Foundations of self-progressive choice models*

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Abstract

Consider a population of heterogenous agents whose choice behaviors are partially *comparable* according to given *primitive orderings*. The set of choice functions admissible in the population specifies a *choice model*. A choice model is *self-progressive* if each aggregate choice behavior consistent with the model is uniquely representable as a probability distribution over admissible choice functions that are comparable. We establish an equivalence between self-progressive choice models and well-known algebraic structures called *lattices*. This equivalence provides for a precise recipe to restrict or extend any choice model for unique orderly representation. To prove out, we characterize the minimal self-progressive extension of rational choice functions, explaining why agents might exhibit *choice overload*. We provide necessary and sufficient conditions for the identification of a (unique) primitive ordering that renders our choice overload representation to a choice model.

Keywords: Random choice, heterogeneity, identification, unique orderly representation, lattice, multiple behavioral characteristics, ternary relations.

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Contents

1 Introduction **3**

2 Self-progressive choice models **7**

2.1 Equivalence between self-progressive models and lattices 8

3 Examples and discussion **11**

3.1 Rational choice and chain lattices 11

3.2 Beyond chain lattices 13

4 Minimal self-progressive extension of rational choice **15**

4.1 Identification of the primitive ordering 16

5 Universally self-progressive choice models **20**

6 Final comments **23**

7 Appendix **27**

7.1 Proof of Theorem 2 27

7.2 Proof of Lemma 2 28

1 Introduction

Random choice models are used successfully to identify heterogeneity in the aggregate choice behavior of a population. The success is achieved despite prominent choice models, such as the random utility model, are underidentified in the sense that the observed choice behavior renders different representations within the model. The typical remedy to this challenging matter has been structuring the model to obtain a unique representation and achieve point-identification.[1] Here, instead of focusing on a specific choice model, we adopt a different approach in which we start with *choice models* as our primitive objects, and assume an "orderliness" in the population that enables partial comparison of agents' choice behaviors.[2] We formulate and analyze *self-progressive choice models* that guarantee a unique orderly representation for each aggregate choice behavior consistent with the model. We believe that using self-progressive choice models could potentially facilitate organization and analysis of random choice data.

In our analysis we first establish an equivalence between self-progressive choice models and well-known algebraic structures called *lattices*. It follows from this equivalence that self-progressive models allow for specification of multiple behavioral characteristics that is critical in explaining economically relevant phenomena. Additionally, we obtain a precise recipe to restrict or extend any choice model to be self-progressive. To prove out, we characterize the minimal self-progressive extension of rational choice functions, which offers an intuitive explanation for why agents might exhibit *choice overload*. We also investigate how to identify the orderliness in the population that renders our choice overload representation to a choice model.

[1] See for example Gul & Pesendorfer (2006) and Dardanoni, Manzini, Mariotti, Petri & Tyson (2022).

[2] See the discussions by Apesteguia, Ballester & Lu (2017) and Filiz-Ozbay & Masatlioglu (2023).

The findings of two recent studies that use the orderliness in the population are precursory for our approach. Apesteguia, Ballester & Lu (2017) observe that if a random utility model is represented as a probability distribution over a set of comparable rational choice functions, then the representation must be unique. Filiz-Ozbay & Masatlioglu (2023) generalizes this observation by showing that each random choice function can be uniquely represented as a probability distribution over a set of choice functions that are comparable to each other. Both studies present intriguing examples in which agents' choices are ordered according to a single characteristic. However, it remains unclear if orderly representations exist for models that capture how agents' choices vary with different behavioral characteristics. A classical example is the equity premium puzzle (Mehra & Prescott 1985) that cannot be explained by maximization of CRRA or CARA utilities parameterized by the risk aversion coefficient. As for an explanation, Epstein & Zin (1989) proposed utility functions in which the coefficient of risk aversion and the elasticity of substitution are separated.[3] Our findings suggest that self-progressive models allow for specifying multiple behavioral characteristics separately. In the rest of the introduction, we describe the concept of self-progressiveness and our results.

Consider a population of agents who rank alternatives according to a *primitive ordering* that depends on the available alternatives, called a *choice set*. In addition to risk attitudes or social preferences that may be choice set independent, primitive orderings can accommodate, for example, the temptation or information processing costs that depend on the availability of more tempting or memorable alternatives. A pair of choice functions are *comparable* if the alternative chosen by one of the choice functions

[3]Another explanation based on agents' choices is Benartzi & Thaler (1995)'s *myopic loss aversion* that combines *loss aversion*–a greater sensitivity to losses than to gains–and a tendency to evaluate outcomes more frequently. Since two parameters should be specified separately, population heterogeneity explained by these models may not be consistent with a fixed set of choice functions ordered according to a single characteristic.

is ranked higher than the alternative chosen by the other for every choice set according to the associated primitive ordering.

The main object of our analysis is a *choice model*, which is simply the set of choice functions that may be adopted by any agent in the population. We call these choice functions *admissible*. Next, we describe our notion of *self-progressiveness*. Suppose that an analyst represents the aggregate choice behavior of a population as a probability distribution over a set of admissible choice functions. The same aggregate choice behavior renders a unique representation as a probability distribution over–possibly different–choice functions that are *comparable* to each other. Self-progressiveness requires these comparable choice functions to be admissible as well. In other words, a self-progressive choice model provides a language to the analyst that allows for orderly representing any aggregate choice behaviour that is consistent with the model.

In Theorem 1, we establish an equivalence between self-progressive choice models and *lattices*. For each pair of choice functions, their *join* (*meet*) is the choice function, choosing from each choice set the higher(lower)-ranked alternative among the ones chosen by the given pair of choice functions. A choice model forms a *lattice* if for each pair of admissible choice functions, their join and meet are admissible as well. By using a simple probabilistic decomposition procedure, we show that self-progressive choice models are the ones that possess a lattice structure. Thus, we present testable foundations of self-progressive choice models. It follows that self-progressive choice models are not limited to models consisting of comparable choice functions. To demonstrate the relevance of this generality, we present examples of choice models in which multiple behavioral characteristics are parametrized.

Theorem 1 provides for a precise recipe to restrict or extend any choice model for unique orderly representation. To prove out, in Theorem 2, we characterize the *minimal self-progressive extension* of rational choice functions via two choice axioms. The resulting model offers an intuitive explanation for why agents might exhibit *choice overload*.[4] In that, the axioms require a more valuable (or the same) alternative be chosen whenever we remove alternatives that are less valuable than the chosen one, or add alternatives that are more valuable than the chosen one.

So far, we assumed that the primitive ordering(s) is specified by the analyst. However, one might wonder how to infer the primitive ordering from the choice model. This would require focusing on a specific model. In Theorem 3, we provide necessary and sufficient conditions for the existence and uniqueness of a primitive ordering that renders our choice overload representation to a choice model. We use classical and modern results from foundational geometry to identify the primitive ordering.

