Persistent Debt and Business Cycles in an Economy with Production Heterogeneity

Aubhik Khan
Ohio State University

Soyoung Lee[1]
Bank of Canada

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ABSTRACT

We study an economy with a time-varying distribution of production to examine the role of debt in amplifying and propagating recessions. In our model, entrepreneurs use risky, long-term debt to finance capital. Liquid assets serve as collateral and transaction costs make debt illiquid.

Debt payments increase the volatility of earnings relative to output, deterring entrepreneurs with insufficient collateral from financing efficient levels of capital. This results in a misallocation of resources.

In a large recession, productive entrepreneurs with high levels of debt deleverage, amplifying the downturn. The model economy exhibits asymmetries over the business cycle. Recessions involve a rapid deterioration of economic activity while expansions are more gradual.

When a recession coincides with a rise in leverage resulting from a fall in assets, fewer producers operate at efficient levels. When aggregate business leverage is ten percentage points above average, the half-life of the recovery doubles.

Keywords: Business cycles, Production heterogeneity, Business debt, Large recessions

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[1]Aubhik Khan: mail@aubhik-khan.net; Soyoung Lee: soyoung.lee.n@gmail.com. We thank Matthias Gnewuch and the participants of the 2021 International Journal of Central Banking and Research conference, the 2021 Oxford workshop on Firm Heterogeneity and the Macroeconomy, the Midwest Macro May 2022 Meetings, and the BSE summer forum 2022. This work previously circulated under the title, 'The Aftermath of the Pandemic'. Any views expressed are our own and not those of the Bank of Canada.

1 Introduction

Debt is large in the balance sheets of US firms. Between 1954 and 2006, aggregate non-financial business leverage averaged 36 percent. Over the Great Recession, it rose by 11 percentage points, remaining near this higher level through the start of the Pandemic. The availability of debt is widely held to be an important determinant of the efficiency with which resources are allocated. Conversely, high levels of existing debt increase the risks associated with further lending, hindering investment.

Beyond the idiosyncratic determinants of default, several channels affect the costs of borrowing. Inflation changes the value of nominal debt causing debt overhang (Gomes et al. (2016)). Fluctuations in the price of collateral constrain new loans (Kiyotaki and Moore (1997)).[1] We focus on a novel channel involving the effect of long-term debt on income. As recurring debt principal repayments increase income volatility, firms reduce their debt following negative income shocks. As income-generating assets are diverted to paying off existing debt, income falls further.

This debt reduction channel has substantive implications for the business cycle. In recessions, firms' efforts to reduce debt amplify aggregate shocks and deepen the downturn. When, at the onset of a recession, average leverage is high, the recession is more severe and the recovery that follows, slower.

We develop a quantitative framework to study the interaction between firm debt and aggregate quantities over the business cycle. Our analysis involves an incomplete markets model where a time varying measure of entrepreneurs produce alongside competitive, risk-neutral firms. Entrepreneurs operate decreasing returns to scale technologies using capital and labor. Capital is rented subject to a collateral constraint. A producer's financial assets serve as collateral for these short-term loans. When an entrepreneur's collateral constraint binds, she may use long-term debt to mitigate its effect. Such loans are costly to adjust, hence illiquid. Moreover, entrepreneurs are subject to idiosyncratic credit shocks which, at times, make new loans unavailable. Finally, borrowers must continuously repay part of their principal balance. Overall, the economy is characterised by a distribution of entrepreneurs that vary in their access to loans and idiosyncratic productivity as well as assets and debt, and a distribution of workers over labor productivity and assets.

Several original elements of our environment distinguish it from existing work on production heterogeneity and debt. Compared to models with risk-neutral firms (for example, Khan

[1] This list is not exhaustive. For example, debt dilution is examined, in sovereign default models, by Hatchondo et al. (2016), Aguiar and Amador (2020), and Chatterjee and Eyigungor (2013). Debt overhang is also studied in Myers (1977), Whited (1992), Philippon (2010), and Crouzet and Tourre (2021) among other papers.

and Thomas (2013)), our entrepreneurs have diminishing marginal utility and have savings rates that are non-linear functions of their assets and debt. In a recession, as the proportion of profits spent on principal payments rises, entrepreneurs use assets to pay off debt. Falling assets reduce output as collateral constraints lower capital. In the aggregate, deleveraging by firms in a large recession drives non-linearities absent in models with one-period debt and risk-neutral firms.

Relative to a model with risk averse entrepreneurs, for example Buera and Shin (2013), our entrepreneurs hold both assets and liabilities; these liabilities are both risky and long-term. Therefore loan rates increase in the level of debt given a borrower's financial assets. Furthermore, debt principal payments play a large role in amplifying the effects of financial constraints on aggregate quantities.

We calibrate the model economy to the size distribution of firms as well as the distribution of size over age. Using shocks to the availability of new loans alongside our equilibrium loan rate schedules and the collateral constraint on capital rentals, we are able to reproduce the relative size of new and young firms. Additionally, we target the aggregate non-financial business debt-to-asset ratio as well as moments from the distribution of debt across firms. Further, the model is consistent with the overall exit rate of firms and targets the average default rate.

When matching the size distribution, we assume a set of large unconstrained risk-neutral firms that have no need for financial assets or debt. These firms limit the potential effect of financial constraints in our economy. Moreover, as more than 86 percent of large firms, with more than five hundred employees, are private, we do not assume that these firms correspond to listed firms in the data (Asker et al. (2014)). Neither do we assume all entrepreneurial production represents non-corporate firms.

While debt offers entrepreneurs the possibility of loosening their collateral constraints, loan repayments may lead to a rise or fall in net income. Even when borrowing raises expected income, entrepreneurs' risk aversion makes the choice of whether to borrow, and how much, non-trivial. Our stationary equilibrium implies far larger losses in aggregate total factor productivity, and GDP, from financial frictions than commonly found in models with production heterogeneity and costly borrowing, for example Khan and Thomas (2013) and Khan et al. (2018). Three properties of our model drive these differences. First, the total mass of entrepreneurs available for production is fixed, and inactive entrepreneurs are slow to enter production. Second, once active, entrepreneurs accumulate assets and increase production slowly. Third, producers that operate at a level of capital that is efficient, given their total factor productivity, may still adjust gradually to an increase in productivity. Below, we discuss these three properties that shape the distribution of production.

Inactive entrepreneurs that wish to start production must pay a cost of entry which is only partly-financed by borrowing. Hence entrepreneurs must accumulate sufficient funds to finance entry costs, and this is a slow process for producers with low resources and high marginal utility of consumption. This reduces the number of active producers which, given decreasing returns to scale, reduces aggregate total factor productivity.

Once an entrepreneur enters production, they grow slowly. This is another implication of diminishing marginal utility; consumption smoothing deters a rapid accumulation of assets. Furthermore, at times entrepreneurs suffer an idiosyncratic shock that leaves them unable to take on a new loan. Unwilling to save at very high rates, and unable to borrow, young firms tend to operate at levels of capital that are far below that associated with their efficient scale, given their productivity.

When they are able to borrow, producers are reluctant to take on large loans to finance efficient levels of investment. Large principal payments increase the volatility of consumption relative to gross income. This implies that when productivity rises, those with insufficient net assets will be slow to adjust capital. Both the slow growth of young firms and gradual increases in investment of poor firms imply an average distance between actual and efficient capital higher than in a model with risk-neutral firms.

In contrast to economies with costly borrowing and heterogeneous, risk-neutral firms, our environment propagates real shocks. We consider persistent, 7.5%, positive and negative total factor productivity shocks and study the dynamics of the economy. We find an asymmetry in how output responds to positive and negative shocks. In a recession, output falls faster and by 1.1 percentage points more than its rise in an expansion.

In a recession, when indebted entrepreneurs experience a loss in their income, debt principal payments lead to a disproportionate fall in their net earnings. Further income losses, from idiosyncratic productivity shocks, would imply a large fall in consumption, especially should they coincide with a credit shock that prevents access to new loans. To avoid default in such circumstances, entrepreneurs use assets to reduce their debt. The loss of assets reduces the capital they may finance for production, hence their income. Across the economy, entrepreneurs with high productivity tend to decrease their debt the most, and this lowers the endogenous component of aggregate total factor productivity. The debt reduction channel propagates a real shock to the economy.

While entrepreneurs' changes to their balance sheets amplify downturns, they gradualize expansions. Over such episodes, an unwillingness to commit a large fraction of income to debt payment deters large increases in borrowing and rapid asset growth. This is reinforced by entrepreneurs' aversion to sharp increases in savings that reduce consumption. Consequently, capital grows slowly, moderating the expansion.

Having seen that firms' deleveraging propagates a downturn, we turn to study the effects of high leverage in a recession. In particular, we examine a recession when non-financial business leverage is 10 percentage points higher than in the steady state of our model. We choose a 10 percentage point rise to capture the gap between the historical average and the level of leverage during the Great Recession and at the onset of the Pandemic recession. We model a rise in leverage resulting from a fall in assets, in keeping with the changes observed in the aggregate data (Figure 8).[2]

We cut entrepreneurs' assets to raise their leverage. To avoid any immediate effect on aggregate capital, we transfer the entire reduction in assets to the government. The government uses these resources to fund capital spending, and all interest income from their investment is lump-sum rebated to workers and entrepreneurs. We follow two economies that differ only in their initial aggregate leverage ratios, each experiencing the same 2.5% fall in TFP.

Compared to a recession that begins with ordinary levels of leverage, the high leverage recession shows an initial fall in output that is 20 percent larger. GDP remains below its level in the ordinary leverage economy 15 years later. Recovering from high-leverage recessions is very slow, the half-life of the recovery more than doubles, rising from 6 quarters to 13.

In the high-leverage recession, as their assets fall, more entrepreneurs enter the recession constrained. Compared to the ordinary recession, their output falls more and takes three times longer to return to its steady-state level.[3] As their income after debt payment falls, entrepreneurs reduce their savings to smooth consumption. Across constrained entrepreneurs, lower savings imply a longer time to overcome collateral constraints. The result is a long period of low output and profits.

The larger real effects of a TFP shock in the high-leverage recession arise not only through the intensive margin but also from the extensive margin. As more entrepreneurs enter the recession constrained, some of them raise debt to fund a partial recovery in their capital. The combination of more debt with a loss in earnings leads to higher bankruptcy and exit. This drives further reductions in real activity. As more firms default on their loans, the eventual fall in the number of producers is 3.5 percentage points larger than in the ordinary recession. This distributional effect is long-lived as defaulting entrepreneurs must go through a period where their credit history prevents them from borrowing to start a new enterprise.

Literature Our work is related to several strands of literature. First, we build on work

[2] The rapid rise of non-financial firms' leverage during the Great Recession is associated with a fall in assets rather than a rise in debt.

[3] The high leverage economy has a larger fall in wages and interest rates which leads to a faster recovery by risk-neutral firms, partly offsetting the effect of the slowdown by entrepreneurs.

using models with financial frictions and a time-varying distribution of firms. In contrast to models of entrepreneurship like Buera and Shin (2013), who focus on economic development, and Buera and Moll (2015), who look at recessions, entrepreneurs in our model have both assets and debt. Compared to business cycle models with risk-neutral firms (Khan and Thomas (2013) and Jo (2020)), we allow for entrepreneurs and risky long-term debt.[4]

Several papers look at the aggregate effects of firm-level debt in models with default. The majority of this work focuses on short-term debt (Arellano et al. (2019), Khan et al. (2021) and Ottonello and Winberry (2020)). Long-term debt is examined by Jungherr and Schott (2022), who focus on sluggish deleveraging during recessions, and Gomes et al. (2016), who study debt overhang arising through the effects of inflation on nominal debt. Both these papers study debt held by risk-neutral firms. Our model also has long-term debt, but held by risk-averse entrepreneurs. This amplifies the effects of leverage on the the business cycle.

Lastly, our work is related to the literature on asymmetries over the business cycle. A body of work has found that contractions are both sharper and shorter than expansions.[5] More recently, papers have found cyclicality in higher moments of the distribution of production. Using firm-level panel data, Salgado et al. (2019) show that skewness in the growth rates of employment, sales, and productivity are procyclical. Examining U.S. durable goods manufacturing at the plant level, Kehrig (2015) finds that the dispersion of total factor productivity is higher in recessions than in booms.

A number of papers study environments leasding to asymmetries.[6] Our work is closest to those that connect aggregate nonlinearities to credit or collateral constraints such as Kocherlakota et al. (2000), Kuhn and George (2019), Jensen et al. (2020), and Fève et al. (2021). These papers study real business cycle models or New Keynesian models, while, in our setting aggregate quantities depend on a time-varying distribution of firms.

2 Model

2.1 Overview

The economy consists of a continuum of infinitely-lived households, risk-neutral firms, banks, and a government. There are two types of households - workers and entrepreneurs -

[4]Seminal papers that explored the role of financial frictions in propagating real shocks include Benanke and Gertler (1989), Bernanke et al. (1999), and Kiyotaki and Moore (1997).

[5]See Neftci (1984), Falk (1986), Hamilton (1989), Sichel (1993), McQueen and Thorley (1993), and Jorda and Taylor (2016).

[6]Adjustment costs are explored by McKay and Reis (2008) and Jovanovic (2006). Learning is studied by Chalkley and Lee (1998), Van Nieuwerburgh and Veldkamp (2006) and Acemoglu and Scott (1997).

differentiated by their ability to organize production; only entrepreneurs can operate a firm. Workers are indexed by their holdings of liquid assets a and their idiosyncratic labor productivity ε. They are subject to uninsurable idiosyncratic shocks to their labor productivity and supply labor inelastically to competitive entrepreneurs. The latter are indexed by their holdings of liquid assets a, long-term debt b, and their idiosyncratic productivity z, which is stochastic. Entrepreneurs can choose between being a worker (inactive entrepreneur) or a business owner (active entrepreneur). When they choose not to have a business, they supply labor for a wage. Depending on their choice, z is their labor productivity or total factor productivity. When an entrepreneur chooses to start a new business, she has to pay an entry cost, $\kappa(z)$.

Entrepreneurs operate firms by renting capital and hiring labor in spot markets. Capital is rented subject to a collateral constraint and entrepreneurs' liquid assets serve as collateral. While all entrepreneurs can rent capital, those operating startups are initially unable to borrow. They randomly gain access to external finance and also may lose the access.

All borrowing is in the form of long-term debt, b, which is defaultable. Default results in a record of bankruptcy in the entrepreneur's credit history. Such bankruptcy flags prevent entrepreneurs from further borrowing and operating a business. Beyond default, active entrepreneurs may choose to exit or be forced to do so after receiving an exit shock.

A unit measure of risk-neutral firms produce goods alongside active entrepreneurs. However, these firms are not subject to idiosyncratic productivity shocks nor collateral constraints on renting capital. Thus they produce efficiently and distributes dividends to households. We assume that all workers and entrepreneurs hold the same quantity of shares in risk-neutral firms. They do not enter or exit, these firms operate continuously.

Banks price debt to reflect entrepreneurs' default risks. The financial sector is competitive and they expect zero profits on each loan. The government collects taxes from households, tax revenues finance transfers and other government spending.

The aggregate states of the economy are g - the distribution of households over $(a, b, \varepsilon, z, o)$ where o indicates entrepreneurs' status - active with the option to borrow ($o = 1$), active without the option to borrow ($o = 2$), inactive ($o = 3$), and inactive with a default flag ($o = 4$). Let g^w be the worker-type household distribution and g^e be the entrepreneur-type household distribution. Time is continuous.

