(\theta_0(k), \theta_i(k))$ is positive and in $O(\zeta)$. As $v_j(\theta_0(k), \theta_j(k)) \geqslant \phi^*(\theta_0(k))$, we now have $\vartheta_i^{*,\zeta} - \vartheta_j^{*,\zeta} = v_i(\theta_0(k), \theta_i(k)) - v_j(\theta_0(k), \theta_j(k))$, which is negative. As we saw in our approximation for $\bar{\pi}_i^{n,\zeta,k}$, $\vartheta_i^{\infty,\zeta} - \vartheta_j^{\infty,\zeta} = v_i(\theta_0(k), \theta_i(k)) - v_j(\theta_0(k), \theta_j(k)) + O(\zeta^2)$, which is again negative and in $O(\zeta)$. Therefore, we can rule out this case as well. Finally, suppose that $k > k_1^*$. Then $\vartheta_{i,k}^* = -\frac{1}{2}\eta\zeta$; $\vartheta_{i,k}^{\infty}$ is also $-\frac{1}{2}\eta\zeta$ if $k > k_1^* + 1$ or $b^{k_1^*}$ is not a market-clearing price in Γ^{∞} under the strategy induced by θ; otherwise it is bounded from above by $-\frac{1}{2}\eta\zeta$. Either way, under the linear homotopy the value is negative and in $O(\zeta)$, which concludes the proof that $\theta_i(k) \neq \underline{\theta}_i(k)$. The argument for the case where $\theta_i(k) = \overline{\theta}_i(k)$ for some i can be handled similarly. □

One could give explicit bounds for $N(\zeta)$ in the above theorem using the following logic. With a more detailed calculation than the one we give in the Appendix, we can see that $\pi_i^{n,\zeta} - \pi_i^{\infty,\zeta}$ is in $O(n^{-\frac{1}{2}}\zeta^{-1})$. As we saw in the proof of the above theorem, the value of $\Upsilon^{\infty,\zeta}$ on the boundary of Θ^{ζ} is in $O(\zeta)$. Therefore, if $n \geqslant L\zeta^{-4}$ for an appropriate constant L, then the linear homotopy between $\Upsilon^{n,\zeta}$ and $\Upsilon^{\infty,\zeta}$ has no zero on the boundary of Θ^{ζ} giving us an equilibrium for $\Gamma^{n,\zeta}$. Unfortunately the distance between such an equilibrium and the limit σ^* would then be in $O(n^{-\frac{1}{4}})$, implying that the we cannot invoke the Central Limit Theorem from Section 5 for this sequence. Of course, this is merely a heuristic argument that the CLT does not hold for such a sequence, but we conjecture that if the games Γ^n do not have an equilibrium for large n, then the grid equilibria will not yield a CLT result.

7. Asymptotic Analysis III

When the effect of the state on the valuations of the agents is not identical around the REE manifold, we can still obtain an existence result if we discretize the state space $\Omega \times X$ as well as the space of bids.

Assume that α, the averaging of the m_0-th and (m_0+1)-st bids in Γ, is strictly between 0 and 1. Let $Z = (\zeta_0, \zeta_1, \ldots, \zeta_I, \zeta) \in \mathbb{R}^{I+2}_{++}$ be a vector of strictly positive numbers. Let $B(Z) = \{0, \zeta, 2\zeta, \ldots\}$ be the grid of admissible bids as in the previous section. Let $\Omega(Z)$ be the set of states in Ω that are of the form $k\zeta_0$, for $0 < k \leqslant \lfloor \frac{1}{\zeta_0} \rfloor$, along with the state $\omega = 1$ (in case it is not an integer multiple of ζ_0). Likewise for each i, let $X_i(Z)$ be the set of signals that are multiples $k\zeta_i$ for $k > 0$ along with $x_i = 1$. Consider a finite approximation $P_i^Z(\cdot \,|\, \omega)$ for each $\omega \in \Omega(Z)$ on $X(Z)$ that assigns mass $P_i([(k-1)\zeta_i, k\zeta_i] \,|\, \omega)$ to $k\zeta_i$ and let P_0^Z be the distribution on $\Omega(Z)$ that assigns mass $P_0([(k-1)\zeta_0, k\zeta_0])$ to $k\zeta_0$. We make the following assumptions on the vector Z.

Assumption 7.1. $\zeta_i \leqslant \zeta_0^2$ for each i. Moreover, for each $(\omega, x) \in \Omega(Z) \times X(Z)$:

(1) $v_i(\omega, x_i) \in B(Z)$ for each i;
(2) $v_i(\omega, x_i) \neq v_j(\omega, x_j) \neq \phi^*(\omega)$ for all i, j;
(3) $\sum_i \sum_{y_i \leqslant x_i} P_i^Z(y_i \,|\, \omega) \neq m_0$.

The first (unnumbered) assumption requires the bid grid on X to be finer than that on Ω. Point (1) is an obvious assumption: players should be allowed to bid any possible realization of their valuation. The other two assumptions are genericity assumptions and hold for "almost every" choice Z.

A behavioral strategy for player-type i is a function $\sigma_i^Z : X_i(Z) \to \Delta(B(Z))$. Payoffs are now defined in the obvious way and we have a finite game $\Gamma^{n,Z}$ for each n. Let Σ^Z be the set of behavioral strategies. We say that a behavioral strategy σ_i^Z of player i is monotone if for $x_i < y_i$, the distribution $\sigma_i^Z(y_i)$ first-order stochastically dominates the distribution $\sigma_i^Z(x_i)$; furthermore, we say that it is strictly monotone if, in addition, every bid in the support of y_i's strategy lies strictly above every bid in the support of x_i's strategy. We will show that $\Gamma^{n,Z}$ has an equilibrium in strictly monotone behavioral strategies.

For each $\omega \in \Omega(Z)$, there exists a unique $i(\omega)$ and $x_{i(\omega)} \in X_{i(\omega)}(Z)$ such that, letting $b(\omega) = v_{i(\omega)}(\omega, x_{i(\omega)})$, we have $b(\omega) > \phi^*(\omega)$ and

$$\sum_j \sum_{y_j : v_j(\omega, y_j) < b(\omega)} P_j^Z(y_j \,|\, \omega) < m_0 < \sum_j \sum_{y_j : v_j(\omega, y_j) < b(\omega)} P_j^Z(y_j \,|\, \omega) + P_{i(\omega)}^Z(x_{i(\omega)} \,|\, \omega).$$

Since $\zeta_i \leqslant \zeta_0^2$, $|x_{i(\omega)} - \chi_{i(\omega)}^*(\omega)| \in O(\zeta_0^2)$. This implies in particular that $b(\omega) - \phi^*(\omega) \in O(\zeta_0^2)$ and thus if $\omega' - \omega = \zeta_0$, then $0 < b(\omega') - v_{i(\omega)}(\omega, x_{i(\omega)}) \in O(\zeta)$.

We will say that two states $\omega_0 < \omega_1$ are adjacent if $\omega_1 - \omega_0 = \zeta_0$; for convenience in notation, we will also include 0 and 2 as states and let the "clearing price" $b(\omega)$ at $\omega = 0$ (resp. $\omega = 2$) be strictly smaller (resp. bigger) than $v_i(\zeta_0, \cdot)$ (resp. $v_i(1, \cdot)$), for all i. The usefulness of this convention is that ζ_0 (resp. 1) has a state below (resp. above) it that is adjacent to it; and for each i and x_i other than the $x_{i(\omega)}$'s, there exists a unique pair of adjacent states $\omega_0(x_i) < \omega_1(x_i)$ such that $b(\omega_0(x_i)) < v_i(\omega_0(x_i), x_i)$ and $v_i(\omega_1(x_i), x_i) < b(\omega_1(x_i))$.