We can also ask, in an abstract way, whether there are any choice models rendering unique orderly representations regardless of the primitive orderings. We observe that a choice model satisfies this stringent property, which we call *universal self-progressiveness*, if and only if the choice model corresponds to the maximizer set of an additively separable value function defined over choice functions.

[4]Choice overload refers to the phenomena that agents tend to deviate from their accurate preferences when they choose from complex environments. See Chernev, Böckenholt & Goodman (2015) for a recent meta-analysis.

2 Self-progressive choice models

Let X be the **alternative set** with n elements. A **choice set** S is a subset of X containing at least two alternatives. The **choice domain** Ω is a nonempty collection of choice sets allowing for limited data sets. A **choice function** is a mapping $c : \Omega \to X$ such that for each $S \in \Omega$, we have $c(S) \in S$. Let $\mathcal{C}$ denote the set of all choice functions. A **choice model** $\mu \subset \mathcal{C}$ is a nonempty set of choice functions. We consider two choice procedures with possibly different formulations as equivalent if these procedures are observationally indistinguishable in the revealed preference framework, that is, two choice procedures rationalize the same set of choice functions.

A **random choice function** (RCF) ρ assigns each choice set $S \in \Omega$ a probability measure over S. We denote by $\rho_x(S)$ the probability that alternative x is chosen from choice set S. A (deterministic) choice function can be represented by an $|\Omega| \times |X|$ matrix with rows indexed by the choice sets and columns indexed by the alternatives, and entries in $\{0, 1\}$ such that each row has exactly one 1. For each $(S, x) \in \Omega \times X$, having 1 in the entry corresponding to row S and column x indicates that x is chosen in S. Similarly, a RCF can be represented by an $|\Omega| \times |X|$ matrix having entries in $[0, 1]$ such that the sum of the entries in each row is 1. For each RCF and each pair $(S, x) \in \Omega \times X$, the associated entry indicates the probability that x is chosen in S. Then, it follows from Birkhoff-von Neumann Theorem (Birkhoff 1946, Von Neumann 1953) that each RCF can be represented as a probability distribution over a set of deterministic choice functions. However, this representation is not necessarily unique. Let $\Delta(\mu)$ be the **random choice model** associated with a choice model μ, which is the set of RCFs that can be represented as a probability distribution over choice functions contained in μ.

For each choice set $S \in \Omega$, a **primitive ordering** $>_S$ is a complete, transitive, and asymmetric binary relation over S. We write $\geq_S$ for its union with the equality relation. Then, we obtain the partial order $\rhd$ from the primitive orderings such that for each pair of choice functions c and c', we have $c \rhd c'$ if and only if $c(S) \geq_S c'(S)$ for each $S \in \Omega$, and $c(S) \neq c'(S)$ for some $S \in \Omega$. We write $c \unrhd c'$ if $c \rhd c'$ or $c = c'$.

Definition. Let $\rhd$ be the partial order over choice functions obtained from the primitive orderings $\{>_S\}_{S\in\Omega}$. Then, a choice model μ is **self-progressive** with respect to $\rhd$ if each RCF $\rho \in \Delta(\mu)$ can be uniquely represented as a probability distribution over a set of choice functions $\{c^1, \ldots, c^k\} \subset \mu$ such that $c^1 \rhd c^2 \cdots \rhd c^k$.

As formulated by Filiz-Ozbay & Masatlioglu (2023), a RCF ρ has a **progressive representation** if it can be represented as a probability distribution over a set of choice functions $\{c^1, \ldots, c^k\} \subset \mu$ such that $c^1 \rhd c^2 \cdots \rhd c^k$. To see that a progressive representation is unique whenever it exists, consider the $\rhd$-best choice function c^1 in a progressive representation. Note that c^1 chooses the $>_S$-best alternative that is assigned positive probability by ρ in each $S \in \Omega$. Therefore, the probability weight of c^1 is the lowest probability of $c^1(S)$ being chosen from any S. Thus, c^1 and its probability weight are determined uniquely. Repeating this argument shows that the whole progressive representation is unique.

2.1 Equivalence between self-progressive models and lattices

Let $\{>_S\}_{S\in\Omega}$ be the primitive orderings and $\rhd$ be the associated partial order over choice functions. For each pair of choice functions c and c', their *join* (*meet*) is the choice function $c \vee c'$ $(c \wedge c')$ that chooses from each choice set S, the $>_S$-best(worst) alternative among the ones chosen by c and c' at S. Then, for each choice model μ, the pair $\langle \mu, \rhd \rangle$ is a **lattice** if for each pair of choice functions c and c' in μ, their join $c \vee c'$ and meet $c \wedge c'$ are contained in μ as well.

Theorem 1. *Let μ be a choice model and $\rhd$ be the partial order over choice functions that is obtained from the primitive orderings $\{>_S\}_{S\in\Omega}$. Then, μ is self-progressive with respect to $\rhd$ if and only if the pair $\langle \mu, \rhd \rangle$ is a lattice.*

To see that the *only if* part holds, let $c, c' \in \mu$. Then, consider the RCF ρ such that for each $S \in \Omega$, $c(S)$ or $c'(S)$ is chosen evenly. Note that ρ has a unique progressive representation in which only $c \vee c'$ and $c \wedge c'$ receive positive probability. Since μ is self-progressive, it follows that $c \vee c' \in \mu$ and $c \wedge c' \in \mu$.

As for the *if* part, suppose that $\langle \mu, \rhd \rangle$ is a lattice, and let $\rho \in \Delta(\mu)$. Next, we present our **uniform decomposition procedure**, which yields the progressive random choice representation for ρ with respect to $\rhd$. Figure 1 demonstrates the procedure.

Step 1: For each choice set S, let $\rho^+(S) = \{x \in S : \rho(x, S) > 0\}$, and partition the $(0, 1]$ interval into $|\rho^+(S)|$ intervals $\{I_{Sx}\}_{\{x\in\rho^+(S)\}}$ such that each interval I_{Sx} is half open of the type $(l_{Sx}, u_{Sx}]$ with length $\rho(x, S)$, and for each $x, y \in \rho^+(S)$ if $x >_S y$, then l_{Sx} is less than l_{Sy}.

Step 2: Pick a real number $r \in (0, 1]$ according to the Uniform distribution on $(0, 1]$. Then, for each choice set and alternative pair (S, x), let $c(S) = x$ if and only if $r \in I_{Sx}$. It is clear that this procedure gives us a unique probability distribution over a set of choice functions $\{c^1, \dots, c^k\}$ such that $c^1 \rhd c^2 \cdots \rhd c^k$.[5] Next, we will show that $\{c^1, \dots, c^k\} \subset \mu$ by using Lemma 1.