2.2 Households

2.2.1 Worker

Each worker's labor productivity follows a Poisson process. Shocks occur with frequency λ_w and result in a new productivity drawn from a time-invariant distribution. Labor productivity shocks arrive independently across workers, and one worker's productivity is independent of others. Workers are also exposed to unemployment shocks which arrive at rate λ_u. When they are unemployed, they receive unemployment benefits $\varepsilon_u w$ where w is the wage rate. We summarize ε_w and ε_u with ε, and summarize λ_w and λ_u with $\pi_{\varepsilon\varepsilon'}$. Across types, households receive utility flow from consuming non-durable goods and their utility function is

$$u(c) = \frac{c^{1-\sigma} - 1}{1-\sigma}, \quad \sigma \geq 0.$$

As worker-type households cannot start a business, they solve the following problem.

$$\begin{aligned} \rho v^w(a,\varepsilon;g) &= \max_c u(c) + \frac{\partial v(a,\varepsilon;g)}{\partial a}\dot{a} + \sum \pi_{\varepsilon\varepsilon'} v^w(a,\varepsilon';g) \\ \dot{a} &= \varepsilon w(g) + r(g)a - c - T(w\varepsilon, r(g)a) + d(g), \quad a \geq \underline{a}. \end{aligned} \tag{1}$$

Above, $d(g)$ is the dividend received from risk-neutral firms.

2.2.2 Entrepreneur

Productivity Entrepreneur-type households have idiosyncratic total factor productivity, z, which follows a Poisson process. Similar to workers, entrepreneurs receive a productivity shock with frequency λ_z. This entails a draw of new productivity from a time-invariant distribution. Active entrepreneurs' productivity is also tied to their ability to adjust debt. When they cannot adjust their debt, their productivity is ψz where $\psi < 1$.

Assets All households can save in a liquid asset a. If a household is a producer, a serves as collateral against rented capital, k; $k \leq \gamma a$. In the absence of the collateral constraint, capital chosen by an entrepreneur would only depend on her productivity and input prices. However, the collateral requirement distorts active entrepreneurs' capital decisions and provides an incentive to take on debt to raise liquid assets.

Debt As noted debt is long-term and defaultable. Entrepreneurial households holding debt, b, are required to pay interest and a fraction of the principal, each instant. Those with the option to borrow can take a new loan by paying a refinancing cost. This cost is proportional to the size of the new loan. Refinancing involves repayment of the remaining balance of the current loan. Due to default risks, the actual amount received is $q(a', b', z, o; g)$ fraction of debt balance, b', at the time of origination; this loan discount function varies with the choice of b', as well as the borrower's state, as these affect the possibility of default. An entrepreneurial household that cannot refinance or prepay its debt is described by $o = 2$.

Whether or not entrepreneurs can adjust their debt or not, they can default. When an entrepreneur chooses to default, she must exit and incurs a utility cost, ξ_d, at the moment. Her debt is erased and she is able to keep a fraction of assets, χa. Moreover, she will have a default record on her credit history, which excludes her from borrowing or entering. For tractability, we assume that this default flag is removed stochastically with intensity λ_d.

Entrants are able to borrow to finance a fraction of their entry cost. Thereafter, they begin production without access to further loans. The option to borrow arrives randomly with a frequency λ_b. Any entrepreneur able to borrow may lose their option at the same rate.

Entreprenеurs' problem

Active entrepreneurs, with $o = 1, 2$ can produce, and those of type $o = 1$ can also adjust b, while those with $o = 2$ cannot. An inactive entrepreneurs without a bankruptcy flag ($o = 3$) can start a business. We summarize individual states as $\omega = (a, b, z, o)$. An entrepreneur solves the following problem,

$$
\begin{aligned}
v(\omega_t; g_t) &= \max_{\{c_t, (k_t, l_t)|_{o\in[1,2]}\}_{t\geq 0}, \tau} \left\{ E_0 \int_0^{\tau} u(c_t) + e^{-\rho\tau} v^*(\omega_\tau; g_\tau) \right\}, \\
v^*(\omega_\tau; g_\tau) &= \begin{cases} \max\{v^r, v^x, v^b\} & \text{if } o = 1 \\ \max\{v^x, v^b\} & \text{if } o = 2 \\ v^e & \text{if } o = 3, \end{cases} \\
(a_0, b_0, z_0, o_0) &= (a, b, z, o).
\end{aligned}
\tag{2}
$$

Above, an entrepreneur chooses non-durable consumption $\{c_t\}$. Active entrepreneurs also choose capital $\{k_t\}$, and labor $\{l_t\}$. Entrepreneurs without default flags choose their optimal stopping time τ. At the stopping date, a household making a discrete choice that brings about a large shift in their asset position, occupation, or credit history. When a household chooses to stop, it chooses one of the following options: i) refinancing debt, ii) exiting production after repaying debt, iii) defaulting on b and exiting, and iv) starting a business; the values corresponding to these choices are v^r, v^x, v^b and v^e, respectively, and defined below. The available options depend on the entrepreneur's current status, o.

The Hamilton-Jacobi-Bellman (HJB) equation corresponding to active entrepreneurs' problems, prior to stopping is,

$$
\begin{aligned}
\rho v(a,b,z,o;g) = \max_{c,(k,l)_{o\in[1,2]}} & u(c) + \frac{\partial v(a,b,z,o;g)}{\partial a}\dot{a} + \sum \pi_{zz'} v(a,b,z',o;g) \\
& + \mathbf{1}_{o\in[1,2]}\Big(\frac{\partial v(a,b,z,o;g)}{\partial b}\dot{b} - \lambda_{10}(v(a,b,z,o;g) - v(a-b-\xi_{b0},0,z,3;g)) \\
& - \lambda_b(v(a,b,z,o;g) - v(a,b,z,o';g))\Big) \\
& \dot{a} = \begin{cases} \pi(z,g) + r(g)a - c - (r(g)+\theta)b - T(\cdot) + d(g), \quad k \leq \gamma a & \text{if } o \in [1,2] \\ zw(g) + r(g)a - c - T(\cdot) + d(g) & \text{if } o \in [3,4] \end{cases} \\
& \dot{b} = -\theta b \\
& v(a,b,z,o;g) \geq v^*(a,b,z,o;g)
\end{aligned}
$$

where $\pi(z,g) = zk^{\alpha}l^{\nu} - w(g)l - (r(g)+\delta)k - \xi(z)$ and $o' = 2$ when $o = 1$ or $o' = 1$ when $o = 2$. $\xi(z)$ is a fixed, operating cost which varies with the level of productivity. The transition probabilities for the productivity process are $\pi_{zz'}$. Beyond productivity shocks, entrepreneurs face an exogenous exiting shock which makes them inactive. When exiting, entrepreneurs need to pay the existing debt and the debt adjusting cost, their assets after exiting will be $a - b - \xi_{b0}$. Thus, the expected change in value, for an active entrepreneur experiencing an exit shock, is $\lambda_{10}(v(a,b,z,o;g) - v(a-b-\xi_{b0},0,z,3;g))$. The last term of the HJB equation, $\lambda_b(v(a,b,z,o;g) - v(a,b,z,o';g))$ reflects the expected change in value from a change in the access to new loans.

The stopping value $v^*(a_\tau,b_\tau,z_\tau,o_\tau;g)$ is the maximum of the available options, which depend on o, all of which are listed below. These options define the value from adjusting an existing loan, exiting or defaulting.

1. Refinancing b

$$v^r(a_\tau,b_\tau,z_\tau,o_\tau;g) = \max_{a',b'} v(a',b',z_\tau,o_\tau;g)$$

$$a' = a_\tau - b_\tau + q(a',b',z_\tau,o_\tau;g)b' - \xi_b|b'| - \xi_{b0}, \ \ \underline{b} \geq b'$$

2. Exiting

$$v^x(a_\tau,b_\tau,z_\tau,o_\tau;g) = v(a-b-\xi_{b0},0,z,3;g)$$

3. Defaulting on b

$$v^b(a_\tau, b_\tau, z_\tau, o_\tau; g) = v(\chi a, 0, z, 4; g) - \xi_d$$

4. Entering

$$v^e(a_\tau, 0, z_\tau, 3; g) = \max_{a',b'} v(a', b', z_\tau, 2; g)$$
$$a' = a_\tau + q(a', b', z_\tau, 2; g)b' - \xi_b|b'| - \xi_{b0} - \kappa(z), \;\; b' \leq \gamma_\kappa \kappa(z)$$

As stated above, available stopping options depend on o.

2.3 Risk-neutral firms

Risk-neutral firms produce goods using a decreasing returns to scale technology with the same factor shares as entrepreneurs' technology. Nonetheless, they are not subject to a collateral constraint on the capital they rent, nor do they experience idiosyncratic productivity shocks.

In contrast to entrepreneurs, these firms rent a capital stock that is predetermined. Each period they decide the rate at which the capital they will rent changes, and pay an adjustment cost to do so. Renting capital, such firms do not directly invest in capital stock directly. Similarly to entrepreneurs, risk-neutral firms hire labor after observing total factor productivity. Unlike entrepreneurs, they do not have a fixed operating costs and distribute dividends. Below is their problem,

$$\begin{aligned} r(g)v^L(k_L) &= \max_{l_L, i_L} z_L k_L^\alpha l_L^\nu - w(g)l_L - (r(g)+\delta)k_L - \frac{\xi_L}{2}\frac{(i_L - \delta k_L)^2}{k_L} + \frac{\partial v^L(k_L)}{\partial k_L}\dot{k}_L, \\ \dot{k}_L &= i_L - \delta k_L. \end{aligned} \tag{3}$$

In the absence of financial constraints, these producers have no reason to accumulate assets or use debt. Below, we map such firms to large firms in the data, thereby fitting the size distribution. Since we abstract from financial constraints faced by these firms, their presence limits the aggregate effects of debt held by entrepreneurs.

2.4 Financial intermediaries

There are risk neutral, competitive financial intermediaries. These intermediaries issue short-term deposits and lend to active entrepreneurs. Given the risk of default on loans, banks offer loan rates based on borrowers' portfolios of assets and debt as well as their income. They earn zero expected profits from each loan.

A borrower's loan rate involves two components. The first is a premium for the probability of default which varies with both individual and aggregate states. The second is the stream of risk-free real rates over the expected duration of the loan, $r_{t,t\in[0,\tau]}$. The risk premium is embodied in a discount factor, $q(a_0, b_0, z_0, o_0; g_0)$ and a borrower with initial debt b_0, receives $q(a_0, b_0, z_0, o_0; g_0)b_0$. Each instant thereafter, she has to pay the risk free real interest rate r_t and a fraction θ of the remaining balance b_t. Therefore, the flow income from a loan is $(r_t + \theta)b_t$. It follows that a financial intermediary's discount rate for a loan is $r_t + \theta$ as the loan matures at the rate θ.

Since the intermediaries earn zero expected profit on each loan, the discounted value of a loan at origination has to be equal to the expected cash flow from it. Hence, the price of the loan region is given by

$$q(a_0, b_0, z_0, o_0; g_0)b_0 = \mathbb{E}\Big[\mathbb{E}_\tau \int_0^\tau e^{-\int_0^s (r_s+\theta)ds}(r_t + \theta)b_0 dt + e^{-\int_0^\tau r_s ds} b(a_\tau, b_\tau, z_\tau, o_\tau; g)\Big].$$

This pricing equation only applies for individual states in the non-default region as a borrowers who would default immediately will not receive a loan.

Recall that when an entrepreneur defaults, she retains χa of her assets; intermediaries recover $min[(1-\chi)a, b]$. Applying the Feynman-Kac formula, the above equation can be written as the following partial differential equation,

$$\begin{aligned}(\theta + r(g))q(a, b, z, o; g) = \theta + r(g) + \frac{\partial q(a, b, z, o; g)}{\partial a}\dot{a} + \frac{\partial q(a, b, z, o; g)}{\partial b}\dot{b} \\ + \sum \pi_{zz'} q(a, b, z', o; g) - \lambda_{10}(q(a, b, z, o; g) - q(a - b - \xi_{b0}, 0, z, 3; g)) \\ -\lambda_b(q(a, b, z, o; g) - q(a, b, z, o'; g)) \quad \text{for} \quad t \in [0, \tau),\end{aligned} \tag{4}$$

$$q(a, b, z, o; g) = \frac{min[(1-\chi)a, b]}{b} \quad \text{for} \quad t = \tau \ . \tag{5}$$

These loan prices only exist for active entrepreneurs, $o \in [1, 2]$, and $o' = 1$ when $o = 2$ and $o' = 2$ when $o = 1$.

2.5 Government

The government collects taxes from households. Taxable income includes labor income of workers and inactive entrepreneurs supplying labor, and profits of active entrepreneurs, as well as interest and dividend income of all households. A fraction of revenues are provide benefits to unemployed households. The remainder is spent on government consumption which is not valued by households.

2.6 Equilibrium

A recursive competitive equilibrium is a set of functions, solving workers, entrepreneurs, risk-neutral firms, and financial intermediaries problems and clearing the markets for assets, debt, labor and output. It is described by the following conditions.

1. Households optimize. Given prices $\{r, w, q\}$, v^w solves (1), v solves (2).
2. Risk-neutral firms maximize profits by solving (3).
3. Competitive financial intermediation implies the debt discount rate q is determined by (4)- (5).
4. Asset market clears: $\int a g^w(a,\varepsilon) d[a \times \varepsilon] + \int (a+b) g^e(a,b,z,o) d[a \times b \times z \times o]$ $= \int_{o \in [1,2]} k g^e(a,b,z,o) d[a \times b \times z \times o] + k_L$.
5. Labor market clears: $\int \varepsilon g^w(a,\varepsilon) d[a \times \varepsilon] + \int_{o \in [3,4]} z g^e(a,b,z,o) d[a \times b \times z \times o]$ $= \int_{o \in [1,2]} l g^e(a,b,z,o) d[a \times b \times z \times o] + l_L$.
6. The government budget constraint holds.

The asset market clearing condition in 4 equates the net assets held by workers and entrepreneurs with the aggregate capital stock, over time. In 5, the aggregate supply of labor is the sum of labor supplied by workers and inactive entrepreneurs. While the former is constant, entrepreneurs' labor supply varies with entry and exit decisions.

3 Mapping Model to Data

The vast majority of US firms are privately held, with little to no data on their balance sheets. However, in our model, employment and assets are correlated for financially constrained firms. Moreover, the severity of collateral constraints determines the rate at which firms grow. In an effort to discipline the effect of debt on aggregate quantities, we match the size distribution of firms as well as the size by age distribution. Further, we add several moments directly related to debt held by firms as well as bankruptcy rate.

We assign values for a subset of parameters before solving the model's steady state. These are either parameters that can be linked to data values independently of the model's solution or parameters that have commonly used values. In Table 1 they are distinguished with N in the column labelled 'Internal'. The remaining 19 parameters, designated Y in the the internal column of the table, as well as the share of entrepreneurs in the population are jointly calibrated. Overall, the number of parameters is 20 and the number of targets is 26.