Let $\tilde{\Sigma}^Z$ be the set of behavioral strategies σ^Z such that each $i \in I$ and $x_i \in X(Z)$: (1) $\sigma_i^Z(x_i) = b(\omega)$ if $x_i = x_{i(\omega)}$ and (2) $\sigma_i^Z(x_i) \in \Delta(\{\, v_i(\omega_0(x_i), x_i), v_i(\omega_1(x_i), x_i)\})$ if $x_i \neq x_{i(\omega)}$. For each $\sigma \in \tilde{\Sigma}^Z$, $b \in B(Z)$ and $\omega \in \Omega(Z)$, let

$$q_0(b; \omega, \sigma) = |I|^{-1} \sum_j P_j^Z(\sigma_j^{-1}([0, b]) \,|\, \omega)$$

and

$$q_1(b, \omega, \sigma) = |I|^{-1} \sum_j P_j^Z(\sigma_j^{-1}([b, \infty))) \,|\, \omega);$$

also define $q_0(b, 0, \sigma) = 2$ and $q_1(b, 2, \sigma) = -2$.

Define a correspondence $\varphi : \tilde{\Sigma}^Z \to \tilde{\Sigma}^Z$ as follows. For each $\sigma \in \tilde{\Sigma}^Z$, $\varphi_i(\sigma)$ is the set of $\tilde{\sigma}_i \in \tilde{\Sigma}_i^Z$ such that: $\tilde{\sigma}_i(x_i) = \sigma_i(x_i)$ if x_i is $x_{i(\omega)}$ for some ω; otherwise, $\tilde{\sigma}_i(x_i)$ assigns positive probability to $v_i(\omega_0(x_i), x_i)$ (resp. $v_i(\omega_1(x_i), x_i)$) only if $q_0(v_i(\omega_0(x_i), x_i); \omega_0(x_i), \sigma) - \mu_0 \leqslant q_1(v_i(\omega_1(x_i), x_i); \omega_1(x_i), \sigma)) - \mu_1$ (resp. if the inequality is reversed). φ is a well-behaved correspondence and, by Kakutani's Fixed-Point Theorem, it has a fixed point $\sigma^{*,Z}$. It is easy to check that $\sigma_i^{*,Z}$ is a monotone strategy profile; moreover, for each i and each pair of adjacent states $\omega_0 < \omega_1$, there is at most one signal, call it $x_i^*(\omega_0, \omega_1)$, that mixes between $v_i(\omega_0, x_i)$ and $v_i(\omega_1, x_i)$; every other signal choose a pure action under $\sigma_i^{*,Z}$.

Even though we forced $x_{i(\omega)}$ to play $b(\omega)$, it is in fact the case that under a fixed point $\sigma^{*,Z}$, $q_0(b(\omega_0); \omega_0, \sigma^{*,Z}) - \mu_0$ is strictly smaller than $q_1(v_i(\omega_1, x_{i(\omega_0)}); \omega_1, \sigma^{*,Z}) - \mu_1$ for adjacent states $\omega_0 < \omega_1$. Indeed, otherwise, for all $x_i > x_{i(\omega_0)}$ with $v_i(\omega_1, x_i) < b(\omega_1)$, $\sigma_i^{*,Z}(x_i) =$

$v_i(\omega_1, x_i)$. As the mass of such x_i's is in $O(\zeta_0)$, $q_1(v_i(\omega_1, x_{i(\omega_0)}); \omega_1, \sigma^{*,Z}) - \mu_1$ is in $O(\zeta_0)$, while $q_0(b(\omega_0); \omega_0, \sigma^{*,Z}) - \mu_0$ is in $O(\zeta_0^2)$, and we have a contradiction.

Let $\sigma^{*,Z}$ be a fixed point of φ. For each i, let $X_i^*(Z)$ be the collection of $x_i^*(\omega_0, \omega_1)$, for adjacent states $\omega_0 < \omega_1$. We now make an additional genericity assumption on Z that makes this fixed point regular.[7]

Assumption 7.2. For each i, $\sigma_i^{*,Z}(x_i)$ is not a pure action if $x_i \in X_i^*(Z)$, and $\varphi(\sigma^{*,Z})(x_i)$ is a singleton if $x_i \notin X_i^*(Z)$.

Thanks Assumption 7.2, there exists some $\varepsilon > 0$ such that for all σ within ε of $\sigma^{*,Z}$ (in the ℓ_∞-norm), $\varphi_i(\sigma)(x_i)$ is locally equal to $\sigma_i(x_i)$ for each i and $x_i \notin X_i^*(Z)$. We are now ready to state the Theorem of this section.

Theorem 7.3. *For large n, the game $\Gamma^{n,Z}$ has an equilibrium $\sigma^{n,Z}$ that is strictly monotone. Moreover, the sequence $\sigma^{n,Z}$ converges to $\sigma^{*,Z}$.*

The way we prove the theorem is to show that $\Gamma^{n,Z}$ has an equilibrium $\sigma^{n,Z}$ when we restrict the players to a certain subset of strategies and then show for large n both that this restriction is irrelevant and that the equilibrium is close to $\sigma^{*,Z}$. Before getting to the proof of the theorem, we need a few definitions and preliminary lemmas.

Let $\Sigma_i^{*,Z}$ be the set of behavioral strategies σ_i that are within ε (obtained above as a bound from regularity) of $\sigma^{*,Z}$ and with the following properties. First, suppose that $x_i \notin X_i^*(Z)$. If i is a buyer (resp. seller), the support of σ_i is: $(\sigma_i^{*,Z}(y_i), \sigma_i^{*,Z}(x_i)]$ (resp. $[\sigma_i^{*,Z}(x_i), \sigma_i^{*,Z}(y_i))$) where $\sigma_i^{*,Z}(y_i)$ is the highest (resp. lowest) bid in the support of $\sigma_i^{*,Z}$ that is strictly below (resp. strictly above) $\sigma_i^{*,Z}(x_i)$, with the understanding that the lower end point (resp. the upper end point) is zero (resp. ∞) if there is no such y_i. Now suppose that $x_i = x_i^*(\omega_0, \omega_1)$ for some pair $\omega_0 < \omega_1$. If i is a buyer, then the support of σ_i is $(\sigma_i^{*,Z}(y_i), v_i(\omega_1, x_i)]$, where y_i, as before, is the highest type below x_i. If he is a seller, the said intervals are left-closed, right-open.