[5]See Theorem 1 by Filiz-Ozbay & Masatlioglu (2023) for an elaborate proof of this fact. It is easy to see that this procedure is applicable even if the choice space is infinite. In a contemporary study, Petri (2023) independently extends Theorem 1 by Filiz-Ozbay & Masatlioglu (2023) to infinite choice spaces.

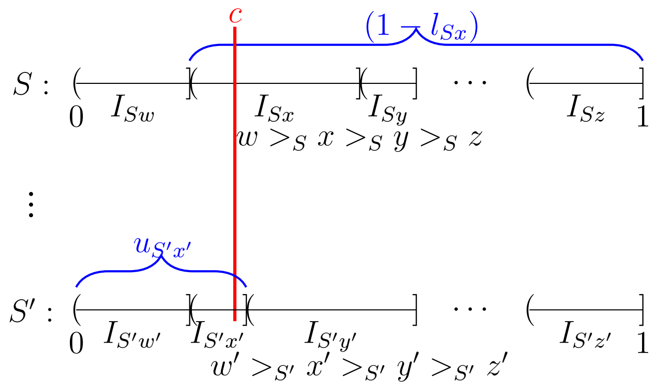


Figure 1

Lemma 1. *Let μ be a choice model such that $\langle \mu, \triangleright \rangle$ is a lattice. Let $c \in \mathcal{C}$ be a choice function. If for each given $S, S' \in \Omega$, there exists a choice function $c^* \in \mu$ such that $c^*(S) = c(S)$ and $c^*(S') = c(S')$, then $c \in \mu$.*

Proof. The result is obtained by applying the following observation inductively. Consider any $\mathbb{S} \subset \Omega$ containing at least three choice sets. Let $c_1, c_2, c_3 \in \mu$ be such that for each $i \in \{1, 2, 3\}$, there exists at most one $S_i \in \mathbb{S}$ with $c_i(S_i) \neq c(S_i)$. Suppose that for each $i, j \in \{1, 2, 3\}$, if such S_i and S_j exist, then $S_i \neq S_j$. Now, for each $S \in \mathbb{S}$, we have $c(S)$ is chosen by the choice function $(c_1 \wedge c_2) \vee (c_1 \wedge c_3) \vee (c_2 \wedge c_3) \in \mu$. To see this, let $S \in \mathbb{S}$, and note that there exist at least two $i, j \in \{1, 2, 3\}$ such that $c_i(S) = c_j(S) = c(S)$. Assume without loss of generality that $i = 1$ and $j = 2$. Now, if $c(S) \geq_S c_3(S)$, then we get $c(S) \vee c_3(S) \vee c_3(S) = c(S)$; if $c_3(S) >_S c(S)$, then we get $c(S) \vee c(S) \vee c(S) = c(S)$. □

Proof of Theorem 1. We proved the *only if* part. For the *if* part, let c^r be a choice function that is assigned positive probability in the uniform decomposition procedure. We show that $c^r \in \mu$ by using Lemma 1. Suppose that $S, S' \in \Omega$ such that $x = c^r(S)$ and $x' = c^r(S')$. Since $\rho \in \Delta(\mu)$, it must be that μ contains a choice function choosing x from S and possibly a different choice function choosing x' from S'. We show that there exists $c^* \in \mu$ such that both $c^*(S) = x$ and $c^*(S') = x'$. Then, it will directly follow from Lemma 1 that $c^r \in \mu$.

First, as demonstrated in Figure 1, we have $(1 - l_{Sx}) + u_{S'x'} > 1$. Thinking probabilistically, this means that making a choice better than x in S and worse than x' in S' are not mutually exclusive events. Since $\rho \in \Delta(\mu)$, it follows that there exists $c_1 \in \mu$ such that $c_1(S) \leq_S x$ and $c_1(S') \geq_{S'} x'$. Symmetrically, since $(1 - l_{S'x'}) + u_{Sx} > 1$, there exists $c_2 \in \mu$ such that $c_2(S) \geq_S x$ and $c_2(S') \leq_{S'} x'$. Next, consider $\{c \in \mu : x \geq_S c(S)\}$ and let c_x be its join. Similarly, consider $\{c \in \mu : x' \geq_{S'} c(S')\}$ and let $c_{x'}$ be its join. By construction, $c_x(S) = x$ and $c_{x'}(S') = x'$. Moreover, c_1 is a member of the former set, while c_2 is a member of the latter one. Now, let $c^* = c_x \wedge c_{x'}$. Then, $c^*(S) = x$ because $c_x(S) = x$ and $c_{x'}(S) \geq_S c_2(S) \geq_S x$. Similarly, $c^*(S') = x'$ because $c_{x'}(S') = x'$ and $c_x(S') \geq_{S'} c_1(S') \geq_{S'} x'$. □

3 Examples and discussion

3.1 Rational choice and chain lattices

We first observe that the rational choice model fails to be self-progressive. To see this, let $X = \{a, b, c\}$ and $\Omega = \{X, \{a, b\}, \{a, c\}, \{b, c\}\}$. Suppose that each primitive ordering is obtained by restricting the ordering $a > b > c$ to a choice set. Figure 2 demonstrates the associated choice functions lattice in which each array specifies the chosen alternatives respectively. The rational choice functions (dark-colored ones) fail to form a lattice. In that, each light-colored choice function is a join or meet of a rational choice function.

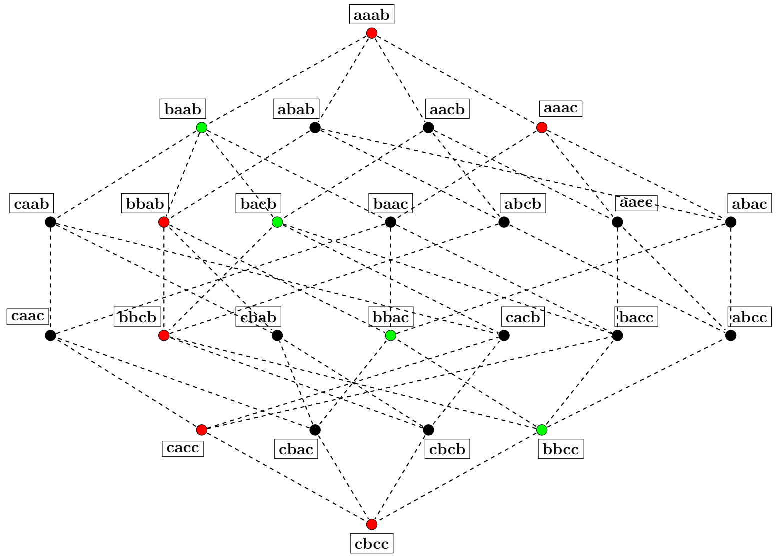


Figure 2: The choice functions lattice.

