School Choice and the Housing Market

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Abstract

We develop a unified framework with schools and residential choices to study the welfare and distributional consequences of public schools' switching from the traditional neighborhood assignment to the Deferred Acceptance mechanism. We show that when families receive higher priorities at neighborhood schools, the Deferred Acceptance mechanism improves aggregate or average welfare compared to neighborhood assignment. Additionally, under general conditions, the Deferred Acceptance mechanism improves the welfare of lowest-income families, both with and without neighborhood priorities. Our work lays theoretical foundations for analyzing assignment games with externalities.

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1 Introduction

According to the Brookings Institution's Center on Children and Families, the proportion of large school districts in the US that allow parental choice over public schools has doubled from 2000 through 2016 (Whitehurst, 2017). Many school districts have replaced the traditional neighborhood assignment with choice-based assignment mechanisms which reflect recent advances in matching theory and market design. A prominent example is the widespread application of the celebrated Deferred Acceptance mechanism of Gale and Shapley (1962). Following a scholarly article by Abdulkadiroğlu and Sönmez (2003), Deferred Acceptance has been adopted for school assignment in New York City, Boston, Chicago, Denver, Washington DC and Newark, among many others.

Under neighborhood assignment, families send their children to the designated neighborhood schools. In contrast, the Deferred Acceptance mechanism assigns children to schools based on families' reported preferences and their priorities at schools. The Deferred Acceptance mechanism and its welfare properties are extensively studied in the matching theory literature. However, previous papers predominantly assume that the preferences and priorities are exogeneously given, while in reality, they depend on families' endogeneous neighborhood choices. It has been empirically documented that families strategically choose where to live, and they do so by taking into account the schooling options (Chung, 2015; Kane, Riegg, and Staiger, 2006; Reback, 2005). Those strategic neighborhood choices affect families' probabilities of being assigned to different schools through their effects on preferences and priorities.[1]

In this paper we develop a unified framework with school choice and a housing market,

[1]Neighborhood choices affect preferences since families prefer schools closer to their homes (e.g., Glazerman (1998), Burgess, Greaves, Vignoles, and Wilson (2011), Abdulkadiroğlu, Agarwal, and Pathak (2017a), Abdulkadiroğlu, Pathak, Schellenberg, and Walters (2020b)), and they affect priorities since schools typically grant higher priorities to neighborhood applicants.

where families make residential choices prior to school admission. The timing of the model is as follows. The city or the school district announces the school assignment mechanism. Then, families make neighborhood choices optimally, given the school assignment mechanism, other families' neighborhood choices and the market-clearing neighborhood prices. Lastly, children are assigned to schools through the announced assignment mechanism. Our goal is to evaluate the welfare and distributional consequences of the widespread education reform of switching from neighborhood assignment to Deferred Acceptance. There are two major findings. First, we show that when there are neighborhood priorities (i.e., when families receive higher priorities at neighborhood schools), the Deferred Acceptance mechanism improves aggregate (or average) welfare compared to neighborhood assignment. Second, we show that under fairly general conditions the lowest-income families prefer the Deferred Acceptance mechanism, both with and without neighborhood priorities, to neighborhood assignment. To the best of our knowledge, ours is the first theoretical analysis of the problem in a general model with endogenous neighborhood choices and unrestricted preference domain. We now elaborate on our findings and their policy implications.

In general, welfare comparison between the Deferred Acceptance and neighborhood assignment is ambiguous. On one hand, the former may generate higher welfare as it gives families more flexibility to reside in their preferred neighborhoods and enroll their children to their preferred schools, potentially outside of the neighborhoods. On the other hand, neighborhood assignment may generate higher welfare as families who value some school the most can 'buy their way in' by purchasing a house in that neighborhood. Despite this trade-off, we show that when the district applies neighborhood priorities, the Deferred Acceptance mechanism generates higher aggregate welfare than neighborhood assignment. Like neighborhood assignment, Deferred Acceptance with neighborhood priorities allows families' with high values to enroll at their preferred schools by purchasing a house in the corresponding neighborhoods. Moreover, it allows families to enroll at a school outside of their neighborhood when there are available seats. We also show that, under some conditions such as identical ordinal preference rankings over neighborhoods and schools, Deferred Acceptance generates higher aggregate welfare with neighborhood priorities than without those.

Neighborhood priorities may be thought of as a compromise between neighborhood assignment and open enrollment without neighborhood priorities (i.e., where all families receive a fair shot at each school). In Boston Public School (BPS) there have been constant debates about using neighborhood assignment or allowing choice (Daley, 1999; Dur, Kominers, Pathak, and Sönmez, 2018; Menino, 2012). Those debates have resulted in redesigning the school assignment system by granting higher priorities to families (at a fraction of seats[2]) at their neighborhood schools (Dur et al., 2018). Neighborhood priorities are applied not only in BPS, but in predominant majority of the US school districts allowing parental choice. Our theoretical results that Deferred Acceptance with neighborhood priorities improves aggregate welfare compared to neighborhood assignment and, under some conditions, compared to Deferred Acceptance without neighborhood priorities, potentially provide a rationale for the widespread application of neighborhood priorities for school assignment. To the best of our knowledge, our findings are the first theoretical justification of using neighborhood priorities for welfare considerations.

Although the aggregate welfare is always larger under Deferred Acceptance with neighborhood priorities compared to neighborhood assignment, some families may be better-off under the latter mechanism. The question that we ask next is how the mechanisms compare in terms of the welfare of lowest-income families. The question is important since low-income and disadvantaged communities' welfare has always been a major consideration for education policy (Fuller, 1996; Orfield and Frankenberg, 2013). In Section 5 we extend our model so that families are differentiated by incomes or budgets. A budget denotes the maximum amount a family can pay for a house. Proponents ar-

[2]In 1999, BPS adopted what is known as the '50-50 seat split', where families are granted higher priorities at only half of the seats at their neighborhood schools.

gue that school choice weakens the links between schools assignment and the housing market, and potentially leads to more equitable outcomes by allowing families in less affluent neighborhoods to apply to higher quality schools outside of their neighborhoods (Bedrick and Burke, 2015; Coons and Sugarman, 1978). Although the argument is intuitive, it has limited theoretical foundation. Papers on the topic typically analyze stylized models where families have identical preferences over neighborhoods and schools. Such an assumption is highly unrealistic for the school choice setting.[3] To the best of our knowledge, ours is the first work to compare distributional effects of the Deferred Acceptance mechanism in a general matching model with rich preferences and residential choices. We show that under general conditions lowest-income families prefer both versions of the Deferred Acceptance mechanism to neighborhood assignment. Our sufficiency conditions have two parts: (1) underdemanded neighborhoods remain underdemanded when the school assignment mechanisms is switched from neighborhood assignment to Deferred Acceptance, (2) underdemanded (or to put it simply, cheapest) neighborhoods have underdemanded (least selective) schools.[4] The former condition is intuitive: an underdemanded neighborhood is unlikely to significantly gain in value when the school assignment mechanism is switched from neighborhood assignment to Deferred Acceptance. The latter condition is consistent with the empirical evidence: (Owens and Candipan, 2019) document that in large metropolitan areas in the US the less affluent neighborhoods typically have underperforming schools. We show that the conditions are satisfied for natural special cases. Thus, our findings provide a theoretical justification for a major argument in favor of school choice, namely, that lowest-income families benefit from choice.

Finally, our work develops a theoretical foundation on studying assignment games with

[3]For example, it has been shown that families prefer schools that are closer to their homes (Abdulkadiroğlu et al., 2017a,2; Burgess et al., 2011; Glazerman, 1998).

[4]We show that (1) and (2) are also 'necessary' conditions for lowest-income families to prefer Deferred Acceptance over neighborhood assignment if we require 'robustness'. The result is formally stated in Theorem 6.

externalities. In our model a family's valuation for a neighborhood depends on other families' neighborhood choices through the latter's effects on the family's school assignment probabilities. These externalities may preclude the existence of a competitive equilibrium in discrete economies. However, we show that a competitive equilibrium always exists in a large economy with a continuum of families. The result builds on that in a large economy a family's school assignment probabilities are continuous in other families' neighborhood choices. This allows us to use a novel application of Schauder-Tychonoff fixed point theorem to establish equilibrium existence. Not only does the continuum model circumvent the equilibrium non-existence issue, but it also buys us tractability. In the continuum model we derive closed-form expressions for school assignment probabilities which are used for proving many of the results. We show that the continuum model is an arbitrarily close approximation of sufficiently large discrete ones. This implies that all results, such as existence of a competitive equilibrium and welfare comparisons across mechanisms, hold in an approximate sense for every sufficiently large discrete economy. Equilibrium existence and large market approximation results extend to general assignment games with externalities, such as complementarities or peer preferences.

The remainder of the paper is organized as follows. Section 2 reviews related literature. Section 3 describes the continuum model and school assignment mechanisms. Section 4 compares aggregate welfare across the mechanisms. Section 5 introduces the model with budget constraints and studies the welfare of lowest-income families. Section 6 shows that the results established for the continuum model hold approximately for sufficiently large discrete economies. Section 7 discusses simulation results. Section 8 concludes. All omitted proofs are in the Appendix. Alternative school assignment mechanisms (such as Top Trading Cycles and Immediate Acceptance) and further extensions are studied in the Supplementary Appendix.

2 Related Literature

Welfare and distributional consequences of school choice have been theoretically analyzed by several earlier works (Avery and Pathak, 2020; Barseghyan, Clark, and Coate, 2013; Calsamiglia, Martínez-Mora, and Miralles, 2015; Epple and Romano, 2003; Lee, 1997; Xu, 2019). These papers feature stylized models, where: (1) types are described by a single parameter which reflects ability or income, (2) families' have no preferences over neighborhoods, (3) schools are ranked by quality and all families prefer the higher quality schools, (4) valuations for schools are supermodular in income and school quality. Some of these assumptions are highly unrealistic in the context of school choice. For example, assumption (3) implies that families have identical ordinal preference rankings over schools. Our work, on the other hand, features a general preference domain with arbitrary valuations over the family's joint assignment to neighborhoods and schools. Such unrestricted heterogeneity is an important aspect in Gale and Shapley (1962) and the vast literature on the two-sided matching literature that followed this seminal work.

The generality of our model allows us to reveal novel insights on welfare and distributional consequences of school assignment mechanisms which are missing from prior papers on the topic. First, in contrast to most works above that are mainly interested in distributional outcomes of school choice, our paper also compares school assignment mechanisms in terms of aggregate or average welfare. Such an analysis is trivial when families have identical ordinal rankings and supermodular valuations: in that setting, neighborhood assignment is the unique aggregate welfare maximizing rule. We show that this is not true in a general model: Deferred Acceptance may create higher aggregate welfare than neighborhood assignment, and it always does so when there are neighborhood priorities. Second, some of the conclusions on lowest-income families' welfare in previous theoretical papers depend on the preference restrictions those works impose. For example, in Calsamiglia et al. (2015) and Xu (2019) lowest-income families always prefer Deferred Acceptance to neighborhood assignment. In contrast, in a model with peer preferences and endogenously priced outside options, Avery and Pathak (2020) show that lowest-income families may be better off under neighborhood assignment. The authors show the result for a stylized example and do not provide general conditions for welfare comparisons across the mechanisms. Unlike the works above, we provide general sufficient conditions which guarantee that lowest-income families prefer Deferred Acceptance to neighborhood assignment.

Our setup corresponds to a two-sided matching problem with endogenous preferences and/or priorities. Papers on the topic, such as Peters and Siow (2002) and Bodoh-Creed and Hickman (2018), typically assume unidimensional family types, supermodular valuations and identical preference rankings for tractability. Bodoh-Creed and Hickman (2018) write that *"the richness of the preferences admitted by most models building on Gale and Shapley ... makes it very difficult to include an element of endogenous student quality"*. Our setting allows general preferences and we apply a continuum framework to gain tractability. Despite the generality of our model, we obtain strong results on welfare comparisons across the mechanisms.

The second part of our work is related to papers that study assignment problems with budget-constrained agents. In particular, Che, Gale, and Kim (2013a) and Che, Gale, and Kim (2013b) show that in that environment a random assignment with resale improves aggregate welfare compared to the market equilibrium. In our model, there is no resale option for school assignment and therefore aggregate welfare comparisons are ambiguous. However, we show that under fairly general conditions random assignment improves the welfare for agents with the smallest budgets.

Our work contributes to the relatively new strand of matching theory literature on 'priority design' (Celebi and Flynn, 2021; Shi, 2021). These papers study optimal priority structures for general assignment mechanisms. In contrast, we are interested the role of neighborhood priorities and its welfare implications for a particular assignment mechanism, namely the Deferred Acceptance. Our results suggest that using priorities may potentially improve aggregate welfare. First, we show that with neighborhood priorities the Deferred Acceptance mechanism always generates higher aggregate welfare than neighborhood assignment. This is not necessarily true without neighborhood priorities. Second, we show that, under some conditions, such as when families have identical ordinal preferences over neighborhoods and schools, the Deferred Acceptance mechanism generates higher aggregate welfare with neighborhood priorities than without them. The last finding is in the spirit papers that show that incorporating 'signaling devices' into matching problems without money may be welfare improving (Abdulkadiroğlu, Che, and Yasuda, 2015; Coles, Cawley, Levine, Niederle, Roth, and Siegfried, 2010; Hylland and Zeckhauser, 1979; Lee and Niederle, 2015). When there are neighborhood priorities, families are allowed to signal their high valuations for schools by choosing the corresponding neighborhoods. Thus, neighborhood choices act as signaling devices in our model, potentially improving aggregate welfare.

Lastly, our work contributes to the literature on large matching markets (Abdulkadiroğlu et al., 2015; Azevedo and Leshno, 2016; Gretsky, Ostroy, and Zame, 1992,9; Kamecke, 1992; Leshno and Lo, 2017) and assignment externalities (Pycia and Yenmez, 2019; Sasaki and Toda, 1996). Unlike the continuum assignment game of Gretsky et al. (1992) and Gretsky et al. (1999), in our model there are assignment externalities: a family cares not only about her own neighborhood choice, but also those of other families since those affect the family's school assignment probabilities. Although externalities preclude the existence of competitive equilibrium in finite discrete markets, we show that a competitive equilibrium always exists in large markets with a continuum of families. Analogous results have been established in alternative matching environments with complementarities and externalities, e.g., Azevedo, Weyl, and White (2013) Azevedo and Hatfield (2018), Che, Kim, and Kojima (2019), Bachas, Fonseca, and Pakzad-Hurson (2021) and Greinecker and Kah (2021). Our model is potentially closest to the last paper. The authors assume that agents have continuous preferences over a superset of assignments to prove the existence of a competitive equilibrium. The assumption is abstract, and the paper does not clarify whether it is satisfied for specific matching problems. We on the other hand, do not impose such an assumption, but instead we prove that in our model families' expected utilities are equicontinuous in neighborhood choices,[5] which is sufficient to guarantee existence of a competitive equilibrium. Our existence result can be applied more broadly to prove existence of equilibria in general assignment games with externalities, such as peer preferences or complementarities.

3 The Continuum Model

There is a unit mass of families with a single child and a finite and equal number of neighborhoods H and schools S. There is a unique school in each neighborhood $h \in H$. We denote that school by $s_h \in S$. The capacity $q_h \in \mathbb{N}$ of neighborhood h is the mass of families that a neighborhood can accommodate. Similarly, the capacity $q_s \in \mathbb{N}$ of school s is the mass of families that can enroll (their children) at school s. Unless mentioned otherwise, we assume that $q_h \leq q_{s_h}$ for all $h \in H$. The assumption is necessary for defining neighborhood assignment, i.e., schools need to have enough capacity to accommodate all neighborhood children.

Each family has a type $v \in [0, 1]^{|H|\times|S|} := V$, where $v(h, s) \in [0, 1]$ denotes the valuation for residing in neighborhood h and enrolling at school s.[6] The economy is described by a (Borel) probability measure η over the type space V.

Valuations induce preference rankings, which are complete, reflexive and anti-symmetric

[5]The last result uses that school assignment probabilities under the Deferred Acceptance mechanism change continuously with families' neighborhood choices.

[6]We assume that families only care about their own assignment to neighborhoods and schools. Although, such valuations profile is very general, it assumes away the possibility of families having peer preferences. As we elaborate in the Discussion section, this may not be without loss of generality.

relations on S. Let P be the space of preference rankings. Conditional on residing in neighborhood h, the preference ranking $\succ_{vh} \in P$ of type v satisfies

$$v(h, s) > v(h, s') \Rightarrow s \succ_{vh} s'. \tag{1}$$

When $v(h, s) = v(h, s')$, ties are broken arbitrarily. For example, we may assume that a fixed ordering over schools is used to break ties.

Let $\bar{H} := H \cup \{0\}$, where 0 denotes the outside option of not buying a house in the school district. Neighborhood choices τ is a probability measure on $V \times \bar{H}$, with the property that

$$\tau\Big((v, h) \in V \times \bar{H} : v \in U, h \in H\Big) = \eta(U),$$

for any measurable $U \subseteq V$. The interpretation of neighborhood choices τ is that for any measurable $U \subseteq V$ and $H' \subseteq \bar{H}$, $\tau\Big((v, h) \in V \times \bar{H} : v \in U, h \in H'\Big)$ denotes the mass of families whose types are in U and who choose to reside in some neighborhood in H'. We denote the space of neighborhood choices by $\mathcal{T}$.

In general, school assignment probabilities depend on the reported preference rankings of families. Throughout this work we consider strategyproof school assignment mechanisms, where each family has a dominant strategy to report preferences truthfully. Thus, assuming truthful reports, valuations and neighborhood choices uniquely pin down the preference reports of families through equation 1 (and the tie-breaker). We denote by $\lambda^{\phi}_{vs}(h, \tau) \in [0, 1]$ the probability that type v is assigned to school s. The probability depends on her neighborhood choice h, the population's neighborhood choices τ and the school assignment mechanism ϕ.

Given school assignment probabilities and neighborhood price vector $p \in [0, 1]^{|H|}$, the expected utility of type v choosing neighborhood $h \in H$ is equal to

$$u^{\phi}_{v}(h, \tau) - p_h.$$

where $u^{\phi}_{v}(h, \tau) := \sum_{s \in S} \lambda^{\phi}_{vs}(h, \tau)\ v(h, s)$. Also, let $u^{\phi}_{v}(0, \tau) := 0$ for all $v \in V$ and $\tau \in \mathcal{T}$. We now define our solution concept.

Definition 1. *For neighborhood choices $\tau \in \mathcal{T}$ and price vector $p \in \mathbb{R}_+^{|H|}$, we say a pair (τ, p) is a* ***competitive equilibrium (CE)*** *of mechanism ϕ if it satisfies the following conditions:*

1. $\tau\Big((v,h) \in V \times \bar{H} : h = \arg\max_{h' \in \bar{H}} \ u_v^{\phi}(h', \tau) - p_{h'}\Big) = 1,$ *where* $p_0 := 0$,
2. $\tau\Big((v,h) \in V \times \bar{H} : h = h'\Big) \leq q_{h'}, \forall h' \in H$,
3. $\tau\Big((v,h) \in V \times \bar{H} : h = h'\Big) < q_{h'} \Rightarrow p_{h'} = 0$.

The definition is standard. The first two conditions in Definition 1 are the optimality and feasibility of neighborhood choices, respectively. The third condition says that neighborhoods with excess capacity are priced at zero. This would guarantee that the sellers of vacant houses in the neighborhood have no incentives to undercut the prices.