Let $\sigma^{n,Z}$ be a sequence of strategies in $\Sigma^{*,Z}$. Let $\sigma^{\infty,Z}$ be a limit point of the sequence. For simplicity in notation, think of the sequence itself as converging to $\sigma^{\infty,Z}$, as what follows now applies to any convergent subsequence. Let F_i^n be the set of all possible empirical frequencies of bids of type $(i,1)$'s opponents that can be observed in the play of $\sigma^{n,Z}$: that is, the frequency of bids in n draws for types $j \neq i$ and $(n-1)$ draws for type i, followed by the randomization prescribed by $\sigma^{n,Z}$. For each $b(\omega_0) \leqslant b_0 < b_1 \leqslant b(\omega_1)$ and each i, x_i

[7]This assumption rules out the case of two types indifferent between $b(\omega_0)$ and $b(\omega_1)$ with the lower type bidding $b(\omega_0)$ and the higher bidding $b(\omega_1)$. Even with this assumption, it is not clear if the fixed point is unique.

and ω, let $E_i^n(b_0, b_1, x_i, \omega)$ the subset of F_i^n consisting of frequencies where there is a payoff difference between bidding b_0 and b_1 in state ω. Let $R_i^n(b_0, b_1, x_i, \omega)$ be the probability of this set and let $\bar{R}_i^n(b_0, b_1, x_i)$ be the expectation of this probability w.r.t. to $P_i^Z(\cdot \mid x_i)$. Assuming that $\bar{R}_i^n(b_0, b_1, x_i) > 0$ for all n, we are interested in the limiting payoff difference between bidding b_1 and b_0:

$$\bar{\pi}_i^\infty(b_0, b_1, x_i) \equiv \lim_n \frac{\pi_i^n(b_1, \sigma^n; x_i) - \pi_i^n(b_0, \sigma^n; x_i)}{\bar{R}_i^n(b_0, b_1, x_i)}.$$

Strictly speaking, the above limit may not exist, so we may have to pass to a subsequence, which as before does not alter the logic of the proof of the theorem of this section. Here are the preliminary lemmas needed, whose proofs can be found in the appendix.

Lemma 7.4. *Let $b(\omega_0) < b_0 < b_1 \leqslant b(\omega_1)$ be admissible bids for (i, x_i) with b_1 being the smallest bid in the support of $\sigma_i^{\infty,Z}(x_i)$ that is greater than b_0. Then, letting ω be such that $v_i(\omega, x_i) = b_1$, we have for any y_i:*

$$\bar{\pi}_i^\infty(b_0, b_1, y_i) - \lim_n \frac{\pi_i^n(b_1, \sigma^n; x_i) - \pi_i^n(b_0, \sigma^n; x_i)}{\bar{R}_i^n(b_0, b_1, y_i)} = v_i(\omega, y_i) - v_i(\omega, x_i).$$

Lemma 7.5. *Let $b(\omega_0) \leqslant b_0 < b_1 \leqslant b(\omega_1)$, i, and $x_i \leqslant y_i$ be such that b_0 is in the support of $\sigma_i^{\infty,Z}(x_i)$ and b_1 is the smallest bid in the support of $\sigma_i^{\infty,Z}(y_i)$ that is greater than b_0. Then, letting ω be such that $v_i(\omega, x_i) = b_1$,*

$$\bar{\pi}_i^\infty(b_0, b_1, y_i) \geqslant (1-\alpha)\zeta \text{ if } i \text{ is a buyer and}$$

$$\bar{\pi}_i^\infty(b_1, b_0, x_i) \geqslant \alpha\zeta \text{ if } i \text{ is a seller.}$$

Lemma 7.6. *Let $b(\omega_0) \leqslant b_0 < b_1 \leqslant b(\omega_1)$, i and x_i be such that b_0 is the highest bid in the support of $\sigma_i^{\infty,Z}(x_i)$ that is smaller than b_1. Then*

$$\bar{\pi}_i^\infty(b_0, b_1, x_i) \leqslant -\zeta.$$

Lemma 7.7. *Suppose $x_i \in X_i^*(Z)$ and $b_0 < b_1$ are in the support of $\sigma^{\infty,Z}(x_i)$, and $\omega(b_0, \sigma^{n,Z}) = \omega_0$ while $\omega(b_1, \sigma^{n,Z}) = \omega_1$. If $q_0(b_0, \sigma^{\infty,Z}) < q_1(b_1, \sigma^{\infty,Z})$, then $\bar{\pi}_i^\infty(b_0, b_1, x_i) < 0$; and if $q_0(b_0, \omega_0 \sigma^{\infty,Z}) > q_1(b_1, \omega_1, \sigma^{\infty,Z})$ then $\bar{\pi}_i^\infty(b_0, b_1, x_i) > 0$.*

With the preliminaries out of the way, we are now ready to prove Theorem 7.3.

Proof of Theorem 7.3. $\Gamma^{n,Z}$ has an equilibrium $\sigma^{n,Z}$ when players are restricted to the strategy space $\Sigma^{*,Z}$. Take a convergent subsequence $\sigma^{n,Z}$ with, say, $\sigma^{\infty,Z}$ as its limit point, and such that the support of $\sigma^{n,Z}$ is constant along the subsequence. We will show both that $\sigma^{\infty,Z} = \sigma^{*,Z}$ and that along the subsequence $\sigma^{n,Z}$ is an equilibrium of $\Gamma^{n,Z}$ for large n, which proves the theorem. All our arguments for a generic i assume that i is a buyer. The arguments for a seller are similar and therefore omitted.

Suppose $\sigma_i^{\infty,Z}(x_i) \neq \sigma_i^{*,Z}(x_i)$ for some i and x_i. If $x_i \notin X_i^*(Z)$, then there exists $b_0 < b_1 \equiv \sigma_i^{*,Z}(x_i)$ that has positive probability under $\sigma_i^{\infty,Z}(x_i)$. By Lemma 7.5 we get that b_1 is a better reply than b_0 against $\sigma^{n,Z}$ for large n, which is impossible. Thus $\sigma_i^{\infty,Z}(x_i) = \sigma_i^{*,Z}(x_i)$. Now take i and $x_i \in X_i^*(Z)$. The argument we just used also shows that $\sigma_i^{\infty,Z}(x_i)$ puts all its weight on the two bids $v_i(\omega_0, x_i)$ and $v_i(\omega_1, x_i)$. To complete the proof of this part, then, we have to show that the probabilities under $\sigma_i^{\infty,Z}(x_i)$ equal those under $\sigma_i^{*,Z}(x_i)$. Suppose $b_0 \equiv v_i(\omega_0, x_i)$ has higher probability under $\sigma_i^{\infty,Z}(x_i)$ than under $\sigma_i^{*,Z}(x_i)$. Then letting $b_1 \equiv v_i(\omega_1, x_i)$ and using Lemma 7.7, b_1 is a better reply than b_0 against $\sigma^{n,Z}$ for large n, which is impossible. Likewise b_1 cannot have a higher probability under $\sigma_i^{\infty,Z}(x_i)$ than under $\sigma_i^{*,Z}(x_i)$. Thus we have shown that $\sigma^{\infty,Z}(x_i) = \sigma^{*,Z}(x_i)$.