Table 1: Parameter values

Parameter	Value	Internal	Description
Preference			
ρ	0.05	N	Discount rate of entrepreneur type
σ	2.0	N	Curvature of the utility function
Tax			
τ_0	0.91	Y	Tax rate
τ_1	0.181	N	Tax progressivity
Labor productivity			
$\overline{\varepsilon}$	3.97	N	Mean of worker productivity
η_ε	1.0	Y	Shape of Pareto distribution
λ_ε	0.1	Y	Shock intensity
$\underline{\varepsilon}$	0.4	N	UI replacement rate
$\lambda_{\varepsilon 10}$	0.238	N	Job separation rate
$\lambda_{\varepsilon 01}$	3.113	N	Job finding rate
Production productivity			
$\overline{z}$	0.85	N	Mean of entrepreneurial productivity
η_z	2.68	Y	Shape of Pareto distribution
λ_z	0.2	Y	Shock intensity to z
ψ	0.85	Y	Size of productivity shock
z_L	1.15	Y	Risk-neutral firm's productivity
Production			
α	0.285	N	Capital share
ν	0.6	N	Labor share
δ	0.07	N	Depreciation rate
$\kappa(z)$	-2.0, 0.0	Y	Entry cost bounds
ξ	0.83	Y	Fixed cost of production
λ_b	0.25	Y	Borrowing option shock frequency
λ_{10}	0.085	Y	Exit shock frequency
ξ_L	2.0	N	Cost of investment of risk-neutral firms
Assets and debts			
θ	0.08	Y	Amortization rate of debt
γ	2.0	Y	Collateral constraint
γ_κ	0.9	Y	Borrowing constraint of entering cost
ξ_b	0.05	Y	Loan origination cost
ξ_{b0}	0.01	Y	Loan origination cost
ξ_d	1.0	Y	Defaulting cost
$\underline{b}$	36.0	Y	Long-term debt borrowing limit
χ	0.1	Y	Retention rate when default
λ_d	0.1667	N	Shock intensity; removal of default flag

Table 1 lists calibrated parameters, and Table 3 reports targeted data moments and model moments. In the following, we associate certain targets with specific parameters, but since the parameters are jointly determined, these associations are loose.

Preference We set the curvature of the utility function σ to 2. The discount rate ρ is set to 0.05.

Tax The income tax function $T(y) = y - \tau_0 y^{1-\tau_1}$ is taken from Heathcote et al. (2017) where y is a taxable income. Taxable income is labor income (wage workers and inactive entrepreneurs) and profits (active entrepreneurs) plus interest and dividend income. The parameter τ_1, determining the degree of progressivity of the tax system, is 0.181 as in Heathcote et al. (2017).[7] Next, τ_0 is set to 0.91 to match a tax revenue-output ratio of 16.7%.[8] The tax function implies transfers when taxable income is low and taxes at higher levels of income. We assume the government consumes the difference between tax revenues and transfer payments. Beyond those implied by the tax function, another source of transfers are unemployment benefits. We set the level of unemployment benefits $\underline{\varepsilon}$ to 40% of workers' median income as most US states replace 30–50 % of workers' lost earnings.

Productivity Both labor and entrepreneurial productivity are assumed to be drawn from a Pareto distribution. At each instant, a shock may arrive and an individual experiences a new productivity draw. To specify this process, we need to choose values for the curvature parameters of the Pareto distributions (η_ε, η_z) and shock intensities (λ_ε, λ_z).[9]

Parameter values for the idiosyncratic total factor productivity process of entrepreneurs largely shape the firm size distribution. We choose the mean of discretized z, $\overline{z}$, to be 0.85. When active entrepreneurs lose their ability to adjusting debt, they also experience a negative productivity shock. We assume that their productivity is reduced by 15%.

We set the productivity for risk-neutral firms z_L to 1.15. We assume that these are large firms and set them as those that hire 48% of labor. The Business Dynamics Statistics (BDS) data reports this as firms with more than 500 employees.[10]

[7] They estimate this parameter using the Panel Study of Income Dynamics (PSID) for survey years 2000, 2002, 2004, and 2006, in combination with the NBER's TAXSIM program.

[8] We use the tax revenue-output ratio between 2000 and 2014 reported by the Congressional Budget Office.

[9] We discretize the support of entrepreneurs productivity distribution into 3 grid points that are not linearly spaced. Instead, the 3 points are chosen to capture the bottom 65.0, 31.0, and 4.0 percentiles of the population of entrepreneur-type households. For labor productivity, we first discretize a grid linearly using 20 points. Then, we add one point to capture the top 1 percentile of the population and one point to capture unemployed households.

[10] To convert entrepreneurs' labor size in the model to the number of employees, we assume that the largest entrepreneur in the steady state hires 499 workers.

We set the parameters for labor productivity to match the earnings distribution (Survey of Consumer Finance, SCF). Since entrepreneurs' business incomes also shape the earnings distribution, labor productivity and idiosyncratic total factor productivity pin down the earnings distribution together. Table 2 shows the data and model fit. We assume that workers are exposed to unemployment shocks. The shock intensities of job separation $\lambda_{\varepsilon_{10}}$ and job finding $\lambda_{\varepsilon_{01}}$ are set to 0.238 and 3.113 to reflect the average duration of staying in a job (4.2 years, BLS) and the average duration of unemployment (16.7 weeks, BLS).

Production We set labor's share, in the production function, ν, to 0.6 and the annual depreciation rate δ to 0.07. These values are similar to Khan and Thomas (2013).

Entry costs are a function of productivity and set to be a fraction of an unconstrained firm's profit (a firm unaffected by the collateral constraint) in the steady state. Specifically, $\kappa(z) = e^{\kappa_z}\pi(z, g*)$ where $\pi(z, g*)$ is an unconstrained firm's profit in the steady state $(g*)$ and κ_z is a evenly spaced vector corresponding to the values of z. We set the lower bound of κ_z to -2.0 and the upper bound to 0.0.[11] Notice that it is more costly for productive entrepreneurs to enter. This helps us to match the relative size of young firms aged 0-5 years and 6-10 years. As higher entry costs for the most productive potential entrants discourage their entry, entrants tend to have lower productivity and an initial size that is small relative to the incumbents.

The operating cost, $\xi(z)$, also depends on total factor productivity, $\xi(z) = \xi z^{\frac{1}{1-\alpha-\nu}}$. This specification sets higher fixed costs for productive entrepreneurs, reducing their earnings and slowing their wealth accumulation and growth.

The frequency of the borrowing option shock λ_b is set to 0.25 which implies that the average duration with and without access to new loans is approximately 4 years. In our model, a key motive for borrowing is to increase the scale of production. When a firm does not have enough savings, its collateral constraint binds. In this case, it can increase its assets by raising debt. This mitigates the effect of the collateral constraint on capital. Therefore the rate at which firms gain access to loans affects the value of entry and the speed of firm growth.

The frequency of exit shocks λ_{10} is 0.085, implying that a firm experiences this shock on average every 12 years. Firms in our model exit as a result of this shock, and also by choice or when they default. Given these additional sources of exit, the shock frequency is set to match the exiting rate of 8.69% (BDS).

As shown in Section 2, risk-neutral firms have the same decreasing returns to scale production function. In the absence of financial constraints, their output would be volatile

[11]For example, when we discretize z as a 3 point grid, the vector of points, $[\kappa_{z1},\kappa_{z2},\kappa_{z3}]$ will be [-2,-1,0]. When this is the case, the entry cost of an entrepreneur with productivity z_1 is$e^{-2}\pi(z, g*)$.

in response to aggregate shocks, without costs of changing the level of capital they rent. We choose a relatively small value for their adjustment costs, setting $\xi_L = 2$. This is sufficient to ensure that highly cyclical changes in production by such firms do not lead to countercyclical movements among entrepreneurs.[12] This is a conservative choice. Higher values of ξ_L move aggregate fluctuations closer to changes in entrepreneurs' production by dampening changes in risk-neutral firms. This amplifies the effects of debt in our model.

We set 85% of the population as workers and 15% as entrepreneurs. In the steady state, 78.3% of entrepreneurs are active which implies that 11.7% of the population engages in entrepreneurship. This falls into the range found by Cagetti and De Nardi (2006) who, using the SCF, showed that the share of entrepreneurs ranged from 7.6%-16.7%.

Assets and debts We match two moments of the distribution of debt across firms using the Small Business Credit Survey (SBCS). This is a national sample of small firms, those with less than 500 employees. Given our calibration to the size distribution of firms which assumes that all firms with more than 500 workers are risk-neutral, small firms in the data correspond to all active entrepreneurs in our model. We match the share of small firms without debt and the share with $1-1,000,000 debt.[13] We also target the aggregate leverage of non-financial businesses (Flow of Funds).

The following parameters play the largest role in matching the moments listed above. The loan origination cost, ξ_b is set to 5% of the size of a new loan size. There is also a fixed cost, ξ_{b0}, set to 0.01. The amortization rate of debt, θ is set to 8% which implies a half-life of 8 years if entrepreneurs do not refinance or prepay the loan. While the loan price schedule for long-term debt b imposes an endogenous borrowing limit, we also set an exogenous borrowing limit, $\underline{b}$.[14]

The collateral constraint parameter γ is set to 2.0 and the fraction of entry costs that must be self financed, $1 - \gamma_k$, is 10 percent. Together with the frequency of the borrowing option shock λ_b, these parameters determine the speed of firm growth which is reflected in relative size of entrants, age 0-5 and age 6-10 firms. The utility cost of default ξ_d is 1 and the retention rate of assets upon default χ is 10 percent. These parameters are important for pinning down default rates. We set the shock intensity for the removal of the default flag λ_d to 0.17 which implies an average duration with default in credit history of 6 years. This

[12] In our calibration, risk-neutral firms would reduce capital and labor by a power of $\frac{1}{1-\alpha-\nu} = 6.7$ of the fall in productivity, absent changes in relative prices, if they rented capital frictionlessly. In equilibrium, this would lead entrepreneurs to actually increase their capital and labor at the onset of a recession.

[13] To convert the real values in the model to dollar values, we use average income in SCF (2010); we multiply (average income in the model)/(average income in the data) to b to have dollar values of debt.

[14] This limit only binds for entrepreneurs with high productivity and large asset levels. In the steady state, 0.7% of active entrepreneurs are at the borrowing limit.

Table 2: Earnings distribution

	Quintiles(%)					Top(%)		
	1q	2q	3q	4q	5q	90-95	95-99	99-100
Data	-0.1	3.5	11.0	20.6	65.0	12.1	18.3	18.0
Model	3.8	8.2	11.3	18.1	58.7	13.1	17.9	11.6

Data: SCF (2010)

is to capture the following features of the US bankruptcy system. Households that file for Chapter 13 bankruptcy enter into repayment plans that last for 3–5 years and those that file for Chapter 7 bankruptcy cannot file again for 6 years.

Calibration results Table 3 shows that our model's match to the targeted moments. The model is over-identified as the number of calibrated parameters is 20 and the number of moments is 26. The model matches entrepreneurs' borrowing behavior well which is reflected in the bankruptcy rate, aggregate leverage, share of small firms with no debt, and those with less than a million dollars of debt. The earnings distribution includes workers' and entrepreneurs' earnings. The model is able to capture the concentration of earnings in the first quintile of the distribution (Table 2) as well as the share of entrepreneurs at the top of the wealth distribution. Our model also comes close to reproducing the firm size (employment) distribution. Further, it matches the relative size of new firms, as well as the relative size of firms aged 0-5 years and 6-10 years. However, it underpredicts the exit rate of young firms.

4 Steady state analysis

Before we study the properties of the model with aggregate shocks, we describe how it works in the steady state. Two features of our model - producers with diminishing marginal utility and long-term debt - are important in shaping the distribution of producers. To understand how these features work, first we describe how entrepreneurs enter, grow and exit by following a cohort over time. Second, we compare four alternative versions of the model in steady states: a version with shorter-duration loan, a version with weaker consumption smoothing motive, a version without long-term debt, and a version with consumption insurance across households.

Table 3: Calibration results

Moments	Data	Model	Source
Capital-Output ratio	2.4	2.7	BEA
Labor share (wn/y)	0.6	0.6	BLS
Tax-Output ratio	0.16	0.18	CBO
Exit rate	8.5%	8.0%	BDS
Bankruptcy rate	3.0%	1.7%	Dun & Brad Street
Corporate profit-Output ratio	0.07	0.05	BEA
Aggregate leverage	37.2%	33.9%	Flow of Funds
Share of small firms without debt	30%	21.6%	SBCS
Share of small firms with $1-1,000,000 debt	65.0%	58.5%	SBCS
Share of earnings in top 20%	65.0%	58.7%	SCF
Share of earnings in bottom 60%	14.4%	23.3%	SCF
Share of entrepreneurs in top 20% of wealth distribution	26.0%	28.1%	Cagetti and De Nardi (2006)
Share of entrepreneurs in top 1% of wealth distribution	62.0%	74.3%	Cagetti and De Nardi (2006)
Share of active entrepreneur	11.1	11.7%	Cagetti and De Nardi (2006)
Share of firms that hire 1-19	88.3%	86.1%	BDS
Share of firms that hire 20-99	9.8%	12.4%	BDS
Share of firms that hire 100-499	1.6%	1.1%	BDS
Share of firms that hire 500+	0.4%	0.4%	BDS
Employment share of firms that hire 1-19	19.6%	23.4%	BDS
Employment share of firms that hire 20-99	17.8%	19.1%	BDS
Employment share of firms that hire 100-499	14.1%	10.1%	BDS
Employment share of firms that hire 500+	48.5%	47.5%	BDS
Relative size of entrant	27.5%	24.1%	BDS
Relative size of age 0-5 firms	37.3%	37.3%	BDS
Relative size of age 6-10 firms	54.5%	56.6%	BDS
Exit rate of age 0-5 firms	54.8%	28.2%	BDS

Note: * In the SBCS, all firms have 500 or less employees. Using only data from firms with employees, the SBCS moments are the means between 2016 and 2019. The BDS data moments are the means over 1984 - 2015 and the BLS data moment is the mean from 1947 to 2016. The tax-output ratio is averaged over 2000-2014. Aggregate leverage is aggregate debt to assets of non-farm, non-financial businesses from the flow of funds (1954-2006). We use the 2010 wave of SCF, and the definition of entrepreneur in Cagetti and De Nardi (2006) is 'self-employed'.

4.1 Cohort in the steady state

We simulate a cohort of entrepreneurs with zero initial asset or debt who begin by working for a wage. The average path of the cohort is shown in Figure 1, solid line. Due to the entry cost, an entrepreneur with zero asset cannot enter right away. Also, she may prefer not to enter because profit can be lower than wage when the collateral constraint binds with low assets. Thus entrepreneurs accumulate assets before entering. There is no active entrepreneur for the first 10 periods, while their assets increase. Once this cohort has a considerable level of savings, there is a surge of entry. Thereafter the number of active entrepreneurs without the option to borrow increases rapidly. As some of them begins to gain the access to debt, the number of active entrepreneurs with the option to borrow increase with a slight lag. Active entrepreneurs receive shocks that remove the option to borrow occasionally with the same frequency as the shock that allows borrowing. However, the share of active entrepreneurs without the option to borrow is larger since all entering entrepreneurs start without the option. The sum of active entrepreneurs with and without the option to borrow is seen in the 'Active' panel in Figure 1.