We now describe the school assignment mechanisms and derive school assignment probabilities under each of those.

Neighborhood Assignment.

Under neighborhood assignment (NA), families are assigned to their neighborhood schools. Then, for all $s \in S, h \in H$ and $\tau \in \mathcal{T}$,

$$\lambda_{vs}^{NA}(h, \tau) = \begin{cases} 1 & \text{if } s = s_h, \\ 0 & \text{otherwise.} \end{cases}$$

Deferred Acceptance.

Deferred Acceptance for the continuum model is defined as in Azevedo and Leshno (2016) and Abdulkadiroğlu et al. (2017a). We consider two versions of the Deferred Acceptance mechanism: in the first version families do not receive higher priorities at neighborhood schools, and in the second version they do.

Deferred Acceptance without Neighborhood Priority (DA).

School assignment under DA is determined based on families' preferences, lottery numbers and market clearing admission cutoffs, or simply cutoffs. Preferences are decided by neighborhood choices through equation 1. Lottery numbers are drawn uniformly and independently from the unit interval. Formally, neighborhood choices τ result in a probability measure G_τ over $P \times [0,1]$, given by

$$G_\tau\Big((\succ, r) \in P \times [0,1] : \succ \in P', r \in (r_0, r_1)\Big)$$

$$= \tau\Big((v,h) \in V \times \bar{H} : \succ_{vh} \in P'\Big) \times \Big(r_1 - r_0\Big),$$

for each $P' \subseteq P$ and $(r_0, r_1) \subseteq [0,1]$. Thus, $G_\tau\Big((\succ, r) \in P \times [0,1] : \succ \in P', r \in (r_0, r_1)\Big)$ equals the mass of types with preferences in P' and lottery numbers in the interval (r_0, r_1).[7]

Cutoffs are derived through an iterative procedure that we describe below. For a vector $c \in [0,1]^{|S|}$, the demand function $D : [0,1]^{|S|} \to [0,1]$ is given by

$$D_s(c) = G_\tau\Big((\succ, r) \in P \times [0,1] : r \geq c_s \text{ and } s \succ s' \text{ for all } s' \text{ with } r \geq c_{s'}\Big).$$

In words, $D_s(c)$ is the mass of families whose lottery numbers exceed c_s, and who prefer s to any other school s' where their lottery numbers exceed $c_{s'}$. For $c \in [0,1]^{|S|}$ and $x \in [0,1]$ we denote by $c(s,x) \in [0,1]^{|S|}$ the vector that differs from c only by that $c_s(s,x) = x$.

We define a sequence of vectors $(c^t)_{t=1}^{\infty}$ recursively by $c^1 = 0$ and

$$c_s^{t+1} = \begin{cases} 0 & \text{if } D_s(c^t) < q_s, \\ \min\Big\{x \in [0,1] : D_s\big(c^t(s,x)\big) \leq q_s\Big\} & \text{otherwise.} \end{cases}$$

[7]The versions of Deferred Acceptance described in this section apply single tie-breaking rules, i.e., all schools use the same lottery number for each family. Our results (with slightly modified proofs) hold for the case of multiple tie-breaking, i.e., when different schools use different lottery numbers for a given family. The extension is discussed in Appendix C.

As shown by Abdulkadiroğlu, Angrist, Narita, and Pathak (2017b), $(c^t)_{t\in\mathbb{N}}$ is convergent. Let $c^{DA} := \lim_{t\to\infty} c^t$ denote the **DA cutoffs**. This cutoffs depend on neighborhood choices τ, but we omit this dependence to keep notation simple. The DA cutoffs determine school assignment as follows. A family is assigned to school s if her lottery number exceeds c_s^{DA}, and she prefers s to any school where her lottery number exceeds the corresponding DA cutoff. The probability of this event is

$$\lambda_{vs}^{DA}(h,\tau) = \min\{c_{s'}^{DA} : s' \succ_{vh} s\} \times \max\left\{0, \frac{\min\left\{c_{s'}^{DA} : s' \succ_{vh} s\right\} - c_s^{DA}}{\min\left\{c_{s'}^{DA} : s' \succ_{vh} s\right\}.}\right\} \quad (2)$$

$$= \max\left\{0, \min\left\{c_{s'}^{DA} : s' \succ_{vh} s\right\} - c_s^{DA}\right\}.$$

The first term in the middle part of equation 2 denotes the probability that v's lottery number does not exceed the cutoff at any school that she prefers more than s. The second term is the probability that her lottery number exceeds the cutoff at s, conditional on it not exceeding those in more preferred schools.

Deferred Acceptance with Neighborhood Priority (DN).

Under DN, school assignment is determined based on families' preferences, lottery numbers, priorities and cutoffs. Again, preferences are decided by neighborhood choices through equation 1 and lottery numbers are drawn uniformly and independently from the unit interval. Families receive priority 1 at neighborhood schools and priority 0 at non-neighborhood ones. Formally, neighborhood choices τ result in a probability measure G_τ on $P \times S \times [0,1]$ satisfying

$$G_\tau\Big((\succ, s, r) \in P \times S \times [0,1] : \succ \in P', s \in S', r \in (r_0, r_1)\Big)$$

$$= \tau\Big((v,h) \in V \times \bar{H} : \succ_{v,h} = \succ, s_h \in S'\Big) \times \Big(r_1 - r_0\Big),$$

for each $P' \subseteq P$, $S' \subseteq S$ and $(r_0, r_1) \subseteq [0,1]$. Thus, $G_\tau\Big((\succ, s, r) \in P \times S \times [0,1] : \succ \in P', s \in S', r \in (r_0, r_1)\Big)$ equals the mass of families with preferences in P', who reside in the neighborhood of some school in $S' \subseteq S$ and whose lottery numbers are in the interval (r_0, r_1). For a vector $c \in [0,2]^{|S|}$ the demand function $D : [0,2]^{|S|} \to [0,1]$ is given by

$$D_s(c) = G_\tau\Big((\succ, s', r) \in P \times S \times [0,1] : r + \mathbb{1}[s' = s] \geq c_s \text{ and}$$
$$s \succ s'' \text{ for all } s'' \text{ with } r + \mathbb{1}[s' = s''] \geq c_{s''}\Big).$$

For $c \in [0,2]^{|S|}$ and $x \in [0,2]$ we denote by $c(s,x) \in [0,2]^{|S|}$ the vector that differ from c by that $c_s(s,x) = x$. Consider the sequence of vectors recursively defined by

$$c_s^{t+1} = \begin{cases} 0 & \text{if } D_s(c^t) < q_s \\ \min\Big\{x \in [0,2] : D_s\big(c^t(s,x)\big) \leq q_s\Big\} & \text{otherwise} \end{cases}$$

Again, as shown by Abdulkadiroğlu et al. (2017b), the sequence is convergent. Let $c^{DN} := \lim_{t\to\infty} c^t$ denotes the **DN cutoffs**. A family is assigned to school s if her priority at s plus her lottery number exceeds c_s^{DN}, and she prefers s to any school where her priority plus lottery number exceeds the corresponding DN cutoff. From the description of DN, it can be verified that the probability of this event for a school s with $c_s^{DN} > 1$ is equal to

$$\lambda_{vs}^{DN}(h,\tau) = \begin{cases} 0 & s_h \neq s, \\ \max\Big\{0, \min\big\{1, c_{s'}^{DN}, s' \succ_{vs} s\big\} - [c_s^{DN} - 1]\Big\} & \text{otherwise.} \end{cases} \tag{3}$$

For a school s with $c_s^{DN} \leq 1$,

$$\lambda_{vs}^{DN}(h,\tau) = \begin{cases} \max\Big\{0, \min\big\{c_{s_h}^{DN} - 1, c_{s'}^{DN}, s' \succ_{vs} s\big\} - c_s^{DN}\Big\} & s_h \succ_{vs} s, \\ \min\big\{1, c_{s'}^{DN}, s' \succ_{vs} s\big\} & s_h = s \\ \max\Big\{0, \min\big\{1, c_{s'}^{DN}, s' \succ_{vs} s\big\} - c_s^{DN}\Big\} & \text{otherwise.} \end{cases} \tag{4}$$

We now compare equilibrium aggregate welfare across the school assignment mechanisms.

4 Aggregate Welfare

In this section we establish the existence of CE of DN, DA and NA, and compare the mechanisms in terms of aggregate (or average) welfare.

Definition 2. *For two mechanisms ϕ and ψ we say that ϕ creates higher* ***aggregate welfare*** *than ψ if for arbitrary CE neighborhood choices τ^ϕ of ϕ and τ^ψ of ψ,*

$$\int u_v^\phi(h,\tau^\phi)d\tau^\phi \geq \int u_v^\psi(h,\tau^\psi)d\tau^\psi.$$

Here, $\int u_v^\phi(h,\tau^\phi)d\tau^\phi$ is a shorter notation for $\int u_v^\phi(h,\tau^\phi)d\tau^\phi(v,h)$.

The definition of aggregate welfare does not account for neighborhood prices. Therefore, it should not be interpreted as the welfare of the families, but that of the entire economy. That is, the aggregate welfare in our model is the 'sum' of utilities of all families and house sellers, who may be thought of as passive agents in our model.

4.1 Existence of CE

We first discuss existence of a CE of NA. Under NA, families' expected utilities of choosing different neighborhoods do not depend on other families' neighborhood choices. Hence, our problem is equivalent to a continuum assignment game without externalities (Gretsky et al., 1992,9). The existence of a (unique) CE of NA is therefore guaranteed by an analogous result for continuum assignment games.

Theorem 1. *When η is non-atomic and has full support, there is a unique CE of NA.*

Proof. For any $\tau \in \mathcal{T}$, $u_v^{NA}(h,\tau) = v(h,s_h)$. Let $\tilde{\eta}$ be a probability measure on $\tilde{V} := [0,1]^{|H|}$ given by

$$\tilde{\eta}(\tilde{U}) = \eta\Big(v \in V : (v(h,s_h))_{h\in H} \in \tilde{U}\Big),$$

for all measurable $\tilde{U} \subseteq \tilde{V}$. Since η has full support, so does $\tilde{\eta}$. A CE of NA corresponds to Walrasian equilibrium of the non-atomic assignment model of Gretsky et al. (1999). Therefore, the existence of a unique CE of NA follows from their Proposition 6. □

Our next result establishes the existence of CE of DN and DA.

Theorem 2. *When η is absolutely continuous and has full support, there is a CE of DN and DA.*

Proof. Appendix A. □

A crucial step for the proof is establishing that school assignment probabilities change continuously with families' neighborhood choices. We use the Schauder-Tychonoff fixed point theorem to establish CE existence.

4.2 DN versus NA

A major result of this paper is that DN creates unambiguously higher aggregate welfare than NA.

Theorem 3. *DN creates higher aggregate welfare than NA.*

Before proving the result we first provide some intuition behind it. Like NA, DN allows families with high valuations to enroll their children to their preferred schools by choosing the corresponding neighborhood. In addition, it provides more flexibility for families to enroll their children to schools outside of their neighborhoods when those have empty seats (i.e., 'unclaimed' by neighborhood families). In a special case where CE prices are equal under both mechanisms, it is immediate that all families prefer DN to NA. However, this observation does not extend to the general case: when CE price of a certain neighborhood is larger under DN than under NA, families choosing the neighborhood in the CE of NA may be worse off because of the price increase. Such an example is provided in Appendix B. Although some families' may prefer NA to DN, Theorem 3 says that the aggregate welfare is always larger under the former mechanism. The proof uses the result that Walrasian equilibria of continuum assignment games maximize aggregate welfare (Gretsky et al., 1992). We outline the proof below.

When fixing school assignment probabilities, our model may be thought of as a continuum assignment game where families valuations for neighborhoods are their expected utilities from choosing them. Consider an arbitrary CE of NA and DN. If families choose neighborhoods according to the CE of NA, but their expected utilities from choosing neighborhoods are calculated as if the other families choose neighborhoods according to DN and the school assignment mechanism is DN, then the corresponding aggregate welfare (in fact, the welfare of each family) would be larger than that under the CE of NA. This is true since under DN each family is guaranteed a school that she weakly prefers to her neighborhood school. Moreover, assuming that the expected utilities are as described above, families' choosing neighborhoods according to DN instead of NA would further improve aggregate welfare. This is true since DN neighborhood choices constitute a Walrasian equilibrium of the corresponding continuum assignment game, and therefore maximize aggregate welfare. For the sake of completeness, we give the formal proof.

Proof. For $\phi \in \{DN, NA\}$, let (τ^ϕ, p^ϕ) be a CE of ϕ. Also, let $\tilde{V} := [0,1]^{|H|}$ and $\tilde{\eta}^\phi$ be a measure on $\tilde{V}$ given by

$$\tilde{\eta}^\phi(\tilde{U}) = \eta\Big(v \in V : (u_v^\phi(h, \tau^\phi))_{h\in H} \in \tilde{U}\Big),$$

for all measurable $\tilde{U} \subseteq \tilde{V}$. Since η has full support, so does $\tilde{\eta}^\phi$. Define a measure $\tilde{\tau}^\phi$ on $\tilde{V} \times \bar{H}$ by

$$\tilde{\tau}^\phi\Big((\tilde{u}, h) \in \tilde{V} \times \bar{H} : \tilde{u} \in \tilde{U}, h \in H'\Big) = \tau^\phi\Big((v, h) \in V \times \bar{H} : \big(u_v^\phi(h, \tau^\phi)\big)_{h\in H} \in \tilde{U}, h \in H'\Big),$$

for all measurable $\tilde{U} \subseteq \tilde{V}$ and $H' \subseteq H$. Then, $(\tilde{\tau}^\phi, p^\phi)$ is a Walrasian equilibrium of the non-atomic assignment game $\tilde{\eta}^\phi$. Hence, by Theorem 4 of Gretsky et al. (1992),

$$\tau^\phi = \arg\max_{\tau \in \mathcal{T}} \int u_v^\phi(h, \tau^\phi) d\tau, \tag{5}$$

$$\text{s.t. } \tau\Big((v,h) \in V \times H : h = h'\Big) \leq q_{h'}, \text{ for all } h' \in H.$$

Hence,

$$\int u_v^{DN}(h, \tau^{DN}) d\tau^{DN} \geq \int u_v^{DN}(h, \tau^{DN}) d\tau^{NA}$$
$$\geq \int v(h, s_h) d\tau^{NA} = \int u_v^{NA}(v, \tau^{NA}) d\tau^{NA},$$

where the first inequality above follows from equation 5 for $\phi = DN$ and the second inequality follows from that each type is assigned to a school she weakly prefers to the neighborhood school under DN. The last property follows from $q_h \leq q_{s_h}$ for all $h \in H$ and from the description of DN. □

In the next subsection we compare aggregate welfare between the two versions of the Deferred Acceptance mechanism.

4.3 DN versus DA

The welfare comparison across DN and DA is less straightforward. Generally, each mechanism can result in a higher aggregate welfare than the other one. We show that DN outperforms DA in two special case of our model.

Assumption 1. *Suppose* $V = \{v_\alpha\}_{\alpha \in [0,1]}$, $H = \{h_i\}_{i=1}^N, S = \{s_i\}_{i=1}^N$ *and for almost all* $\alpha \in [0,1]$,

- $v_\alpha(h_i, s_m) \geq v_\alpha(h_j, s_n)$ *for all* $h_i, h_j \in H, i \geq j$ *and* $s_m, s_n \in S, m \geq n$,
- $v_\alpha(h_1, s_1) = 0$ *and* $v_\alpha(h_i, s_m) - v_\alpha(h_j, s_n)$ *is increasing in* α *for all* $h_i, h_j \in H, i \geq j$ *and* $s_m, s_n \in S, m \geq n$.

In words, Assumption 1 says that families, neighborhoods and schools are indexed, all families have a higher valuation for higher indexed neighborhoods and schools and these valuations have increasing differences in $(\alpha; i, j)$.[8] The index of the family reflects the child's ability, parent's education level, family income or some combination of those. The index of a neighborhood or a school reflects its quality.

Assumption 2. *Suppose* $H = \{h_i\}_{i=1}^N, S = \{s_i\}_{i=1}^N$, *and there are constants* $(e_i)_{i=1}^N$ *such that for almost all* $v \in V$,

- $v(h, s_m) \geq v(h, s_n)$ *for all* $h \in H$ *and* $s_m, s_n \in S, m \geq n$,
- $v(h_i, s) - v(h_j, s) = e_i - e_j \geq 0$ *for all* $h_i, h_j \in H, i \geq j$ *and* $s \in S$.

Thus, Assumption 2 relaxes increasing differences, but assumes that families have common and additively separable valuations for neighborhoods.

Theorem 4. *Suppose either Assumption 1 or 2 holds. Then, DN creates higher aggregate welfare than DA.*

Proof. Appendix A.2. □

It is important to note that Assumptions 1 and 2 are restrictive as they imply that families have common ordinal preferences over neighborhoods and schools. Such a restrictive preference structure has been commonly imposed by the previous works on the topic to gain tractability (Avery and Pathak, 2020; Calsamiglia et al., 2015; Xu, 2019). Our work too uses the assumption for tractability, and we do not provide more general conditions to give stronger welfare comparisons across DN and DA. Therefore,

[8] The assumptions of same ordinal preference rankings and increasing differences of valuations are also made by Xu (2019) and Avery and Pathak (2020). However, they assume that families only care about schools, while we allow valuations for neighborhoods, too. None of these papers provide welfare comparisons between the two version of the Deferred Acceptance mechanism.

the superior welfare performance of DN compared to DA in Theorem 4, should be interpreted with care. In fact, we provide two counterexamples, where Assumption 1 and 2 fail, and DA outperforms DN in terms of aggregate welfare. The first example relaxes common valuations of neighborhoods and increasing differences of valuations.

Example 1. *There are two neighborhoods* $H = \{h_1, h_2\}$ *and two schools* $S = \{s_1, s_2\}$*. Each neighborhood and school has a capacity* 0.5*. Economy* η *is supported at only two points* v_1 *and* v_2*, with*

$$\eta\Big(v \in V : v = v_1\Big) = \eta\Big(v \in V : v = v_2\Big) = 0.5.$$

Valuations are given in in Table 6.

	(h_1, s_1)	(h_1, s_2)	(h_2, s_1)	(h_2, s_2)
v_1	0	0.3	0	0.3
v_2	0.1	0	0.5	0.6

Table 1: Valuations

It is easy to verify that prices $p^{\phi}_{h_1} = 0$ *and* $p^{\phi}_{h_2} = 2$ *support CE* (τ^{ϕ}, p^{ϕ}) *of* $\phi \in \{DN, DA\}$*, satisfying*

$$\tau^{\phi}\Big((v, h) \in V \times \bar{H} : v = v_i, h = h_i\Big) = \eta\Big(v \in V : v = v_i\Big) \text{ for all } i \in \{1, 2\}.$$

Under DN, type v_2 *receives a higher priority at* s_2 *and therefore she is assigned there with probability one. Under DA, each type has an equal probability of being assigned to* s_2*. Expected utilities are*

$$\begin{array}{ll} u^{DN}_{v_1}(h_1, \tau^{DN}) = 0 & u^{DA}_{v_1}(h_1, \tau^{DA}) = 0.15 \\ u^{DN}_{v_2}(h_2, \tau^{DN}) = 0.6 & u^{DA}_{v_2}(h_2, \tau^{DA}) = 0.55 \end{array}.$$

Therefore,

$$\int u^{DN}_{v}(h, \tau^{DN}) d\tau^{DN} = \frac{1}{2} \times u^{DN}_{v_1}(h_1, \tau^{DN}) + \frac{1}{2} \times u^{DN}_{v_2}(h_2, \tau^{DN}) = 0.3$$

$$< 0.35 = \frac{1}{2} \times u^{DA}_{v_1}(h_1, \tau^{DA}) + \frac{1}{2} \times u^{DA}_{v_2}(h_2, \tau^{DA}) = \int u^{DA}_{v}(h, \tau^{DA}) d\tau^{DA}.$$

Our second example maintains the assumption of common and additively separable valuation over neighborhoods, but relaxes the assumption of identical ordinal preferences over schools.