We now show that $\sigma^{n,Z}$ is an equilibrium of $\Gamma^{n,Z}$ for large n. Fix i and x_i. Any bid above $b(\omega_1)$ or less than $b(\omega_0)$ is clearly suboptimal against the limit and hence also against $\sigma^{n,Z}$ for large n. There remains to only show that a bid between $b(\omega_0)$ and $b(\omega_1)$ that is not admissible is suboptimal for x_i. Suppose first that $x_i \notin X_i^*(Z)$. Let $b = \sigma_i^{*,Z}(x_i)$. Suppose $v_i(\omega_0, x_i) = b$. For any $b_0 < b$, there is some y_i for whom it is an admissible bid. Applying Lemma 7.4 with $b_1 = \sigma_i^{*,Z}(y_i)$, we get that b_1 is a better reply than b_0 against $\sigma^{n,Z}$ for x_i if n is large. Now apply Lemma 7.5 with $\sigma_i^{*,Z}(y_i)$ as b_0 and b as b_1 to obtain that b is better than $\sigma_i^{*,Z}(y_i)$ and hence than b_0. For any $b_1 > b_0$, apply Lemma 7.6 to get that b_1 is not optimal. Thus, no bid that is inadmissible for i does better if $x_i \notin X_i^*(Z)$ and $v_i(\omega_0, x_i) = \sigma_i^{*,Z}(x_i)$. The case where $x_i \notin X_i^*(Z)$ and $v_i(\omega_1, x_i) = \sigma_i^{*,Z}(x_i)$ is similar and thus omitted.

Finally, suppose that $x_i \in X_i^*(Z)$. If $b_0 < v_i(\omega_0, x_i)$ or $b_1 > v_i(\omega_1, x_i)$ the arguments of the previous paragraph apply. Observe that any b between $v_i(\omega_0, x_i)$ and $v_i(\omega_1, x_i)$ is admissible for x_i, which completes the proof.

8. Total Monotonicity

We relied on the total monotonicity property of the REE in the previous sections to obtain existence results for large auctions. Proposition 3.5 established that average crossing is a sufficient condition for total monotonicity and thus total monotonicity is a robust property. Unfortunately, it is not a generic property. And we now show by means of an example that when total monotonicity fails, the corresponding large auction may not have an equilibrium and that this feature is robust.

Let Γ be a single-unit auction with two bidder-types, with valuations given by $v_1(\omega, x_1) = a + x_1$ and $v_2(\omega, x_2) = x_2 + \omega$ and conditional CDFs given by $P_1(x_1 \,|\, \omega) = \min\{\max\{\beta x_1 - \omega, 0\}, 1\}$ and $P_2(x_2 \,|\, \omega) = x_2$, for $\beta > 2$. While the functional forms for the CDFs do not satisfy the assumptions we have made on priors in the paper, we can approximate the prior with a function that does and our conclusions hold for such a perturbation. For a.e. (ω, x) we must have $\beta x_1 - \omega \in (0,1)$ and hence $Q^*(x) = \beta x_1 + x_2 - 1$ for $1 - \beta x_1 < x_2 < 2 - \beta x_1$. It follows that $v_1^*(x) = a + x_1$ and $v_2^*(x) = 2x_2 + \beta x_1 - 1$, so the average crossing property fails. Total monotonicity also fails, as $\chi_2(\omega) = \frac{1}{1+\beta}[1 + \beta a - (\beta - 1)\omega]$.

We claim that Γ^∞ does not admit a nontrivial equilibrium, even allowing for players to play behavioral strategies. Indeed, let σ be an equilibrium of Γ^∞. Because player 1 has private values and x_2 is independent of ω, best replies are monotone, and hence so must σ_i be. Let $\underline{x}_i = \inf\{\, x_i \,|\, \sigma_i(x_i) \text{ has positive probability of trading}\,\}$. We claim that σ_i is strictly monotone and continuous for $x_i > \underline{x}_i$. This is immediate for player 1 because of private values—indeed, we must have $\sigma_1(x_1) = a + x_1$. For player 2, as $\varrho^\infty(\omega, \sigma)$ is strictly increasing in ω, σ_2 cannot have flat portions. Moreover, the support of $\sigma_2(x_2)$ must be a singleton: if not, for b and b' in the support of $\sigma_2(x_2)$, there would exist ω and ω' such that $\frac{b'-b}{\omega'-\omega} = \frac{1}{\beta}$, but we also need $\varrho^\infty(\omega, \sigma) - \omega = \varrho^\infty(\omega', \sigma) - \omega'$, that is, $\frac{b'-b}{\omega'-\omega} = 1$. Letting now $x_i(b) = \sigma_i^{-1}(b)$, by the definition of an equilibrium we must have $a + x_1(b) = x_2(b) + \omega = b$ and $\beta(b - a) - \omega + x_2(b) = 1$, which readily yields the same conclusion as in the previous paragraph; in particular, $x_2(b)$ must be decreasing in b, contradicting the monotonicity of σ_2.

Observe that the preceding analysis holds for all other high-bid auction formats: with I^∞ as the player set, the optimality condition for each player type is the same for other pricing rules, so no known auction format will have a non-trivial equilibrium. In particular, not even the information advantage of the open ascending English auction will be enough to restore existence of a non-trivial solution.[8]

The upper semicontinuity result for Nash equilibria in Section 5 applies to equilibria in behavioral strategies for this game, i.e. the limit of a sequence σ^n of ε-equilibria of Γ^n, as $n \to \infty$ and $\varepsilon \to 0$ is an equilibrium of Γ^∞. Therefore, we can now conclude that when n is large, Γ^n does not have an ε-equilibrium in behavioral strategies. In particular, if we restrict players to a bid-grid, then while an equilibrium for Γ^n does exist for any n (including $n = \infty$), such equilibria are not approximate equilibria of the game where the players are unconstrained.

[8] Indeed, when average-crossing fails, it follows from Krishna [13] that Γ^∞ has no efficient equilibrium.

The example also allows us to explore further the non-existence of equilibria for finite auctions. Assume that $\alpha = 0$, so that Γ is a second-price auction. Then we do have trivial equilibria of Γ^n for all n including $n = \infty$, where a player type bids a very high number and the other bids a very low number, regardless of their signals—triviality here refers to the fact all agents of player-type 2 get an object with probability one, and no agent of type 1 gets an object. If player type 2 is relatively strong (a is small), then we have another family of trivial equilibria of Γ^n, which are reasonable in that all agents of type 1 bid their value, $\sigma_1(x_1) = a + x_1$. In fact, assume that the marginal on Ω is uniform. The expected value of the first-order statistic of x_1 given ω is $\frac{\omega}{\beta} + \frac{n}{\beta(n+1)}$, so we have an equilibrium σ of Γ^1 with $\sigma_1(x_1) = a + x_1$ and $\sigma_2(x_2) \geq a + \frac{2}{\beta}$ for $x_2 \geq a + \frac{2-\beta}{2\beta}$ and $\sigma_2(x_2) \leq a$ otherwise. However, if $a > \frac{2-\beta}{2\beta}$, then the profile above is not an equilibrium of Γ^2, as player $(2,1)$ with $x_2 \in [0, \frac{1}{2\beta}]$ will want to beat player $(2,2)$ with signals below his x_2, so will not bid less than a. Building on this, for $a < \frac{2-\beta}{2\beta}$, the profile σ described above is a trivial equilibrium of Γ^n with player 2 always winning iff $\frac{n}{n+1} < \frac{\beta-1-2\beta a}{2}$. We do not know, however, whether other equilibria could exist for Γ^n, or even for Γ^∞.

9. Directions for Future Research

In the Introduction, we discussed the technical difficulties we encountered in trying to obtain an equilibrium in a large auction by solving the differential equations arising from first-order-conditions. Even an assumption of smoothness did not suffice to overcome these problems. But perhaps something stronger, like analyticity with very strong bounds on the derivatives, might help? Or maybe, even under those assumptions, there are counterexamples. Settling this question one way or another seems a hard but worthy endeavor.