We can use the equivalence between self-progressiveness and lattices as a guide to restrict or extend rational choice model to be self-progressive. In this vein, a particularly simple lattice is a *chain lattice,* which is a set of choice functions $\{c_i\}_{i=1}^k$ that are comparable: $c_1 \rhd c_2 \cdots \rhd c_n$. Suppose that each primitive ordering $>_S$ is obtained by restricting the ordering $>_X$ to the choice set S. Then, there is a one-to-one correspondence between the chain lattices of rational choice model and the preferences with *single-crossing property* defined by Apesteguia, Ballester & Lu (2017).[6] To see this, let $\mu = \{c_i\}_{i=1}^k$ be a choice model consisting of choice functions rationalized by maximization of preferences $\{\succ_i\}_{i=1}^k$. Then, $\{\succ_i\}_{i=1}^k$ is *single-crossing with respect to* $>_X$ means: for each alternative pair $x >_X y$, if $x \succ_i y$, then $x \succ_j y$ for every $i > j$. It is easy to see that $\langle \mu, \rhd \rangle$ is a chain lattice if and only if $\{\succ_i\}_{i=1}^k$ is single-crossing with respect to $>_X$.[7]

[6]Additionally, if the choice domain Ω contains every choice set, then every lattice $\langle \mu, \rhd \rangle$ is a chain lattice. This is not true for a general domain of choice sets. For a simple example, suppose that the choice domain consists of disjoint binary choice sets. Then, every choice function is rational, thus every sublattice of choice functions is a set of rational choice functions.

[7]See also Lemma 1 by Filiz-Ozbay & Masatlioglu (2023).

Apesteguia, Ballester & Lu (2017) present economic examples of rational choice functions that form chain lattices.[8] Filiz-Ozbay & Masatlioglu (2023) present choice models that are not rational. However, these choice models also form chain lattices because the choice functions are ordered according to a single behavioral characteristic.

3.2 Beyond chain lattices

It follows from our Theorem 1 that self-progressive choice models are not limited to chain lattices, thus capture multiple behavioral characteristics of agents. We demonstrate this point with examples. In the first example, we present a choice model in which agents deviate from their accurate preferences when they choose from larger choice sets.

Example 1. *(Smaller-is-better)* Let $\mathcal{P}$ be a set of faulty preferences that are single-crossing with respect to the accurate preference $>$. Then, a choice function $c \in \mu$ if for each choice set S, the alternative $c(S)$ is the $\succ_S$-maximal one in S for some $\succ_S \in \mathcal{P}$. If S is a subset of S', then $\succ_S$ is more aligned with $>$ (less faulty) than $\succ_{S'}$. Note that μ is self-progressive with respect to the comparison relation obtained from $>$, since the join and meet of each $c^i, c^j \in \mu$ are the choice functions obtained by maximization of the preferences $max(\{\succ^i_S, \succ^j_S\}, \geq)$ and $min(\{\succ^i_S, \succ^j_S\}, \geq)$.

Example 2. *(Limited attention meets satisficing)*[9] Consider a population with primitive orderings $\{>_S\}_{S\in\Omega}$ in which each agent i has the same preference relation $\succ^*$, but a possibly different *threshold alternative* x^i_S for each choice set S. Then, for given choice set S, agent i chooses the $\succ^*$-best alternative in the consideration set $\{x \in S : x \geq_S x^i_S\}$. Let

[8]See also Curello & Sinander (2019) who characterize when a common primitive ordering over alternatives allows for preferences form a lattice according to single-crossing dominance, and provide several applications.

[9]See Simon (1955), Tyson (2008), Rubinstein & Salant (2008), and Masatlioglu, Nakajima & Ozbay (2012).

μ be the set of associated choice functions. Then, $\langle \mu, \rhd \rangle$–where $\rhd$ is obtained from $\succ^*$–is a lattice, since the join and meet of each $c^i, c^j \in \mu$ are the choice functions described by threshold alternatives $max(\{x^i_S, x^j_S\}, \geq_S)$ and $min(\{x^i_S, x^j_S\}, \geq_S)$.[10]

Example 3. *(Similarity-based choice)* Let (m, p) denote a *lottery* giving a monetary prize $m \in (0, 1]$ with probability $p \in (0, 1]$ and the prize 0 with the remaining probability. Consider a population of agents choosing from binary lottery sets[11] such that each agent i has a *perception of similarity* described by (ϵ^i, δ^i) with $\delta^i \geq \epsilon^i$ as follows: for each $t_1, t_2 \in (0, 1]$, "t_1 is similar to t_2" if $|t_1 - t_2| < \epsilon^i$ and "t_1 is different from t_2" if $|t_1 - t_2| > \delta^i$. Then, in the vein of Rubinstein (1988), to choose between two lotteries (m_1, p_1) and (m_2, p_2), agent i first checks if "m_1 is similar to m_2 and p_1 is different from p_2", or vice versa.[12] If one of these two statements is true, for instance, m_1 is similar to m_2 and p_1 is different from p_2, then the probability dimension becomes the decisive factor, and i chooses the lottery with the higher probability. Otherwise, each agent chooses the higher-ranked lottery according to a given primitive ordering $>^*$.

Let μ be the set of associated choice functions. Then, $\langle \mu, \rhd \rangle$–where $\rhd$ is generated from $>^*$–is a lattice, since for each $c^i, c^j \in \mu$, their join and meet are the choice functions whose perceptions of similarity are described by $(min(\epsilon^i, \epsilon^j), max(\delta^i, \delta^j))$ and $(max(\epsilon^i, \epsilon^j), min(\delta^i, \delta^j))$.[13]

[10] As a special case, consider agents who faces temptation with limited willpower formulated as by Masatlioglu, Nakajima & Ozdenoren (2020). Each agent i chooses the alternative that maximizes the common *commitment ranking* u from the set of alternatives where agent i overcomes *temptation*, represented by v^i, with his *willpower stock* w^i. Suppose that the primitive orderings are aligned with the commitment ranking u. Then, for each choice set S, let the threshold alternative x^i_S be the $>_S$-worst one such that $v^i(x) - max_{z \in S} v^i(z) \leq w^i$. As demonstrated by Filiz-Ozbay & Masatlioglu (2023) if we only allow agents' willpower stock to differ, then we obtain a choice model forming a chain lattice.

[11] One can assume that the monetary prizes and probability values have a finite domain.

[12] Rubinstein (1988) additionally requires one of these two statements be true. The slight difference is that our "t_1 is different from t_2" statement implies the negation of "t_1 is similar to t_2", while the converse does not necessarily hold. Both versions of the procedure provide explanations to the Allais paradox.