Example 2. *There are three neighborhoods* $H = \{h_1, h_2, h_3\}$ *and three schools* $S = \{s_1, s_2, s_3\}$*. Capacities are* $q_{h_1} = 0.6$*,* $q_{h_2} = q_{h_3} = 0.2$ *and* $q_{s_2} = q_{s_3} = 0.3$*. Economy* η *is supported at only three points* v_1, v_2 *and* v_3*, with*

$$\eta\Big(v \in V : v = v_1\Big) = 0.6, \eta\Big(v \in V : v = v_2\Big) = \eta\Big(v \in V : v = v_3\Big) = 0.2.$$

We assume that families only care about schools. Formally, $v_i(h_j, s) = v_i(h_k, s)$ *for all* $i, j, k \in \{1, 2, 3\}$ *and* $s \in S$*. Thus, a type can be described by its valuation for schools. Valuations are given in Table 2.*

	s_1	s_2	s_3
v_1	0	1	0.9
v_2	0	0.9	0
v_3	0	0	0.9

Table 2: Valuations

We first compute aggregate welfare under DN. We prove that prices $p^{DN}_{h_1} = 0$*,* $p^{DN}_{h_2} = 0.7$*, and* $p^{DN}_{h_3} = 0.6$ *supports CE neighborhood choices*

$$\tau^{DN}\Big((v, h) \in V \times \bar{H} : v = v_i, h = h_i\Big) = \eta\Big(v \in V : v = v_i\Big) \text{ for all } i \in \{1, 2, 3\}.$$

First, let us show optimality of type v_1 *families' neighborhood choices at* (τ^{DN}, p^{DN})*. Since families receive higher priorities at neighborhood schools, almost all* v_2 *type families are assigned to* s_2 *and almost all* v_3 *type families are assigned to* s_3*. The remaining* 0.2 *cumulative capacity at schools* s_2 *and* s_3 *are assigned to highest ranked families of type* v_1*. Thus, the probability that* v_1 *is assigned to either* s_2 *or* s_3 *is equal to* $\frac{1}{3}$*. Therefore,*

$$u^{DN}_{v_1}(h_1, \tau^{DN}) - p^{DN}_{h_1} \geq \frac{1}{3} \times 0.9 = u^{DN}_{v_1}(h_2, \tau^{DN}) - p^{DN}_{h_2} = u^{DN}_{v_1}(h_3, \tau^{DN}) - p^{DN}_{h_3}.$$

Now consider a type v_2 family. If a v_2 family chooses neighborhood h_1 or h_3, she is assigned to s_2 only if she has one of the 0.1 *highest lottery numbers among* 0.6 *mass of type v_1 families. The probability of this event is $\frac{1}{6}$. Therefore,*

$$u_{v_2}^{DN}(h_2,\tau^{DN})-p_{h_2}^{DN} = 0.9-0.7 > \frac{1}{6}\times 0.9 = u_{v_2}^{DN}(h_1,\tau^{DN})-p_{h_1}^{DN} > u_{v_2}^{DN}(h_3,\tau^{DN})-p_{h_3}^{DN}.$$

Finally, consider a type v_3 family. Conditional on being assigned to h_1 or h_2, type v_3 is assigned to s_3 only she has one of the highest 0.1 *highest lottery numbers among mass* 0.5 *type v_1 families who do not have a high enough lottery number to be assigned to s_2. The conditional lottery numbers' distribution of families not assigned to s_2 is uniform in $\left[0,\frac{5}{6}\right]$, and the probability that v_3 is assigned to s_3 is $\frac{1}{6}+\frac{5}{6}\times\frac{1}{5}=\frac{1}{3}$. Therefore,*

$$u_{v_3}^{DN}(h_2,\tau^{DN})-p_{h_3}^{DN} = 0.9-0.6 = \frac{1}{3}\times 0.9 = u_{v_3}^{DN}(h_1,\tau^{DN})-p_{h_1}^{DN} > u_{v_3}^{DN}(h_2,\tau^{DN})-p_{h_2}^{DN}.$$

Aggregate welfare under DN is

$$\int u_v^{DN}(h,\tau^{DN})d\tau^{DN} = 0.1\times 1 + 0.1\times 0.9 + 0.2\times 0.9 + 0.2\times 0.9 = 0.550.$$

Now consider DA. Since families only care about schools, any neighborhood choice $\tau^{DA}\in\mathcal{T}$ is supported as a CE with prices $p_{h_1}^{DA}=p_{h_2}^{DA}=p_{h_3}^{DA}$. Then, a mass 0.15 *of type v_1 families who have valuation* 1 *for s_2 are assigned to the school. The remaining* 0.15 *capacity at s_2 is filled with families who have valuation* 0.9 *for s_2. The entire* 0.3 *capacity of school s_3 is filled with families who have valuation* 0.9 *for s_3.*[9] *Thus, aggregate welfare under DA is*

$$\int u_v^{DA}(h,\tau^{DA})d\tau^{DA} = 0.15\times 1 + 0.15\times 0.9 + 0.3\times 0.9 = 0.555.$$

Our results that DN creates higher welfare than DA in certain special cases is a potential justification of the fact that school district typically grant higher priorities to neighborhood students.

[9] We implicitly assume that types v_2 and v_3 apply to s_1 before s_3 and s_2, respectively. This is without loss of generality, as alternatively we could slightly increase valuations at s_1 for all families and adjust prices accordingly.

5 Budget Constraints: Distributional Consequences of School Choice

In this section a family's type is her valuations $v \in [0,1]^{|H| \times |S|} := V$ for neighborhoods and schools and her budget $b \in [0,1]$, which denotes the maximum amount she can pay for a neighborhood. The economy is described by a probability measure η on $V \times [0,1]$. In this section we are interested in the welfare of lowest-income families, i.e., those whose budget is zero (or sufficiently close to zero).[10]

Neighborhood choices τ is a probability measure on $V \times [0,1] \times \bar{H}$ satisfying $\tau\big(U \times I \times \bar{H}\big) = \eta\big(U \times I\big)$ for all measurable $U \times I \subseteq V \times [0,1]$.

Definition 3. *For neighborhood choices τ and price vector $p \in \mathbb{R}_+^{|H|}$, we say a pair (τ, p) is a **competitive equilibrium (CE)** of mechanism ϕ if it satisfies the following conditions:*

1. $\tau\Big((v,b,h) \in V \times [0,1] \times \bar{H} : h = \arg\max_{h' \in \bar{H}_b} \ u^{\phi}_{vb}(h', \tau) - p_{h'}\Big) = 1,$
   where $p_0 := 0$ and $\bar{H}_b := \{h \in \bar{H} : p_h \leq b\}$,
2. $\tau\Big((v,b,h) \in V \times [0,1] \times \bar{H} : h = h'\Big) \leq q_{h'}, \forall h' \in H.$
3. $\tau\Big((v,b,h) \in V \times [0,1] \times \bar{H} : h = h'\Big) < q_{h'} \Rightarrow p_{h'} = 0.$

Throughout this section we assume that $\sum_{h \in H} q_h \geq 1$. This is without loss of generality, since otherwise there will be no zero-priced neighborhood in equilibrium, making

[10]Instead of modelling budget constraints, an alternative way of incorporating income levels would be through assuming that families are differentiated by an income parameter, and those with a smaller income parameter have a 'higher valuation for money' (Avery and Pathak, 2020; Epple and Romano, 1998; Xu, 2019). Our results would extend to that environment if lowest-income families would have sufficiently high valuation for money so that they would choose the cheapest neighborhood in equilibrium. We find that modelling income levels through budget constraints is natural, and therefore, stick to that setup.

the analysis for lowest-income family welfare trivial. We restrict attention to economies that admit a CE.[11]

Definition 4. *Let (τ^ϕ, p^ϕ) be a CE of $\phi \in \{DN, DA, NA\}$. We say h is an* ***underdemanded neighborhood*** *if $p_h^\phi = 0$. Similarly, for $\phi \in \{DN, DA\}$, we say s is an* ***underdemanded school*** *if $c_s^\phi = 0$.*

Our next result gives a sufficient condition for lowest-income families preferring DN and DA to NA.

Theorem 5. *Let (τ^ϕ, p^ϕ) be CE of $\phi \in \{DN, DA, NA\}$. Also, let H_-^ϕ and S_-^ϕ be the set of underdemanded neighborhoods and schools at (τ^ϕ, p^ϕ), respectively.*

1. *If underdemanded neighborhoods under NA are also underdemanded under DN, then lowest-income families prefer DN to NA. Formally, if $H_-^{NA} \subseteq H_-^{DN}$, then there is a $\bar{b} > 0$ such that for any measurable $U \times I \subseteq V \times [0, \bar{b}]$,*

$$\int_{U\times I} \left[u_{vb}^{DN}(h, \tau^{DN}) - p_h^{DN}\right] d\tau^{DN} \geq \int_{U\times I} \left[u_{vb}^{NA}(h, \tau^{NA}) - p_h^{NA}\right] d\tau^{NA}.$$

2. *If underdemanded neighborhoods under NA are also underdemanded under DA, and they have underdemanded schools, then lowest-income families prefer DA to NA. Formally, if $H_-^{NA} \subseteq H_-^{DA}$ and $\left\{s_h \in S : h \in H_-^{NA}\right\} \subseteq S_-^{DA}$, then there is a $\bar{b} > 0$ such that for any measurable $U \times I \subseteq V \times [0, \bar{b}]$,*

$$\int_{U\times I} \left[u_{vb}^{DA}(h, \tau^{DA}) - p_{H_-}^{DA}\right] d\tau^{DA} \geq \int_{U\times I} \left[u_{vb}^{NA}(h, \tau^{NA}) - p_h^{NA}\right] d\tau^{NA}.$$

Proof. First, we prove point 1. Let $\bar{b} := \min_{h \in H \setminus H_-^{NA}} p_h^{NA}/2$. Then, by Definition 3,

$$\tau^{NA}\Big((v, b, h) \in V \times [0,1] \times \bar{H} : b \in [0, \bar{b}], h \in H_-^{NA}\Big)$$

[11] In general, existence of CE is not guaranteed under any of the studied mechanisms. This is an immediate consequence of an analogous result for Walrasian equilibria of assignment games with budget constraints (e.g., see van der Laan, Talman, and Yang (2018)).

$$= \eta\Big((v,b) \in V \times [0,1] : b \in [0,\bar{b}]\Big). \tag{6}$$

In words, equations 6 says that almost all families with budgets in $[0,\bar{b}]$ choose a neighborhood in H_{-}^{NA} under τ^{NA}. Consider an arbitrary measurable $U \times I \subseteq V \times [0,\bar{b}]$. Then,

$$\int_{U\times I} \left[u_{vb}^{DN}(h,\tau^{DN}) - p_h^{DN}\right] d\tau^{DN} \geq \int_{U\times I} \left[u_{vb}^{DN}(h,\tau^{DN}) - p_h^{DN}\right] d\tau^{NA}$$
$$= \int_{U\times I} u_{vb}^{DN}(h,\tau^{DN}) d\tau^{NA} \geq \int_{U\times I} v(h,s_h) d\tau^{NA}$$
$$= \int_{U\times I} u_{vb}^{NA}(h,\tau^{NA}) d\tau^{NA} = \int_{U\times I} \left[u_{vb}^{NA}(h,\tau^{NA}) - p_h^{NA}\right] d\tau^{NA}. \tag{7}$$

The first inequality in equation 7 follows from equation 6 and the optimality of neighborhood choices. The first equality follows from that $H_{-}^{NA} \subseteq H_{-}^{DN}$. The second inequality follows from that under DN each family is guaranteed a weakly better school than the neighborhood school. The last equality again follows from equation 6.

We now prove point 2. Let $\bar{b}$ be as before and consider an arbitrary measurable $U \times I \subseteq V \times [0,\bar{b}]$. Then,

$$\int_{U\times I} \left[u_{vb}^{DA}(h,\tau^{DA}) - p_h^{DA}\right] d\tau^{DA} \geq \int_{U\times I} \left[u_{vb}^{DA}(h,\tau^{DA}) - p_h^{DA}\right] d\tau^{NA}$$
$$= \int_{U\times I} u_{vb}^{DA}(h,\tau^{DA}) d\tau^{NA} \geq \int_{U\times I} v(h,s_h) d\tau^{DA}$$
$$= \int_{U\times I} u_{vb}^{NA}(h,\tau^{NA}) d\tau^{NA} = \int_{U\times I} \left[u_{vb}^{NA}(h,\tau^{NA}) - p_h^{NA}\right] d\tau^{NA}. \tag{8}$$

The second inequality in equation 8 follows from that $\{s_h \in S : h \in H_{-}^{NA}\} \subseteq \bar{S}^{DA}$. The condition along with equation 6 implies that schools at neighborhood chosen by types in $U \times I$ are underdemanded. Thus, under DA each of these families is guaranteed a weakly better school than the neighborhood school. The arguments for other steps in equation 8 are as in point 1. □

In other words, the conditions in in Theorem 5 say the following. First, they say that underdemanded neighborhoods remain underdemanded once we switch from a CE of

NA to a CE of DN or DA. Second, they say that schools in underdemanded neighborhoods are underdemanded. The conditions are intuitive, and more importantly, later in this section we show that they are satisfied for some natural special cases (Corollaries 1 and 2).

Conditions in Theorem 5 are sufficient, but not necessary. That is, there are economies that do not satisfy the conditions, but where all lowest-income families prefer DN or DA to NA. However, for any such economy, or for any economy in general, we can find another economy that is arbitrarily close to the original one, such that either the conditions in Theorem 5 hold, or a positive measure of zero-income families prefer NA to DN or DA. Thus, in a sense, the conditions in Theorem 5 are necessary if we also require 'robustness' of lowest-income family welfare comparisons to small perturbations.

Theorem 6. *Consider an arbitrary economy η' and $\epsilon > 0$.*

1. *There is an economy η satisfying,*

$$\|\eta - \eta'\|_2 < \epsilon$$

(where $\|\cdot\|_2$ denotes the L^2 norm), such that $H_{-}^{NA} \subseteq H_{-}^{DN}$, or for some $U \subseteq V$ with $\eta\big(U \times \{0\}\big) > 0$,

$$\int_{U\times I} \left[u_{vb}^{NA}(h, \tau^{NA}) - p_h^{NA}\right] d\tau^{NA} \geq \int_{U\times I} \left[u_{vb}^{DN}(h, \tau^{DN}) - p_h^{DN}\right] d\tau^{DN}.$$

2. *There is an economy η satisfying*

$$\|\eta - \eta'\|_2 < \epsilon,$$

such that $H_{-}^{NA} \subseteq H_{-}^{DA}$ and $\left\{s_h \in S : h \in H_{-}^{NA}\right\} \subseteq S_{-}^{DA}$, or for some $U \subseteq V$ with $\eta\big(U \times \{0\}\big) > 0$,

$$\int_{U\times I} \left[u_{vb}^{NA}(h, \tau^{NA}) - p_h^{NA}\right] d\tau^{NA} \geq \int_{U\times I} \left[u_{vb}^{DA}(h, \tau^{DA}) - p_h^{DA}\right] d\tau^{DA}.$$

Proof. Appendix A.3. □

We finish this section by showing that the conditions is Theorem 5 are satisfied for natural special cases. The first case assumes common ordinal preference rankings over neighborhoods and schools. The assumption is common in the literature (e.g., Avery and Pathak (2020); Calsamiglia et al. (2015); Xu (2019)).

Assumption 3. *Suppose* $H = \{h_i\}_{i=1}^N, S = \{s_i\}_{i=1}^N$, *and for almost all* $v \in V$,

- $v(h, s_m) \geq v(h, s_n)$ *for all* $h \in H$ *and* $s_m, s_n \in S, m \geq n$,
- $v(h_i, s) \geq v(h_j, s)$ *for all* $h_i, h_j \in H, i \geq j$ *and* $s \in S$.

Note that Assumption 3 is weaker than Assumptions 1 and 2.

Corollary 1. *Suppose Assumption 3 is satisfied. Then, there is a* $\bar{b} > 0$ *such that for any measurable* $U \times I \subseteq V \times [0, \bar{b}]$,

$$\int_{U\times I} \left[u_{vb}^{DN}(h, \tau^{DN}) - p_h^{DN}\right] d\tau^{DN} \geq \int_{U\times I} \left[u_{vb}^{NA}(h, \tau^{NA}) - p_h^{NA}\right] d\tau^{NA}, \text{ and}$$

$$\int_{U\times I} \left[u_{vb}^{DA}(h, \tau^{DA}) - p_h^{DA}\right] d\tau^{DA} \geq \int_{U\times I} \left[u_{vb}^{NA}(h, \tau^{NA}) - p_h^{NA}\right] d\tau^{NA}.$$

Proof. Appendix A.4. □

The result is intuitive: with a common ordinal preferences rankings over neighborhoods and schools, the least preferred neighborhoods and schools are the underdemanded ones for all CE and school assignment mechanisms. Thus, the conditions Theorem 5 are satisfied.

Although widely applied for tractability (e.g., Abdulkadiroğlu, Che, and Yasuda (2011); Avery and Pathak (2020); Calsamiglia et al. (2015); Neilson, Akbarpour, Kapor, van Dijk, and Zimmerman (2020); Xu (2019)), the assumption of common ordinal preference rankings is restrictive. Our next result (Corollary 2) provides another condition that guarantees that families prefer DA over NA. We say an economy is uniform if each valuation profile is equally likely. Formally, η is a **uniform economy** if for each measurable $U \times I \subseteq V \times [0,1]$ and $U' \times I \subseteq V \times [0,1]$, U and U' have the same Lebesgue measure only if $\eta\big(U \times I\big) = \eta\big(U' \times I\big)$.

Suppose $\sum_{h \in H} q_h = 1$ and $q_{s_{\bar{h}}} \geq q_{s_h}$ for all $\bar{h} \in \arg\max_{h \in H} q_h$. In other words, the first condition says that the total capacity at neighborhoods equals to the total mass of families, and the last condition says that largest neighborhoods have the largest schools. We show that the uniform economy satisfies the second part of conditions in Theorem 5 for this special case.

Corollary 2. *Let η be a uniform economy. Then, there is a $\bar{b} > 0$ such that for any measurable $U \times I \subseteq V \times [0, \bar{b}]$,*

$$\int_{U \times I} \big[u_{vb}^{DA}(h, \tau^{DA}) - p_h^{DA}\big] d\tau^{DA} \geq \int_{U \times I} \big[u_{vb}^{NA}(h, \tau^{NA}) - p_h^{NA}\big] d\tau^{NA}.$$

Proof. Appendix A.5. □

The result follows from that in the uniform economy the underdemanded neighborhoods and schools are the ones with the largest capacities. The result is intuitive, but proving it formally requires some effort.