On a somewhat more modest scale, what can be said about the existence of equilibria with bid-grids? We were unable to use the degree-theoretic approach of Section 6 to obtain a positive result without Assumption 6.1. Does the result extend to the more general case? Or, are there counterexamples to existence when player types are sufficiently asymmetric? This issue seems to be a version of the problem that we already encounter in finite auctions. As Reny and Zamir [22] have shown, for a second-price auction, even assuming affiliation, existence of nontrivial equilibria is not guaranteed when players are asymmetric.

Turning to the result of Section 7, could we allow for a continuous bid and type spaces, while retaining the finite grids for the state space? Or perhaps even just allowing for continuous bids? These questions get at the role that discreteness plays in existence results. The picture that emerges, in our view, and in light of our own attempts, is that a trifecta of factors—asymmetric types, interdependent valuations (through the state variable) and a continuum of states and signals—present major hurdles to establishing existence theorems for auctions.

As we saw in the last section, when the REE of the limit economy is not totally monotone, the associated limit auction, and hence also each corresponding large auction, may not have an equilibrium. Even worse, these large auctions may not even admit an ε-equilibrium for small $\varepsilon > 0$. Thus, auction theory cannot provide an explanation for price-formation in such economies.[9] However, do there exist other mechanisms that can implement the REE outcome? It is simple to construct an incentive compatible and individually rational direct mechanism for the limit model that would yield the same allocation and prices as the REE: have agents report their signals and compute the state from the frequency of signals; given the state, use the REE clearing price to determine the allocation from the reports. Ex-post incentive compatibility is trivial because individual reports do not affect the clearing price. Observe that this is also an ε-IC and IR mechanism for a large but finite model.[10] Potentially, therefore, one could construct a implementable (in the sense of being close enough to real-world mechanisms) indirect mechanism that would outperform auctions in providing foundations for price-formation in REE.

Could we describe the set of equilibria of Γ^∞? In the presence of total monotonicity, we have identified one equilibrium; and with the average-crossing property, it is the unique monotone equilibrium. But, are there other equilibria in Γ^∞ that are limits of ε-equilibria of Γ^n? If so, how reasonable are those in relation to the REE? More intriguing is the question of whether Γ^∞ has a Nash equilibrium when total monotonicity fails. As it is a discontinuous game, the answer is not clear—the techniques developed by the literature on discontinuous games (cf. Reny [20] and followers) do not seem to be applicable here. However, if the answer is yes, then it would point to a basic inconsistency between the two theories, strategic and competitive.

More results in a positive direction seem possible. This paper assumes that agents want or are endowed with one unit of a single homogenous good and that they are risk-neutral. It seems somewhat straightforward to relax the risk-neutrality assumption. A more challenging problem is to obtain results for cases with multiple commodities and multi-unit demands, and removing the restriction that each agent operate on only one side of the market—which

[9]Paul Milgrom has suggested to us (private communication) that, quite possibly, a dynamic auction could refute this conclusion.

[10]In an Arrow-Debreu framework with private information, Gul and Postlewaite [5] proposed a somewhat similar, albeit stochastic due to rationining, direct mechanism and showed that it is IC, IR, and ex post ε-efficient.

would amount to moving from a partial to a general equilibrium setting. While the tools developed here should generalize to higher dimensions, what is not completely evident is the nature of the assumptions concerning substitutability/complementarity that would deliver a counterpart to our findings.

10. Conclusion

Before summarizing our results, we recall that the study of market mechanisms via game theory over the past half-century originated as a program to elucidate the role of dispersed information and limited observability on incentives. In 1945 Hayek [9, p. 526] observed that "The mere fact that there is one price for any commodity [reflects] all the information . . . dispersed among all the people involved in the process. . . . We must look at the price system as such a mechanism for communicating information if we want to understand its real function." Then in 1973 Hurwicz [10] modeled markets as mechanisms for eliciting offers from traders to obtain allocations that reflect information dispersed among them, and proposed that outcomes result from Nash equilibria. This posed the question, studied by the authors cited in Section 1, of when prices produced by familiar mechanisms, such as (double) auctions, summarize *all* information among traders, formalized as rational expectations equilibria with fully revealing market-clearing prices. This paper contributes to that literature.

The model in Section 2 generalizes previous work by allowing any finite set of types of traders, but still requires replication of traders of each type. Our main result is existence of ε-equilibria for sufficiently numerous replicates, that then converge to the unique rational expectations equilibrium, for which the clearing price is a sufficient statistic for all the signals observed by traders.

We provide two technical innovations. One is a limit economy with a countable set of traders, and the corresponding limit auction. The other is proof that the ε-equilibrium correspondence is continuous at the rational expectations equilibrium. The degree theory we use to establish lower-semi-continuity suggests applications to other mechanisms with large-but-finite sets of participants. In particular, a notable goal is to use an extension to multiple commodities and multi-unit demands with multidimensional state and signal spaces (as in Gul and Stacchetti [6, 7] or Kelso and Crawford [12]) to apply degree theory to establish lower-semi-continuity at Walrasian equilibria of general models with countable sets of agents so that properties of limits of Nash equilibria of large-but-finite economies can be obtained, including the role of rational expectations and the extent to which dispersed information is reflected in prices.

Appendix A. Asymptotic Probabilities

This appendix first lays out a few key computations concerning asymptotic probabilities, both in the case where the Central Limit Theorem applies as well as in the case of large deviations. The asymptotics in Section 6 derive from getting n draws from an (I-fold) sum of trinomial or binomial variables. These results are then used to complete the proofs that were omitted in the text. The asymptotics in Section 7 follow from direct relative-entropy calculations for random variables with finite support.

A.1. **Trinomial Probabilities.** For each player i, a pair of signals $0 \leqslant x_i^1 \leqslant x_i^2 \leqslant 1$ generates a trinomial distribution with outcomes 0, 1, and 2, with their respective probabilities being $P_i(x_i^1 \,|\, \omega)$, $P_i([x_i^1, x_i^2] \,|\, \omega)$, and $1 - P_i(x_i^2 \,|\, \omega)$. For each n, and a triple of integers $k_i = (k_{i,0}, k_{i,1}, k_{i,2})$ that sum to n, let $P_i(\omega, x^1, x^2, k_i, n)$ be the probability that in n trials, exactly $k_{i,0}$ draws are below x_i^1, and exactly $k_{i,2}$ draws are above x_i^2. Letting $\kappa_i = n^{-1} k_i$, we have:

$$P_i(\omega, x_i^1, x_i^2, k_i, n) = \binom{n!}{k_{i,0}!k_{i,1}!k_{i,2}!} \exp(-nD(\omega, x_i^1, x_i^2, \kappa_i)) \exp(-nH(\kappa_i)),$$

where $H(\cdot)$ is the entropy of the trinomial distribution $\kappa_i = (\kappa_{i,0}, \kappa_{i,1}, \kappa_{i,2})$ and $D(\cdot)$ is its relative entropy w.r.t. to the given trinomial.