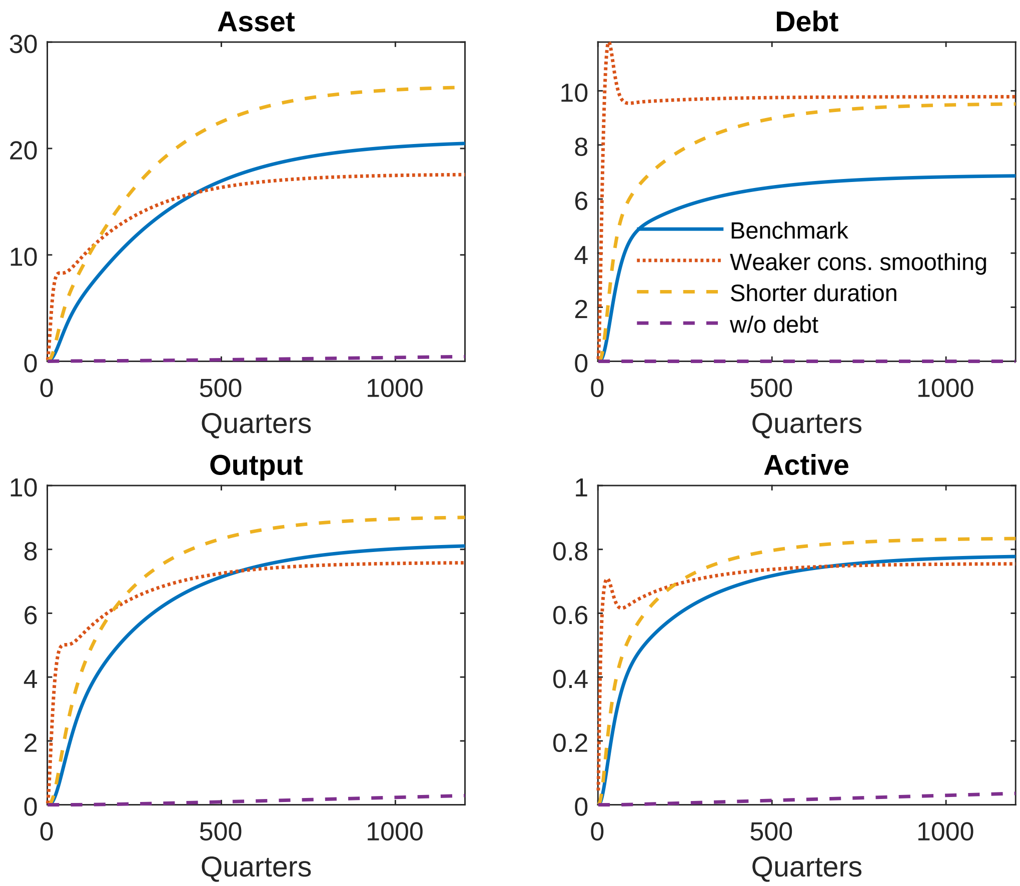
After entry, an entrepreneur produces by renting capital and hiring labor. Her capital and labor decisions are affected by the collateral constraint, $k \leq \gamma a$. Since producers do not have the borrowing option when they enter, they produce and save. Entrepreneurs without the borrowing option have strong incentive to save since they can rent more capital as they accumulate wealth. Before they earn the borrowing option, they can exit or default but we find that voluntary exits are very rare. Once they earn the borrowing option, those who have not accumulated enough savings raise debt quickly; debt and assets rise together as this allows a producer to rent more capital. This is seen in the upper panels of Figure 1. All active entrepreneurs experience exit shocks. Entrepreneurs choose whether or not to default when they receive a negative total factor productivity shock or exit shock. If they do not default, they repay their debt before exiting. This allows them to retain assets net of debt and re-enter more quickly by avoiding the no-credit default state.

4.2 Role of long-term debt and consumption smoothing

While Figure 1 shows the average values of a cohort, there is considerable dispersion over assets and debt in the steady state.[15] One important aspect of entrepreneurs' distribution is a degree of misallocation. The collateral constraint prevents entrepreneurs from renting an efficient level of capital if they do not have enough assets. Therefore the distribution of entrepreneurs over assets affects output. The distribution of entrepreneurs over debt also

[15] Figure 15 in Appendix C shows steady state distribution.

Figure 1: Cohort in steady state



Note: 'Shorter duration' is a version of the model with a smaller refinancing cost. 'Weaker consumption smoothing' is a version of the model with a utility function, $u(c + c_0)$ where $c_0 = 1$. 'Without debt' is a version without the long-term debt b and stopping options involving the long-term debt. The output and assets are those of entrepreneur's only.

matters because debt balances affect the disposable income, thus affect changes of assets.

Two features of our model - producers with diminishing marginal utility and long-term debt - are important in shaping the distribution of producers over assets and debt. Compared to the model of firms such as Khan and Thomas (2013), entrepreneurs need to consume and they may dis-save even if they are constrained. Unlike entrepreneurs in our model, risk-neutral firms in Khan and Thomas (2013) delay dividend payout to accumulate capital if they are constrained. Compared to models of entrepreneurs such as Buera et al. (2011) and Buera and Shin (2013), long-term debt reduces misallocation by providing an instrument to overcome the collateral constraint. Moreover, having long-term debt allows us to study debt-related topics such as leverage and recovery.

To better understand the role of long-term debt and diminishing marginal utility, we

Table 4: Role of long-term debt and consumption smoothing

	Benchmark	**Weaker cons. smoothing**	**Shorter duration**	**No debt**	**Complete market**
Output	1.00	0.97	1.02	0.92	1.99
Entrepreneur's output	1.00	0.93	1.10	0.62	3.47
Interest rate	1.00	1.25	0.91	1.23	2.24
Wage	1.00	0.97	1.02	0.93	1.08
Measured TFP	1.00	0.99	1.01	0.95	1.19
Entrepreneur's MTFP	1.00	0.99	1.01	0.93	1.22
Active entrepreneurs (%)	78.28	76.54	84.58	59.32	99.77
Aggregate leverage	0.34	0.57	0.38	-	-
Loan duration (quarters)	21.8	22.2	14.1	-	-

Note: 'Shorter duration' is a version of the model with a smaller refinancing cost. 'Weaker consumption smoothing' is a version of the model with a utility function, $u(c+c_0)$ where $c_0 = 1$. 'Without debt' is a version without the long-term debt b and stopping options involving the long-term debt. 'Complete market' is a version with consumption insurance across households, without collateral constraint and debt b. Output, interest rate, wage, and measured TFP are normalized to the benchmark model value. The loan duration is average of time computed from simulating a panel of 10,000 entrepreneurs for 1,600 quarters.

compare four alternative versions of the model in a steady state.[16] First, we lower refinancing cost χ_b from 0.05 to 0.0 to create a cheaper to adjust, shorter-duration loan.[17,18] In principle, the duration of a loan is infinite if a borrower does not refinance, prepay or default. However, entrepreneurs with debt choose to adjust or terminate their debt contracts as various shocks change their desired portfolio. In Table 4, we report the average loan duration; the number of quarters from getting a new loan b to loan adjustment; refinance, prepay or default.

Second, we reduce entrepreneurs' consumption smoothing motive by changing the utility function from $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$ to $u(c) = \frac{(c+1)^{1-\sigma}}{1-\sigma}$ with the constraint $c \geq 0$. This bounds marginal utility above by 1, thereby reducing fluctuations in marginal utility from changes in consumption, and thus entrepreneurs' consumption smoothing motive. This, in turn, allows for faster asset accumulation and moves our model with entrepreneurs closer to an alternative with risk-neutral firms.

Third, we remove long-term debt entirely. As a result, entrepreneurs do not have the borrowing option to raise their assets and do not have an option to default. Entrepreneurs stopping choices are reduced to entering and exiting. Also, we added a utility cost of exiting,

[16] We do not recalibrate these models.

[17] The refinancing cost is $\chi_b|b'| + \chi_{b_0}$. Therefore refinancing is not free when χ_b becomes zero.

[18] It would be ideal to compare our model to a model with a short-term debt model. However, replacing long-term debt with short-term debt requires other changes that will reduce comparability.

2, for a computational reason.[19] This model is close to Buera et al. (2011) and Buera and Shin (2013) and can highlight the role of long-term debt.

Lastly, we remove the collateral constraint and allow consumption insurance across households. While all other features regarding technology and taxes - decreasing returns to scale, entry cost, fixed cost of production, progressive tax - are present in this version of the model, consumption insurance and lack of collateral constraint lead to an efficient allocation of resources. While the model with a shorter loan duration and weaker consumption smoothing motive alleviate the effects of the long-term debt and risk-averse producers, this model removes idiosyncratic risks and need for long-term debt. Thus, the complete markets model shows the degree of misallocation caused by financial frictions, long-term debt, and consumption smoothing motive. Full characterization of the model is in Appendix A.

Table 4 shows statistics of four alternative models with those of the benchmark model and Figure 1 shows a cohort simulation of three alternative models with that of the benchmark model. The complete markets model is the furthest from the benchmark model. Output in the complete markets model is almost twice that of the benchmark despite higher input prices. Interest rates are more than double the benchmark's and the wage is 8% higher. Also, the allocation of resources between entrepreneurs and the risk-neutral firm is starkly different. In the benchmark, 47.5% of labor is used by risk-neutral firms but they use only 15% in the complete markets model. Measured total factor productivity is 19% higher and 99% of entrepreneurs are actively producing.

The model without debt and the model with a shorter duration show the role of long-term debt. While the nature of our debt contract (costly to adjust, principal repaid continuously) may lead to sluggish adjustment in response to shocks, the model without debt shows that the availability of debt significantly reduces the misallocation caused by the collateral constraint. When entrepreneurs cannot use long-term debt, they need to save to pay entry costs. Once entered, they also need to rely on assets to overcome the collateral constraint. As a result, only 59% of entrepreneurs engage in production, output is 8% lower, and measured productivity is 5% lower than in the benchmark. Not only is output lower, but the speed of growth is also very slow (Figure 1). When we start a cohort of entrepreneurs with zero initial assets or debt who begin by working for a wage, while the benchmark reaches its steady state after 1,000 periods, the model without debt takes 60,000 periods to reach the steady state.

As we have seen, long-term debt is an instrument to alleviate the collateral constraint.

[19] Without the cost of exiting, there are points, given productivity and assets, where inactive entrepreneurs choose to enter while active entrepreneurs choose to exit. These points appear at the lowest productivity level and high level of wealth, as their entering cost is low compared to their assets.

However, its illiquidity causes some inefficiency. The lower cost of adjusting debt leads to more frequent refinancing and prepayment; the effective duration of the loan falls to 14.1 quarters from 21.8 quarters. As entrepreneurs become more flexible in adjusting their financial positions, output is 2% higher and misallocation is lower.

In Figure 1, the entrepreneurs in the model with a weaker consumption smoothing motive behave more like risk-neutral firms. Inactive entrepreneurs save more when assets are low and enter faster. They also raise debt rapidly and grow more quickly. However, this model does not reach a more efficient allocation or higher output. Once entrepreneurs are away from a very low level of assets, they are less willing to save given the weaker consumption smoothing motive reduces precautionary savings. Therefore they end up having a lower level of average assets and, as a result, produce less when compared to the benchmark.

In summary, the illiquidity of debt and consumption smoothing motive lead to slow growth but their effects on misallocation are mixed. The availability of debt reduces misallocation but its illiquidity makes it less effective. Less consumption smoothing leads to fast growth when entrepreneurs' assets are low, but lowers long-run aggregate savings and output. This suggests that these features will affect the speed of recovery from recessionary shocks. We will explore the dynamic behavior of our model in the next section.

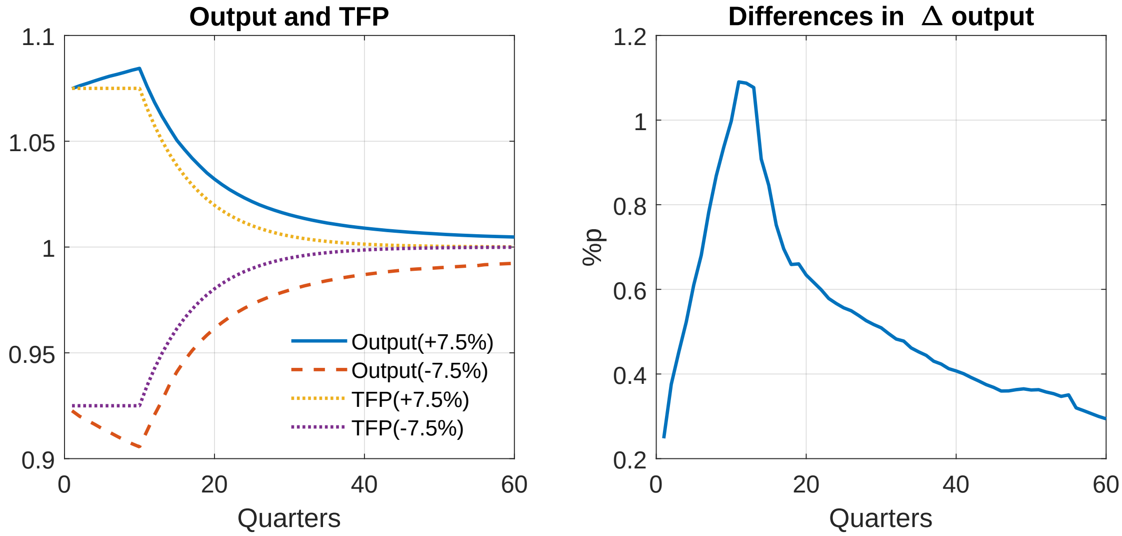
5 Results

Recent approaches to understanding the business cycle implications of costly external finance have emphasized the role of firm heterogeneity.[20] Compared to such models, our setting has long-term debt held by entrepreneurs. We show that the principal payments associated with long-lived debt create non-linear individual responses to aggregate shocks. The resulting changes in the distribution of producers deliver new results on the role of debt in shaping aggregate fluctuations.

Below, we focus on how debt propagates recessions and slows the speed of recovery. Our first result is that aggregate nonlinearities arise in our environment when comparing recessions to expansions. The second result shows that high levels of leverage slow the recovery from a recession. We obtain these findings using perfect foresight equilibrium paths following persistent changes in aggregate total factor productivity.

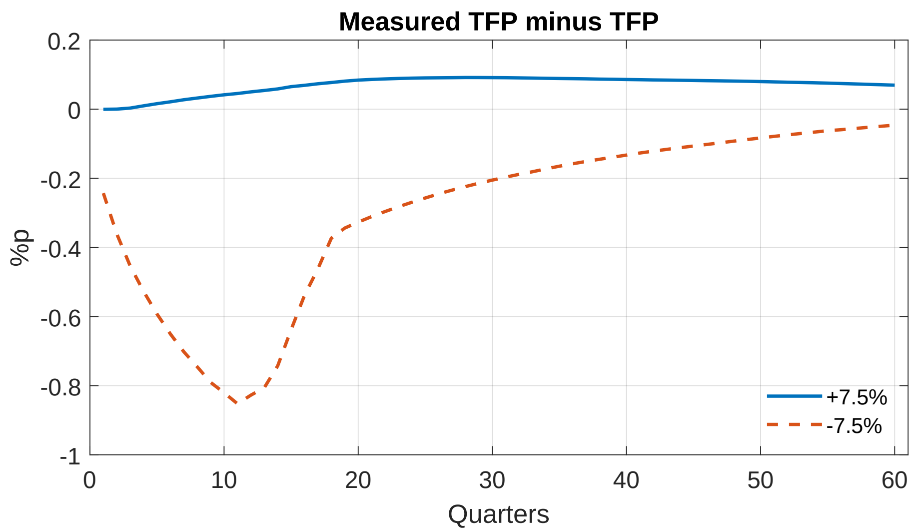
[20] Khan and Thomas (2013) argue that the aggregate debt-to-asset ratio in the flow of funds suggests that a representative firm have no need for external financing of investment. Models with a distribution of firms are consistent with aggregate leverage while generating a non-trivial role of leverage for some firms.

Figure 2: GDP and TFP in expansions and recessions



Note: Output is normalized at steady state level. The right panel shows the difference in the distance from steady state GDP between the recession and the expansion; $abs(\frac{y_t(recession)-y^*}{y^*}) - \frac{y_t(expansion)-y^*}{y^*}$, where y^* is steady state output.