The uniform economy framework is commonly applied in matching theory literature to obtain analytical results without strong restrictions on preferences and priorities (Abdulkadiroğlu, Che, Pathak, Roth, and Tercieux, 2020a; Che and Tercieux, 2017; Grigoryan, 2020). The uniform economy can be thought of as an ‘average’ economy. Hence, the result may be interpreted as that ‘on average’ lowest-income income families prefer DA over NA.

6 Continuum Economy as a Limit of Discrete Economies

6.1 The Discrete Model

There is a finite set of families F with a single child and equal number of neighborhoods H and schools S. There is a unique school in neighborhood $h \in H$, which we denote by $s_h \in S$. Each neighborhood h has capacity $q_h \in \mathbb{N}$ which denotes the maximum number of families it can accommodate. Similarly, each school s has a capacity $q_s \in \mathbb{N}$, which denotes the maximum number of families that can enrol at the school. Each family $f \in F$ has a valuation $v_f(h, s) \in [0, 1]$ for residing in neighborhood h and enrolling at school s. Valuations of all families are commonly known. Families' valuations induce preference rankings over schools. Conditional on living in neighborhood h, the preference ranking of family f satisfies

$$v_f(h, s) > v_f(h, s') \Rightarrow s \succ_{fh} s'. \tag{9}$$

When $v_f(h, s) = v_f(h, s')$, ties are broken arbitrarily.

Let $\bar{H} := H \cup \{0\}$. Neighborhood choices of families is a mapping $\sigma : F \to \bar{H}$.

Family's expected utilities of choosing a certain neighborhood depend on other families' neighborhood choices and the school assignment mechanism, as they jointly determine the family's school assignment probabilities. For a school assignment mechanism ϕ, let $\lambda^{\phi}_{fs}(h, \sigma) \in [0, 1]$ denote the probability that family f is assigned to school s when she chooses neighborhood h and other families' choose neighborhoods according σ.

Given the school assignment probabilities and neighborhood price vector $p \in [0, 1]^{|H|}$, the expected utility of family f choosing neighborhood h is equal to

$$u^{\phi}_f(h, \sigma) - p_h.$$

where $u^{\phi}_f(h, \sigma) := \sum_{s \in S} \lambda^{\phi}_{fs}(h, \sigma) v_f(h, s)$. Also, let $u^{\phi}_f(0, \sigma) := 0$. The housing market is competitive: families choose neighborhoods to maximize expected utilities, given other families neighborhood choices and the market clearing neighborhood prices.

Definition 5. *For a neighborhood choices σ and price vector $p \in \mathbb{R}_+^{|H|}$, we say a pair (σ, p) is a* ***competitive equilibrium (CE)*** *of ϕ is it satisfies the following conditions:*

1. $u_f^\phi(\sigma(f), \sigma) - p_{\sigma(f)} = \arg\max_{h \in \bar{H}} \; u_f^\phi(h, \sigma) - p_h, \forall f \in F$, where $p_0 := 0$,
2. $|\sigma^{-1}(h)| \leq q_h, \forall h \in H$,
3. $|\sigma^{-1}(h)| < q_h \Rightarrow p_h = 0$.

We now discuss the discrete analogs of school assignment mechanisms of previous sections.

Neighborhood Assignment.

Under neighborhood assignment (NA), families are assigned to the neighborhood schools. Therefore, school assignment probabilities are trivial:

$$\lambda_{fs}^{NA}(h, \sigma) = \begin{cases} 1 & \text{if } s = s_h, \\ 0 & \text{otherwise} \end{cases}$$

Deferred Acceptance.

As in the continuum model, we study two versions of DA, which differ on how schools' priority ranking is determined.

Deferred Acceptance without Neighborhood Priority (DA).

School assignment under DA is determined based on families' preferences and lottery number. Preferences are induced by neighborhood choices through equation 9. A lottery number for each family is uniformly and independently drawn from the unit interval. All schools rank families according to their lottery numbers, i.e., a higher lottery numbers denotes a higher rank. The assignment is determined through the following algorithm by Gale and Shapley (1962): until there are no more rejections,

- each family f with $\sigma(f) \neq 0$ applies to her most preferred school that has not rejected her,
- each school tentatively accepts up to q_s of all its highest ranked applicants and rejects the rest.

Deferred Acceptance with Neighborhood Priority (DN).

Under DN, school assignment is determined based on families' preferences, lottery number and priorities. Preferences and lottery numbers are decided as under DA. Families receive priority 1 at neighborhood schools and priority 0 at non-neighborhood ones. Schools rank families according to their lottery numbers plus the priority. Again, DN assignment is determined by the Gale and Shapley (1962) algorithm.

6.2 (Non)Existence of CE in the Discrete Model

Under NA, the existence of CE follows from Shapley and Shubik (1971). In contrast, assignment externalities may preclude the existence of CE under DN and DA. The example below demonstrates the nonexistence result for DN. The one for DA is in Appendix B.

Example 3. *Suppose there are two families $F = \{f_1, f_2\}$, two neighborhoods $H = \{h_1, h_2\}$ and two schools $S = \{s_1, s_2\}$. Each neighborhood and school has a unit capacity. Families' valuations are shown in Table 3.*

Suppose, for the sake of contradiction, that there is a CE (σ, p) of DN. Consider cases:

(i) Suppose $\sigma(f_1) = h_1$ and $\sigma(f_2) = h_2$. Then, f_1's utility is 0, as she is rejected by s_2,

	(h_1, s_1)	(h_1, s_2)	(h_2, s_1)	(h_2, s_2)
f_1	0	0.3	0	0.1
f_2	0.1	0.2	0	0.1

Table 3: Valuations

where f_2 has a higher priority. If f_1 chooses h_2 instead of h_1, her utility is $\frac{1}{2} \times 0.1 = 0.05$ as she has $\frac{1}{2}$ change of being assigned to s_2. Thus, $\sigma(f_1) = h_1$ implies $p_{h_2} - p_{h_1} \geq 0.05$. Also, f_2's utility is 0.1 as she is guaranteed being assigned to s_2. If f_2 chooses h_1, she has a $\frac{1}{2}$ chance of being assigned to s_1 and $\frac{1}{2}$ chance of being assigned to s_2, thus her utility is $\frac{1}{2} \times 0.1 + \frac{1}{2} \times 0.2 = 0.15$. Thus, $\sigma(f_2) = h_2$ implies $p_{h_2} - p_{h_1} \leq -0.05$, a contradiction.

(ii) Now suppose $\sigma(f_1) = h_2$ and $\sigma(f_2) = h_1$. Then, f_1's utility is 0.1. If f_1 chooses h_1 instead of h_2, her utility is $\frac{1}{2} \times 0.3 = 0.15$. Thus, $\sigma(f_1) = h_2$ implies $p_{h_2} - p_{h_1} \leq -0.05$. Also, f_2's utility is 0.1 as she is rejected by s_2. If f_2 chooses h_2 instead of h_1, her utility is $\frac{1}{2} \times 0.1 = 0.05$. Thus, $\sigma(f_2) = h_1$ implies $p_{h_2} - p_{h_1} \geq 0.05$, a contradiction.

As the proof demonstrates, the nonexistence result is due to assignment externalities. A family's expected utility from different neighborhood choices depend on other families' neighborhood choices through the latter's effect on school assignment probabilities. In contrast, as shown in Section 3, competitive equilibria always exist in the continuum model. In Section 6.3, we show that continuum economies are arbitrarily good approximations of finite discrete economies when the number of families is sufficiently large. In particular, this implies that approximate CE exist in sufficiently large discrete economies, and welfare comparisons for the continuum model carry over to the discrete one.

6.3 Existence of Approximate CE in Large Markets

Let η be an absolutely continuous and fully supported probability measure on $V := [0,1]^{|H|\times|S|}$. For a fixed $k \in \mathbb{N}$ let $\{v_f\}_{f\in F}, |F| = k$, be k independent draws from V according to η. Suppose neighborhood $h \in H$ has a capacity $\lfloor q_h k \rfloor$ and each school $s \in S$ has a capacity $\lfloor q_s k \rfloor$.

Definition 6. *For an $\epsilon > 0$, neighborhood choices $\sigma : F \to \bar{H}$ and a price vector $p \in \mathbb{R}_+^{|H|}$, we say a pair (σ, p) is an $\epsilon-$**competitive equilibrium (ϵ-CE)** of ϕ if it satisfies the following conditions:*

1. $u_f^\phi(\sigma(f), \sigma) - p_{\sigma(f)} + \epsilon \geq \max\left\{u_f^\phi(h, \sigma) - p_h, 0\right\}, \forall f \in F, h \in H,$
2. $|\sigma^{-1}(h)| \leq (q_h + \epsilon)k, \forall h \in H,$
3. $|\sigma^{-1}(h)| < (q_h - \epsilon)k \Rightarrow p_h = 0.$

Let (τ^ϕ, p^ϕ) denote a CE of a continuum economy η for $\phi \in \{DN, DA, NA\}$. For each size k discrete economy $(v_f)_{f\in F}$, consider neighborhood choices $\sigma_k : F \to \bar{H}$ satisfying

$$\sigma_k(f) = \arg\max_{h\in\bar{H}} \; u_f^\phi(h, \tau^\phi) - p_h^\phi.$$

Theorem 7. *Let (τ^ϕ, p^ϕ) be a competitive equilibrium of the continuum economy η. Then for any $\epsilon > 0$ the probability that (σ_k, p) is an ϵ-competitive equilibrium of the discrete economy converges to one as k goes to infinity.*

Proof. Appendix A.6. □

In other words, Theorem 7 says that in a sufficiently large market approximate equilibria exist with a probability that is arbitrarily close to one.

To prove Theorem 7 uses the fact that expected utilities in the discrete markets converge to their continuum analogous. This also implies that all welfare comparisons that we establish for the continuum economy, hold 'approximately' for the corresponding sufficiently large discrete ones.

7 Simulations

We compare school assignment mechanisms in a simulated environment 1000 students, 10 neighborhoods and 10 schools. The valuation of family f for the joint assignment to neighborhood h and school s is equal to

$$v_f(h, s) = \alpha U_h + (1 - \alpha)U_s + \beta \mathbb{1}[s = s_h] + \epsilon_{fhs},$$

where

- U_h is the common valuation for neighborhood h,
- U_s is the common valuation for school s,
- ϵ_{fhs} is the idiosyncratic valuation of family f for the joint assignment to h and s,
- α and β are parameters.

Values of U_h, U_s and ϵ_{fhs} are iid uniform draws from the unit interval. The capacity of school s is equal to $100 + \kappa_s$, where κ_s is a random draw from the set $\{1, 2, ..., 100/\gamma\}$. Thus, a larger value of γ means a smaller variance in schools' capacities. We report simulations results for the following parameters: $\alpha \in \{0, 0.5, 1\}$, $\beta \in \{0.1, 0.2\}$ and $\gamma \in \{2, 4\}$.

Table 4 reports the percentage gains or losses in aggregate welfare under DN and DA compared to NA. As the theory predicts, DN always generate larger aggregate welfare than NA. The average aggregate welfare gains are 2.40%. Those gains are larger when neighborhood schools are less desirable (i.e., β is smaller) and when schools have more

α	β	γ	DN	DA
0	0	2	4.96	−3.75
		4	3.17	−7.93
	0.1	2	2.73	−9.88
		4	1.54	−14.53
	0.2	2	2.01	−13.30
		4	0.69	−18.26
0.5	0	2	4.32	1.50
		4	3.19	−0.43
	0.1	2	2.10	−3.87
		4	1.13	−5.52
	0.2	2	0.98	−5.79
		4	0.49	−7.82
1	0	2	5.57	5.65
		4	5.55	5.58
	0.1	2	1.82	1.61
		4	1.55	1.54
	0.2	2	0.71	0.55
		4	0.70	0.55
Average			2.40	−4.12

Table 4: Aggregate welfare, % gains/losses compared to NA

seats (i.e., γ is larger). The table also illustrates that NA and DA are not comparable in terms of aggregate welfare: the former mechanism performs better for smaller values of α, while the latter mechanism performs better for larger ones. Here is the intuition behind the comparative statics. When $\alpha = 0$, families' preferences for schools are very aligned. In that case, NA (and also DN) allows families' with highest cardinal valuations for schools to guarantee admission there by choosing the corresponding neighborhood. Hence, NA generates higher aggregate welfare than DA. In contrast, when $\alpha = 1$, families have no common valuations for schools, and preferences for schools are not aligned. Hence, DA (and also DN) manages to assign almost all families to their most preferred schools, and generates higher aggregate welfare than NA.

Finally, Table 5 illustrates how DN and DA compare to NA in terms of welfare of lowest-income family.

The welfare of lowest-income families is computed by assuming that 10 out of 1000 individuals have budgets of 0.05 and the remaining ones have infinite budgets.[12] As the table illustrates, DN and DA create larger welfare for lowest-income families compared to NA. The average gains are 26.51% and 38.25%, respectively. Thus, simulations show that the superior performance of the Deferred Acceptance mechanism in terms of lowest-income families' welfare extends beyond the special cases in Corollaries 1 and 2.

[12] We restrict attention to this simple case for tractability: in general, as discussed in Section 5, CE may not even exist.

α	β	γ	DN	DA
0	0	2	47.80	68.40
		4	37.54	62.58
	0.1	2	29.24	47.16
		4	21.68	41.23
	0.2	2	24.43	41.62
		4	15.75	35.64
0.5	0	2	35.78	46.67
		4	29.80	43.11
	0.1	2	24.22	31.55
		4	13.64	31.14
	0.2	2	13.36	19.85
		4	8.94	18.71
1	0	2	47.05	50.64
		4	46.77	50.34
	0.1	2	28.93	31.72
		4	28.45	32.42
	0.2	2	12.10	17.49
		4	11.62	18.19
Average			26.51	38.25

Table 5: Lowest-income welfare, % gains compared to NA

8 Discussion

Our results suggest that the Deferred Acceptance mechanism has superior welfare and distributional properties compared to neighborhood assignment. The results potentially justify the mechanisms' widespread application for school assignment.

School choice programs may take diverse forms, including open enrollment, expansion of magnet or charter schools, private schools and voucher programs. Hence, our findings should not be interpreted as arguments for school choice programs in general, but arguments for open enrollment, potentially through the Deferred Acceptance mechanism. Arguments against (and for) other school choice programs are numerous. For example, Epple and Romano (2003) show that voucher programs may lead to higher ability stratification (at schools) compared to neighborhood assignment, which, in turn, leads to more stratification compared to open enrollment. This, as the authors describe it as another example of the truism that *"all choice programs are not alike"*. A different argument has been made against charter schools by Zheng (2019). She shows that opening a charter school may hurt low-income families through its effect on neighborhood prices. Other papers provide arguments against open enrollment through some alternative school assignment mechanism. For example, the well-studied Immediate Acceptance mechanism, also known as the 'Boston' mechanism, has been criticized on the grounds that it is not strategyproof: families have incentives to 'game the system' by misreporting preferences to obtain better choices (Abdulkadiroğlu and Sönmez, 2003). Moreover, the Immediate Acceptance mechanism may exacerbate inequalities as low-income families might be disproportionately hurt if they are worse at 'gaming the system' (Pathak and Sönmez, 2008) or if they have worse outside options (Calsamiglia et al., 2015; Neilson et al., 2020). In our Supplementary Appendix A.3 we show that, when there are neighborhood priorities, the lowest-income families may prefer Deferred Acceptance to Immediate Acceptance. Although we do not directly model heterogeneous outside options, neighborhood schools act as outside options in our model. This is because those schools are guaranteed for neighborhood applicants due to neighborhood priorities. Therefore, our result is analogous to those in Calsamiglia et al. (2015) and Neilson et al. (2020).

In our model a family only cares about her own final assignment to a neighborhood and a school. In other words, there are no peer preferences. Although peer preferences are oftentimes assumed in the economics of education literature, those papers typically study stylized models with unidimentional family types, simple peer preferences and identical ordinal preference rankings over neighborhoods and schools (e.g., Calsamiglia et al. (2015), Barseghyan et al. (2013) and Avery and Pathak (2020)). Matching models with general peer preferences and rich preference domains are intractable due to the non-existence of desirable solution concepts (Sasaki and Toda, 1996) or the computational complexity issues (Ronn, 1990). Therefore, we follow the standard approach in theoretical two-sided matching and school choice literature (Abdulkadiroğlu and Sönmez, 2003; Gale and Shapley, 1962) and abstract away from peer preferences. It is important to note though that this abstraction may not be without loss of generality for our welfare analysis. For example, Barseghyan et al. (2013) show that when there are peer preferences and endogenous school quality, neighborhood assignment may result in higher aggregate welfare compared to open enrollment. Avery and Pathak (2020) show that when there are peer preferences and endogenously priced outside options, neighborhood assignment may result in higher welfare for lowest-income families compared to open enrollment. These papers do not provide general conditions, but only give examples or cases where neighborhood assignment may be welfare enhancing compared to open enrollment. In contrast, we give general results and conditions when the Deferred Acceptance mechanism performs better than neighborhood assignment.

There are other arguments (both in favor or against) on school choice that our work does not address. For example, we do not consider schools' incentives to improve education quality, whereas proponents consider it as a major argument in favor of school choice. They argue that parental choice enhances school quality through competitive pressures (Chubb and Moe, 1990; Friedman, 1962; Hoxby, 2003). Our work also ignores the possibility of sorting on dimensions other than income (O'Neil, 1996; Smith, 1995). For example, when families have same-race preferences, parental choice may exacerbate racial segregation, which is another major concern in public policy of school choice.

Despite the potential limitations above, our results provide a unique theoretical justification on welfare and distributional grounds for using the Deferred Acceptance mechanism, either with or without neighborhood priorities, as an alternative to neighborhood assignment. Additionally, we develop a theoretical framework that can be used for future research and potential extensions which would address the limitations.

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A Omitted Proofs

A.1 Proof of Theorem 2

The proof below is for DN. The result for DA is proved analogously.

In what follows, whenever we talk about continuity and convergence on measure spaces, the topology under consideration is the topology of weak convergence of measures. We use $\tau_n \rightharpoonup \tau$ to denote that the sequence $(\tau_n)_{n\in\mathbb{N}}$ converges to τ in that topology.

As mentioned in Section 3, each $\tau \in \mathcal{T}$ results in a measure G_τ on $P \times S \times [0,1]$ given by

$$G_\tau\Big((\succ, s, r) \in P \times S \times [0,1] : \succ \in P', s \in S', r \in (r_0, r_1)\Big)$$
$$= \tau\Big((v,h) \in V \times \bar{H} : \succ_{vh} \in P', s_h \in S'\Big) \times \Big(r_1 - r_0\Big),$$

for each $P' \subseteq P, S' \subseteq S$ and $(r_0, r_1) \subseteq [0,1]$.

For two measures G and G' on $P \times S \times [0,1]$, we define a distance between them by

$$d(G,G') := \sup_{P' \subseteq P, S' \subseteq S, r_0, r_1 \in [0,1]} \Big| G\Big((\succ, s, r) \in P \times S \times [0,1] : \succ \in P', s \in S', r \in (r_0, r_1)\Big)$$
$$- G'\Big((\succ, s, r) \in P \times S \times [0,1] : \succ \in P', s \in S', r \in (r_0, r_1)\Big)\Big|.$$

Lemma 1. *Let (τ, p) be an arbitrary competitive equilibrium of DN and let c denote the corresponding cutoffs vector. Consider a sequence of economies $(\tau_n)_{n\in\mathbb{N}}$ converging to τ and the corresponding sequence of DN cutoffs $(c_n)_{n\in\mathbb{N}}$. Then, $c_n \to c$.*

Proof. The proof has two part.