[13] Note that $\langle \mu, \rhd \rangle$ may not be a chain lattice since we can have $\epsilon^i > \epsilon^j$, while $\delta^i < \delta^j$.

4 Minimal self-progressive extension of rational choice

We will follow the guide provided by Theorem 1 to discover the "minimal" self-progressive extension of the rational choice model. We assume that there is a single primitive ordering $>$ rankings of which reflect alternatives' "accurate values" and Ω contains every choice set. The comparison relation $\rhd$ over choice functions is obtained from $>$ as usual. An extension is minimal if we are parsimonious in adding nonrational choice functions so that each choice model containing all rational choice functions and is contained in the extension fails to be self-progressive with respect to $\rhd$.[14] Next, we characterize the minimal self-progressive extension of the rational choice model in terms of two choice axioms. Figure 3 demonstrates the minimal extension when there are three alternatives.

Theorem 2. *Let μ^θ be the minimal self-progressive extension of the rational choice model with respect to $\rhd$. Then, a choice function $c \in \mu^\theta$ if and only if for each $S \in \Omega$ and $x \in S$,*

$\theta 1.$ *if $c(S) > x$ then $c(S \setminus x) \geq c(S)$, and*

$\theta 2.$ *if $x > c(S)$ then $c(S) \geq c(S \setminus x)$.*[15]

Proof. Please see Section 7.1 in the Appendix. □

Axioms $\theta 1$ and $\theta 2$ require a more valuable (or the same) alternative be chosen whenever we remove alternatives that are less valuable than the chosen one, or add alternatives that are more valuable than the chosen one. Along these lines–in an attempt to unravel the choice overload phenomena–Chernev & Hamilton (2009) experimentally demonstrate that consumers' selection among choice sets is driven by the value of the

[14] It follows from Theorem 1 that the minimal extension of any choice model is unique.

[15] *Independence from preferred alternative* formulated by Masatlioglu, Nakajima & Ozdenoren (2020) similarly require choice remain unchanged whenever unchosen better options are removed.

alternatives constituting the choice sets. In that, the smaller choice set is more likely to be selected when the value of the alternatives is high than when it is low. The proof of Theorem 2 demonstrates how to use Theorem 1 and Lemma 1 to obtain similar results.

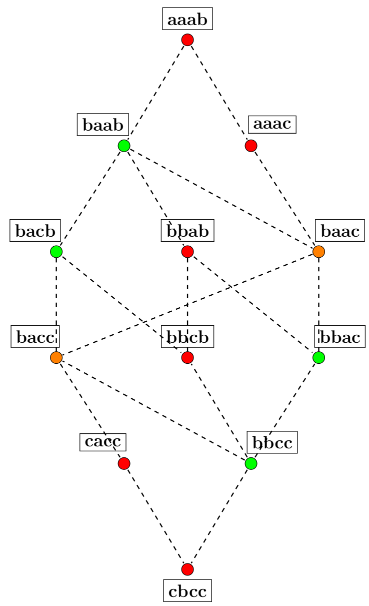


Figure 3: A demonstration of $\langle \mu^\theta, \triangleright \rangle$, where $X = \{a, b, c\}$, $\Omega = \{X, \{a, b\}, \{a, c\}, \{b, c\}\}$, and each array specifies the respective choices. The rational choice functions are colored in red, their joins and meets are colored in green, and the additional ones–obtained as a join or meet of the previous ones–are colored in orange.

4.1 Identification of the primitive ordering

So far, we assumed that the analyst has specified the primitive ordering. We next focus on how to identify a primitive ordering that renders a choice overload representation to a choice model. Formally, an (primitive) ordering $>$ *renders a choice overload representation* to a choice model μ if $\mu \subset \mu^\theta(>)$. Thus, the observed choice functions can be interpreted as the sample choice behavior of a population whose choices comply with $\theta 1$ and $\theta 2$ according to the primitive ordering $>$.

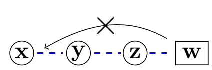
We first show how to infer from a given choice model that an alternative lies "between" two other alternatives according to every primitive ordering that renders a choice overload representation to the choice model. Let μ be a given choice model and $x, y, z \in X$ be a triple. Then, y is **revealed to be between** x and z–denoted by $y \ \mathcal{B}_\mu \ \{x, z\}$–if there exists a choice function $c \in \mu$ such that $c(S) = y$ and $c(S \setminus z) = x$ for some choice set S. We refer to $\mathcal{B}_\mu$ as the **betweenness relation** associated to the choice model μ. Then, we have $y \ \mathcal{B}_\mu \ \{x, z\}$ if and only if $x > y > z$ or $x < y < z$ for every primitive ordering $>$ that renders a choice overload representation to μ.

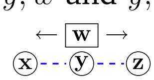
We next introduce conditions structuring the betweenness relation associated with a choice model. We show that these conditions are necessary and sufficient for the existence and uniqueness of a primitive ordering that renders a choice overload representation to the choice model. As a corollary, we conclude that the primitive ordering corresponding to the minimal extension of rational choice functions is identified unique up to its inverse. Let μ be a choice model and $\mathcal{B}_\mu$ be the associated betweenness relation.

$B1$. Each triple $x, y, z \in X$ appear in at most one $\mathcal{B}_\mu$-comparison.

$sB1$. Each triple $x, y, z \in X$ appear in exactly one $\mathcal{B}_\mu$-comparison.

For axioms $B2$ and $B3$, let $x, y, z, w \in X$ be distinct and $y \ \mathcal{B}_\mu \ \{x, z\}$.

$B2$. If $z \ \mathcal{B}_\mu \ \{x, w\}$, then it is not $w \ \mathcal{B}_\mu \ \{x, y\}$. 

$B3$. If x, y, w and y, z, w appear in $\mathcal{B}_\mu$-comparison, then $y \ \mathcal{B}_\mu \ \{x, w\}$ or $y \ \mathcal{B}_\mu \ \{z, w\}$ but not both. 

We can think of $B1$ as an "asymmetry" and $B2$ as an "3-acylicity" requirement for the betweenness relation. In the vein of *negative transitivity* (Kreps 1988), $B3$ requires that if y lies between x and z, then each w should lie either on the x- or z-side of y.

Finally, $sB1$ strengthes $B1$ by additionally requiring that for each triple x, y, z, we have $x \; \mathcal{B}_\mu \; \{y, z\}$, $y \; \mathcal{B}_\mu \; \{x, z\}$ or $z \; \mathcal{B}_\mu \; \{y, x\}$. Since these conditions are to be satisfied by a choice model, different choice functions may provide for different triples related according to the betweenness relation. Thus, we can interpret $sB1$ as a "richness" requirement for the choice model, which may be hard for a single choice function to satisfy.