Figure 3: Measured TFP in expansions and recessions

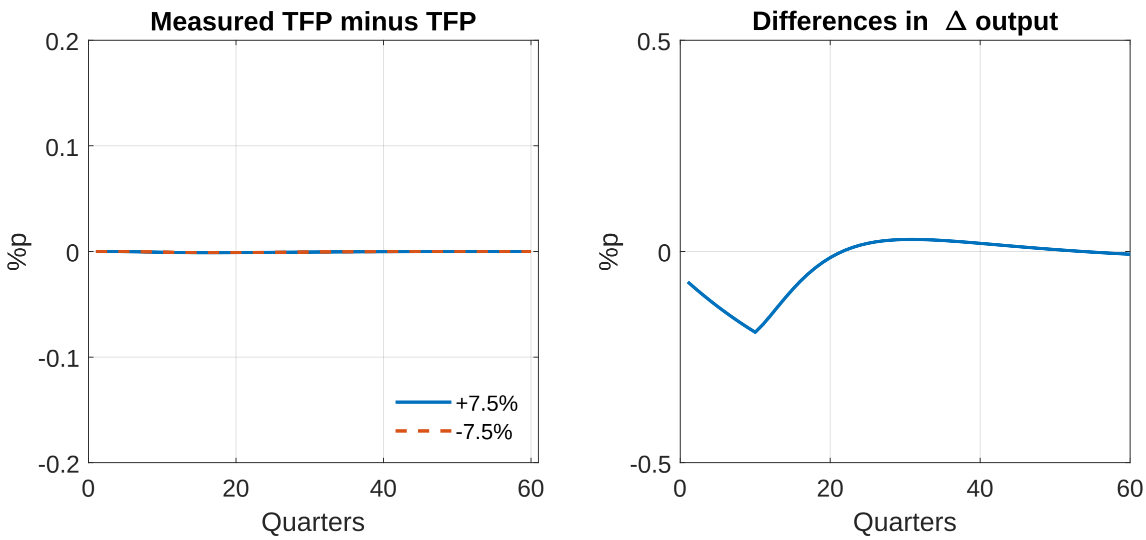


Note: Measured TFP is $\frac{Y}{K^{\alpha}L^{\nu}}$ where Y is GDP and K and L are the aggregate stocks of capital and labor.

5.1 Asymmetries over the business cycle

Our model shows asymmetries over the business cycle that arise as entrepreneurs adjust their balance sheets following aggregate shocks. We uncover these properties of the model by considering large shocks to TFP. As changes in the distribution are gradual, we study a recession that last several periods before the start of a recovery. Afterwards we allow a fairly rapid return in TFP. Specifically, we study positive and negative 7.5% shocks to aggregate TFP that are maintained for 10 quarters, then return to the mean with a persistence of 0.5. While the shock itself has time-varying persistence, its overall half life is comparable to a shock with more common persistence of about 0.9.[21] We provide details of computation in Appendix B.

Figure 4: GDP and TFP in expansions and recessions: Complete markets



Note: Measured TFP is $\frac{Y}{K^{\alpha}L^{\nu}}$ where Y is GDP and K and L are the aggregate stocks of capital and labor. The right panel shows the difference in the distance from steady state GDP between the recession and the expansion; $abs(\frac{y_t(recession)-y^*}{y^*}) - \frac{y_t(expansion)-y^*}{y^*}$, where y^* is steady state output. In the complete markets model, households have no income risks. The full description of the complete markets model is in the Appendix A.

Figure 2 shows the aggregate TFP over expansions (in yellow) and recessions (in purple). GDP in the expansion is in blue, while its path over the recession is red. We see that the negative TFP shock leads to a deeper recession than the rise in GDP over the expansion. This is clear in the right panel which shows the difference in the distance from steady state GDP between the recession and the expansion. One percentage point implies that the recession has exhibited a one percent greater fall in GDP than the rise in the expansion. This is the rather pronounced aggregate non-linearity. As we see in Figure 4, such asymmetry is absent in a complete markets version of the model. Similarly, we do not observe asymmetry in a model without debt, which is a comparable to Buera et al. (2011) (shown in Figure 16 in Appendix C). This suggests all aggregate nonlinearities arise as entrepreneurs adjust their non-contingent debt.

[21]Unanticipated shocks may generate aggregate profits or losses in the financial sector; we assume that these are absorbed by the government. This allows us to avoid introducing a market for shares in financial intermediaries.

The sharper fall of GDP in recessions, is associated with a rise in misallocation across firms. This is seen in Figure 3 which plots measured TFP in the recession, in red, and expansion, in blue.[22] We see that the recession involves a pronounced fall in measured TFP beyond the exogenous fall in aggregate TFP. By the end of the initial ten periods, measured TFP has fallen by about 0.8 percentage points more than the 7.5 percent fall in the exogenous component of aggregate TFP. In contrast, there is a very mild rise in the endogenous component of aggregate TFP in the expansion, and it is more gradual than the drop seen in the recession.

These changes in the endogenous component of TFP arise through redistribution of resources across entrepreneurs. The allocation of resources will respond to changes in the choice of k and l by an entrepreneur, (z, a, b), as well as changes in their distribution. We explore how changes in decision rules and the distribution of entrepreneurs contribute to the overall change in production.

Let i index the distribution of producers over (z_i, a_i, b_i). Define g_{i*} as the steady state density and g_{it} as the actual time t density. An active entrepreneur of type i produces $z_i k_{it}^{\alpha} l_{it}^{\nu}$. In any period, aggregate output y_t can be decomposed as follows,

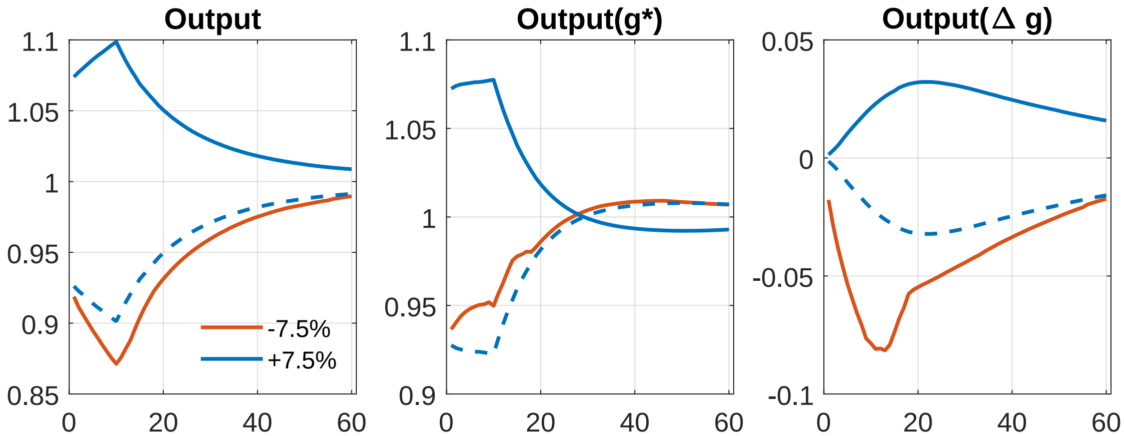
$$\begin{aligned} y_t &= A_t \int z_i k_{it}^{\alpha} l_{it}^{\nu} g_{it} di = A_t \int z_i k_{it}^{\alpha} l_{it}^{\nu} (g_{i*} + \Delta g_{it}) di \\ &= A_t \int z_i k_{it}^{\alpha} l_{it}^{\nu} g_{i*} di + A_t \int z_i k_{it}^{\alpha} l_{it}^{\nu} \Delta g_{it} di \end{aligned} \tag{6}$$

The left panel of Figure 5 plots actual output in expansions and recessions. Our decomposition implies that the middle and right panels of the figure sum to output in the left panel.

The middle panel shows the first term in the decomposition, aggregating actual production at each point, $y_{it} = A_t z_i k_{it}^{\alpha} l_{it}^{\nu}$ using the steady state density g_{i*}. Entrepreneurs' capital and labor decisions change following a change in aggregate productivity and associated movements in interest rates and wages. We see the change in GDP that would have occurred from time-varying changes in capital and labor, at each level of productivity, debt and assets, if there had been no change in the distribution of entrepreneurs over these variables. The right panel illustrates the effect of the change in the distribution, $\Delta g_{it} = g_{it} - g_{i*}$ on output. It sums equilibrium production at each point, weighted by the change in the distribution of entrepreneurs distribution, at that point, relative to the steady state.

[22] Measured TFP is $\frac{Y}{K^{\alpha} L^{\nu}}$ where Y is GDP and K and L are the aggregate stocks of capital and labor. We provide a measure of total factor productivity as if there was a representative firm. Therefore any difference between measured and exogenous TFP is attributable to changes in the allocation of capital and labor.

Figure 5: Decomposing changes in output of entrepreneurs



Note: The blue dashed lines illustrate the negative of the response of each series in the expansion, for easier comparability with the recession. Output series are normalized at steady state level. The middle panel shows the first term in Equation 6 which captures the effects of price changes to output and the right panel shows the second term in Equation 6.

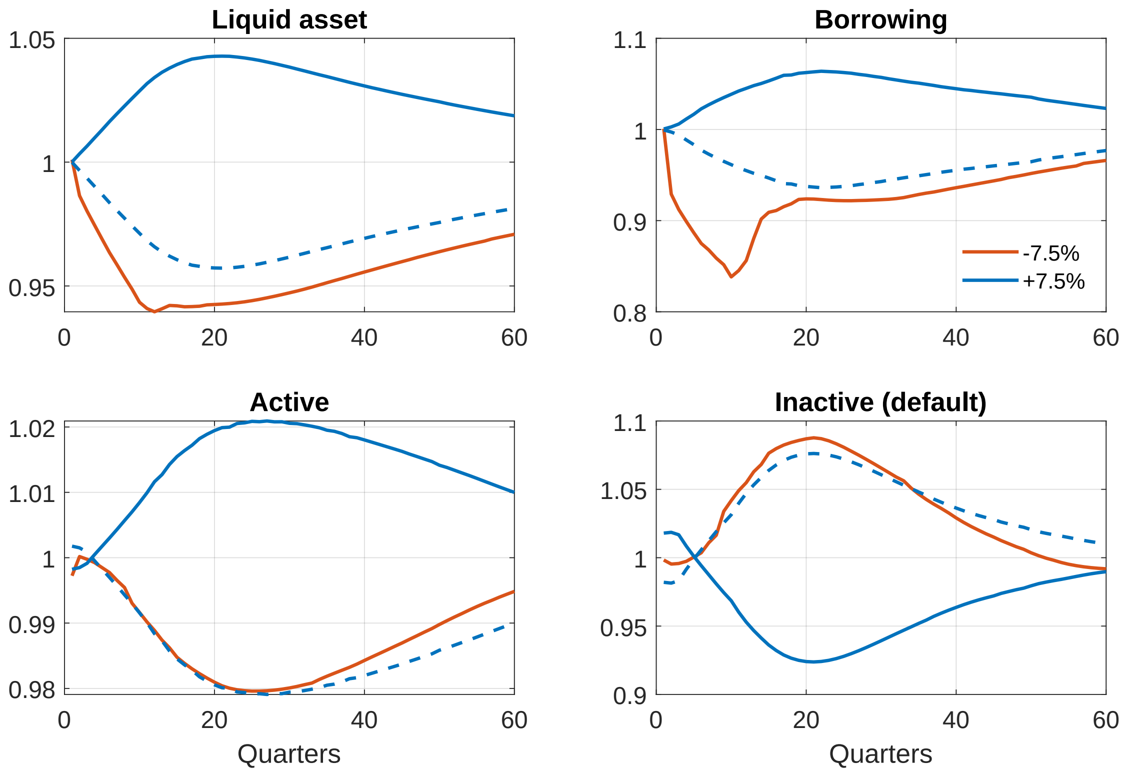
Comparing recessions with expansions, in the middle panel, we see that the fluctuation in output from changes in individual decision rules is largely similar. However, the recession sees a smaller decrease in output compared to the exogenous fall in aggregate TFP as falling wages and interest rates partially offset the effects of the productivity shock.

The right panel of Figure 5 shows that the change in output arising from the change in the measure of entrepreneurs, at each point in their individual state space, weighted by their equilibrium decision rules. Changes in the distribution of active entrepreneurs have almost three times greater an effect in reducing output in a recession as they do in increasing output in an expansion. The aggregate non-linearity we are seeing arises from this asymmetry in the effect of the distribution.

We see that the sharper downturn seen in Figure 2 and associated with a fall in measured TFP in recessions (Figure 3) is a caused by changes in the distribution of entrepreneurs. This distribution will change if entrepreneurs disproportionately exit during recessions and if entrepreneurs adjust their assets and debt. We find that the changes in distribution is mainly caused by the latter. The lower panels of Figure 6 show the stock of active entrepreneurs as well as those are inactive and have default in their credit history. There is little difference between recessions and expansions when comparing the percentage deviation from steady state for active and inactive entrepreneurs.

Turning to assets and debt, we see large differences between expansions and recessions. Figure 6 shows two percentage points larger fall in assets in the recession than the rise in the expansion. Moreover, the fall in assets is more rapid. This larger, faster decrease during

Figure 6: Assets, debt and entrepreneurship



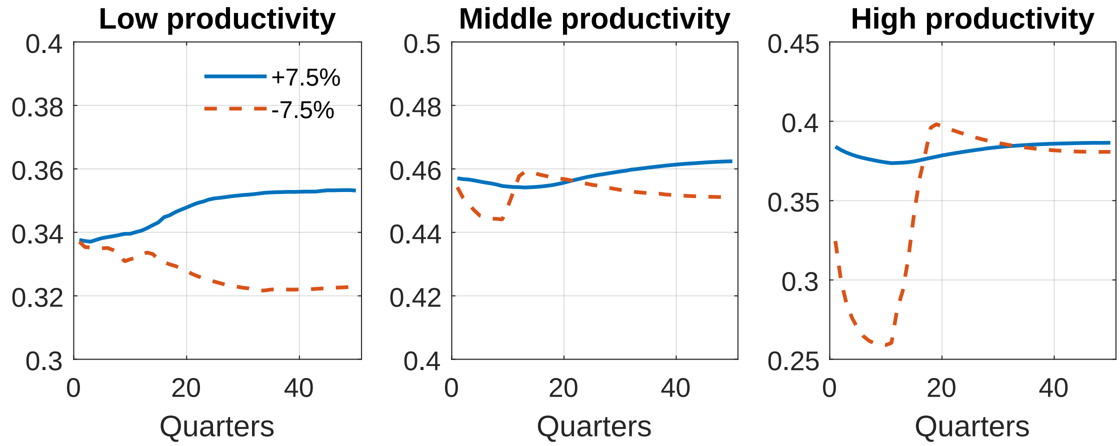
Note: The blue dashed lines illustrate the negative of the response of each series in the expansion, for easier comparability with the recession. All series are normalized at their steady state level.

the recession is driven by entrepreneurs who use assets to reduce their debt. The upper right panel shows a sharp fall in debt of more than 15% in the recession. In contrast, the expansion involves a gradual rise of 5%. As a significant share of entrepreneurs (43% in the steady state) are constrained by their collateral, the decline in assets reduces their capital relative to their productivity. This increases misallocation and reduces measured TFP.

As the aggregate debt-to-asset ratio is less than one, assets fall by a lesser percentage than debt in recessions, and there is a decline in aggregate leverage. Figure 7 shows changes in leverage across entrepreneurs. The three panels show aggregate leverage by productivity type, the series in recessions are red and blue indicates their levels over an expansion. We see that firms adjust debt more rapidly in recessions than in expansions. Low productivity firms experience a moderate decline in their leverage in a recession, and a small rise in an expansion. The fall in recessions is more pronounced for middle productivity firms. They see little change in expansions. What is striking is the large deleveraging by high productivity firms, in a recession, compared to the small change in expansions.

Entrepreneurs reduce debt to lower income risk. Entrepreneurs with large debt payments

Figure 7: Changes in leverage across entrepreneurs



run the risk of a large fall in income following a negative idiosyncratic productivity shock. While such a productivity shock lowers profits, debt principal payments remain unchanged. This implies that net disposable income falls by more when an entrepreneur has long-term debt. Figure 17 in Appendix C shows that high productivity entrepreneurs, with leverage of 80 percent, experience between 20 p.p. and 27 p.p. larger fall in their net income, following a negative idiosyncratic productivity shock, compared to an entrepreneur without debt. While heavily indebted entrepreneurs may be able to lower debt using assets once their productivity falls, such events often coincide with a shock to their ability to obtain new loans. As many highly productive entrepreneurs hold large levels of debt (Figure 15 and Figure 18 in Appendix C), they deleverage to avoid the risk of a large drop in their income.