Part 1. First, we show that $G_{\tau_n} \xrightarrow{d} G_\tau$. By definitions of the distance function d and measures G_{τ_n} and G_τ,

$$d(G_{\tau_n}, G_\tau) =$$
$$\sup_{P' \subseteq P, S' \subseteq S, r_0, r_1 \in [0,1]} \Big(r_1 - r_0\Big) \times \Big| \tau_n\Big((v,h) \in V \times \bar{H} : \succ_{vh} \in P', s_h \in S'\Big)$$

$$-\tau\Big((v,h) \in V \times \bar{H} : \succ_{vh} \in P', s_h \in S'\Big)\Big|$$

$$\leq \max_{P' \subseteq P, S' \subseteq S} \Big|\tau_n\Big((v,h) \in V \times \bar{H} : \succ_{vh} \in P', s_h \in S'\Big)$$

$$- \tau\Big((v,h) \in V \times \bar{H} : \succ_{vh} \in P', s_h \in S'\Big)\Big|. \tag{10}$$

Since $\tau_n \rightharpoonup \tau$, by Portmanteau theorem (Billingsley, 1968) the last term in equation 10 converges to zero. This establishes Part 1.

Part 2. We say G_τ has rich support if a positive measure of each preference type resides in each neighborhood. Formally,

Definition 7. *G_τ has **rich support** if for all $\succ \in P$ and $s \in S$,*

$$\tau\Big((v,h) \in V \times \bar{H} : \succ_{vh} = \succ, s_h = s\Big) > 0.$$

By Lemma 3 in Abdulkadiroğlu et al. (2017b), in order to have $c_n \to c$, it is sufficient to show that G_τ has rich support. In what follows, we establish this result.

First, we prove that $p_h < 1$ for all $h \in H$. Suppose, for the sake of contradiction, that $p_h \geq 1$ for some $h \in H$. Then,

$$0 < q_h = \eta\Big(v \in V : u_v(h,\tau) - p_h = \arg\max_{h' \in \bar{H}} \ u_v(h',\tau) - p_{h'}\Big)$$

$$\leq \eta\Big(v \in V : u_v(h,\tau) - p_h \geq 0\Big) \leq \eta\Big(v \in V : \max_{s \in S} v(h,s) - p_h \geq 0\Big) = 0,$$

a contradiction.

Now, consider an arbitrary $h \in H$ and let $\epsilon := 1 - p_h > 0$. Define a subset $V_h \subseteq V$ by

$$V_h := \Big\{v \in V : v(h,s) > 1 - \epsilon/2, v(h',s) < \epsilon/2, \forall s \in S, h' \in H \setminus \{h\}\Big\}.$$

When choosing h, type $v \in V_h$ guarantees a payoff strictly larger than

$$1 - \epsilon/2 - p_h = 1 - \epsilon/2 - (1-\epsilon) = \epsilon/2,$$

and when choosing $h' \in H \setminus \{h\}$, she can obtain at most

$$\epsilon/2 - p_{h'} \leq \epsilon/2.$$

Thus, almost all types in V_h choose h at any CE and

$$\tau\big(v \in V_h : s_h = s\big) = \eta(V_h) > 0.$$

The last inequality follows from that η has full support. Again, by full support of η, for each $\succ \in P$ there is a positive measure of types in V_h whose preferences are $\succ$. Denoting by δ the smallest of these measures, we obtain the desired result. $\square$

As Part 2 of Lemma 1 demonstrates, all types in V_h choose neighborhood h in any CE (τ, p). Consider an arbitrary $\tau \in \mathcal{T}$ satisfying

$$\tau\Big((v,h) \in V \times \bar{H} : v \in V_{h'}, h = h'\Big) = \eta\big(V_{h'}\big) \text{ for all } h' \in H. \tag{11}$$

Let $\tau_n \rightharpoonup \tau$ be an arbitrary sequences of neighborhood choices with cutoffs the corresponding cutoffs sequence c_n. Then, with similar arguments as in Lemma 1 we establish that $c_n \to c$, where c denotes the cutoff of τ. Thus, the continuity of cutoffs hold for any neighborhood choices τ satisfying equation 11. For the rest of the proof, we restrict attention to such neighborhood choices. With abuse of notation, we denote this set by $\mathcal{T}$.

Lemma 2. *The collection of functions $(u_v(h,\tau))_{v \in V, h \in H}$ is equicontinuous in τ.*

Proof. Recall that $u_v(h,\tau) = \sum_{s \in S} \lambda_s(\succ_{v,h}, h, \tau) v(h,s)$. First, we show that $\lambda_s(\succ, h, \tau)$ is continuous in τ. That $\lambda_{vs}(h,\tau)$ is continuous in c is immediate from equations 3 and 4. Thus, by Lemma 1, $\lambda_{vs}(h,\tau)$ is continuous in τ.

Since $\big\{\lambda_{vs}(h,\tau)\big\}_{v \in V, h \in H}$ is a finite collection of functions, it is equicontinuous. Since v is bounded, $\big\{u_v(h,\tau)\big\}_{h \in H, v \in V}$ is equicontinuous, too. $\square$

Lemma 3. *For any $\tau \in \mathcal{T}$, there is a unique price vector $\mathcal{P}(\tau) \in \mathbb{R}_+^{|H|}$ such that for all $h \in H$,*

$$\eta\Big(v \in V : u_v(h,\tau) - \mathcal{P}_h(\tau) = \underset{h' \in \bar{H}}{\arg\max}\ u_v(h',\tau) - \mathcal{P}_{h'}(\tau)\Big) \leq q_h. \tag{12}$$

and the equality is strict only if $\mathcal{P}_h(\tau) = 0$. Moreover, $\mathcal{P}(\tau)$ is continuous in τ.

Proof. Let $\tilde{V} := [0,1]^{|H|}$ and define a measure $\tilde{\eta}$ over $\tilde{V}$ by

$$\tilde{\eta}(\tilde{U}) = \eta\Big(v \in V : (u_v(h,\tau))_{h \in H} \in \tilde{U}\Big),$$

for all measurable $\tilde{U} \subseteq \tilde{V}$. Since $\tilde{\eta}$ is absolutely continuous and full support, the existence of the unique vector $\mathcal{P}(\tau) \in \mathbb{R}^N$ satisfying equation 12 follows from Gretsky et al. (1999).

We divide the proof of continuity of $\mathcal{P} : \mathcal{T} \to \mathbb{R}_+^{|H|}$ to two parts.

Part 1. Suppose $\tau_n \rightharpoonup \tau$. We show that $\tilde{\eta}_n \rightharpoonup \tilde{\eta}$. Consider an arbitrary $\epsilon > 0$ and $\tilde{U} \subseteq \tilde{V}$ with a measure zero boundary $\partial\tilde{U}$. By Portmanteau theorem, it is sufficient to show that $\tilde{\eta}_n(\tilde{U}) \to \tilde{\eta}(\tilde{U})$.

By absolute continuity of $\tilde{\eta}$, there is an open cover $\{\mathcal{O}\}_{i \in I}$ of $\partial\tilde{U}$ such that $\tilde{\eta}\Big(\cup_{i\in I}\mathcal{O}_i\Big) < \epsilon$. Since $\partial\tilde{U}$ is a compact set, there is $\underline{\delta} > 0$ such that for any $\tilde{u} \in \partial\tilde{U}$, the $\underline{\delta}$-ball around $\tilde{u}$ is contained in some element of the open cover $\{\mathcal{O}_i\}_{i\in I}$. For any $\delta \in [0, \underline{\delta}]$, let $\tilde{E}^\delta$ denote the union of δ-balls around each point in $\partial\tilde{U}$. Then, $\partial\tilde{U} \subseteq \tilde{E}^\delta$ and

$$\tilde{\eta}(\tilde{E}^\delta) \leq \tilde{\eta}\Big(\cup_{i\in I}\mathcal{O}_i\Big) < \epsilon. \tag{13}$$

By Lemma 2, for any sufficiently large $n \in \mathbb{N}$,

$$\begin{aligned}\tilde{\eta}_n(\tilde{E}^{\delta/2}) = \eta\Big(v \in V : \big(u_v(h,\tau_n)\big)_{h\in H} \in \tilde{E}^{\delta/2}\Big) &\leq \eta\Big(v \in V : \big(u_v(h,\tau)\big)_{h\in H} \in \tilde{E}^{\delta}\Big) \\ &= \tilde{\eta}(\tilde{E}^\delta) < \epsilon.\end{aligned} \tag{14}$$

By equation 13,

$$\tilde{\eta}(\tilde{U}) \leq \tilde{\eta}\big(\tilde{U} \setminus \tilde{E}^\delta\big) + \tilde{\eta}(\tilde{E}^\delta) < \tilde{\eta}\big(\tilde{U} \setminus \tilde{E}^\delta\big) + \epsilon. \tag{15}$$

By Lemma 2, potentially for a larger $n \in \mathbb{N}$,

$$\tilde{\eta}\big(\tilde{U} \setminus \tilde{E}^{\delta}\big) = \eta\Big(v \in V : \big(u_v(h,\tau)\big)_{h\in H} \in \tilde{U} \setminus \tilde{E}^{\delta}\Big)$$
$$\leq \eta\Big(v \in V : \big(u_v(h,\tau_n)\big)_{h\in H} \in \tilde{U}\Big) = \tilde{\eta}_n(\tilde{U}). \tag{16}$$

Combining equations 15 and 16,

$$\tilde{\eta}(\tilde{U}) < \tilde{\eta}_n(\tilde{U}) + \epsilon.$$

Similarly, by equation 14,

$$\tilde{\eta}_n(\tilde{U}) \leq \tilde{\eta}_n\big(\tilde{U} \setminus \tilde{E}^{\delta/2}\big) + \tilde{\eta}_n(\tilde{E}^{\delta/2}) < \tilde{\eta}_n\big(\tilde{U} \setminus \tilde{E}^{\delta/2}\big) + \epsilon, \tag{17}$$

and by Lemma 2,

$$\tilde{\eta}_n\big(\tilde{U} \setminus \tilde{E}^{\delta/2}\big) = \eta\Big(v \in V : \big(u_v(h,\tau_n)\big)_{h\in H} \in \tilde{U} \setminus \tilde{E}^{\delta/2}\Big)$$
$$\leq \eta\Big(v \in V : \big(u_v(h,\tau)\big)_{h\in H} \in \tilde{U}\Big) = \tilde{\eta}(\tilde{U}). \tag{18}$$

Combining 17 and 18,

$$\tilde{\eta}_n(\tilde{U}) < \tilde{\eta}(\tilde{U}) + \epsilon.$$

Part 2. Now, we show that $\mathcal{P}$ is continuous in $\tilde{\eta}$. Let $\tilde{\eta}_n \rightharpoonup \tilde{\eta}$ and $(\mathcal{P}^n)_{n\in\mathbb{N}}$ be the corresponding sequence of prices. Note that $\mathcal{P}^n_h < 1$ for all $h \in H$. Suppose, for the sake of contradiction, that $\mathcal{P}^n_h \geq 1$ for some $h \in H$ and $n \in \mathbb{N}$. Then,

$$0 < q_h = \eta\Big(v \in V : u_v(h,\tau^n) - \mathcal{P}^n_h = \arg\max_{h'\in\bar{H}} u_v(h',\tau^n) - \mathcal{P}^n_{h'}\Big)$$
$$\leq \eta\Big(v \in V : u_v(h,\tau^n) - p_h \geq 0\Big) \leq \eta\Big(v \in V : \max_{s\in S} v(h,s) - \mathcal{P}^n_h \geq 0\Big) = 0,$$

a contradiction. By Bolzano-Weierstrass theorem, $(\mathcal{P}^n)_{n\in\mathbb{N}}$ has a convergent subsequence. Without loss of generality, suppose $\mathcal{P}^n \to \mathcal{P}^*$. It is sufficient to show that $\mathcal{P}^*$ satisfies equation 12. By uniqueness, this would imply $\mathcal{P}^* = \mathcal{P}$ and the desired continuity result. For all $h \in H$ define

$$\tilde{V}_h := \Big\{\tilde{v} \in \tilde{V} : \tilde{v}(h) - \mathcal{P}^*_h = \arg\max_{h\in\bar{H}}\ \tilde{v}(h') - \mathcal{P}^*_{h'}\Big\}.$$

Suppose, for the sake of contradiction, that $\tilde{\eta}(\tilde{V}_h) \neq q_h$ for some $h \in H$. Without loss of generality, let $\tilde{\eta}(\tilde{V}_h) > q_h$. Since $\tilde{\eta}_n \rightharpoonup \tilde{\eta}$ there are $\delta > 0$ and $M \in \mathbb{N}$ such that $\tilde{\eta}_n(\tilde{V}_h) > q_h + \delta, \forall n > M$. By picking n large enough we can make $\mathcal{P}^n$ arbitrarily close to $\mathcal{P}^*$ and therefore

$$\tilde{\eta}_n\Big(\tilde{v} \in \tilde{V} : \tilde{v}(h) - \mathcal{P}^n_h = \arg\max_{h' \in \bar{H}} \ \tilde{v}(h') - \mathcal{P}^n_{h'}\Big) > q_h,$$

a contradiction. This completes the proof of Lemma 3. □

Define families' best response mapping $\mathcal{B} : \mathcal{T} \to \mathcal{T}$ by

$$\mathcal{B}_\tau\big(U, h\big) = \eta\Big(v \in U : h = \arg\max_{h' \in \bar{H}} \ u_v(h', \tau) - \mathcal{P}_{h'}(\tau)\Big),$$

for all $h \in H$ and measurable $U \subseteq V$.

Lemma 4. *$\mathcal{B}$ is continuous.*

Proof. Suppose $\tau_n \rightharpoonup \tau$ and $U \subseteq V$ is a measure zero boundary set.

For any $v \in V$ and $h \in H$, define $\mathcal{F}_\tau(v, h) : \mathcal{T} \to \mathbb{R}$ by

$$\mathcal{F}_\tau(v, h) = u_v(h, \tau) - \mathcal{P}_h(\tau) - \max_{h' \in \bar{H} \setminus \{h\}} \big(u_v(h', \tau) - \mathcal{P}_{h'}(\tau)\big).$$

By Portmanteau theorem, it is sufficient to show

$$\mathcal{B}_{\tau_n}(U, h) = \eta\Big(v \in U : \mathcal{F}_{\tau_n}(v, h) \geq 0\Big) \to \eta\Big(v \in V : \mathcal{F}_\tau(v, h) \geq 0\Big) = \mathcal{B}_\tau(U, h).$$

By Lemmas 2 and 3, the collection of functions $\{\mathcal{F}_\tau(v, h)\}_{v \in V, h \in H}$ is equicontinuous. Fix an arbitrary $\epsilon > 0$. By absolute continuity of η, there is $\delta > 0$ such that

$$\eta\Big(v \in U : \mathcal{F}_\tau(v, h) \in [0, \delta)\Big) < \epsilon. \tag{19}$$

By equicontinuity of $\{\mathcal{F}_\tau(v, h)\}_{v \in V, h \in H}$, for any sufficiently large $n \in \mathbb{N}$,

$$\eta\Big(v \in U : \mathcal{F}_{\tau_n}(v, h) \in [0, \delta/2)\Big) < \eta\Big(v \in U : \mathcal{F}_\tau(v, h) \in [0, \delta)\Big) < \epsilon. \tag{20}$$

By equation 19,

$$\eta\Big(v \in U : \mathcal{F}_\tau(v,h) \geq 0\Big) = \eta\Big(v \in U : \mathcal{F}_\tau(v,h) \geq \delta\Big) + \eta\Big(v \in U : \mathcal{F}_\tau(v,h) \in [0,\delta)\Big)$$

$$< \eta\Big(v \in U : \mathcal{F}_\tau(v,h) \geq \delta\Big) + \epsilon. \tag{21}$$

By equicontinuity of $\{\mathcal{F}_\tau(v,h)\}_{v\in V, h\in H}$, and potentially larger $n \in \mathbb{N}$,

$$\eta\Big(v \in U : \mathcal{F}_\tau(v,h) \geq \delta\Big) < \eta\Big(v \in U : \mathcal{F}_{\tau_n}(v,h) \geq 0\Big). \tag{22}$$

Combining 21 and 22,

$$\eta\Big(v \in U : \mathcal{F}_\tau(v,h) \geq 0\Big) < \eta\Big(v \in U : \mathcal{F}_{\tau_n}(v,h) \geq 0\Big) + \epsilon.$$

Similarly, by equation 20,

$$\eta\Big(v \in U : \mathcal{F}_{\tau_n}(v,h) \geq 0\Big) = \eta\Big(v \in U : \mathcal{F}_{\tau_n}(v,h) \geq \delta/2\Big) + \eta\Big(v \in U : \mathcal{F}_{\tau_n}(v,h) \in [0,\delta/2)\Big)$$

$$< \eta\Big(v \in U : \mathcal{F}_{\tau_n}(v,h) \geq \delta/2\Big) + \epsilon, \tag{23}$$

and by equicontinuity of $\{\mathcal{F}_\tau(v,h)\}_{v\in V, h\in H}$,

$$\eta\Big(v \in U : \mathcal{F}_{\tau_n}(v,h) \geq \delta/2\Big) < \eta\Big(v \in U : \mathcal{F}_\tau(v,h) \geq 0\Big). \tag{24}$$

Combining 23 and 24,

$$\eta\Big(v \in U : \mathcal{F}_{\tau_n}(v,h) \geq 0\Big) < \eta\Big(v \in U : \mathcal{F}_\tau(v,h) \geq 0\Big) + \epsilon.$$

This completes the proof of Lemma 4. □

Each fixed point τ^* of $\mathcal{B}$ corresponds to a CE $\big(\tau^*, \mathcal{P}(\tau^*)\big)$ of DN. Since $\mathcal{T}$ is (weakly) compact and $\mathcal{B} : \mathcal{T} \to \mathcal{T}$ is (weakly) continuous, the existence of CE of DN follows from Schauder-Tychonoff fixed point theorem.

A.2 Proof of Theorem 4

The proof has two parts. Part 1 establishes the result for Assumption 1 and Part 2 establishes the result for Assumption 2. Note that η does not have full support under either of the assumptions. Nevertheless, CE exist in both cases.

In the proof we assume that $\sum_{i=1}^{N} q_{h_i} = 1$. The assumption is without loss of generality, since when $\sum_{i=1}^{N} q_{h_i} < 1$ we can add a neighborhood that no one family likes, and when $\sum_{i=1}^{N} q_{h_i} > 1$ we can add families who are indifferent across all schools and neighborhoods, and who in equilibrium will choose lower indexed neighborhoods and schools.

Part 1. Define the numbers $0 = a_0 \leq a_1 \leq ... \leq a_N = 1$ by

$$\eta\Big(\{v_\alpha\}_{\alpha\in[a_{k-1},a_k]}\Big) = q_{h_k}, \forall k \in \{1, 2, ..., N\}.$$

Then, it is immediate from the increasing differences property of valuations that in any CE (τ^ϕ, p^ϕ) of $\phi \in \{DN, DA\}$,

$$\tau^\phi\Big(\{v_\alpha\}_{\alpha\in[a_{k-1},a_k]} \times \{h_k\}\Big) = q_{h_k}, \forall k \in \{1, 2, ..., N\}.$$

We now compute school assignment probabilities and expected utilities under DA and DN.

Under DA, school assignment is solely determined by lottery numbers. Any type v_α is assigned to a school she prefers weakly more than s_k if and only if her lottery number is in the interval $\big[1 - \sum_{j=1}^{k} q_{s_j}, 1\big]$, the probability of which is $\min\big\{\sum_{j=1}^{k} q_{s_j}, 1\big\}$.