Theorem 3. *Let μ be a choice model and $\mathcal{B}_\mu$ be the associated betweenness relation. Then,*

i. $\mathcal{B}_\mu$ satisfies $B1 - B3$ if and only if there exists a primitive ordering $>$ such that μ is contained in $\mu^\theta(>)$.

ii. $\mathcal{B}_\mu$ satisfies $sB1$ and $B3$ if and only if there exists a unique (up to its inverse) primitive ordering $>$ such that μ is contained in $\mu^\theta(>)$.

For the proof, it is critical to identify an ordering that **agrees with the betweeness relation**, in the sense that if $y \; \mathcal{B}_\mu \; \{x, z\}$ then $x > y > z$ or $x < y < z$ for each triple $x, y, z \in X$. Betweenness relation is a ternary one, interest in which stems from its use in axiomatizations of geometry. For example, Huntington & Kline (1917) proposed eleven different sets of axioms to characterize the usual betweenness on a real line; most of which can be translated to replace $sB1$ and $B3$. Our $sB1$ appears almost directly in these axiomatizations, whereas $B3$ is most similar to the axioms used in more succinct characterizations provided by Huntington (1924) and Fishburn (1971).[16]

To prove part i of Theorem 3, we have $B1$ instead of $sB1$. To fill this gap, we use a recent result by Biró, Lehel & Tóth (2023) who provide a unified view to existing results. They show that: "if there is an agreeing linear order for every subsytem on four points, then there is an agreeing linear order for the whole system."[17] To use their result, in

[16]See axiom 9 used by Huntington (1924) and axiom $A3$ used by Fishburn (1971).

[17]This result is in line with the fact that no axiom appeared in the aforementioned characterizations uses more than four elements.

Lemma 2, we show that $B1 - B3$ suffice for the existence of orderings that "locally" agree with our betweenness relation. We also use a characterization by Fishburn (1971) to prove Lemma 2 that does not readily follow from existing results.

Lemma 2. *Let μ be a choice model such that the associated betweenness relation $\mathcal{B}_\mu$ satisfies $B1 - B3$. Then, for each distinct $x, y, z, w \in X$, there exists an ordering $>_L$ such that for each triple $a, b, c \in \{x, y, z, w\}$, if $b \; \mathcal{B}_\mu \; \{a, c\}$ then $a >_L b >_L c$ or $a <_L b <_L c$.*

Proof. Please see Section 2 in the Appendix. □

Proof of Theorem 3. Part i: It directly follows from $\theta 1$ and $\theta 2$ that the if part holds. To prove the only if part, suppose that $\mathcal{B}_\mu$ satisfies $B1 - B3$. Then, it follows from our Lemma 2 and Theorem 1 by Biró, Lehel & Tóth (2023) that there exists an ordering $>$ over X such that for each triple $x, y, z \in X$, if $y \; \mathcal{B}_\mu \; \{x, z\}$ then $x > y > z$ or $x < y < z$. Thus, we conclude that μ is contained in $\mu^\theta(>)$.

Part ii: Since by $sB1$, each triple $x, y, z \in X$ appears in an $\mathcal{B}_\mu$-comparison, it follows from the proof of part i that there is an ordering $>$ over X such that for each triple $x, y, z \in X$, we have $y \; \mathcal{B}_\mu \; \{x, z\}$ if and only if $x > y > z$ or $x < y < z$. Then, by the only if part, μ is contained in $\mu^\theta(>)$. By the if part, $>$ and its inverse are the only such orderings. □

A choice model can comprise a single choice function as well as a collection of choice functions representing the revealed choice behavior of a population. To best of our knowledge, identifying primitives from a choice model in this way is novel. Pursuing this approach further, suppose that a choice model μ coincides with the minimal extension of rational choice functions with respect to a primitive ordering $>$, i.e. $\mu = \mu^\theta(>)$. Then, it follows from our Theorem 3 that we can identify the underlying primitive ordering unique up to its inverse.

Corollary 1. *Let μ be a choice model. Then, $\mu = \mu^{\theta}(>)$ and $\mu = \mu^{\theta}(>')$ if and only if $>'$ is the inverse of $>$.*

Proof. Suppose that $\mu = \mu^{\theta}(>)$ for some primitive ordering $>$. Then, we show that μ satisfies $sB1$. To see this, let $x, y, z \in X$ be a triple such that $x > y > z$. Consider the choice function c such that $c(\{x, y, z\}) = y$ and $c(\{x, y\}) = x$, and $c(S)$ is the $>$-best alternative in S for every other choice set S. Since c satisfies $\theta 1$ and $\theta 2$ according to $>$, we have $c \in \mu$ and $y \ \mathcal{B}_{\mu} \ \{x, z\}$. Thus, μ satisfies $sB1$ and the conclusion follows from part ii of Theorem 3. □

5 Universally self-progressive choice models

A natural question is whether there are choice models that yield unique orderly representations for any primitive orderings. We then define and examine this strong condition.

Definition. *A choice model μ is **universally self-progressive** if μ is self-progressive with respect to every partial order $\triangleright$ obtained from a set of primitive orderings $\{>_S\}_{S \in \Omega}$.*

To characterize the universally self-progressive choice models, we first offer a new perspective about choice functions. A choice function can be interpreted as a complete contingent plan to be implemented upon observing available alternatives.[18] Then, suppose that a population of agents evaluate choice functions via a common *value function*, which can be thought of as an indirect utility function associated with the problem of optimally adopting a choice function. The population is homogeneous in the sense that each agent evaluates choice functions via the same value function. The unique source of

[18]Here, a choice function is analogous to a "worldview" as described by Bernheim, Braghieri, Martínez-Marquina & Zuckerman (2021) who offer a dynamic model of endogenous preference formation.

heterogeneity is the maximizers' multiplicity. Then, the question is: What sort of choice heterogeneity allows for universal self-progressiveness? In Proposition 1, we show that additive separability of the value function over set contingent utilities is the answer. For each $S \in \Omega$ and $x \in S$, let $U(x, S)$ be the *set contingent utility* of choosing x. In addition to the intrinsic utility of alternative x that may be menu independent, $U(x, S)$ can accommodate the likelihood of S being available or the temptation cost due to choosing x in the presence of more tempting alternatives.[19]

Proposition 1. *A choice model μ is universally self-progressive if and only if for each $S \in \Omega$, there exist a set contingent utility function $U(\cdot, S)$ such that μ is the set of choice functions that maximize their sum, that is $\mu = \text{argmax}_{c\in\mathcal{C}} \sum_{S\in\Omega} U(c(S), S)$.*

Proof. If part: Note that if c^* is obtained as a "mixture" of some $c, c' \in \mu$ in the sense that $c^*(S) \in \{c(S), c'(S)\}$ for every $S \in \Omega$, then $c^* \in \mu$ as well. Since meet and join are special mixtures, $\langle \mu, \rhd \rangle$ is a lattice for any partial order $\rhd$ obtained from a set of primitive orderings. Then, it follows from Theorem 1 that μ is universally self-progressive.