Small firms' deleveraging during recessions Our results have shown that deleveraging amplifies the fall in output during a recession. Since entrepreneurs are relatively small, we seek evidence on changes in small firms' debt over recessions. As our model shows that there is heterogeneity in deleveraging among small firms, we focus on differences within them.

While a growing number of studies have used data on private and public firms' balance sheets, they have mostly focused on the difference between large and small or public and private firms.[23] These papers find that, compared to large firms, small, young firms are more responsive to monetary policy shocks (Caglio et al. (2021)) and more cyclical (Crouzet and Mehrotra (2020) and Clymo and Rozsypal (2019)). Dinlersoz et al. (2018) find that private

[23] Caglio et al. (2021) and Chodorow-Reich et al. (2022) use FR Y-14Q data, Crouzet and Mehrotra (2020) use the Quarterly Financial Report (QFR) of the US Census Bureau. Moody's Bureau van Dijk data is used in Dinlersoz et al. (2018), Kalemli-Özcan et al. (2018) and Zetlin-Jones and Shourideh (2017)). Clymo and Rozsypal (2019) use firm data, which includes private firms, from Statistics Denmark and Huynh and Petrunia (2010) use Statistics Canada.

firms deleverage during recessions while public firms do not. However, to the best of our knowledge, there is no work focusing on differences within smaller firms.

The following is suggestive. We compare debt growth rates by firm size in our model to the Small Business Credit Survey (SBCS) data, which covers businesses that hire less than 500 employees. This corresponds to entrepreneurs in our model. The SBCS started in 2016, therefore the only recession it includes involved the Pandemic. The SBCS reports shares of small businesses that have debt balances in specific ranges. We approximate debt by employment size using the median value of each range.[24] When we compute balances between 2020 and 2021, relatively large businesses (employment size 50-499) lowered debt by 7.7% while smaller firms raised their debt. Table 5 in Appendix C shows that our model has large firms reducing their debt by more than small firms. Computing the change in debt between the first and second year of a recession, we find that firms that hire 50-499 employees reduce their debt by 4.2%, while smaller firms either increase their debt or reduce it by less.

Table 5: Change of debt during a recession (%)

Employment size	1-4	5-9	10-19	20-49	50-499
Data	4.0	11.8	12.7	2.8	-7.7
Model	1.9	-2.9	-2.3	-1.9	-4.2

Note: Data is from SBCS, it is the growth rate between 2020 and 2021. The data provide share of firms that has a particular range of debt ($25,000 or less, $25,001-$50,000, $50,001-$100,000, $100,001-$250,000, $250,001-$1,000,000, and more than $1,000,000). We use median value of each range to approximate debt balance, and only count those who have positive debt balance (2020 survey does not provide the share of firms with zero balance.). The approximated debt balance is deflated by Consumer price index for all urban consumers (BLS). For model, we compute the growth rate of debt between the first and second year of a recession.

Skewness Until now, we have focused on asymmetries in aggregate quantities. Recently, a few papers have found asymmetries over business cycle in higher moments. For example, Salgado et al. (2019) find procyclical skewness in the distribution of firm employment, sales, and productivity growth rates, and Kehrig (2015) finds counter-cyclical dispersion in their productivity.

To see whether our model is consistent with these findings, we simulate a panel of a large number of entrepreneurs (80,000) and risk-neutral firms (337) over the transition along positive and negative TFP shocks.[25] Our model generates procyclical skewness in the growth

[24] Figure 7 reported entrepreneurs' leverage by their productivity level. However, as employment and productivity are positively correlated in our model, we explore its consistency with the empirical employment-leverage correlation.

[25] In calibration, risk neutral firms is 0.42% of all firms which maps to firms that hire more than 500

rate of employment.[26] We compare the first four periods of transition of a recession (expansion). Skewness falls by 0.38 in the recession, relative to its long-run level, while it remains close to this level during the expansion. In Salgado et al. (2019), skewness falls to -0.1, in a typical recession, from an average value of 0.11. However it changed by 0.31 in the Great Recession. The latter decrease is perhaps most comparable to ours as we study a large recession driven by a 7.5% fall in TFP. Our model is able to match the cyclicality of firms skewness as entrepreneurs' adjustments of assets and debt change resource allocation and therefore the distribution of firm production, without imposing cyclical idiosyncratic shocks.

5.2 Leverage and recoveries from large recessions

We have seen that firms' deleveraging contributes to a downturn in aggregate economic activity. Now, we show that high levels of debt can slow down an economic recovery. We examine an economy where non-financial business leverage is 10 percentage points higher than in the steady state of our model. This is the difference in the level of aggregate leverage, relative to its historical average, during the Great Recession and at the onset of the last recession.[27] The right panel of Figure 8 shows that the steep increase in leverage over the Great Recession was associated with a fall in asset values and not with a rise in debt. In keeping with the changes observed in the aggregate data, below we study a rise in leverage driven by a fall in assets.

In our model, where the relative price of assets in terms of consumption does not vary, we increase leverage by reducing entrepreneurs' assets while holding their debt constant.[28] We avoid any immediate effect on aggregate capital by transferring the reduction in entrepreneurs' assets to the government. The government uses these assets to fund capital spending and all interest income from this activity is lump-sum rebated to workers and entrepreneurs. Thus, the shift in the ownership of capital from entrepreneurs to the government increases business leverage, and implies redistribution from wealthy entrepreneurs, who lose

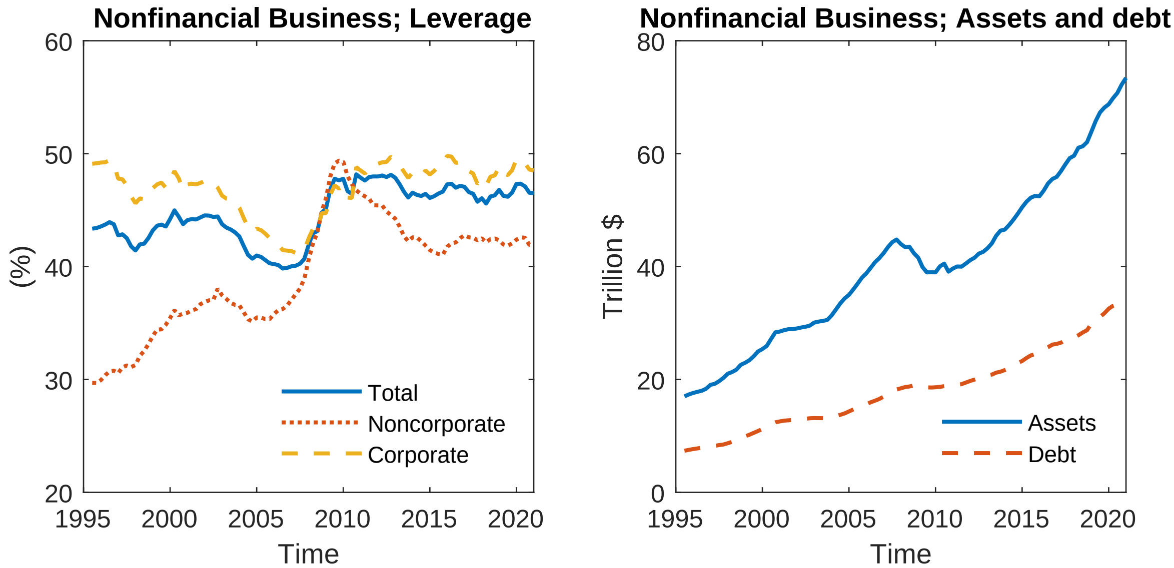
employees in the BDS. The number of risk-neutral firms in our simulation, 337, is approximately 0.42% of 80,337.

[26]Following Salgado et al. (2019), we calculate the growth rate of employment using the arc-percentage change between periods t and $t+1$ which is given by $g_{i,t} = \frac{l_{t+1}-l_t}{0.5(l_{t+1}+l_t)}$. Then we compute weighted the Kelly Skewness $(\frac{p90-p50}{p50-p10} - \frac{p50-p10}{p50-p10})$ of employment growth rates, using $\frac{1}{2}(l_t + l_{t+1})$ for weights.

[27]The average of non-financial firm leverage between 1954 and 2006 was 37.2%. In 2007Q4, at the start of the Great Recession, it stood at 40.7 percent, rising to 47.6 percent by the end of 2009. It is remained at this level and at the onset of the pandemic when, in 2020Q1, aggregate leverage was 47.3 percent.

[28]We raise leverage 10 percentage point by reducing each entrepreneur's assets by 23 percentage, or until they are at the boundary of their default region, whichever is less. This avoids any sharp increase in bankruptcy as we increase leverage.

Figure 8: Non-financial business leverage



Source: Board of Governors

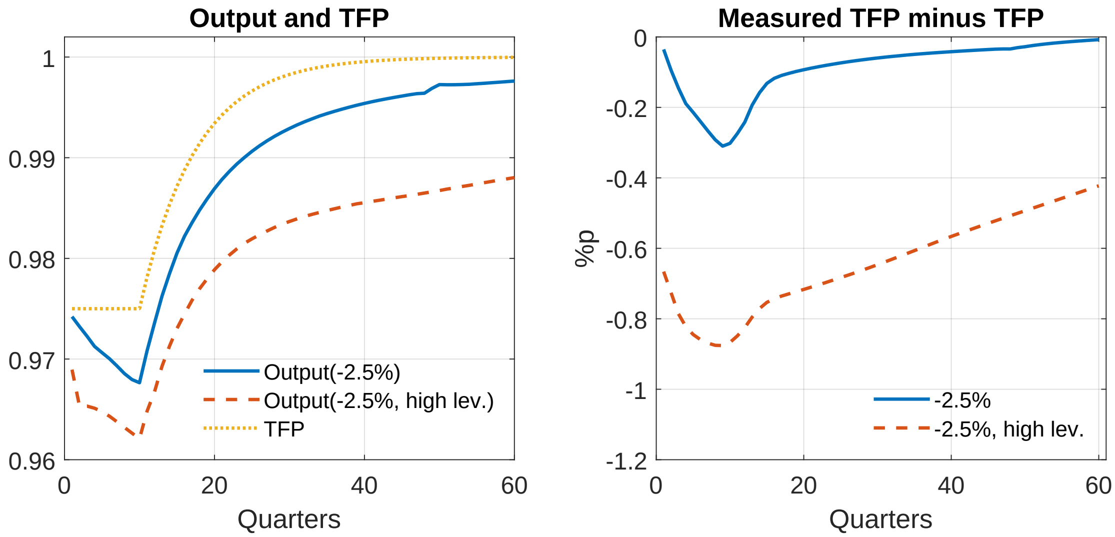
a substantial quantity of assets, to poorer entrepreneurs, who see a smaller reduction in their level of assets, and workers.

As seen in Figure 9, higher initial leverage implies a sharper fall in GDP following a persistent 2.5 percent shock to aggregate TFP. GDP in the high leverage economy falls by roughly a half percentage point more than in the ordinary economy where initial leverage equals its steady state level. Startlingly, the half-life of the recovery more than doubles, rising from 6 quarters to 13. The left panel of the figure shows this slower recovery involves a far larger, and persistent, fall in measured TFP. Falling by about 0.3 percent in the ordinary leverage model, measured TFP falls by more than 0.8 percent with high leverage. Sixty quarters after the start of the recession, the measured TFP of the high leverage economy remains below its lowest level in the ordinary economy. Evidently, higher leverage is associated with a rise in misallocation which prolongs the recession.

Figure 10 shows that the larger falls in aggregate output, alongside its slower recovery, is attributable to production by entrepreneurs. In the right panel, we see that, compared to its response in the ordinary leverage economy, risk-neutral firms actually see a milder fall in production then rebound past their steady state output level. This is a result of the sharper fall in factor prices that occurs in the recession with higher leverage.

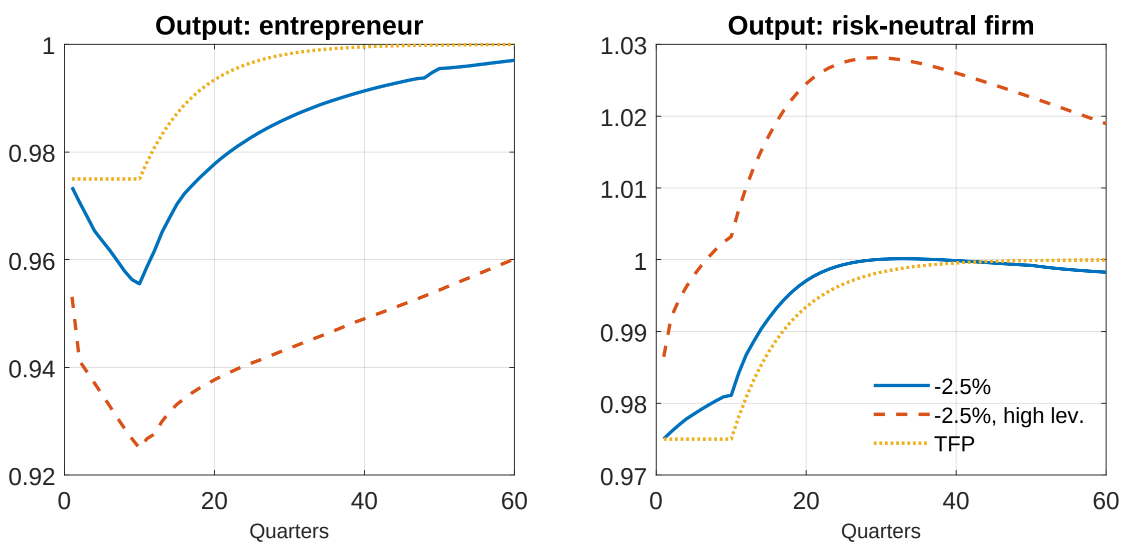
The effects of the fall in factor prices are also seen in middle panel of Figure 11. Here and in the right panel, we decompose entrepreneurs' output following Equation 6. As before, the middle panel shows the effect of changes in decisions rules using the steady state distribution

Figure 9: GDP and TFP



Note: Output is shown relative to its steady state level. Measured TFP is $\frac{Y}{K^{\alpha}L^{\nu}}$ where Y is GDP and K and L are the aggregate stocks of capital and labor.