Under DN, school assignment is determined based on both neighborhood choice and lottery numbers. DN assignment can be given by the following procedure.

<u>Round 1</u>: Let $V_N = V$ and $\bar{V}_N = \big\{v_\alpha \in V_N : \alpha \in [a_{N-1}, a_N]\big\}$. Each family in $\bar{V}_N$ is assigned to s_N with probability one. Remaining seats at s_N are assigned to families $q_{s_N} - \eta(\bar{V}_N) = q_{s_N} - q_{h_N}$ highest lottery numbers among the remaining ones.

Round $k > 1$: Let V_{N-k+1} denote the set of families that are unassigned by Round k and $\bar{V}_{N-k+1} = \{v_\alpha \in V_{N-k+1} : \alpha \in [a_{N-k}, a_{N-k+1}]\}$. If $\eta(V_{N-k+1}) < q_{s_{N-k+1}}$, all remaining families are assigned to s_{N-k+1}. Otherwise, families in $\bar{V}_{N-k+1}$ is assigned to s_{N-k+1} with probability one and remaining seats at s_{N-k+1} are assigned to families with $q_{s_{N-k+1}} - \eta(\bar{V}_{N-k+1}) = q_{s_{N-k+1}} - q_{s_{N-k+1}}$ highest lottery numbers among the remaining ones.

Consider an alternative school assignment procedure, where we apply only Round 1 of DN, and assign remaining students to schools uniform randomly. By an induction argument, in order to show that DN creates higher aggregate welfare than DA, it is sufficient to the alternative procedure creates higher aggregate welfare than DA.

The alternative procedure is equivalent to applying DA first, then switching the assignment of types in $\bar{V}_N$ who are not assigned to s_N, with types not in $\bar{V}_N$ who are assigned to s_N. By increasing differences assumption, this reallocation improves aggregate welfare. This completes the proof of Part 1.

Part 2. Let (τ^{DN}, p^{DN}) and (τ^{DA}, p^{DA}) be CE of DN and DA, respectively. Under DA, a family is assigned to a school she weakly prefers to s_k (satisfying $\sum_{j=k}^{N} s_k \leq 1$) if and only if her lottery number is in the interval $[1 - \sum_{j=1}^{k} q_{s_j}, 1]$, the probability of which is $\{ \sum_{j=1}^{k} q_{s_j}$. Now, consider DN. A family with neighborhood choice h_k is assigned to a school weakly better than $s_i, i > k$ if and only if she has one of the $\sum_{j=1}^{k}(q_{s_j} - q_{h_j})$ highest lottery numbers among individuals who live in a neighborhood $h_j, j < i$. The probability of the latter event is $\frac{\sum_{j=1}^{i}(q_{s_j} - q_{h_j})}{1 - \sum_{j=1}^{i} q_{h_j}}$. Consider the 'strategy' of type v, where she chooses neighborhood h_j with probability q_{h_j} for all $j \in \{1, 2, ..., N\}$. Then, the probability that she is assigned to a school she prefers weakly more than $s_i, i > k$, is equal to

$$\sum_{j=1}^{i} q_{h_j} + \big(1 - \sum_{j=1}^{i} q_{h_j}\big) \frac{\sum_{j=1}^{i}(q_{s_j} - q_{h_j})}{1 - \sum_{j=1}^{i} q_{h_j}} = \sum_{j=1}^{i} q_{s_j}.$$

The first term on the left hand side is the probability of choosing a neighborhood with an index weakly larger than i, in which case the assignment to a school weakly preferred to s_i is guaranteed. Thus, under any CE of DN, type v can replicate the DA assignment probabilities for any school she prefers strictly more than s_k by playing the strategy above. Neighborhood choices τ corresponding to almost all types playing this strategy is given by

$$\tau\big(U \times \{h_j\}\big) = \eta(U) \times q_{h_j},$$

for all measurable $U \subseteq V$ and $h_j \in H$. Then,

$$\int u_v^{DN}(h, \tau^{DN}) d\tau^{DN} \geq \int u_v^{DN}(h, \tau^{DN}) d\tau \geq \int u_v^{DA}(h, \tau^{DA}) d\tau^{DA},$$

where the first inequality follows equation 5 and the second inequality follows from strategy τ replicates DA assignment probabilities for any school that a family prefers to her neighborhood school. This complete the proof of Part 2.

A.3 Proof of Theorem 6

First, we prove point 1. Consider an arbitrary economy η' and $\epsilon > 0$. Suppose $H_-^{NA} \not\subseteq H_-^{DN}$ for all η with $\|\eta - \eta'\|_2 < \epsilon$. Consider the economy

$$\eta = \Big(1 - \frac{\epsilon}{|H| + 1}\Big) \times \eta + \sum_{h \in H} \frac{\epsilon}{|H| + 1} \times \delta_{(v_h, 0)},$$

where $\delta_{(v_h,0)}$ is the Dirac measure that puts all probability mass on the point $(v_h, 0)$, and for each $h \in H$,

$$v_h(h', s') = \begin{cases} 1 & \text{if } (h', s') = (h, s_h), \\ 0 & \text{otherwise.} \end{cases}$$

Consider a neighborhood $h \in H_-^{NA} \setminus H_-^{DN}$. It is immediate that all families with type $(v_h, 0)$ prefer NA to DN.

Now we prove point 2. Consider an arbitrary economy η' and $\epsilon > 0$. Suppose $H_-^{NA} \not\subseteq H_-^{DA}$ or $\big\{s_h \in S : h \in H_-^{NA}\big\} \not\subseteq S_-^{DA}$ for all η with $\|\eta - \eta'\|_2 < \epsilon$. Consider the economy

$$\eta = \Big(1 - \frac{\epsilon}{2|H| + 1}\Big) \times \eta + \sum_{h \in H} \frac{\epsilon}{2|H| + 1} \times \delta_{(v_h, 0)} + \sum_{h \in H} \frac{\epsilon}{2|H| + 1} \times \delta_{(v_s, 0)},$$

where

$$v_h(h', s') = \begin{cases} 1 & \text{if } (h', s') = (h, s_h), \\ 0 & \text{otherwise,} \end{cases}$$

and

$$v_s(h, s') = \begin{cases} 1 & \text{if } s' = s, \\ 0 & \text{otherwise.} \end{cases}$$

If there is a neighborhood $h \in H_-^{NA} \setminus H_-^{DA}$, then all families with type $(v_h, 0)$ prefer NA to DN. Otherwise, consider a school $s \in \{s_h \in S : h \in H_-^{NA}\} \setminus S_-^{\prime DA}$. It is immediate that all families with type $(v_s, 0)$ prefer NA to DA.

A.4 Proof of Corollary 1

Since neighborhoods and schools are ranked, the set of underdemanded neighborhoods are those with index k satisfying $\sum_{j=k}^{N} q_{h_j} \geq 1$ under all three mechanisms. Moreover, the set of underdemanded schools under DN and DA are those with index k satisfying $\sum_{j=k}^{N} q_{s_j} \geq 1$. The result therefore follows from that $q_{s_j} \geq q_{h_j}$ for all $j \in \{1, 2, ..., N\}$.

A.5 Proof of Corollary 2

The proof has two parts. Part 1 establishes that $H_-^{NA} = H_- := \arg\max_{h \in H} q_h \subseteq H_-^{DA}$, and Part 2 establishes that $\{s_h \in S : h \in H_-^{NA}\} \subseteq S_-^{DA}$. By Theorem 5, these two conditions are sufficient to prove Corollary 2.

Part 1. We first show that $H_-^{NA} \subseteq H_-$. Suppose, for the sake of contradiction, that $p_h^{NA} = 0$ for some $h \in H \setminus H_-$. Consider an arbitrary $\bar{h} \in H_-$. Then,

$$q_h = \eta\Big((v, b) \in V \times B : h = \arg\max_{h' \in \bar{H}_b} v(h', s_{h'}) - p_{h'}^{NA}\Big)$$

$$= \eta\Big((v, b) \in V \times B : v(h, s_h) \geq v(\bar{h}, s_{\bar{h}}) - p_{\bar{h}}^{NA}$$

$$\text{and } v(h,s_h) \geq v(h',s_{h'}) - p_{h'}^{NA}, \forall h' \in \bar{H}_b \setminus \{h,\bar{h}\}\Big)$$

$$\geq \eta\Big((v,b) \in V \times B : v(h,s_h) \geq v(\bar{h},s_{\bar{h}}) \text{ and } v(h,s_h) \geq v(h',s_{h'}) - p_{h'}^{NA}, \forall h' \in \bar{H}_b \setminus \{h,\bar{h}\}\Big)$$

$$= \eta\Big((v,b) \in V \times B : v(\bar{h},s_{\bar{h}}) \geq v(h,s_h) \text{ and } v(\bar{h},s_{\bar{h}}) \geq v(h',s_{h'}) - p_{h'}^{NA}, \forall h' \in \bar{H}_b \setminus \{h,\bar{h}\}\Big)$$

$$\geq \eta\Big((v,b) \in V \times B : v(\bar{h},s_{\bar{h}}) - p_{\bar{h}}^{NA} \geq v(h,s_h)$$

$$\text{and } v(\bar{h},s_{\bar{h}}) - p_{\bar{h}}^{NA} \geq v(h',s_{h'}) - p_{h'}^{NA}, \forall h' \in \bar{H}_b \setminus \{h,\bar{h}\}\Big)$$

$$\geq \eta\Big((v,b) \in V \times B : b \geq p_{\bar{h}}^{NA}, v(\bar{h},s_{\bar{h}}) - p_{\bar{h}}^{NA} \geq v(h,s_h)$$

$$\text{and } v(\bar{h},s_{\bar{h}}) - p_{\bar{h}}^{NA} \geq v(h',s_{h'}) - p_{h'}^{NA}, \forall h' \in \bar{H}_b \setminus \{h,\bar{h}\}\Big)$$

$$\eta\Big((v,b) \in V \times B : \bar{h} = \arg\max_{h' \in \bar{H}_b} v(h',s_{h'}) - p_{h'}^{NA}\Big) = q_{\bar{h}} > q_h,$$

a contradiction. The second equality above follows from uniformity (therefore, symmetry) of η. The remaining steps are immediate. It is left to show that $H_- \subseteq H_-^{NA}$, or equivalently, $p_h^{NA} = 0$ for all $h \in H_-$. Let $\bar{h} \in H_-$ be such that $p_{\bar{h}}^{NA} = 0$. Such a neighborhood exists since $H_- \supseteq H_-^{NA} \neq \emptyset$. Suppose, for the sake of contradiction, that $p_h^{NA} > 0$. Then,

$$q_h = \eta\Big((v,b) \in V \times B : h = \arg\max_{h' \in \bar{H}_b} v(h',s_{h'}) - p_{h'}^{NA}\Big)$$

$$= \eta\Big((v,b) \in V \times B : v(h,s_h) - p_h^{NA} \geq v(\bar{h},s_{\bar{h}}) - p_{\bar{h}}^{NA}$$

$$\text{and } v(h,s_h) - p_h^{NA} \geq v(h',s_{h'}) - p_{h'}^{NA}, \forall h' \in \bar{H}_b \setminus \{h,\bar{h}\}\Big)$$

$$< \eta\Big((v,b) \in V \times B : v(\bar{h},s_{\bar{h}}) - p_{\bar{h}}^{NA} \geq v(\bar{h},s_h) - p_h^{NA}$$

$$\text{and } v(\bar{h},s_{\bar{h}}) - p_{\bar{h}}^{NA} \geq v(h',s_{h'}) - p_{h'}^{NA}, \forall h' \in \bar{H}_b \setminus \{h,\bar{h}\}\Big)$$

$$= \eta\Big((v,b) \in V \times B : b \geq p_{\bar{h}}^{NA}, v(\bar{h},s_{\bar{h}}) - p_{\bar{h}}^{NA} \geq v(h,s_h)$$

$$\text{and } v(\bar{h},s_{\bar{h}}) - p_{\bar{h}}^{NA} \geq v(h',s_{h'}) - p_{h'}^{NA}, \forall h' \in \bar{H}_b \setminus \{h,\bar{h}\}\Big)$$

$$\eta\Big((v,b) \in V \times B : \bar{h} = \arg\max_{h' \in \bar{H}_b} v(h',s_{h'}) - p_{h'}^{NA}\Big) = q_{\bar{h}},$$

a contradiction. The strict inequality above follows from uniformity of η and from that $p_h^{NA} > 0 = p_{\bar{h}}^{NA}$.

We now show that $H_- \subseteq H_-^{DA}$.

Let the mapping $\pi_{h\leftrightarrow h'} : V \to V$ be such that for all $s \in S$,

$$\pi_{h\leftrightarrow h'}(v)(h,s) = v(h',s), \pi_{h\leftrightarrow h'}(v)(h',s) = v(h,s), \text{ and}$$

$$\pi_{h\leftrightarrow h'}(v)(h'',s) = v(h'',s), \text{ for all } h'' \in H \setminus \{h,h'\}.$$

By uniformity of η, for any measurable $U \subseteq V$,

$$\eta(U) = \eta\Big((v,b) \in V \times B : \pi_{h\leftrightarrow h'}(v) \in U\Big). \tag{25}$$

Suppose, for the sake of contradiction, that $p_h^{DA} > 0$ for some $h \in H_-$. Consider an arbitrary $\bar{h} \in H_-^{DA}$. Then,

$$\begin{aligned}
q_h = \eta\Big((v,b) \in V \times B : h = \arg\max_{h' \in \bar{H}_b} u_v^{DA}(h',\tau^{DA}) - p_{h'}^{DA}\Big)& \\
= \eta\Big((v,b) \in V \times B : u_v^{DA}(h,\tau^{DA}) - p_h^{DA} \geq u_v^{DA}(\bar{h},\tau^{DA}) - p_{\bar{h}}^{DA}& \\
\text{and } u_v^{DA}(h,\tau^{DA}) - p_h^{DA} \geq u_v^{DA}(h',\tau^{DA}) - p_{h'}^{DA}, \forall h' \in \bar{H}_b \setminus \{h,\bar{h}\}\Big)& \\
= \eta\Big((v,b) \in V \times B : u_v^{DA}(\bar{h},\tau^{DA}) - p_h^{DA} \geq u_v^{DA}(h,\tau^{DA}) - p_{\bar{h}}^{DA}& \\
\text{and } u_v^{DA}(h,\tau^{DA}) - p_h^{DA} \geq u_v^{DA}(h',\tau^{DA}) - p_{h'}^{DA}, \forall h' \in \bar{H}_b \setminus \{h,\bar{h}\}\Big)& \\
< \eta\Big((v,b) \in V \times B : u_v^{DA}(\bar{h},\tau^{DA}) - p_{\bar{h}}^{DA} \geq u_v^{DA}(h,\tau^{DA}) - p_h^{DA}& \\
\text{and } u_v^{DA}(\bar{h},\tau^{DA}) - p_{\bar{h}}^{DA} \geq u_v^{DA}(h',\tau^{DA}) - p_{h'}^{DA}, \forall h' \in \bar{H}_b \setminus \{h,\bar{h}\}\Big)& \\
= \eta\Big((v,b) \in V \times B : \bar{h} = \arg\max_{h' \in \bar{H}_b} u_v^{DA}(h',\tau^{DA}) - p_{h'}^{DA}\Big) = q_{\bar{h}},&
\end{aligned}$$

a contradiction. The third equality follows from equation 25, and the strict inequality follows $p_h^{DA} > 0 = p_{\bar{h}}^{DA}$.

Part 2. Consider an arbitrary $\bar{h} \in H_-$. We show that $\bar{s} := s_{\bar{h}} \in S_-^{DA}$. Suppose, for the sake of contradiction, that

$$\lambda := \lambda_{\bar{v}\bar{s}}^{\tau^{DA}}(h,\tau^{DA}) < \lambda_{vs}^{\tau^{DA}}(h,\tau^{DA}) = 1,$$

for some $s \in S$ and types $\bar{v}$ and v that rank $\bar{s}$ and s as first choices, respectively. Note that these probabilities are the same for all $h \in H$.

For each $h \in H$, let $U_h \subset V_h$ denote the set of types that rank $\bar{s}$ and s as the first two choices, in arbitrary order. Then,

$$\sum_{h \in H} \eta\Big((v,b) \in U_h \times B : v(h,\bar{s}) > v(h,s) \text{ and } h = \arg\max_{h' \in \bar{H}_b} u_v^{DA}(h',\tau^{DA}) - p_{h'}^{DA}\Big)$$

$$= \sum_{h \in H} \eta\Big((v,b) \in U_h \times B : v(h,\bar{s}) > v(h,s)$$

$$\text{and } \lambda v(h,\bar{s}) + (1-\lambda)v(h,s) - p_h^{DA} \geq \lambda v(h',\bar{s}) + (1-\lambda)v(h',s) - p_{h'}^{DA}, \forall h' \in H\Big)$$

$$< \sum_{h \in H} \eta\Big((v,b) \in U_h \times B : v(h,s) > v(h,\bar{s})$$

$$\text{and } v(h,s) - p_h^{DA} \geq h = v(h',\bar{s}) - p_{h'}^{DA}, \forall h' \in H\Big)$$

$$= \sum_{h \in H} \eta\Big((v,b) \in U_h \times B : v(h,s) > v(h,\bar{s}) \text{ and } h = \arg\max_{h' \in \bar{H}_b} u_v^{DA}(h',\tau^{DA}) - p_{h'}^{DA}\Big).$$

The strict inequality follows from uniformity of η and from that $\lambda < 1$. Thus, in equilibrium the mass of types that rank $\bar{s}$ as first choice and s as second choice is larger than the mass of types that rank s as first choice and $\bar{s}$ as second choice. With analogous arguments, we can show that this is true for any position of choices for $\bar{s}$ and s. This contradicts that the probability of being assigned to $\bar{s}$ is smaller than the probability of being assigned to s. This completes the proof.

A.6 Proof of Theorem 7

First, consider $\phi = NA$. Let (τ, p) be a CE of NA for each size $k \in \mathbb{N}$ economy $(v_f)_{f \in F_k}$, define $\sigma_k : F_k \to H$ by

$$\sigma_k(f) = \arg\max_{h \in H} \; v_f(h, s_h) - p_h.$$

We show that for a sufficiently large $k \in \mathbb{N}$, the probability that (σ_k, p) satisfies the conditions of Definition 6 of ϵ-CE approaches to one.

1. The first point is immediate from the definition of σ_k.

2. Let $F_{kh} = \Big\{ f \in F : v_f(h, s_h) - p_h = \arg\max_{h' \in H} v_f(h', s'_h) - p_{h'} \Big\}$ denote the set of families in F whose optimal choice is h (ties broken arbitrarily). Then,

$$\begin{aligned} \frac{\big|\sigma_k^{-1}(h)\big|}{k} = \frac{|F_{kh}|}{k} &\xrightarrow{p} \eta\Big(v \in V : h = \arg\max_{h' \in H} v(h', s_{h'}) - p_{h'}\Big) \\ &= \tau\big(V \times \{h\}\big) \leq q_h < q_h + \epsilon, \end{aligned} \tag{26}$$

where the convergence in probability follows from low of large numbers. Multiplying the first and last terms of equation 26 by k, we obtain the desired condition.