Only if part: Let μ be a universally–self-progressive choice model. Then, we first show that μ is *convex*: for each $c_1, c_2 \in \mu$, if $c(S) \in \{c_1(S), c_2(S)\}$ for every $S \in \Omega$, then $c \in \mu$. By contradiction, suppose that $c \notin \mu$. Then, for each $S \in \Omega$, define the primitive ordering $>_S$ such that $c(S)$ is highest-ranked. Thus, we have $c = c_1 \vee c_2$, and $\langle \mu, \rhd \rangle$ is a not a lattice. By Theorem 1, this contradicts that μ is universally self-progressive. Next, we define the set contingent utilities for each $S \in \Omega$ such that $U(x, S) = 1$ if there exists $c \in \mu$ with $c(S) = x$, and $U(x, S) = 0$ otherwise. Since μ is convex, a choice function $c \in \mu$ if and only if $U(x, S) = 1$ for each $S \in \Omega$. It follows that μ is the set of choice functions that maximize $\sum_{S\in\Omega} U(c(S), S)$. □

[19]For example, in the vein of Gul & Pesendorfer (2001), one can set $U(x, S) = u(x)+v(x)-max_{z\in S}v(z)$, where u represents the *commitment* ranking and v represents the *temptation ranking*.

Proposition 1 shows how to modify a choice model for universal self-progression, while reflecting its demanding nature. To see this, consider a choice model μ consisting of two choice functions rationalized by maximizing preference relations $\succ_1$ and $\succ_2$. For fixed primitive orderings, we can extend μ into a self-progressive model by adding at most two choice functions. In contrast, to extend μ to be universally self-progressive we must add every choice function choosing the $\succ_1$- or $\succ_2$-maximal alternative in each choice set. More generally–assuming that the choice domain contains every choice set–to extend the rational choice model into a universally self-progressive one, we must add every choice function. In contrast, by Theorem 2, we know that the minimal self-progressive extension of the rational choice is a structured model. We finally present Example 4 demonstrating that Proposition 1 facilitates verifying if a choice model is universal self-progressive.

Example 4. *Kalai, Rubinstein & Spiegler (2002)* Let $\{\succ_k\}_{k=1}^{K}$ be a K-tuple of strict preference relations on X. A choice function $c \in \mu$ if for each $S \in \Omega$, the alternative $c(S)$ is the $\succ_k$-maximal one in S for some k. To see that μ is universally self-progressive, define $U(x, S) = 1$ if x is the $\succ_k$-maximal alternative in S for some k; and $U(x, S) = 0$ otherwise. It follows that μ is the set of choice functions that maximize $\sum_{S\in\Omega} U(c(S), S)$. To see that every universally self-progressive choice model is not representable in this way, let $U(x, S) = 1$ and $U(x, T) = 0$ for a pair of choice sets S and T with $x \in T \subset S$. Then, there is a strict preference relation $\succ_k$ such that x is the $\succ_k$-maximal alternative in S. Thus, we obtain $c \in \mu$ with $c(T) = x$, contradicting that c maximizes $\sum_{S\in\Omega} U(c(S), S)$.

6 Final comments

We have explored a new approach for analyzing heterogeneity in the aggregate choice behavior of a population. We take choice models as our primitive objects, thus leaving the details of the agents' choices unspecified. Our analysis proposes testable foundations of choice models that ensure each aggregate choice behavior, that follows the model, has a unique orderly representation.

We conjectured that self-progressive choice models could be economically valuable in terms of organizing random choice data. In this vein, we observe that they are in a one-to-one correspondence with lattices. It follows that self-progressive models are not limited to comparable choice functions, but allow for a generality that is significant. As an advantage of our model-free approach, we provide a guide for restricting or extending any choice model to be self-progressive. We characterize the minimal self-progressive extension of the rational choice model. The resulting model offers an experimentally supported explanation for choice overload phenomena, and allows for identification of the primitive ordering. It demonstrates that–in addition to their analytical properties–self-progressive choice models may be fruitful in crafting models explaining economically relevant phenomena.

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7 Appendix

7.1 Proof of Theorem 2

Since μ^θ is self-progressive, it follows from Theorem 1 that $\langle \mu^\theta, \rhd \rangle$ is a lattice such that there is no $\mu \subsetneq \mu^\theta$ that contains every rational choice function and $\langle \mu, \rhd \rangle$ is a lattice. Let μ^* be the choice model comprising choice functions that satisfy $\theta 1$ and $\theta 2$.

We first show that $\mu^\theta \subset \mu^*$. To see this, first note that each rational choice function $c \in \mu^*$, since for each $S \in \Omega$ and $x \in S$, rationality of c implies that $c(S) \neq c(S \setminus x)$ only if $x = c(S)$. Next, we show that $\langle \mu^*, \rhd \rangle$ is a lattice. Let $c^1, c^2 \in \mu^*$ and $c = c^1 \vee c^2$. Then, to see that c satisfies $\theta 1$ and $\theta 2$, assume w.l.o.g. that $c(S) = c^1(S)$. Now, if $c^1(S) > x$ then, since c^1 satisfies $\theta 1$, we have $c^1(S \setminus x) \geq c^1(S)$. It follows that $c(S \setminus x) \geq c(S)$. If $x > c^1(S)$, then $x > c^2(S)$. Since c^1 and c^2 satisfy $\theta 2$, we have $c(S) \geq c(S \setminus x)$. Thus, we conclude that $c^1 \vee c^2 \in \mu^*$. Symmetric arguments show that $c^1 \wedge c^2 \in \mu^*$ as well.

Next, we show that $\mu^* \subset \mu^\theta$. To see this, let $c \in \mu^*$. Since $\langle \mu^\theta, \rhd \rangle$ is a lattice, by Lemma 1, it suffices to show that for each $S, S' \in \Omega$, there exists $c^* \in \mu^\theta$ such that $c^*(S) = c(S)$ and $c^*(S') = c(S')$. Let $S, S' \in \Omega$ such that $c(S) = a$ and $c(S') = a'$. If $a = a'$, then $c(S)$ and $c(S')$ are obtained by maximizing a preference relation that top-ranks a. If $a \neq a'$, then assume w.l.o.g. that $a > a'$. Now, there are two cases.

Case 1: Suppose that $\{a, a'\} \not\subset S \cap S'$. Then, let c_1 be a choice function maximizing a preference relation that top-ranks first a then a', and c_2 be a choice function maximizing a preference relation that top-ranks first a' then a. Next, if $a \notin S'$ then let $c^* = c_1 \vee c_2$, if $a' \notin S$ then let $c^* = c_1 \wedge c_2$. For both cases, $c^*(S) = a$ and $c^*(S') = a'$, and $c^* \in \mu^\theta$ since $\langle \mu^\theta, \rhd \rangle$ is a lattice containing every rational choice function.