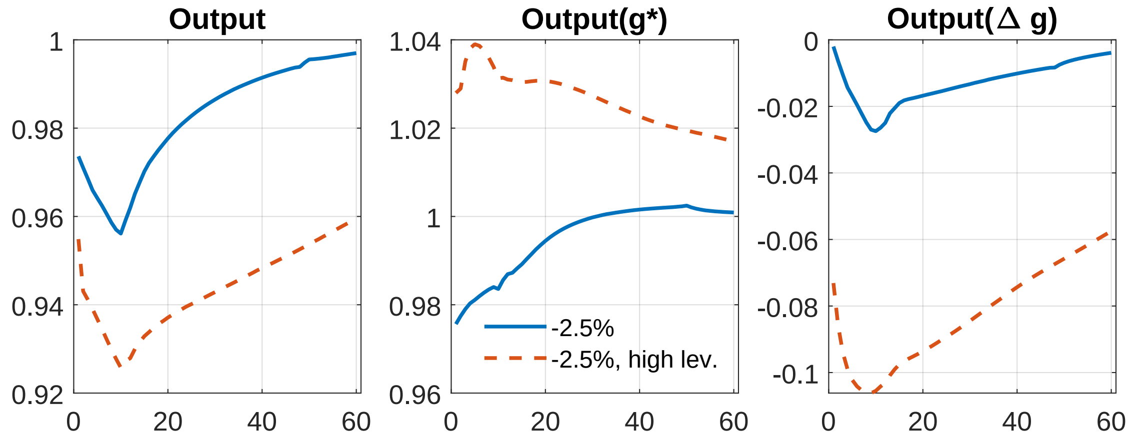
Figure 10: Output of entrepreneurs and risk-neutral firms



Note: The left panel shows aggregate output of entrepreneurs and the right panel shows those of the risk neutral firm. All series are normalized by their steady state level.

of entrepreneurs, while the right panel captures the effect on aggregate output of changes in this distribution. The more pronounced fall in wages and real interest rates, that follows the larger fall in measured TFP in the economy with high leverage, leads to higher levels of capital and labor chosen by firms at any given (z, a, b). This is seen in the red dashed line of the middle panel where changes in decision rules drive output above its steady state

Figure 11: Decomposing changes in output of entrepreneurs



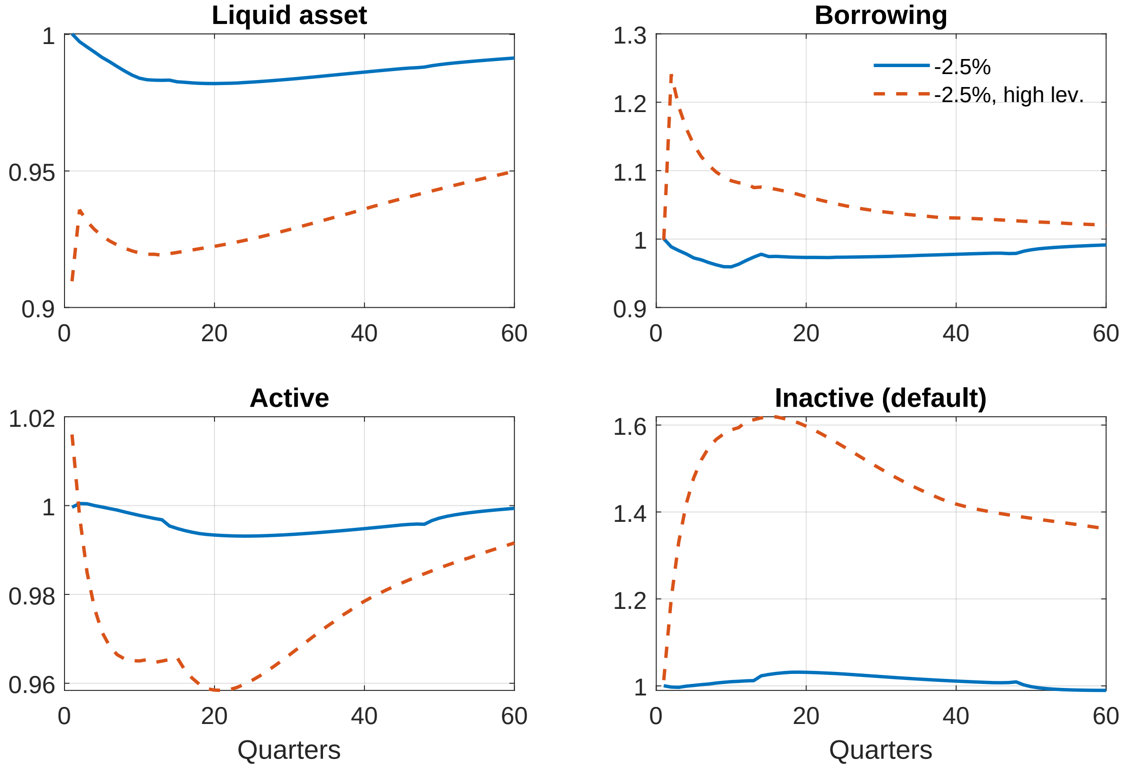
Note: Output series are normalized by their steady state levels. The middle panel shows the first term in Equation 6 which captures the effects of price changes to output and the right panel shows the second term in Equation 6.

level. Nonetheless, the left panel of the Figure 11 shows entrepreneurial production falling by almost twice as much as in the economy with higher debt relative to assets. Reconciling this with the effects of decision rules requires large changes in the distribution of firms.

As entrepreneurs' assets fall, they increase borrowing in an effort to dampen the fall in their collateral and capital (see the top right panel of Figure 12). The rise in borrowing over a period of low earnings eventually leads to an increase in default (lower right panel of Figure 12). Eventually, higher indebtedness drives balance sheet dynamics that increase the number of inactive entrepreneurs, with bankruptcy in their credit history, by 60 percent. Alongside the rise in the misallocation of resources amongst those continuing production, the result is a large change in the distribution of entrepreneurs. In the right panel of Figure 11, we see that these changes in the distribution are entirely responsible for the larger fall in production by entrepreneurs. This distributional effect is long-lived. One reason is that defaulting entrepreneurs must go through a period where their credit history prevents them from borrowing to start a new enterprise. Below we discuss additional reasons for the slow recovery.

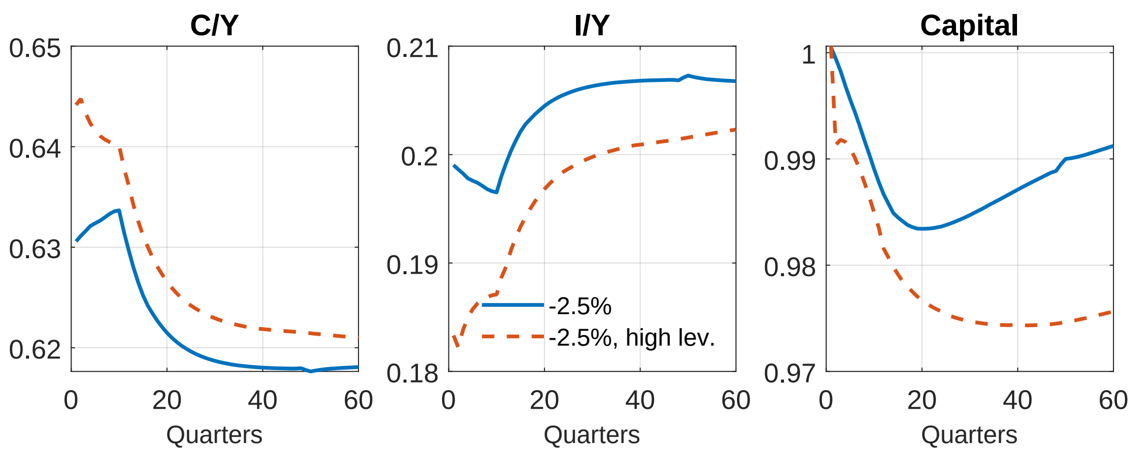
In Figure 12 we see that there is an initial jump down in total assets, held by workers and entrepreneurs, and a sharp rise in debt, in the economy with higher initial leverage. After these initial jumps, both assets and debt move slowly. In the ordinary leverage economy, liquid assets evolve smoothly and there is a small decline in debt and a mild rise in default. Examining the source of the large initial jump in debt, Figure 14 shows that not all entrepreneurs increase debt in the high leverage economy. In the right panels, we see that the sharp rise in debt at the beginning of the recession is driven by low-productivity

Figure 12: Assets, debt and entrepreneurship



Note: All series are normalized at their steady state level.

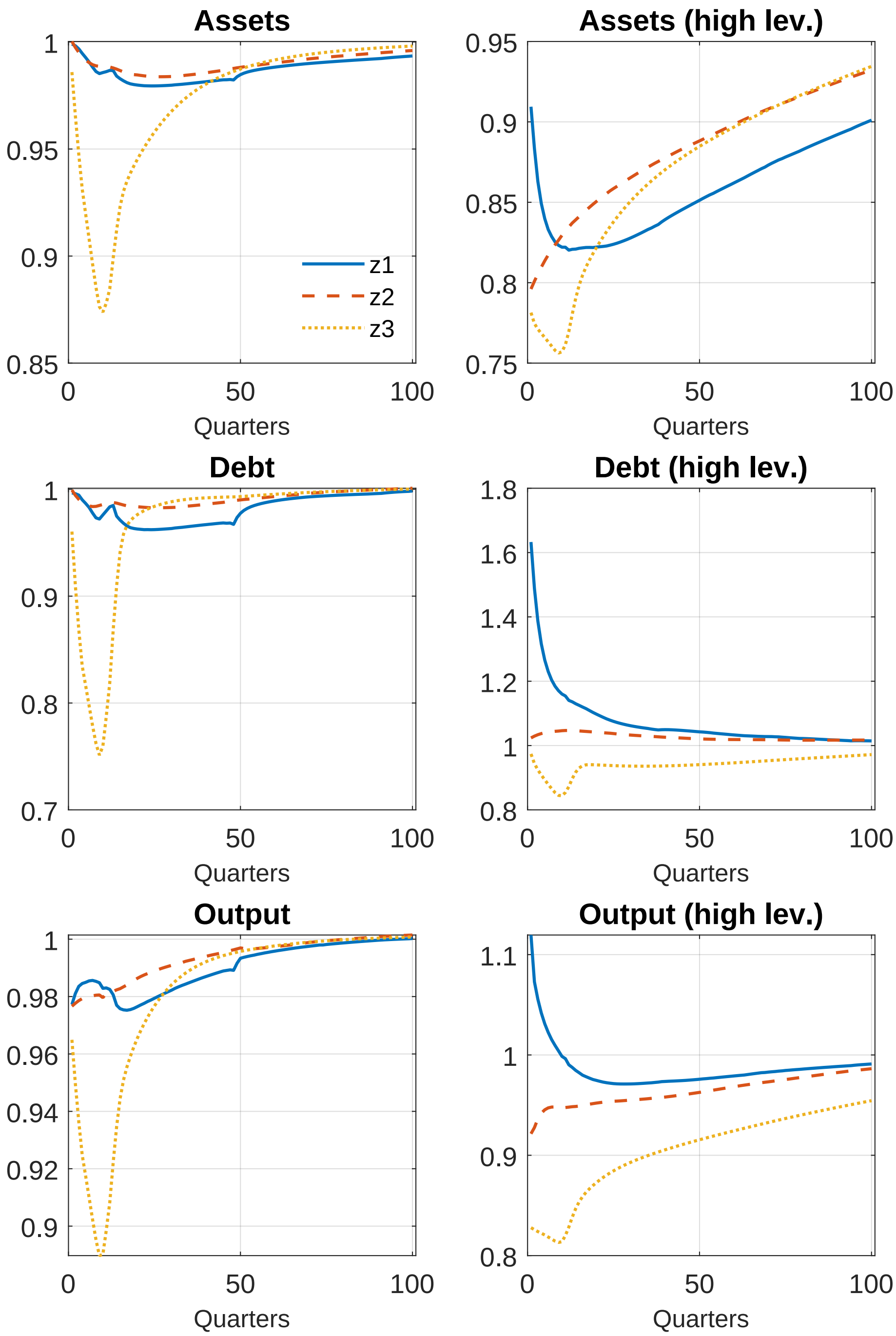
Figure 13: Consumption and investment shares of GDP and capital



Note: Capital series are normalized at their steady state level.

entrepreneurs. Their output initially increases as they take advantage of low factor prices by raising debt to overcome the drop in their assets. In contrast, as in the previous section on aggregate nonlinearities, high-productivity entrepreneurs lower debt and assets causing a

Figure 14: Assets, debt and output of entrepreneurs by productivity



large fall in their output. This explains most of the larger fall in output in the high leverage economy.

The slow rebuilding of assets in the higher than ordinary leverage economy implies a long-lived recession as entrepreneurs' consumption smoothing slows economic recovery. As their flow income after debt payment falls, entrepreneurs reduce savings relative to the economy without higher than ordinary leverage (Figure 13 left panel). Their assets fall from an already low level which, through the collateral constraint, leads to a reduction in the capital they use. As entrepreneurs decrease capital, investment falls. In Figure 13, we see that investment as a share of GDP falls by more in the high leverage case, compared to the ordinary leverage case. This leads to a weak recovery in capital taking 50 quarters to begin (Figure 13 right panel). The larger drop in aggregate capital stock in the economy with high leverage amplifies the downturn.

6 Concluding remarks

We have developed an incomplete markets model where production is managed by entrepreneurs who vary in their productivity, financial assets and debt. Entrepreneurs use financial assets as collateral for short-term production loans used to finance capital expenditures. Long-term, illiquid debt, which is costly to refinance, allows producers to increase their assets and thus capital. Entrepreneurs may default on these loans. We solve for equilibrium loan rate schedules consistent with individual borrowers' probability of repayment.

The distribution of debt implies large departures in the aggregate response of the economy when compared to a frictionless economy where entrepreneurs rent capital and hire workers. In recessions, as profits fall, existing principal repayments amplify the reduction in net earnings. In an effort to avoid large decreases in consumption or costly default, risk-averse entrepreneurs use financial assets to reduce debt. This decreases their ability to finance capital used in production. Across the economy, the allocation of resources worsens, endogenously reducing aggregate total factor productivity.

If many businesses are highly indebted at the beginning of a recession, insolvencies rise, businesses exit, and the number of continuing producers falls. This reduces efficiency in the allocation of capital and labor, and propagates an initial aggregate shock. As default implies a period when producers cannot obtain further loans, the number of firms rises gradually. This slows the economic recovery, doubling the half-life of output.

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A Complete markets model

We describe a complete markets model using a large household that pools all earnings risk over its members and allocates consumption equally across them. A fraction of members supply labor as workers, and the remainder are entrepreneurs.

As in the incomplete markets model, an entrepreneur may be active and operating a firm, or it may supply labor. The value of having an entrepreneur supply labor is $W^N(z,g)$ when the aggregate state g implies real interest rates and wages of $r(g)$ and $w(g)$. If the entrepreneur operates a firm, the value of having her do so is $W^E(z,g)$. The start of period stopping choice for an active entrepreneur is $W_0^E(z,g)$.

$$W_0^E(z,g) = \max\left\{W^E(z,g), W^N(z,g)\right\}$$

If the entrepreneur operates a firm, its value satisfies the following HJB equation,

$$r(g)\,W^E(z,g) = \pi(z,g) + \sum \pi_{zz'} W^E(z',g) - \lambda_{10}(W^E(z,g) - W^N(z,g)). \tag{7}$$

Above, profits, $\pi(z,g)$, is derived in (14) below.

If the entrepreneur has been inactive, its start of period stopping choice is

$$W_0^N(z,g) = \max\left\{W^E(z,g) - \kappa(z,g), W^N(z,g)\right\}.$$

Should the entrepreneur remain inactive, its value follows the HJB equation

$$r(g)\,W^N(z,g) = zw(g) + \sum \pi_{zz'} W^N(z',g). \tag{8}$$

A.1 Production with complete markets

Production by an entrepreneur operating a firm with idiosyncratic productivity z is $y = zk^\alpha l^\nu$. As capital and labor are rented, and there is no collateral constraint, profits are

$$\pi(z;g) \equiv \max_{k,l} zk^\alpha l^\nu - (r+\delta)k - wl - \xi(z), \tag{9}$$

where $\xi(z)$ is the operating cost. ptimal choices satisfy

$$\alpha z k^{\alpha-1} l^\nu = r + \delta \tag{10}$$

$$\nu z k^\alpha l^{\nu-1} = w, \tag{11}$$

which imply the following solutions for k and l,

$$k = \left(z\alpha^{1-\nu}\nu^{\nu}\left(r+\delta\right)^{\nu-1}w^{-\nu}\right)^{\frac{1}{1-(\alpha+\nu)}}, \tag{12}$$

$$l = \left(z\alpha^{\alpha}\nu^{1-\alpha}\left(r+\delta\right)^{-\alpha}w^{\alpha-1}\right)^{\frac{1}{1-(\alpha+\nu)}}. \tag{13}$$

Together, the solutions for capital and labor in terms of r, w and z, imply

$$\pi\left(z,g\right) = \left(z\right)^{\frac{1}{1-(\alpha+\nu)}}\alpha^{\frac{\alpha}{1-(\alpha+\nu)}}\nu^{\frac{\nu}{1-(\alpha+\nu)}}\left(r+\delta\right)^{\frac{-\alpha}{1-(\alpha+\nu)}}w^{\frac{-\nu}{1-(\alpha+\nu)}}\left(1-\alpha-\nu\right)-\xi(z). \tag{14}$$

The risk-neutral firms' problem is unaltered and described by 3. All dividends now are paid to the large household. This complete markets model will operate as if there is a representative household. Given σ, the coefficient of relative risk aversion, aggregate consumption will satisfy,

$$\dot{C}/C = \frac{r\left(g\right)-\rho}{\sigma}. \tag{15}$$

B Computation

This section describes the computational method used to solve the model. The way we solve the value function is based on the finite difference method described in Achdou et al. (2017) with several differences. First, there are multiple stopping choices including default on long-term debt, refinancing, entry and exit. We solve the steady state first and solve perfect foresight paths to study recessions and expansions. We find the equilibrium prices using the Broyden algorithm.