3. The proof is by contrapositive. Suppose $p_h \neq 0$. Then,

$$\begin{aligned} \frac{\big|\sigma_k^{-1}(h)\big|}{k} = \frac{|F_{k,h}|}{k} &\xrightarrow{p} \eta\Big(v \in V : h = \arg\max_{h' \in H} v(h', s_{h'}) - p_{h'}\Big) \\ &= \tau\big(V \times \{h\}\big) = q_h > q_h - \epsilon, \end{aligned} \tag{27}$$

where the last equality follows from $p_h \neq 0$. Multiplying the first and last terms of equation 27 by k, we obtain $\big|\sigma_k^{-1}(h)\big| > (q_h - \epsilon)k$ with probability approaching to one.

Now consider $\phi = DN$. The proof for $\phi = DA$ is similar. Let (τ, p) be a CE of NA for each size $k \in \mathbb{N}$ economy $(v_f)_{f \in F_k}$, define $\sigma_k : F_N \to H$ by

$$\sigma_k(f) = \arg\max_{h \in H} \; u_f(h, \tau) - p_h.$$

We show that for a sufficiently large $k \in \mathbb{N}$, the probability that (σ_k, p) satisfies the conditions of Definition 6 of ϵ-CE approaches to one.

1. Let $F_{kh} = \Big\{ f \in F : h = \arg\max_{h' \in H} u_f(h', \sigma_k) - p_{h'} \Big\}$ denote the set of families in F whose optimal choice is h (ties broken arbitrarily). Then, by law of large number,

$$\frac{\big|\sigma_k^{-1}(h)\big|}{k} = \frac{|F_{kh}|}{k} \xrightarrow{p} \eta\Big(v \in V : h = \arg\max_{h' \in H} u_f(h', \sigma_k) - p_{h'}\Big) = \tau\big(V \times \{h\}\big).$$

Thus, the proportion of individuals with given preferences and priorities in the discrete economy converges to its continuum analog. This, by Lemma 3 of Abdulkadiroğlu et al. (2017b), implies that $u_f(h, \sigma_k)$ converges to $u_f(h, \tau)$ in probability for all $h \in H$, establishing the desired result.

2. The proof for this part is similar to that for $\phi = NA$.

3. The proof for this part is similar to that for $\phi = NA$.

B More Examples

The following example demonstrates that some families may prefer NA to DN.

Example 4. *There are two neighborhoods* $H = \{h_1, h_2\}$ *and two schools* $S = \{s_1, s_2\}$. *Each neighborhood and school has a capacity* 0.5. *Economy* η *is supported at only two points* v_1 *and* v_2, *with*

$$\eta\Big(v \in V : v = v_1\Big) = \eta\Big(v \in V : v = v_2\Big) = 0.5.$$

Valuations are given in in Table 6.

	(h_1, s_1)	(h_1, s_2)	(h_2, s_1)	(h_2, s_2)
v_1	0	0.1	0.2	0.3
v_2	0.5	0	0.8	0.3

Table 6: Valuations

It is easy to verify that prices $p^{NA}_{h_1} = p^{NA}_{h_2} = 0$ *support CE* (τ^{NA}, p^{NA}) *of* NA, *satisfying*

$$\tau^{NA}\Big((v, h) \in V \times \bar{H} : v = v_1, h = h_2\Big) = \eta\Big(v \in V : v = v_1\Big),$$

and

$$\tau^{NA}\Big((v, h) \in V \times \bar{H} : v = v_2, h = h_1\Big) = \eta\Big(v \in V : v = v_2\Big).$$

Also, prices $p_{h_1}^{DN} = 0, p_{h_2}^{DN} = 0.2$ *support* (τ^{DN}, p^{DN}) *of* DN*, satisfying*

$$\tau^{DN}\Big((v,h) \in V \times \bar{H} : v = v_1, h = h_1\Big) = \eta\Big(v \in V : v = v_1\Big),$$

and

$$\tau^{DN}\Big((v,h) \in V \times \bar{H} : v = v_2, h = h_2\Big) = \eta\Big(v \in V : v = v_2\Big).$$

Thus,

$$u_{v_1}^{NA}(h_2, \tau^{NA}) = 0.3 > 0.1 = u_{v_1}^{DN}(h_1, \tau^{DN}),$$

and type v_1 *families prefer NA to DN.*

The next example demonstrates that in a discrete economy there may exist no CE of DA.

Example 5. *Consider a discrete economy with two families* $F = \{f_1, f_2\}$*, two neighborhoods* $H = \{h_1, h_2\}$ *and two schools* $S = \{s_1, s_2\}$*. Assume* $q_{h_1} = q_{h_2} = q_{s_1} = 1$ *and* $q_{s_2} = 2$*. Valuations are given in Table 5.*

	(h_1, s_1)	(h_1, s_2)	(h_2, s_1)	(h_2, s_2)
f_1	0.5	0	0	0.4
f_2	0	0.1	0.3	0

Table 7: Valuations

Suppose, for the sake of contradiction, that (σ, p) *is a CE. Consider cases:*

(i) Suppose $\sigma(f_1) = h_1$ *and* $\sigma(f_2) = h_2$*. Then,* f_1*'s utility is* $\frac{1}{2} \times 0.5 = 0.25$ *when choosing* h_1 *and* 0.4 *when choosing* h_2*. Thus,* $\sigma(f_1) = h_1$ *implies* $p_2 - p_1 \geq 0.15$*. Also,* f_2*'s utility is* 0.1 *when choosing* h_1 *and* $\frac{1}{2} \times 0.3 = 0.15$ *when choosing* h_2*. Thus,* $\sigma(f_2) = h_2$ *implies* $p_2 - p_1 \leq 0.05$*, a contradiction.*

(ii) Now suppose $\sigma(f_1) = h_2$ *and* $\sigma(f_2) = h_1$*. Then,* f_1*'s utility is* 0.5 *when choosing* h_1 *and* 0.4 *when choosing* h_2*. Thus,* $\sigma(f_1) = h_2$ *implies* $p_1 - p_2 \geq 0.1$*. Also,* f_2*'s*

utility is 0.1 *when choosing* h_1 *and* 0.3 *when choosing* h_2. *Thus,* $\sigma(f_2) = h_1$ *implies* $p_1 - p_2 \leq -0.2$, *a contradiction.*

Theorem 3 shows that aggregate welfare is unambiguously larger under DN than under NA. However, this does not necessarily mean that all families prefer DN to NA. Our next example demonstrates this fact. To keep things simpler, we consider a discrete economy.

Example 6. *There are two families* $F = \{f_1, f_2\}$, *two neighborhoods* $H = \{h_1, h_2\}$ *and two schools* $S = \{s_1, s_2\}$, *each with a unit capacity.*

Valuations are given in Table 6.

	(h_1, s_1)	(h_1, s_2)	(h_2, s_1)	(h_2, s_2)
f_1	0.2	0.1	0.1	0
f_2	0.4	0.9	0	0.5

Table 8: Valuations

It is easy to verify that $\sigma^{NA}(f_1) = h_1, \sigma^{NA}(f_2) = h_2$ *and* $p^{NA} = (0, 0)$ *is a CE of NA and* $p^{DN} = (0.2, 0), \sigma^{DN}(f_1) = h_2$ *and* $p^{DN} = (0.2, 0)$ *is a CE of DN. Moreover,*

$$u_{f_1}^{NA}\big(\sigma^{NA}(f_1), \sigma^{NA}\big) = 0.2 > 0.1 = u_{f_1}^{DN}\big(\sigma^{DN}(f_1), \sigma^{DN}\big),$$

which shows that family f_1 *prefers NA to DN.*

C Multiple Tie-breaking

Consider the model in Section 3. We first define the Deferred Acceptance mechanisms for multiple tie-breaking. With abuse of terminology, we maintain the names DA and DN.

C.1 School Assignment Mechanisms

Deferred Acceptance without Neighborhood Priority (DA).

School assignment under DA is determined based on families' preferences, school-specific lottery numbers and market clearing cutoffs, or simply cutoffs. Preferences are decided by neighborhood choices through equation 1. School-specific lottery numbers are drawn uniformly and independently from the unit interval. Formally, neighborhood choices τ result in a probability measure G_τ over $P \times [0,1]^{|S|}$, given by

$$G_\tau\Big((\succ, r) \in P \times [0,1]^{|S|} : \succ \in P', r_s \in (r_{s0}, r_{s1}), \forall s \in S\Big)$$
$$= \tau\Big((v,h) \in V \times \bar{H} : \succ_{vh} \in P'\Big) \times \prod_{s \in S}\Big(r_{s1} - r_{s0}\Big),$$

for each $P' \in P$ and $(r_{s0}, r_{s1}) \subseteq [0,1]$. Thus, $G_\tau\Big((\succ, r) \in P \times [0,1] : \succ \in P', r_s \in (r_{s0}, r_{s1}), \forall s \in S\Big)$ equals the mass of types with preferences in P' and school s lottery numbers in the interval (r_{s0}, r_{s1}) for each $s \in S$.

Cutoffs are derived through an iterative procedure that we describe below. For a vector $c \in [0,1]^{|S|}$, the demand function $D : [0,1]^{|S|} \to [0,1]$ is given by

$$D_s(c) = G_\tau\Big((\succ, r) \in P \times [0,1]^{|S|} : r_s \geq c_s \text{ and } s \succ s' \text{ for all } s' \text{ with } r_{s'} \geq c_{s'}\Big).$$

In words, $D_s(c)$ is the mass of families whose school s lottery numbers exceed c_s, and who prefer s to any other school s' where their school s' lottery numbers exceed $c_{s'}$. For $c \in [0,1]^{|S|}$ and $x \in [0,1]$ we denote by $c(s,x) \in [0,1]^{|S|}$ the vector that differs from c only by that $c_s(s,x) = x$.

We define a sequence of vectors $(c^t)_{t=1}^{\infty}$ recursively by $c^1 = 0$ and

$$c_s^{t+1} = \begin{cases} 0 & \text{if } D_s(c^t) < q_s, \\ \min\Big\{x \in [0,1] : D_s\big(c^t(s,x)\big) \leq q_s\Big\} & \text{otherwise.} \end{cases}$$

As shown by Abdulkadiroğlu et al. (2017b), $(c^t)_{t \in \mathbb{N}}$ is convergent. Let $c^{DA} := \lim_{t \to \infty} c^t$ denote the **DA cutoffs**.

The DA cutoffs determine school assignment as follows. A family is assigned to school s if her school s lottery number exceeds c_s^{DA}, and she prefers s to any school where her school-specific lottery number exceeds the corresponding DA cutoff. The probability of this event is

$$\lambda_{vs}^{DA}(h,\tau) = \prod_{s':s'\succ s} c_{s'}^{DA} \times (1 - c_s^{DA}). \tag{28}$$

The first term in the right-hand side of equation 28 is the probability that the family does not clear the cutoffs at choices more preferred to s, and the second term is the probability that the family clears the school s cutoff.

Deferred Acceptance with Neighborhood Priority (DN).

Under DN, school assignment is determined based on families' preferences, school-specific lottery numbers, priorities and cutoffs. Again, preferences are decided by neighborhood choices through equation 1 and school-specific lottery numbers are drawn uniformly and independently from the unit interval. Families receive priority 1 at neighborhood schools and priority 0 at non-neighborhood ones. Formally, neighborhood choices τ result in a probability measure G_τ on $P \times S \times [0,1]^{|S|}$ satisfying

$$\begin{aligned} G_\tau\Big((\succ, s, r) \in P \times S \times [0,1]^{|S|} : \succ \in P', s \in S', r_{s'} \in (r_{s'0}, r_{s'0}), \forall s' \in S\Big) \\ = \tau\Big((v,h) \in V \times \bar{H} : \succ_{v,h} = \succ, s_h \in S'\Big) \times \prod_{s \in S} \Big(r_{s1} - r_{s0}\Big), \end{aligned}$$

for each $P' \subseteq P$, $S' \subseteq S$ and $(r_{s0}, r_{s1}) \subseteq [0,1]$. For a vector $c \in [0,2]^{|S|}$ the demand function $D : [0,2]^{|S|} \to [0,1]$ is given by

$$\begin{aligned} D_s(c) = G_\tau\Big((\succ, s', r) \in P \times S \times [0,1]^{|S|} : r_s + \mathbb{1}[s' = s] \geq c_s \text{ and} \\ s \succ s'' \text{ for all } s'' \text{ with } r_{s''} + \mathbb{1}[s' = s''] \geq c_{s''}\Big). \end{aligned}$$

Consider the sequence of vectors recursively defined by

$$c_s^{t+1} = \begin{cases} 0 & \text{if } D_s(c^t) < q_s \\ \min\Big\{x \in [0,2] : D_s\big(c^t(s,x)\big) \leq q_s\Big\} & \text{otherwise} \end{cases}$$

and let $c^{DN} := \lim_{t\to\infty} c^t$ denotes the **DN cutoffs**. A family is assigned to school s if her priority at s plus her school s lottery number exceeds c_s^{DN}, and she prefers s to any school where her priority plus the school-specific lottery number exceeds the corresponding DN cutoff.

For a school s with $c_s^{DN} > 1$ is equal to

$$\lambda_{vs}^{DN}(h,\tau) = \begin{cases} 0 & s_h \neq s, \\ \prod_{s':s' \succ s} \min\left\{c_{s'}^{DN}, 1\right\} \times \left(2 - c_s^{DN}\right) & \text{otherwise.} \end{cases}$$

For a school s with $c_s^{DN} \leq 1$,

$$\lambda_{vs}^{DN}(h,\tau) = \begin{cases} \prod_{s':s' \succ s, s' \neq s_h} \min\left\{c_{s'}^{DN}, 1\right\} \times \left(c_{s_h}^{DN} - 1\right) \times c_s^{DN} & s_h \succ_{vs} s, \\ \prod_{s':s' \succ s} \min\left\{c_{s'}^{DN}, 1\right\} & s_h = s \\ \prod_{s':s' \succ s} \min\left\{c_{s'}^{DN}, 1\right\} \times c_s^{DN} & \text{otherwise.} \end{cases}$$

C.2 Results for Multiple-tie Breaking

Like in the single tie-breaking case, school assignment probabilities under multiple tie-breaking are continuous in the cutoffs. Therefore, Theorem 2 holds for this latter setting, too, and the proof is (almost) identical to its single-tie breaking analog. Theorem 5, Corollaries 1 and 2 and Theorem 7 are also analogously proved.

The remaining results and examples require modified proofs for multiple tie-breaking. However, to avoid unnecessarily lengthy discussions, we are agnostic about extending those to the multiple tie-breaking case.

Supplementary Appendix

A Alternative School Assignment Mechanisms

A.1 Overview

This section studies two additional school assignment mechanisms: Top Trading Cycles (TTC) and Immediate Acceptance (IA).

The TTC mechanism has been originally formulated by Shapley and Scarf (1974) and has been introduced for public school assignment by Abdulkadiroğlu and Sönmez (2003). For our continuum economy model, we use the formulation of TTC developed by Leshno and Lo (2021). We show that most of our results on how Deferred Acceptance mechanism compares to neighborhood assignment in terms of welfare also hold for TTC.

The IA mechanism, also known as the 'Boston' mechanism, is widely applied for public school admissions in the US and around the world. The mechanism has often been criticized on the grounds of being manipulable, i.e., families have incentives to misreport their true preferences to improve their school assignments (Abdulkadiroğlu and Sönmez, 2003). Despite this potential shortcoming, it is also known that IA may improve families' welfare as it allows to 'signal' their valuations by preference manipulation. For example, Abdulkadiroğlu, Che, and Yasuda (2011) show that in a setting without neighborhood priorities and where families have common ordinal preference rankings over schools, all families prefer any (symmetric Bayesian) equilibrium outcome of IA to the outcome of DA. This result does not extend to the setting with neighborhood priorities. An important observation in our analysis of IA is that families who reside in the neighborhoods of the least preferred schools may be worse-off under IA with neighborhood priorities compared to DN. The reason is that when the more preferred schools are sufficiently demanded, the families in the neighborhood of the least preferred schools have no 'safe option' other than the least preferred schools. Therefore, under IA with neighborhood priorities they may find it optimal to rank a moderate school as a first choice to avoid the possibility of being rejected by all higher ranked choices and being assigned to their preferred one. As a consequence, those families are worse off. This observation is analogous to the ones in Calsamiglia, Martínez-Mora, and Miralles (2015) and Neilson, Akbarpour, Kapor, van Dijk, and Zimmerman (2020). Those papers demonstrate that families without outside options may prefer DA to IA. We do not explicitly model outside options, but because of neighborhood priorities, in our setting neighborhood schools correspond to outside options.

In what follows we talk about two versions of TTC and IA: one where families do not receive higher priorities at neighborhood schools, and one where they do so. When it is clear from the context, we do not mention which version of the mechanism we are talking about.

A.2 TTC

A.2.1 Overview

Consider the model in Section 3.

TTC without neighborhood priorities.

Neighborhood choices $\tau \in \mathcal{T}$ uniquely determine a probability measure G_τ over $P \times [0,1]$, satisfying

$$G_\tau\Big((\succ, r) \in P \times [0,1] : \succ \in P', r \in (r_0, r_1)\Big)$$
$$= \tau\Big((v,h) \in V \times \bar{H} : \succ_{vh} \in P'\Big) \times \Big(r_1 - r_0\Big),$$

for each $P' \subseteq P$ and $(r_0, r_1) \subseteq [0,1]$.

For the resulting measure G_τ the TTC assignment (without neighborhood priorities) is found by the procedure given by Leshno and Lo (2021). We omit the technical details for the sake of brevity. In a nutshell, TTC assignment is determined by cutoffs $c^{TTC} := (c^{TTC}_{ss'})_{s,s' \in S} \in [0,1]^{|S|^2}$, such that a family is assigned to school s if and only if her lottery number is larger than $\min_{s' \in S} c^{TTC}_{s's}$ and she prefers s to any school $s'' \in S \setminus \{s\}$ such that her lottery number is larger than $\min_{s' \in S} c^{TTC}_{s's''}$. Let $N := |S|$ and suppose the schools are indexed as follows,

$$\min_{s \in S} c^{TTC}_{ss_i} > \min_{s \in S} c^{TTC}_{ss_j} \text{ if and only if } i > j$$

. Also, let $c^{TTC}_{ss_0} := 0, \forall s \in S$. Then, it follows from the TTC description by Leshno and Lo (2021) that the cutoffs c^{TTC} should satisfy

$$\sum_{i=1}^{k} G_\tau\Big((\succ, r) \in P \times [0,1] : \; s_k \succ s, \forall s \in S \backslash \{s_i, \ldots, s_N\}, r \in \big[\min_{s \in S} c^{TTC}_{ss_{i-1}}, \min_{s \in S} c^{TTC}_{ss_i}\big)\Big) \leq q_{s_k},$$

and the equation holds with equality whenever $\min_{s \in S} c^{TTC}_{ss_k} > 0$. It then follows from the description of DA cutoffs c^{DA}, that $c^{DA}_s = \min_{s' \in S} c^{TTC}_{s's}$ for all $s \in S$. Thus, we obtain the following equivalence result.

Proposition 1. *For any $v \in V, h \in H$ and $\tau \in \mathcal{T}$,*

$$u^{DA}_v(h, \tau) = u^{TTC}_v(h, \tau).$$

The result extends an earlier finding about the equivalence of DA (random serial dictatorship) and TTC (core from random endowments) by Abdulkadiroğlu and Sönmez

(1998) to the continuum one-to-many matching model. To the best of our knowledge, out Proposition 1 is the first documentation of this observation.

TTC with neighborhood priorities.