A.2. **Relative-Entropy Minimization.** Let $0 \leqslant x_i^1 \leqslant x_i^2 \leqslant 1$ for each i be such that: (a) letting J_0 (resp. J_1) be the set of types i for whom $x_i^1 = 1$ (resp. $x_i^2 = 0$), we have $|J_0| \leqslant m_0 - 1$ and $|J_1| \leqslant m_1 - 1$; (b) $x^1 < x^2$ for all $i \notin J_0 \cup J_1$. Let K be the set of $\kappa \in [0,1]^{3I}$ such that: for each i, $\sum_{l=0}^{2} \kappa_{i,l} = 1$, with $\kappa_{i,l} = 0$ if l is not in the support of the trinomial given by (x_i^1, x_i^2); $\sum_i \kappa_{i,0} < m_0$; and $\sum_i \kappa_{i,2} < m_1$. Let $\bar{K}$ be its closure. For each κ, let $D(\omega, x^1, x^2, \kappa) = \sum_i D_i(\omega, x_i^1, x_i^2, \kappa_i)$ and consider the minimization problem:

$$\min_{\kappa \in \bar{K}} D(\omega, x^1, x^2, \kappa).$$

Let $D^*(\omega, x^1, x^2)$ be the optimal value and $\kappa^*(\omega, x^1, x^2)$ the minimizer. The norm of the gradient $d^*(\omega, x^1, x^2)$ at the optimal solution is non-negative and finite. Denote by $\Omega(x^1, x^2)$ the set of ω such that $\sum_i P_i(x \,|\, \omega) = m_0$ for some $x \in [x^1, x^2]$. $\Omega(x^1, x^2)$ is then an interval. If $\Omega(x^1, x^2)$ is a singleton, then it is either the state $\omega = 0$ or $\omega = 1$. If $\Omega(x^1, x^2)$ is an interval, then for each $\omega \in \Omega(x^1, x^2)$, in the optimal solution $\kappa^*(\omega, x^1, x^2)$, the i-th coordinates, for each i, are the trinomial probabilities of the distribution derived from (x_i^1, x_i^2) and the optimal relative entropy is zero. Outside this interval of states, the relative entropy $D^*(\cdot, x^1, x^2)$ is strictly increasing in the distance from $\Omega(x^1, x^2)$ and convex. If $\Omega(x^1, x^2)$ is empty then $D^*(\cdot, x^1, x^2)$ is minimized at either $\omega = 0$ or $\omega = 1$ depending on whether

$\sum_i P_i(x_i^1 \,|\, 1) < m_0$ or $\sum_i (1 - P_i(x^2 \,|\, 0)) > m_1$ and then $D^*(\cdot, x^1, x^2)$ is C^2 and accordingly either strictly increasing or decreasing.

A.3. **Expected Probability of Winning in a Tie at a Bid.** Take pairs $0 \leqslant x_i^1 \leqslant x_i^2 \leqslant 1$ for each i as in the previous subsection. Suppose types in $[x_i^1, x_i^2]$ bid b, types above (resp. below) x_i^2 (resp. x_i^1) bid above b (resp. below b) in the game Γ^n. A tie at bid b in state ω occurs if the total number of players with signals above x^2 is strictly less than $m_1 n$ and those with signals less than x^1 is strictly less than $m_0 n$. Thus, it occurs when the empirical frequency of the trinomial draws falls in K. The probability $G^n(\omega, x^1, x^2)$ of a tie (at bid b) in state ω is computed as:

$$G^n(\omega, x^1, x^2) \equiv \sum_{\kappa \in K^n} \prod_i P_i(\omega, x^1, x^2, \kappa_i)$$

where K^n is the subset of K consisting of κ such that $n\kappa$ is a vector of integers. As we only have to move at most n^{-1} in each coordinate in K to be in K^n, and K is convex, the usual inequalities from the method of types employed in proving Sanov's Theorem give the following bounds:

$$\frac{1}{(n+1)^{3|I|}} \exp(-\sqrt{3|I|}d^*(\omega, x^1, x^2) + O(n^{-2})) \leqslant \frac{G^n(\omega, x^1, x^2)}{\exp(-nD^*(\omega, x^1, x^2))} \leqslant 1,$$

where $d^*(\omega, x^1, x^2)$ is the norm of the gradient of $D(\omega, x^1, x^2)$ at the entropy minimizer. Moreover, suppose $(x^{1,n}, x^{2,n}) \to (x^1, x^2)$. If ω is in the interior of $\Omega(x^1, x^2)$, then by the Uniform Law of Large Numbers, $\lim_n G^n(\omega, x^{1,n} x^{2,n}) = 1$. If ω falls outside this interval, then we get $\lim_n n^{-1} \ln(G^n(\omega, x^{1,n}, x^{2,n})) = -D^*(\omega, x^1, x^2)$ by Sanov's Theorem. For a boundary point ω of the set $\Omega(x^1, x^2)$, the limit probability falls somewhere in $[0, 1]$.

When there is a tie, the probability of winning is determined by the number of agents involved in the tie and is thus a random variable defined as follows. Let $\bar{\tau}^\infty : \bar{K} \to [0, 1]$ be given by:

$$\bar{\tau}^\infty(\kappa) = \frac{m_1 - \bar{\kappa}_2}{\bar{\kappa}_1}$$

where $\bar{\kappa}_l = \sum_i \kappa_{i,l}$ for $l = 1, 2$. Then, the expected probability of winning a tie in state ω is:

$$\bar{\tau}^n(\omega, x^1, x^2) = \sum_{\kappa \in K^n} \prod_i P_i(\omega, x^1, x^2, \kappa_i) \bar{\tau}^\infty(\kappa).$$

Also, let

$$\hat{\tau}^n(\omega, x^1, x^2) = \sum_{\kappa \in K^n} \prod_i P_i(\omega, x^1, x^2, \kappa_i)(1 - \bar{\tau}^\infty(\kappa)).$$

As $\bar{\tau}^{\infty}(\kappa)$ is at least $(nI)^{-1}$ and, of course, no more than $\frac{1}{2}$, we have the following bounds for this probability:

$$\frac{1}{n|I|}G^n(\omega, x^1, x^2) \leqslant \bar{\tau}^n(\omega, x^1, x^2) \leqslant \frac{1}{2}G^n(\omega, x^1, x^2),$$

and similarly,

$$\frac{1}{2}G^n(\omega, x^1, x^2) \leqslant \hat{\tau}^n(\omega, x^1, x^2) \leqslant \frac{n|I|-1}{n|I|}G^n(\omega, x^1, x^2),$$

Suppose $(x^{1,n}, x^{2,n}) \to (x^1, x^2)$. If ω is in the interior of $\Omega(x^1, x^2)$, then by the Uniform Law of Large Numbers we have $\lim_n \bar{\tau}^n(\omega, x^{1,n}, x^{2,n}) = \bar{\tau}^{\infty}(\kappa^*(\omega, x^1, x^2))$.

Finally, let $\tilde{K}^n$ be the subset of $\bar{K}$ consisting of κ such that $n\kappa$ is a vector of integers, $\sum_i \kappa_{i,0} = m_0$, and $\sum_i \kappa_{i,2} = m_1$. Using $\tilde{K}^n$ in the place of K^n we compute $\tilde{G}^n(\omega, x^1, x^2)$, the probability of the event that the number of players with signals less than x^1 is equal to $m_0 n$ and that the number of players with signals above x^2 is equal to $m_1 n$. As above, $\tilde{G}^n$ is driven by the minimum relative entropy $\tilde{D}^*(\omega, x^1, x^2)$. Observe that $\tilde{D}^*(\omega, x^1, x^2) < D^*(\omega, x^1, x^2)$, and hence that $\frac{\tilde{G}^n(\omega, x^1, x^2)}{G^n(\omega, x^1, x^2)} \to 0$ exponentially in n and uniformly in (ω, x^1, x^2).