Case 2: Suppose that $\{a, a'\} \subset S \cap S'$. First, we show that either (i) there exists $x \in S \setminus S'$ with $x > a$ or (ii) there exists $y \in S' \setminus S$ with $a' > y$. If not, then consider $S \cap S'$. Suppose that we remove each $x \in S \setminus S'$ from S one-by-one. Since $c \in \mu^\theta$, by applying $\theta 1$ at each step, we get $c(S \cap S') \geq c(S)$. Similarly, suppose that we remove each $y \in S' \setminus S$ from S' one-by-one. Then, by applying $\theta 2$ at each step, we get $c(S') \geq c(S \cap S')$. Therefore, we must have $a' \geq a$, a contradiction. Thus, we conclude that (i) or (ii) holds.

Suppose that (i) holds. Then, let $c^* = c_1 \wedge c_2$, where c_1 maximizes a preference relation that top-ranks first x then a', and c_2 maximizes a preference relation that top-ranks a. Suppose that (ii) holds. Then, let $c^* = c_1 \vee c_2$, where c_1 maximizes a preference relation that top-ranks first y then a, and c_2 maximizes a preference relation that top-ranks a'. For both cases, $c^*(S) = a$ and $c^*(S') = a'$, and $c^* \in \mu^\theta$ since $\langle \mu^\theta, \rhd \rangle$ is a lattice such that μ^θ contains every rational choice function.

7.2 Proof of Lemma 2

If there no triple among $x, y, z, w \in X$ appear in $\mathcal{B}_\mu$, then let $>_L$ be any ordering of these alternatives. For what follows, assume w.l.o.g that $y \; \mathcal{B}_\mu \; \{x, z\}$. If no other triple appear in $\mathcal{B}_\mu$, then let $>_L$ be any ordering such that $x >_L y >_L z$. If neither x, y, w nor y, z, w appear in $\mathcal{B}_\mu$, then let $>_L$ be any ordering such that $x >_L y >_L z$ and w is ordered depending on how x, z, w appear in $\mathcal{B}_\mu$. It is easy to see that for these cases the selected $>_L$ agrees with $\mathcal{B}_\mu$.

Suppose that x, y, w and y, z, w appear in $\mathcal{B}_\mu$. Then, it follows from $B3$ that w lies either on the x- or z-side of y. If x, z, w fail to appear in $\mathcal{B}_\mu$, then we can choose $>_L$ such that $x >_L y >_L z$ and w is ordered depending on the side of y in which w is located.

If x, z, w appear in $\mathcal{B}_\mu$, then $\mathcal{B}_\mu$ satisfies $sB1$. Then, by Theorem 4 of Fishburn (1971), there is an ordering that agrees with $\mathcal{B}_\mu$.

Finally, suppose that only one of the triples x, y, w or y, z, w fail to appear in $\mathcal{B}_\mu$. Assume w.l.o.g. that it is y, z, w. If x, z, w also fail to appear in $\mathcal{B}_\mu$, then we can choose $>_L$ such that $x >_L y >_L z$ and w is ordered depending on how x, y, w appear in $\mathcal{B}_\mu$. If x, z, w appear in $\mathcal{B}_\mu$, then there are three cases that we will consider separately.

Case 1: Suppose that $z \ \mathcal{B}_\mu \ \{x, w\}$. Since we also have $y \ \mathcal{B}_\mu \ \{x, z\}$, we can construct an ordering $>_L$ that agrees with $\mathcal{B}_\mu$ only if $y \ \mathcal{B}_\mu \ \{x, w\}$. To see that $y \ \mathcal{B}_\mu \ \{x, w\}$, by contradiction suppose that $w \ \mathcal{B}_\mu \ \{x, y\}$ or $x \ \mathcal{B}_\mu \ \{w, y\}$. First, since $y \ \mathcal{B}_\mu \ \{x, z\}$ and $z \ \mathcal{B}_\mu \ \{x, w\}$, it directly follows from $B2$ that it is not $w \ \mathcal{B}_\mu \ \{x, y\}$. If $w \ \mathcal{B}_\mu \ \{x, y\}$, then since x, z, w and y, z, w appear in $\mathcal{B}_\mu$, it follows from $B3$ that $w \ \mathcal{B}_\mu \ \{x, z\}$ or $w \ \mathcal{B}_\mu \ \{y, z\}$ but not both. However, since we supposed $z \ \mathcal{B}_\mu \ \{x, w\}$, by $B1$, it is not $w \ \mathcal{B}_\mu \ \{x, z\}$. Since we supposed y, z, w fail to appear in $\mathcal{B}_\mu$ it is not $w \ \mathcal{B}_\mu \ \{x, z\}$ either.

Case 2: Suppose that $w \ \mathcal{B}_\mu \ \{x, z\}$. Since we also have $y \ \mathcal{B}_\mu \ \{x, z\}$, we can construct an ordering $>_L$ that agrees with $\mathcal{B}_\mu$ unless $x \ \mathcal{B}_\mu \ \{w, y\}$. To see that it is not $x \ \mathcal{B}_\mu \ \{w, y\}$, by contradiction, suppose that $x \ \mathcal{B}_\mu \ \{w, y\}$. Then, since x, y, z and x, w, z appear in $\mathcal{B}_\mu$, it follows from $B3$ that $x \ \mathcal{B}_\mu \ \{y, z\}$ or $x \ \mathcal{B}_\mu \ \{w, z\}$ but not both. But, by $B1$, this is not possible since we already have $y \ \mathcal{B}_\mu \ \{x, z\}$ and $x \ \mathcal{B}_\mu \ \{w, z\}$.

Case 3: Suppose that $x \ \mathcal{B}_\mu \ \{w, z\}$. Since $y \ \mathcal{B}_\mu \ \{x, z\}$, an ordering $>_L$ agrees with $\mathcal{B}_\mu$ only if $x \ \mathcal{B}_\mu \ \{w, y\}$. To see that $x \ \mathcal{B}_\mu \ \{w, y\}$, first notice x, y, z and x, y, w appear in $\mathcal{B}_\mu$. Then, since $x \ \mathcal{B}_\mu \ \{w, z\}$, it follows from $B3$ that $x \ \mathcal{B}_\mu \ \{y, z\}$ or $x \ \mathcal{B}_\mu \ \{w, y\}$ but not both. Since we already have $y \ \mathcal{B}_\mu \ \{x, z\}$, by $B1$, it is not $x \ \mathcal{B}_\mu \ \{y, z\}$, thus we must have $x \ \mathcal{B}_\mu \ \{w, y\}$.