Neighborhood choices τ uniquely determine a probability measure G_τ on $P \times S \times [0,1]$, satisfying

$$G_\tau\Big((\succ, s, r) \in P \times S \times [0,1] : \succ \in P', s \in S', r \in (r_0, r_1)\Big)$$
$$= \tau\Big((v,h) \in V \times \bar{H} : \succ_{v,h} = \succ, s_h \in S'\Big) \times \Big(r_1 - r_0\Big),$$

for each $P' \subseteq P$, $S' \subseteq S$ and $(r_0, r_1) \subseteq [0,1]$.

For the resulting measure G_τ the TTC assignment (with neighborhood priorities) is given by cutoffs $c^{TTC} := (c^{TTC}_{ss'})_{s,s' \in S} \in [0,2]^{|S|^2}$, such that a family, choosing neighborhood of school s', is assigned to school s if and only if her lottery number plus $\mathbb{1}[s' = s]$ is larger than $\min_{s'' \in S} c^{TTC}_{s''s}$ and she prefers s to any school $s''' \in S \setminus \{s\}$ such that her lottery number plus $\mathbb{1}[s' = s''']$ is larger than $\min_{s'' \in S} c^{TTC}_{s''s'''}$.

Therefore, it follows from the TTC description by Leshno and Lo (2021) that the cutoffs c^{TTC} should satisfy

$$\sum_{i=1}^{k} G_\tau\Big((\succ, s', r) \in P \times S \times [0,1] : \; s_k \succ s, \forall s \in S \setminus \{s_i, \ldots, s_N\},$$
$$r \in \big[\min_{s \in S} c^{TTC}_{ss_{i-1}}, \min_{s \in S} c^{TTC}_{ss_i}\big) \cup \big[\max\{0, c^{TTC}_{s's_{i-1}}\}, \max\{0, c^{TTC}_{s's_i}\}\big)\Big)$$
$$\leq q_{s_k},$$

and the equation holds with equality whenever $\min_{s \in S} c^{TTC}_{ss_k} > 0$.

When there are neighborhood priorities, TTC is no longer equivalent to the Deferred Acceptance mechanism.[1] However, the equivalence holds for a special case of our

[1] In fact, Calsamiglia and Miralles (2020) show that with neighborhood priorities TTC may be a better alternative to Deferred Acceptance with regard to providing better access to non-neighborhood schools. Unlike in our work, the authors take neighborhood choices as exogenously given.

problem, where families have common ordinal preference rankings over schools. This observation is important for establishing some of the further results.

Proposition 2. *Let* $S = \{s_i\}_{i=1}^{|S|}$ *and suppose there is a* $V' \subseteq V$ *with* $\eta(V') = 1$*, such that* $v(h, s_i) \geq v(h, s_j)$ *for all* $v \in V, h \in H$ *and* $s_i, s_j \in S, i \geq j$*. Then, for any* $v \in V, h \in H$ *and* $\tau \in \mathcal{T}$*,*

$$u_v^{DN}(h, \tau) = u_v^{TTC}(h, \tau).$$

We now discuss which results established for the Deferred Acceptance mechanism extend to TTC.

The proofs of Theorem 3 and Theorem 5 directly apply to TTC. Moreover, Assumptions 1 and 2 imply same ordinal rankings. Therefore, it is immediate from Propositions 1 and 2, that Theorem 4, Corollary 1 and 2 apply to TTC.

A.3 IA

Unlike the Deferred Acceptance and TTC, the IA mechanism is not strategyproof, i.e., truthfully reporting preferences is not a dominant strategy for families. Since preferences are typically unknown to the central planner, it is realistic to extend the model to allow families to choose not only where to reside, but also what preference ranking to report. Therefore we model families choices τ as a (Borel) probability measure over $V \times \bar{H} \times \mathcal{P}$. Let $\mathcal{T}$ be the space of such measures.

For a given mechanism ϕ and choices τ, we denote by $\lambda_{vs}^{\phi}(h, \succ, \tau) \in [0, 1]$ the probability that type v is assigned to school s conditional on choosing neighborhood h and submitting a preference ranking $\succ$. Later in this section, we derive school assignment probabilities for IA with or without neighborhood priorities. Before that, we define competitive equilibrium in this extended model.

Given school assignment probabilities and neighborhood price vector $p \in [0, 1]^{|H|}$, the expected utility of type v choosing neighborhood $h \in H$ and submitting preference ranking $\succ$ is equal to

$$u_v^\phi(h, \succ, \tau) - p_h.$$

where $u_v^\phi(h, \succ, \tau) := \sum_{s \in S} \lambda_{vs}^\phi(h, \succ, \tau)\ v(h, s)$. Also, let $u_v^\phi(0, \succ, \tau) := 0$ for all $v \in V, \succ \in P$ and $\tau \in \mathcal{T}$.

Definition 1. *For neighborhood choices $\tau \in \mathcal{T}$ and price vector $p \in \mathbb{R}_+^{|H|}$, we say a pair (τ, p) is a **competitive equilibrium (CE)** of mechanism ϕ if it satisfies the following conditions:*

1. $\tau\Big((v, h, \succ) \in V \times \bar{H} \times P : h = \arg\max_{h' \in \bar{H}}\ u_v^\phi(h', \succ, \tau) - p_{h'}\Big) = 1$, *where* $p_0 := 0$,
2. $\tau\Big((v, h, \succ) \in V \times \bar{H} \times P : h = h'\Big) \leq q_{h'}, \forall h' \in H$,
3. $\tau\Big((v, h, \succ) \in V \times \bar{H} \times P : h = h'\Big) < q_{h'} \Rightarrow p_{h'} = 0$.

We now derive school assignment probabilities two versions of IA mechanism. To the best of our knowledge, this is the first description of IA for the continuum economy model.

IA without neighborhood priorities.

Neighborhood choices $\tau \in \mathcal{T}$ uniquely determines a probability measure G_τ over $P^2 \times [0, 1]$, given by

$$G_\tau\Big((\succ, \succ', r) \in P^2 \times [0,1] : \succ \in P', \succ' \in P'', r \in (r_0, r_1)\Big)$$
$$= \tau\Big((v, h, \succ') \in V \times \bar{H} \times P : \succ_{vh} \in P', \succ' \in P''\Big) \times \Big(r_1 - r_0\Big),$$

for each $P', P'' \subseteq P$ and $(r_0, r_1) \subseteq [0, 1]$.

For any $\succ \in P$ and $s \in S$, let $rk_\succ(s)$ denote the rank of school s in the preference ranking P in reverse order (i.e., $rk_\succ(s) = |S|$ when s the highest ranked according to $\succ$, and $rk_\succ(s) = 1$ if it is the lowest ranked).

Like with the Deferred Acceptance mechanism, the IA assignment can be given by cutoffs. Cutoffs are derived through an iterative procedure that we describe below. For a vector $c \in [1, |S|+1]^{|S|}$, the demand function $D : [1, |S|+1]^{|S|} \to [0, 1]$ is given by

$$D_s(c) = G_\tau\Big((\succ, \succ', r) \in P^2 \times [0,1] : r + rk_{\succ'}(s) \geq c_s \text{ and } s \succ s' \text{ for all } s' \text{ with } r + rk_{\succ'}(s') \geq c_{s'}\Big).$$

In other words, we may think of families having scores at schools which equals their lottery number plus the ranking of the school in their reported preferences. Thus, in this way families receive higher 'priorities' at IA when they rank it higher. Then, $D_s(c)$ is the mass of families whose scores exceed c_s, and who prefer s to any other school s' where their scores exceed $c_{s'}$. For $c \in [0, 1]^{|S|}$ and $x \in [1, |S|+1]$ we denote by $c(s, x) \in [1, |S|+1]^{|S|}$ the vector that differs from c only by that $c_s(s, x) = x$.

We define a sequence of vectors $(c^t)_{t=1}^\infty$ recursively by $c^1 = 0$ and

$$c_s^{t+1} = \begin{cases} 0 & \text{if } D_s(c^t) < q_s, \\ \min\Big\{x \in [0,1] : D_s\big(c^t(s,x)\big) \leq q_s\Big\} & \text{otherwise.} \end{cases}$$

It follows from similar arguments as in Abdulkadiroğlu, Angrist, Narita, and Pathak (2017), that the sequence $(c^t)_{t \in \mathbb{N}}$ is convergent. Let $c^{IA} := \lim_{t \to \infty} c^t$ denote the **IA cutoffs**.

For cutoffs c^{IA} and preference ranking $\succ$, let $\bar{s}$ denote the most preferred school with $rk(\underline{s}) \geq c_{\underline{s}}^{IA}$. Also, let $\bar{S} \subseteq S$ be the largest set such that for each $s \in \bar{S}$, $s \succ \bar{s}$ or $s = \bar{s}$ and $rk(s) \geq c_s^{IA} - 1$. Then, the probability that type v is assigned to school s when choosing neighborhood h and reporting preference ranking $\succ$ is equal to $\lambda_{vs}^{IA}(h, \succ, \tau) = 0$ if $s \notin \bar{S}$, and otherwise,

$$\lambda_{vs}^{IA}(h, \succ, \tau) = \max\Big\{0, \min\Big\{c_{s'}^{IA} : s' \succ s, s' \in \bar{S}\Big\} - c_s^{DA}\Big\}.$$

IA with neighborhood priorities.

Neighborhood choices $\tau \in \mathcal{T}$ uniquely determines a probability measure G_τ over $P^2 \times S \times [0,1]$, given by

$$G_\tau\Big((\succ, \succ', s, r) \in P^2 \times S \times [0,1] : \succ \in P', \succ' \in P'', s \in S', r \in (r_0, r_1)\Big)$$
$$= \tau\Big((v, h, \succ') \in V \times \bar{H} \times P : \succ_{vh} \in P', s_h \in S', \succ' \in P''\Big) \times \Big(r_1 - r_0\Big),$$

for each $P', P'' \subseteq P$, $S' \subseteq S$ and $(r_0, r_1) \subseteq [0,1]$.

For a vector $c \in [1, 2(|S|+1)]^{|S|}$ consider the demand function $D : [1, 2(|S|+1)]^{|S|} \to [0,1]$ given by

$$D_s(c) = G_\tau\Big((\succ, \succ', s', r) \in P^2 \times S \times [0,1] : r + 2rk_{\succ'}(s) + \mathbb{1}[s' = s] \geq c_s$$
$$\text{and } s \succ s'' \text{ for all } s'' \text{ with } r + 2rk_{\succ'}(s'') + \mathbb{1}[s' = s''] \geq c_{s'}\Big).$$

Define a sequence of vectors $(c^t)_{t=1}^\infty$ recursively by $c^1 = 0$ and

$$c_s^{t+1} = \begin{cases} 0 & \text{if } D_s(c^t) < q_s, \\ \min\Big\{x \in [0,1] : D_s\big(c^t(s,x)\big) \leq q_s\Big\} & \text{otherwise,} \end{cases}$$

and let $c^{IA} := \lim_{t\to\infty} c^t$ be the IA cutoffs.

Again, for cutoffs c^{IA}, preference ranking $\succ$ and neighborhood choice h, let $\bar{s}$ denote the most preferred school with $2rk(\underline{s}) + \mathbb{1}[s_h = \underline{s}] \geq c_{\underline{s}}^{IA}$. Also, let $\bar{S} \subseteq S$ be the largest set such that for each $s \in \bar{S}$, $s \succ \bar{s}$ or $s = \bar{s}$ and $rk(s) + \mathbb{1}[s_h = s] \geq c_s^{IA} - 1$. Then, the probability that type v is assigned to school s when choosing neighborhood h and reporting preference ranking $\succ$ is equal to $\lambda_{vs}^{IA}(h, \succ, \tau) = 0$ if $s \notin \bar{S}$, and otherwise,

$$\lambda_{vs}^{IA}(h, \succ, \tau) = \max\Big\{0, \min\Big\{c_{s'}^{IA} : s' \succ s, s' \in \bar{S}\Big\} - c_s^{IA}\Big\}.$$

We briefly discuss how some of the results we established for the Deferred Acceptance mechanism extend to the setting with IA. Theorem 3 applies to IA with neighborhood priorities since a family can guarantee a neighborhood school by ranking it as a first choice. Since lowest-income families can guarantee underdemanded neighborhoods and schools for both versions of IA, Theorem 5 applies to IA as well.

We finish this section by discussing how IA compares to DA in terms of families' welfare. When there are no neighborhood priorities and families have identical ordinal preferences over schools, Abdulkadiroğlu et al. (2011) show that all families prefer IA to DA. We illustrate by an example that this is not necessarily the case when there are neighborhood priorities. In what follows we use IA to denote the version with neighborhood priorities.

Example 1. *There are three neighborhoods* $H = \{h_1, h_2, h_3\}$ *and three schools* $S = \{s_1, s_2, s_3\}$*, where* $q_{h_1} = 2$ *and* $q_{h_2} = q_{h_3} = 0.4$*,* $q_{s_1} = 0.4$ *and* $q_{s_2} = q_{s_3} = 0.58$*. Economy* η *is supported at only three points* v_1, v_2 *and* v_3*, with*

$$\eta\Big(v \in V : v = v_1\Big) = 0.2$$

and

$$\eta\Big(v \in V : v = v_2\Big) = \eta\Big(v \in V : v = v_3\Big) = 0.4,$$

where v_1, v_2 *and* v_3 *are shown in Table 1.*

	(h_1, s_1)	(h_1, s_2)	(h_1, s_3)	(h_2, s_1)	(h_2, s_2)	(h_2, s_3)	(h_3, s_1)	(h_3, s_2)	(h_3, s_3)
v_1	0.95	0.9	0.8	0.15	0.1	0	0.15	0.1	0
v_2	0.95	0.9	0.8	0.95	0.9	0.8	0.15	0.1	0
v_3	0.15	0.1	0	0.15	0.1	0	0.15	0.1	0

Table 1: Valuations

There is a CE of DN, where

- $p^{IA}_{h_1} = 0.5$*,* $p^{IA}_{h_2} = 0.2$ *and* $p^{IA}_{h_3} = 0$*,*

- *all type* v_1 *families choose neighborhood* h_1 *and submit their true preference rankings,*
- *all type* v_2 *families choose neighborhood* h_2 *and submit their true preference rankings,*
- *all type* v_3 *families choose neighborhood* h_3 *and submit preference ranking* $s_2 \succ s_1 \succ s_3$*.*

and a CE of IA, where

- $p_{h_1}^{IA} = 0.5$*,* $p_{h_2}^{IA} = 0.2$ *and* $p_{h_3}^{IA} = 0$*,*
- *all type* v_1 *families choose neighborhood* h_1 *and submit their true preference rankings,*
- *all type* v_2 *families choose neighborhood* h_2 *and submit their true preference rankings,*
- *all type* v_3 *families choose neighborhood* h_3 *and submit preference ranking* $s_2 \succ s_1 \succ s_3$*.*

The expected utility of type v_3 *under IA is* 0.1*, whereas, under DN her expected utility is*

$$\frac{1}{4} \times 0.15 + \frac{3}{4} \times \frac{58}{60} \times 0.1 > 0.1.$$

Thus, when there are neighborhood priorities, families in less preferred neighborhoods may prefer DN to IA.

B Aggregate Welfare and (In)Efficiency

B.1 Overview

None of the studied assignment mechanisms maximizes aggregate welfare. The goal of this section is to quantify the inefficiencies admitted by DN, DA and NA by comparing them to two benchmarks: (1) welfare maximizing assignment (first best), and (2) welfare maximizing stable assignment. All subsequent analysis builds on the discrete economy model of Section 6.1.

B.2 Results

Our first benchmark is the (aggregate) **welfare maximizing assignment**. A joint neighborhood-school assignment of families is given by a mapping $\mu : F \to H \times S$, satisfying

- $\sum_{s \in S} \left|\mu^{-1}(h, s)\right| \leq q_h, \forall h \in H,$
- $\sum_{h \in H} \left|\mu^{-1}(h, s)\right| \leq q_s, \forall s \in S.$

Let $\mathcal{M}$ denote the set of all assignments. We say assignment μ^* is welfare maximizing assignment if it solves,

$$\mu^* = \arg\max_{\mu \in \mathcal{M}} \sum_{f \in F} v_f(\mu(f)).$$

When families valuations for joint neighborhood-school assignment take values of either zero or one, finding welfare maximizing assignment reduces to the NP-complete 3-dimensional matching problem (Karp, 1972). Therefore, finding a welfare maximizing assignment is NP-hard problem. The problem is tractable in the special case where families' valuations for neighborhood schools are additively separable.

Our second benchmark is the **welfare maximizing stable assignment**. When the school district grants neighborhood priorities, stability (also known as elimination of justified-envy) requires that a family is assigned to a school she prefers less than her neighborhood school only if the latter does not admit any family residing outside of the school's neighborhood. Formally, assignment μ is stable if there are no families $f, f' \in F$, such that $\mu(f) = (h, s)$, $\mu(f') = (h', s_h)$, $v_f(h, s_h) > v_f(\mu(f))$ and $h' \neq h$. To simplify the analysis, we consider additively separable valuations for neighborhoods and schools. Moreover, instead of maximizing welfare in the entire set of stable assignments, we first fix families' neighborhood choices $\sigma^* : F \to \bar{H}$ to maximize the sum of neighborhood valuations, and then maximize aggregate welfare in the set of stable assignments that 'agree' with σ^*, i.e., for all $f \in F$, there is an $s \in S$, such that $\mu(f) = (\sigma^*(f), s)$.

Even with the simplifications above, finding a welfare maximizing stable assignment is an NP-hard problem. However, we solve this problem in our simulated environment using the methodology developed by Abdulkadiroğlu, Dur, and Grigoryan (2021). The authors provide an algorithm that is polynomial time in the number of students, but potentially exponential time in the number of schools. Since the number of school districts is typically much smaller than the number of students, the algorithm is tractable for real-life problems.

In the remainder of this section, we compare welfare across assignment mechanisms through simulations. There 1000 students, 10 neighborhoods and 10 schools. The valuation of family f for the joint assignment to neighborhood h and school s is equal to

$$v_f(h, s) = \alpha U_h + (1 - \alpha) U_s + 0.5\epsilon_{fh} + 0.5\epsilon_{fs},$$

where

- U_h and U_s are the common valuation for neighborhood h and schools s, respectively,

- ϵ_{fh} and ϵ_{fs} are the idiosyncratic valuations of family f for neighborhood h and schools s, respectively,

- α is a parameter.

Values of U_h, U_s, ϵ_{fh} and ϵ_{fs} are iid uniform draws from the unit interval. The capacity of school s is $100 + \kappa_s$, where κ_s is a random draw from the set $\{1, 2, ..., 100/\gamma\}$. We report results for the following values for our parameters: $\alpha \in \{0, 0.5, 1\}$ and $\gamma \in \{2.4\}$.

α	γ	DN	DA	WM	WMS
0	2	5.28	−7.47	14.15	12.04
	4	3.15	−13.79	10.85	8.19
0.5	2	4.73	0.33	10.28	9.45
	4	3.28	−2.56	8.79	7.74
1	2	7.15	7.15	7.17	7.17
	4	7.14	7.14	7.17	7.17
Average		5.12	−1.58	9.74	8.63

Table 2: Aggregate welfare, % gains/losses compared to NA

The last two columns show the percentage gains in aggregate welfare from the welfare maximizing assignment and welfare maximizing stable assignment compared to NA. The numbers are 9.74% and 8.63%, respectively. Those gains are less than twice as large as those under DN. Therefore, DN eliminates more than half of the inefficiency admitted by NA (and DA).

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