For the first-order difference equations for a player i, we need to compute the probability $G_i^n(\omega, x^1, x^2)$ of a tie at a bid b if he were to submit it, as well as the corresponding expected probabilities $\hat{\tau}_i^n(\omega, x^1, x^2)$ and $\hat{\tau}_i^n(\omega, x^1, x^2)$. The probabilities are obtainable by a small modification of the above computations. Fix i. We get n trinomial trials for $j \neq i$ and $n-1$ for i. Let K_i^n be the set of $\kappa \in K$ such that: (a) for $j \neq i$, $n\kappa_j$ is a vector of integers; (b) $(n-1)\kappa_i$ is a vector of integers; $\sum_{j \neq i} n\kappa_{j,0} + (n-1)\kappa_{i,0} \leqslant m_0 n - 1$, $\sum_{j \neq i} n\kappa_{j,2} + (n-1)\kappa_{i,2} \leqslant m_1 n - 1$, $\sum_{j \neq i} n\kappa_{j,1} + (n-1)\kappa_{i,1} \geqslant 1$. Replace K^n with K_i^n to get the probability $G_i^n(\omega, x^1, x^2)$ of a tie involving i at bid b and also $\bar{\tau}_i^{\infty}$, with the denominator being $\bar{\kappa}_1 + n^{-1}$ (to include i). One way to leverage the previous computations is to to take $n-1$ trials for all j (including i) and then have an extra trial for players $(j, 1)$, $j \neq i$. Thus, we get the bound below for $G_i^n(\omega, x^1, x^2)$:

$$\frac{1}{n^{3|I|}} \exp(-\sqrt{3}|I|d^*(\omega, x^1, x^2) + O(n^{-2})) \leqslant \frac{G_i^n(\omega, x^1, x^2)}{\exp(-(n-1)D^*(\omega, x^1, x^2))} \leqslant 1,$$

and $\bar{\tau}_i^n(\omega, x^2, x^1)$ is derived as before, using now $G_i^n(\omega, x^1, , x^2)$ instead of $G^n(\omega, x^1, x^2)$. We can similarly compute the probability $\tilde{G}_i^n(\omega, x^1, x^2)$ that i is the only player with a signal in $[x^1, x^2]$.

A.4. **Proof of Lemma 6.2.** As $\Theta^\zeta \times X_i$ is compact and $\bar{\pi}^{n,\zeta}_{i,k}$ is continuous in (θ, y_i) for all n (including $n = \infty$), the result is proved if we show that for a sequence $(\theta^n, y_i^n) \to (\theta, y_i)$, we have $\bar{\pi}_i^{n,\zeta,k}(\theta^n, y_i^n) \to \bar{\pi}_i^{\infty,\zeta}(\theta, y_i, k)$ and similarly for the derivatives.

Let $(x^{0,n}, x^{1,n}, x^{2,n}) = (\theta^n_{-0}(k-1), \theta^n_{-0}(k), \theta^n_{-0}(k+1))$ for each n and let (x^0, x^1, x^2) be its limit. Let $y_i^n \to y_i$.

For each n, and for the case of a buyer, we decompose $\bar{\pi}_i^{n,\zeta,k}(\theta^n, y_i^n)$ as

$$\lambda^{0,n} \int_\Omega (v_i(\omega, y_i^n) - b^{k-1}) q^{0,n}(\omega) d\omega + \lambda^{1,n} \int_\Omega (v_i(\omega, y_i^n) - b^k) q^{1,n}(\omega) d\omega - (1 - \lambda^{0,n} - \lambda^{1,n})\alpha\zeta$$

(for a seller we have $-(1-\alpha)\zeta$ in the place of $\alpha\zeta$),

$$q^{0,n}(\omega) = \frac{\hat{\tau}_i^n(\omega, x^{0,n}, x^{1,n}) p(\omega \mid y_i^n)}{\int_\Omega \hat{\tau}_i^n(\omega', x^{0,n}, x^{1,n})) p(\omega' \mid y_i^n) d\omega'};$$

$q^{1,n}$ is defined similarly using $(x^{1,n}, x^{2,n})$ and also replacing $\hat{\tau}_i^n$ with $\bar{\tau}_i^n$; and

$$\lambda^{\ell,n} = \frac{\int_\Omega q^{\ell,n}(\omega) d\omega}{\int_\Omega [q^{0,n}(\omega) + q^{1,n}(\omega) + \tilde{G}_i^n(\omega, x^{0,n}, x^{1,n})] d\omega}, \ \ell = 0, 1.$$

Suppose $\Omega(x^1, x^2)$ has a nonempty interior. Then, $q^{1,n}$ converges pointwise to the density q^1 given by: $q^1(\omega) = \bar{\tau}^\infty(\kappa^*(\omega, x^1, x^2)) p(\omega \mid y_i)$ if ω belongs to the interior of $\Omega(x^1, x^2)$ and is zero if it does not belong to $\Omega(x^1, x^2)$. Hence, the expectation under $q^{1,n}$ corresponds to the payoff from a tie at bid b^k for types in $[x^1, x^2]$. If $\Omega(x^1, x^2)$ has an empty interior, then $D^*(\omega, x^1, x^2)$ is the lowest at either $\omega = 0$ or $\omega = 1$. Assume the former. Then, for each $\omega < \omega'$,

$$\lim_{n \to \infty} \frac{q^{1,n}(\omega')}{q^{1,n}(\omega)} = \lim_{n \to \infty} \exp[-n(D^*(\omega', x^{1,n}, x^{2,n}) - D^*(\omega, x^{1,n}, x^{2,n}))] = 0.$$

and we have that the limit of the probability measures $Q^{1,n}$ is point mass at $\omega = 0$. Hence the expectation converges to $v_i(0, y_i) - b$, which is what we impute under $\bar{\pi}_i^{\infty,\zeta}$. A similar computation holds for $q^{0,n}$. To finish the proof, we need to get the convergences of $\lambda^{\ell,n}$. Observe first that because $\tilde{G}_i^n(\omega, x^0, x^1)$ is dominated by $G_i^n(\omega, x^0, x^1)$, $\lambda^{0,n} + \lambda^{1,n} \to 1$. If both $\Omega(x^0, x^1)$ and $\Omega(x^1, x^2)$ have nonempty interiors, then the limit of $\lambda^{0,n}$ exists and is in $(0,1)$ as $q^{0,n}$ and $q^{1,n}$ converge pointwise; if $\Omega(x^0, x^1)$ has an empty interior but $D^*(\omega, x^0, x^1)$ is lowest at $\omega = 0$, then

$$\lim_n \lambda^{0,n} = \lim_n \exp(-n(D^*(0, x^{0,n}, x^{1,n}) - D^*(0, x^{1,n}, x^{2,n})) = 0.$$

All other cases are handled similarly and we get the appropriate convergence.

The logic for the functions involving the derivatives is similar and, therefore, omitted.