B.1 Solving steady state

Solving the steady state involves finding an equilibrium interest rate, r, and a wage, w, that clear the capital and the labor markets. Our model does not have the structure of Aiyagari (1994) model where aggregate capital determines real interest and wage rate through a representative firm. Thus, we need to check whether the prices we impose imply excess supply (or demand) in the capital and labor market. To find two equilibrium prices, we use the Broyden algorithm. Given $\mathbf{x}^n$, $\mathbf{B}^n$, and $f(\mathbf{x}^n)$,

- Update the prices, $\mathbf{x}^{n+1} = \mathbf{x}^n. - \mathbf{B}^n f(\mathbf{x}^n)$ where $\mathbf{x}^n = [r, w]^T$
- Solve the value functions of households and risk neutral firms.

- Solve the stationary distribution, and compute the aggregate supply and demand for capital and labor.
- Compute $f(\mathbf{x}^{n+1}) = [\text{excess supply of capital}, \text{excess supply of labor}|\mathbf{x}^{n+1}]^T$. If $\max(|f(\mathbf{x}^{n+1})|)$ is not small enough, return to the first step with updated $\mathbf{B}^{n+1}$.
  - Update the $\mathbf{B}^{n+1}$ using the Broyden algorithm,

$$\mathbf{B}^{n+1} = \mathbf{B}^n + \frac{(d^n - u^n)(d^n)^T\mathbf{B}^n}{(d^n)^T u^n}$$

$$d^n = \mathbf{x}^{n+1} - \mathbf{x}^n$$

$$u^n = \mathbf{B}^n(f(\mathbf{x}^{n+1}) - f(\mathbf{x}^n)).$$

Above, n means the n^{th} iteration. The initial $\mathbf{B}^0$ is the inverse of the numerical gradient, $\begin{bmatrix} \frac{\partial f_1(x^0)}{\partial x_1^0} & \frac{\partial f_1(x^0)}{\partial x_2^0} \\ \frac{\partial f_2(x^0)}{\partial x_1^0} & \frac{\partial f_2(x^0)}{\partial x_2^0} \end{bmatrix}$. We guess the prices, $\mathbf{x}^0 = [r, w]^T$, to compute $f(\mathbf{x}^0)$ and $\mathbf{B}^0$.

In the second step, we need to solve the value functions of worker-type households and entrepreneur-type households. Entrepreneur households' problems depend on their status, o: active with the option to borrow, active without the option to borrow, inactive, and inactive with a default flag. Solving the value function of the worker type and entrepreneur type with the default flag involves finding their consumption-savings decision only, here we follow Achdou et al. (2017). The remaining value functions - those of active and inactive entrepreneurs - involve finding stopping choices. For these problems, we follow Lee (2022) Appendix C. We rewrite our HJBVIs as linear complementarity problems (LCPs) and solve these using an available solver.[29]

We discretize worker-type households productivity into 21 points and assets into 500 points. For entrepreneur-type households, we use 3 grid points for productivity, 137 points for assets, and 29 points for debt. All assets and debt grids have more points at low levels.

B.2 Solving perfect foresight paths

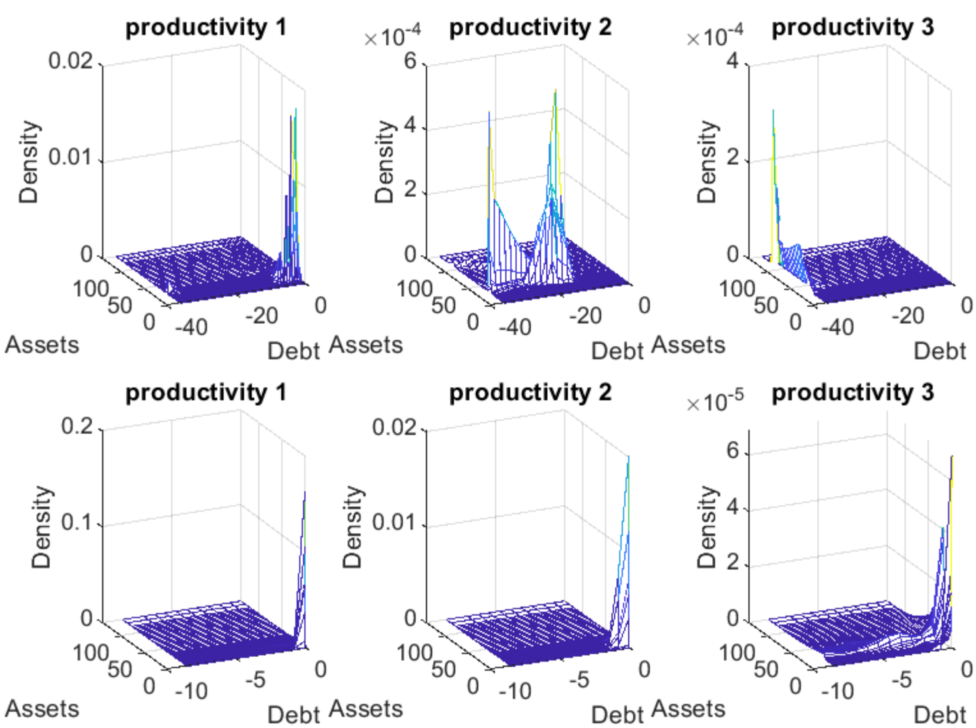
As above, we use the Broyden algorithm to find equilibrium price paths that clear the capital and the labor market in each period. Since we need to find paths of prices over simulation periods, nt, $\mathbf{x}^n$ becomes $[r_1, r_2, ...r_{nt}, w_1, w_2, ...w_{nt}]^T$, a vector of size $2 \times nt$ by 1. Accordingly, $f(\mathbf{x}^n)$ becomes a vector of the same size and each element is excess supply in the capital and the labor market from period 1 to nt. Except that, the algorithm is similar to the one for solving steady state.

[29] One difference is that our model does not have short-term debt default. We do not allow households to have short-term debt. Thus, we do not need to find endogenous borrowing limits for a given (b, z, o).

C Additional figures and tables

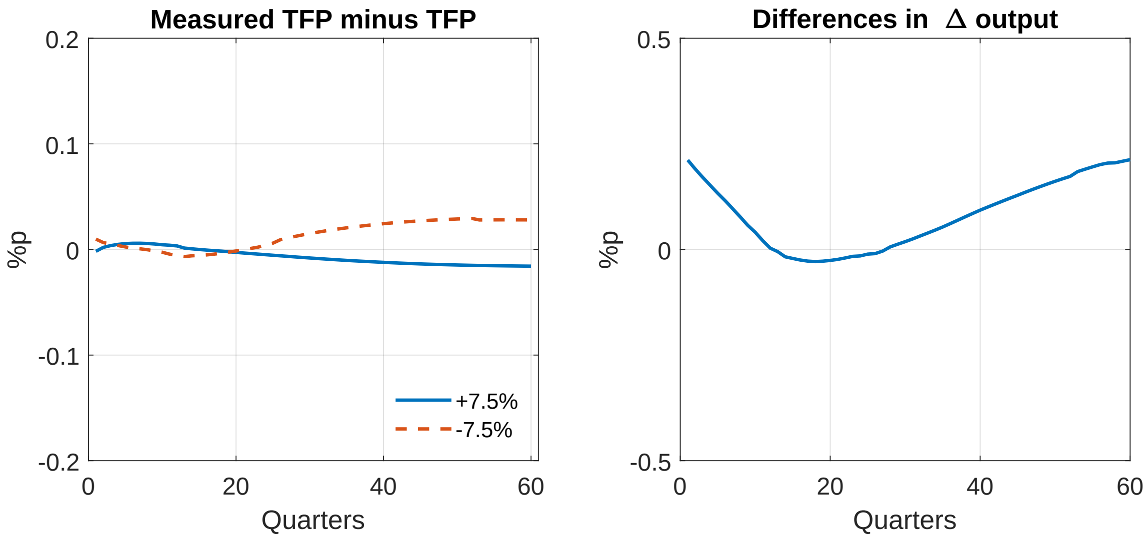
Figure 15 mentioned in Section 4. It shows the distribution of entrepreneurs over assets and debt, conditional on each productivity level. The top row shows entrepreneurs with the the ability to take new loans, while the bottom are those with a credit shock. As mentioned in Section 5.1, the difference between measured and exogenous TFP is small in a model without debt, seen in the left panel of Figure 16. The difference in output in recessions versus expansions, seen in the right panel, is far less than that seen in our benchmark model with debt. As mentioned in Section 5.2, Figure 17 shows the change in the income of high productivity entrepreneurs, after the aggregate shock. The top line represents those with the option to take new loans, and the dashed red line are those with the credit shock. Figure 18 is mentioned in Section 5.1. The figure shows densities of active entrepreneurs over leverage in steady state.

Figure 15: Steady state distribution by productivity z



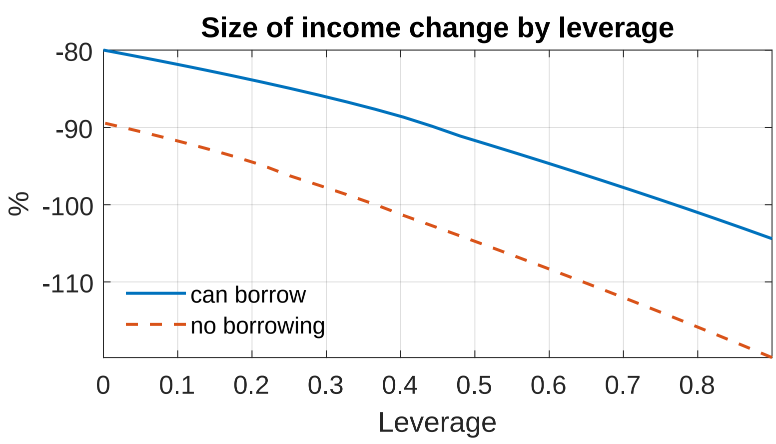
Note: The upper three panels are distributions of producers with the borrowing option and the lower three panels are distributions of producers without the ability to take new loans.

Figure 16: GDP and TFP in expansions and recessions: Without debt



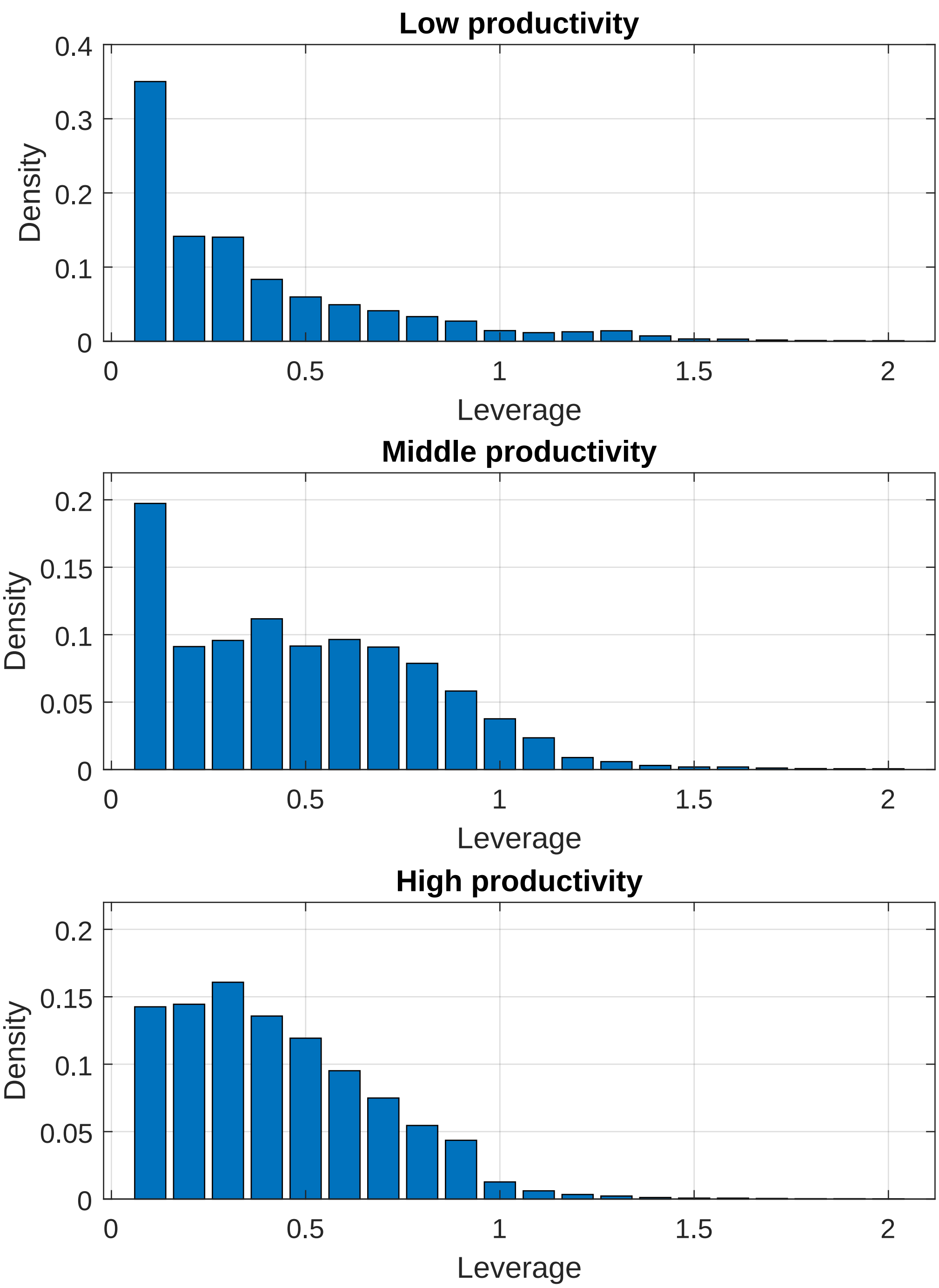
Note: Measured TFP is $\frac{Y}{K^{\alpha}L^{\nu}}$ where Y is GDP and K and L are the aggregate stocks of capital and labor. The right panel shows the difference in the distance from steady state GDP between the recession and the expansion; $abs(\frac{y_t(recession)-y^*}{y^*}) - \frac{y_t(expansion)-y^*}{y^*}$, where y^* is steady state output.

Figure 17: Income change of high productivity entrepreneur by leverage



Note: Expected income change of entrepreneurs with the highest productivity conditional on whether or not they can borrow. The series are computed at the net worth level with the largest mass of entrepreneurs at the beginning of the recession.

Figure 18: Distribution of active entrepreneurs over leverage



Note: Distribution of active entrepreneurs over leverage, by productivity type in steady state.