A.5. **Relative-Entropy Calculations for Section 7.** We recall the set up from Section 7. For each i and n, F_i^n is set of empirical frequencies in $\Delta(B(Z))$ of $(i,1)$'s opponents that are observable from the play of $\sigma^{n,Z}$. Given $b(\omega_0) \leqslant b_0 < b_1 \leqslant b(\omega_1)$, let $E_i^n(b_0,b_1,x_i,\omega)$ be the set of frequencies in F_i^n where there is a payoff difference between bidding b_0 and b_1 for x_i. Let $R_i^n(b_0,b_1,x_i,\omega)$ be the probability of this event and let $\pi_i^n(b_0,b_1,x_i,\omega)$ be the expectation of the difference in payoffs conditional on $E_i^n(b_0,b_1,x_i,\omega)$. Let $\bar{R}_i^n$ be the expectation of $R_i^n(b_0,b_1,x_i,\cdot)$ w.r.t. $P_i^Z(\cdot\,|\,x_i)$ and let $\bar{\pi}_i^n(b_0,b_1,x_i)$ be the corresponding expectation of the payoff difference. Observe that

$$\bar{\pi}_i^\infty(b_0,b_1,x_i) = \lim_n \bar{\pi}_i^n(b_0,b_1,x_i) = \lim_n \sum_{\omega\in\Omega(Z)} \pi_i^n(b_0,b_1,x_i,\omega)\frac{R_i^n(b_0,b_1,x_i,\omega)}{\bar{R}_i^n(b_0,b_1,x_i)}.$$

Asymptotically, then, the payoff differences are driven by the relative likelihoods of the events $E_i^n(b_0,b_1,x_i,\omega)$, which in turn are determined by the associated relative entropy estimates. The relative entropy of an empirical frequency $\hat{\beta}_i^n$ relative to the true distribution β^n is $\sum_{b\in B(Z)} \hat{\beta}_i^n(b)[\ln(\hat{\beta}_i^n(b)) - \ln(\beta_i^n(b))]$. If we let $d_i^n(b_0,b_1,x_i,\omega)$ be the minimal entropy between points in $E_i^n(b_0,b_1,x_i,\omega)$ and the distribution $\beta_i^n(\cdot\,|\,\omega)$ of bids in state ω induced by $\sigma^{n,Z}$, then

$$n^{-|B(Z)|}\exp(-nd_i^n(b_0,b_1,x_i,\omega)) \leqslant R_i^n(b_0,b_1,x_i,\omega) \leqslant n^{|B(Z)|}\exp(-nd_i^n(b_0,b_1,x_i,\omega)).$$

Let $\hat{\beta}_i^n(b_0,b_i,x_i,\omega) \in E_i^n(b_0,b_1,x_i,\omega)$ be a frequency that achieves the minimal entropy, and let $\hat{\beta}_i^\infty(b_0,b_i,x_i,\omega)$ be defined as follows. First for $\omega \leqslant \omega_0$, (a) $\sum_{b<b_0} \hat{\beta}_i^\infty(b_0,b_i,x_i,\omega)(b) = \mu_0$ (resp. $\sum_{b\leqslant b_0} \hat{\beta}_i^\infty(b_0,b_i,x_i,\omega)(b) = \mu_0$) if $v_i(\omega,x_i) \neq b_0$ (resp. $v_i(\omega,b_0) = b_0$); (b) the ratio of the probabilities of $b < b'$ is the same as in $\sigma^{\infty,Z}$ if $b' < b_0$ (resp. $b' \leqslant b_0$) $b < b_0$ (resp. $b > b_0$). For $\omega \geqslant \omega_1$, we get a similar set of conditions except that the defining equation in (a) sets $\sum_{b>b_0} \hat{\beta}_i^\infty(b_0,b_i,x_i,\omega)(b) = \mu_0$ (resp. $\sum_{b\geqslant b_1} \hat{\beta}_i^\infty(b_0,b_i,x_i,\omega)(b) = \mu_1$) if $v_i(\omega,x_i) \neq b_1$ (resp. $v_i(\omega,x_i) = b_1$). By construction, $\hat{\beta}_i^n(b_0,b_1,x_i,\omega)$ converges to $\hat{\beta}_i^\infty(b_0,b_1,x_i,\omega)$. Also the entropy $d_i^n(b_0,b_1,x_i,\omega)$ converges to the relative entropy $d_i^\infty(b_0,b_1,x_i,\omega)$ of $\beta_i^\infty(b_0,b_1,x_i,\omega)$ w.r.t. the distribution β^∞ induced by $\sigma^{\infty,Z}$. $d_i^\infty(b_0,b_1,x_i,\cdot)$ is strictly monotonically decreasing (resp. increasing) over the interval $[\zeta_0,\omega_0]$ (resp. $[\omega_1,1]$). Thus, asymptotically, x_i's choice between b_0 and b_1 is determined by which of the two states, ω_0 and ω_1, has the lower entropy. This computation gives us all the lemmas for Section 7 as we now show.

A.6. **Proofs for Section 7.** For Lemma 7.4, $d_i^\infty(b_0,b_1,x_i,\omega_0)$ is strictly smaller or strictly larger than $d_i^\infty(b_0,b_1,x_i,\omega_1)$ depending on whether ω is ω_0 or not. Either way, the payoff difference between b_0 and b_1 for x_i is determined in state ω. And, relative entropy under

$d_i^\infty(b_0, b_1, \cdot, \omega_0)$ continues to be minimized in ω if we replace x_i with y_i. Thus, we are basically in a private values model in state ω. The difference of differences, i.e., the difference between y_i and x_i of their payoff differences between b_0 and b_1 depends on $E_i^n(b_0, b_1, y_i, \omega)$ and equals their value differences in this state which gives us the result.

For Lemma 7.5, suppose first that there exists one state ω such that $v_i(\omega, x_i) = b_0$ and $v_i(\omega, y_i) = b_1$. Then $d_i^\infty(b_0, b_1, y_i, \cdot)$ is the smallest in state ω and hence $\bar{\pi}_i^\infty(b_0, b_1, y_i) = b_1 - b_0$ if $\omega = \omega_0$ and we can say that $\bar{\pi}_i^\infty(b_0, b_1, y_i)$ is at least $(1-\alpha)\zeta$ if $\omega = \omega_1$. Now suppose that $v_i(\omega_0, x_i) = b_0$ while $v_i(\omega_1, y_i) = b_1$. Then, as in the previous case, $\bar{\pi}_i^\infty(b_0, b_1, y_i, \omega_0) = v_i(\omega_0, y_i) - b_0 \geqslant \zeta$ and $\bar{\pi}_i^\infty(b_0, b_1, y_i, \omega_1) \geqslant (1-\alpha)\zeta$ and regardless of which of the two states, ω_0 and ω_1 has a lower limit entropy, the payoff difference is at least ζ.

For the Lemma 7.6, if $b_1 > v_i(\omega_1, x_i)$ then, regardless of whether ω_0 or ω_1 has a lower entropy, $\bar{\pi}_i^\infty(b_0, b_1, x_i) \leqslant -\zeta$. If $b_1 \leqslant v_i(\omega_1, x_i)$ and $b_0 = v_i(\omega_0, x_i)$, then ω_0 has a lower entropy, implying that $\bar{\pi}_i^\infty(b_0, b_1, x_i) \leqslant -\zeta$, again. The logic for the proof of Lemma 7.7 is the same as that for the second case of Lemma 7.6: $q_0(b_0, \sigma^{\infty,Z}) < q_1(b_1, \sigma^{\infty Z})$ implies that ω_0 has a lower entropy so that b_1 is inferior by at least ζ; when the inequality is reversed, b_1 is a better reply by at least ζ.

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